FINITENESS AND DUALITY FOR THE COHOMOLOGY OF PRISMATIC CRYSTALS

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Abstract. Let (A, I) be a bounded prism, and X be a smooth p-adic formal scheme over $\mathrm{Spf}(A/I)$. We consider the notion of crystals on Bhatt–Scholze's prismatic site $(X/A)_{\Delta}$ of X relative to A. We prove that if X is proper over $\mathrm{Spf}(A/I)$ of relative dimension n, then the cohomology of a prismatic crystal is a perfect complex of A-modules with tor-amplitude in degrees $[0, 2n]$. We also establish a Poincaré duality for the reduced prismatic crystals, i.e. the crystals over the reduced structural sheaf of $(X/A)_{\Delta}$. The key ingredient is an explicit local description of reduced prismatic crystals in terms of Higgs modules.

0. Introduction

In a recent ground breaking work [6], Bhatt and Scholze introduced the prismatic site for p-adic formal schemes (with p a fixed prime), and they studied the cohomology of the natural structural sheaf on the prismatic site, called the prismatic cohomology. This new cohomology theory seems to occupy a central role in the study of cohomological properties of p-adic formal schemes, since it is naturally related to various previously known p-adic cohomology theories, and thus provides new insight on the p-adic comparison theorems with integral coefficients. For instance, it gives a natural site theoretic construction of the A_{inf}-cohomology which was previously constructed by Bhatt–Morrow–Scholze [4] in an ad-hoc way using the magical $L\eta$-functor.

As analogues of classical crystalline crystals, there is a natural notion of crystals on prismatic site. Indeed, such objects have already been considered in some special cases by many authors such as Anschütz–Le Bras [1], Gros–Le Strum–Quirós [8], Li [9] and Morrow–Tsuji [10]. In this article, we will consider the cohomology of prismatic crystals on rather general prismatic sites and prove some finiteness and duality theorems for such crystals.

Let us explain the main results of this article in more detail. Let (A, I) be a bounded prism, and X be a smooth p-adic formal scheme over A/I. We denote by $(X/A)_{\Delta}$ Bhatt–Scholze's prismatic site of X relative to A [6, Def. 4.1]. Roughly speaking, the category $(X/A)_{\Delta}$ consists of bounded prisms (B, J) over (A, I) together with a structural map $\mathrm{Spf}(B/J) \to X$, and coverings in $(X/A)_{\Delta}$ are (p, I)-completely faithfully flat maps of such bounded prisms. We have the structural sheaf $\mathcal{O}_{\Delta}$ (resp. the reduced structural sheaf $\overline{\mathcal{O}}_{\Delta}$) which sends each object (B, J) to B (resp. to B/J). An $\mathcal{O}_{\Delta}$-crystal (resp. an $\overline{\mathcal{O}}_{\Delta}$-crystal) is a (p, I)-completely flat and derived (p, I)-complete $\mathcal{O}_{\Delta}$-module (resp. $\overline{\mathcal{O}}_{\Delta}$-module) that satisfies similar properties as classical crystals on crystalline sites (cf. Def. 2.3). Here, we insist to impose the flatness condition in order to avoid some technical difficulties in faithfully flat descent for derived (I, p)-complete modules (cf. Prop. 1.8).

The first main result of this article is a finiteness theorem for $\mathcal{O}_{\Delta}$-crystals (Theorem 2.9), which claims that, if X is proper and smooth of relative dimension n over A/I and $\mathcal{F}$ is an

$\mathcal{O}_{\Delta}$-crystal locally free of finite rank, then $R\Gamma((X/A)_{\Delta}, \mathcal{F})$ is a perfect complex of A-modules with tor-amplitude in degree $[0, 2n]$. Moreover, the formation of $R\Gamma((X/A)_{\Delta}, \mathcal{F})$ commutes with arbitrary base change in A. This finiteness theorem for $\mathcal{O}_{\Delta}$-crystals is a consequence of a similar result for $\overline{\mathcal{O}}_{\Delta}$-crystals. Actually, if $\mathcal{E}$ is an $\overline{\mathcal{O}}_{\Delta}$-crystal, we will show in Theorem 2.8 that the derived push-forward of $\mathcal{E}$ to the étale topos of X is a perfect complex of $\mathcal{O}_X$-modules with tor-amplitude in $[0, n]$.

We are thus reduced to the study of $\overline{\mathcal{O}}_{\Delta}$-crystals. Since the problem is local for the étale topology of X, we may assume that $X = \mathrm{Spf}(R)$ is affine such that R is p-completely étale over the convergent power series ring $A/I\langle T_1, \dots, T_n\rangle$. In this case, we can give a rather explicit description of $\mathcal{O}_{\Delta}$-crystals in terms of Higgs modules. More precisely, after choosing a smooth lift $\widetilde{R}$ over A of R together with a δ-structure on $\widetilde{R}$ compatible with that on A, we can show that there exists an equivalence between the category of $\overline{\mathcal{O}}_{\Delta}$-crystals and that of topologically quasi-nilpotent Higgs modules over R (cf. Theorem 4.10); furthermore, the cohomology of an $\overline{\mathcal{O}}_{\Delta}$-crystal is computed by the de Rham complex of its associated Higgs module (cf. Theorem 4.12). From this description, our finiteness theorem for $\overline{\mathcal{O}}_{\Delta}$-crystals follows easily.

As another application of the local description of $\overline{\mathcal{O}}_{\Delta}$-crystals, we establish also in Theorem 5.3 a Poincaré duality for the cohomology of $\overline{\mathcal{O}}_{\Delta}$-crystals. This can be viewed as a combination of the duality for de Rham complexes of Higgs modules and the classical Grothendieck–Serre duality. If one can construct a trace map for the prismatic cohomology of proper and smooth formal schemes, our results imply also a Poincaré duality for $\mathcal{O}_{\Delta}$-crystals as well (cf. Remark 5.4).

The organization of this article is as follows. In Section 1, we prove some preliminary results in commutative algebra. The main result of this section is a descent result on derived I-complete and I-completely flat modules (cf. Prop. 1.8). In Section 2, we discuss the notion of prismatic crystals and state the main finiteness theorems. Section 3 is devoted to the local study of prismatic crystals. When $X = \mathrm{Spf}(R)$ is affine equipped with a lifting $\widetilde{R}$ over A together with a δ-structure, we show that the category of prismatic crystals is equivalent to that of modules over $\widetilde{R}$ equipped with a certain stratification (cf. Prop. 3.7). In Section 4, we work in the local situation of affine formal schemes equipped with étale local coordinates, and we related $\overline{\mathcal{O}}_{\Delta}$-crystals to Higgs modules as mentioned above. Then we finish the proof of finiteness theorems at the end of Section 4. Finally, in Section 5, we prove a Poincaré duality for $\overline{\mathcal{O}}_{\Delta}$.

It should be pointed out that Frobenius structures and filtrations on prismatic crystals are ignored in this article, even though these aspects should play important roles in many applications.

The results of this article was announced in a conference in honor of Luc Illusie in June 2021. At this conference, I learned that similar results in this article were also obtained independently by Ogus for crystalline prisms and Bhatt–Lurie for absolute prismatic crystals.

0.1. **Notation.** Let A be a commutative ring. We denote by $\mathbf{Mod}(A)$ the abelian category of A-modules, and by $D(A)$ the derived category of $\mathbf{Mod}(A)$. If M is an A-module and $f \in A$ is an element, we denote by $M[f]$ the kernel of the multiplication by f on M. We put also $M[f^\infty] = \bigcup_{n \geqslant 1} M[f^n]$.

Let $J \subset A$ be a PD-ideal. For $x \in J$ and an integer $n \geqslant 1$, we denote by $x^{[n]}$ the n-th divided power of x.

0.2. **Sign Conventions.** We will use the following conventions on the signs of complexes:

- For a naïve double complex $K^{\bullet,\bullet}$, its associated simplex $\underline{s}(K^{\bullet,\bullet})$ is defined by putting the sign $(-1)^i$ on the differentials $K^{i,j} \to K^{i,j+1}$.
- For a complex K and an integer $n \in \mathbb{Z}$, the i-th differential of the shift $K[n]$ is obtained by multiplying $(-1)^n$ with the $(i+n)$-th differential of K.
- For two complexes K and L, their tensor product $K \otimes L$ is the simplex complex attached to the naïve double complex with (i,j)-component given by $K^i \otimes L^j$ (whenever such a tensor product is well defined). The hom-complex $\mathrm{Hom}(K,L)$ is the simplex complex attached to the naïve double complex with (i,j)-component given by $\mathrm{Hom}(K^{-j}, L^i)$ and with the sign $(-1)^j$ on the canonical map $\mathrm{Hom}(K^{-j}, L^i) \to \mathrm{Hom}(K^{-j-1}, L^i)$.

1. Preliminaries in commutative algebra

We recall first some general facts on derived completion, for which the main reference is [11, Tag 091N].

We consider a pair (A, J), where A is a commutative ring, and $J \subset A$ is an ideal. A complex K of A-modules is called derived J-complete, if $R\,\mathrm{Hom}(A_f, K) = 0$ for all $f \in J$, where $A_f = A[\frac{1}{f}]$. Then K is derived J-complete if and only if so are $H^q(K)$ for all $q \in \mathbb{Z}$. The derived J-complete A-modules form an abelian full subcategory of $\mathbf{Mod}(A)$ that is stable under kernels, cokernels, images and extensions. Any classically J-adically complete A-module is derived J-complete, but the converse is not necessarily true. Let let $D_{comp}(A)$ be the full subcategory consisting of $D(A)$ consisting of derived J-complete objects.

Assume from now on that J is finitely generated. Then the natural inclusion functor $D_{comp}(A) \to D(A)$ admits a left adjoint $K \mapsto \widehat{K}$, called the *derived J-completion*. An explicit construction of $\widehat{K}$ is given as follows. Write $J = (f_1, \ldots, f_r)$. Let $\mathrm{Kos}(A; f_1, \ldots, f_r)$ be the homological Koszul complex sitting in degrees $[-r, 0]$:

$$\wedge^r A^{\oplus r} \to \wedge^{r-1} A^{\oplus r-1} \to \cdots \to A^{\oplus r} \xrightarrow{(f_1,\ldots,f_r)} A.$$

For an integer $n \geqslant 1$, we have a transition map of complexes

$$\mathrm{Kos}(A; f_1^{n+1}, \ldots, f_r^{n+1}) \to \mathrm{Kos}(A; f_1^n, \ldots, f_r^n)$$

given by the multiplication by $f_{i_1} \cdots f_{i_m}$ on a basis element

$$e_{i_1} \wedge \cdots \wedge e_{i_m} \in \mathrm{Kos}^{-m}(A; f_1^{n+1}, \ldots, f_r^{n+1}) = \wedge^m A^{\oplus r}.$$

Then for an object $K \in D(A)$, we have

$$\widehat{K} = R\lim_n \big(K \otimes_A^L \mathrm{Kos}(A; f_1^n, \ldots, f_r^n)\big). \tag{1.0.1}$$

Note that, in general, the canonical map $\widehat{K} \to R\lim_n (K \otimes_A^L A/J^n)$ is not an isomorphism in $D(A)$. However, we will see later (Prop. 1.4) that this indeed holds in an important special case.

Lemma 1.1. *For an object K of $D(A)$ and an integer $n \geqslant 1$, the canonical maps*

$$K \otimes^L \mathrm{Kos}(A; f_1^n, \dots, f_r^n) \xrightarrow{\sim} \widehat{K} \otimes^L \mathrm{Kos}(A; f_1^n, \dots, f_r^n),$$
$$K \otimes_A^L A/J^n \xrightarrow{\sim} \widehat{K} \otimes_A^L A/J^n$$

are isomorphisms.

Proof. Since the cohomology groups of $\mathrm{Kos}(A; f_1^n, \dots, f_r^n)$ are annihilated by $(f_1^n, \cdots, f_r^n)$ and hence by J^n, it follows from [11, Tag 091W] that $K \otimes_A^L \mathrm{Kos}(A; f_1^n, \dots, f_r^n)$ is already derived J-complete. Hence, we have

$$K \otimes_A^L \mathrm{Kos}(A; f_1^n, \dots, f_r^n) = (K \otimes_A^L \mathrm{Kos}(A; f_1^n, \dots, f_r^n))^\wedge \simeq \widehat{K} \otimes_A^L \mathrm{Kos}(A; f_1^n, \dots, f_r^n).$$

The Lemma follows by further base change via the canonical map $\mathrm{Kos}(A; f_1^n, \dots, f_r^n) \to A/J^n$. $\square$

A complex of A-modules K is called *J-completely flat* (resp. *J-completely faithfully flat*) if $K \otimes_A^L A/J$ is concentrated in degree 0 and is a flat (resp. faithfully flat) A/J-module. An object $K \in D(A)$ is called *J-completely locally free* if it is J-completely flat and $H^0(K \otimes_A^L A/J)$ is a locally free A/J-module. It is clear that J-complete flatness (or J-complete locally freeness) is stable under (derived) base change, and by Lemma 1.1 an object $K \in D(A)$ is J-completely flat if and only if so is $\widehat{K}$.

Lemma 1.2. *Let $A \to B$ be a J-completely faithfully flat map of rings.*

(1) *An object K of $D(A)$ is J-completely flat (or J-completely locally free) if and only if so is $K_B := (K \otimes_A^L B)^\wedge$.*
(2) *Let $\phi : M \to N$ be a morphism in $D(A)$ with M, N derived J-complete. Then ϕ is a quasi-isomorphism if and only if so is $\phi_B : M_B \to N_B$.*

Proof. (1) By Lemma 1.1, we have

$$K_B \otimes_B^L B/JB = K \otimes_A^L B/JB = (K \otimes_A^L A/J) \otimes_{A/J}^L B/JB. \tag{1.2.1}$$

Statement (1) follows immediately from the usual fpqc-descent of modules via the faithfully flat map $A/J \to B/JB$.

(2) Pick a distinguished triangle $M \xrightarrow{\phi} N \to K \to M[1]$ in $D(A)$, from which we deduce a distinguished triangle in $D(B)$: $M_B \xrightarrow{\phi_B} N_B \to K_B \to M_B[1]$. We need to prove that $K = 0$ if and only if $K_B = 0$. In view of (1.2.1) and the faithful flatness of B/JB over A/J, one has $K \otimes_A^L A/J = 0$ if and only if $K_B \otimes_B^L B/JB = 0$. Then we conclude by the derived Nakayama Lemma [11, Tag 0G1U]. $\square$

The following definition is motivated by the notion of bounded prism [6, Def. 3.2].

Definition 1.3. We say that a pair (A, J) is of *reduced prismatic type* if $J = (f)$ for some $f \in A$, A is derived J-complete and has bounded f^∞-torsion, i.e. there exists an integer $c \geqslant 0$ such that $A[f^c] = A[f^\infty]$.

We say that a pair (A, J) is of *prismatic type* if $J = I + fA$ and A is derived J-complete, where $I \subseteq A$ is a finitely generated ideal which is locally generated by a nonzero divisor and A/I has bounded f^∞-torsion.

Proposition 1.4. *Let (A, J) ba a pair of (reduced) prismatic type.*

(1) *For an object $K \in D(A)$, the canonical map*

$$\widehat{K} \xrightarrow{\sim} R\lim_n (K \otimes_A^L A/J^n)$$

deduced from the universal property of $\widehat{K}$ is an isomorphism.

(2) *If M is J-completely (faithfully) flat complex of A-modules, then $\widehat{M}$ is concentrated in degree 0 and a classically J-adically complete A-module such that $\widehat{M}/J^n\widehat{M}$ is (faithfully) flat over A/J^n.*

(3) *Conversely, if N is a classically J-adically complete A-module such that N/J^nN is (faithfully) flat over A/J^n, then N is an J-completely (faithfully) flat and derived J-complete object in $D(A)$.*

Proof. For (1), we will treat only the case when (A, J) is a pair of prismatic type, the arguments for the case of reduced prismatic type being similar and much simpler. By Lemma 1.2, it suffices to show the Lemma after flat localization of A. Hence, we may assume that $J = (\xi, f)$ such that $\xi \in A$ is a nonzero divisor and $A/\xi A$ has bounded f^∞-torsion.

The arguments follows from the same lines as [6, Lemma 3.7(1)]. By (1.0.1) and a simple cofinality argument, we have

$$\widehat{K} = R\lim_n R\lim_m (K \otimes_A^L \mathrm{Kos}(A; \xi^n, f^m)).$$

Since ξ is a nonzero divisor, we have a quasi-isomorphism

$$\mathrm{Kos}(A; \xi^n, f^m) \simeq \mathrm{Kos}(A/\xi^n; f^m).$$

By dévissage, the assumption that $A/\xi A$ has bounded f^∞-torsion implies that so does $A/\xi^n A$, which in turn implies that the pro-object $\{\mathrm{Kos}(A/\xi^n A; f^m) : m \geqslant 1\}$ is isomorphic to $\{A/(\xi^n, f^m) : m \geqslant 1\}$ by [11, Tag 091X]. Hence, we get

$$\begin{aligned}
\widehat{K} &= R\lim_n R\lim_m (K \otimes_A^L \mathrm{Kos}(A; \xi^n, f^m)) \\
&\simeq R\lim_n R\lim_m (K \otimes_A^L \mathrm{Kos}(A/\xi^n; f^m)) \\
&\simeq R\lim_n R\lim_m (K \otimes_A^L A/(\xi^n, f^m)) \\
&\simeq R\lim_n (K \otimes_A^L A/J^n),
\end{aligned}$$

where the last step follows from a cofinality argument.

Statement (2) follows from the same argument as [6, Lemma 3.7(2)].

For (3), we consider first the case when (A, J) is a pair of reduced prismatic type. We write thus $J = (f)$ with $f \in A$ such that A has bounded f^∞-torsion. By the same proof as [5, Lemma 4.7] (with p replaced by f), it suffices to show that N has bounded f^∞-torsion. By assumption, there exists an integer $c \geqslant 1$ such that $A[f^\infty] = A[f^c]$. Put $A' = A/A[f^c]$, which is f-torsion free. Put $N' = N \otimes_A A'$. Then by the exact sequence

$$A[f^c] \otimes_A N \to N \to N' \to 0,$$

it is enough to show that N' is f-torsion free. For any integer $n \geqslant 1$, let $A'_n = A'/f^nA'$ and $N'_n = N'/f^nN'$. By the f-torsion freeness of A', we have an exact sequence

$$0 \to A'_{n-1} \xrightarrow{\times f} A'_n \to A'_1 \to 0.$$

Note that the flatness of N/f^nN over A/f^nA implies the flatness of N'_n over A'_n. Tensoring with N'_n, we get

$$0 \to N'_{n-1} \xrightarrow{\times f} N'_n \to N'_1 \to 0.$$

Now let $x \in N'$ with $fx = 0$. Then the above exact sequence implies that $x \in f^{n-1}N'$ for any integer $n \geqslant 1$. But N' is f-adically complete, we get thus $x = 0$. This finishes the proof of (3) in the case of reduced prismatic type.

Assume now that (A, J) is of prismatic type. Up to flat localization, we may assume that $J = (\xi, f)$ such that A is ξ-torsion free and $A/\xi A$ has bounded f^∞-torsion. We prove first that N is ξ-torsion free. Let $c \geqslant 0$ be an integer such that $(A/\xi A)[f^\infty] = (A/\xi A)[f^c]$. For any integer $n \geqslant 1$, we put $A_n = A/(f^n, \xi^n)$ and $N_n = N \otimes_A A_n$. Let $\bar{x} \in A_n[\xi]$ with a lift $x \in A$. Then one has $\xi x = \xi^n y + f^n z$ for some $y, z \in A$. By our choice of c, there exists $z_1 \in A$ such that $f^c z = \xi z_1$. Hence, if $n \geqslant c$, we get $\xi x = \xi^n y + f^{n-c}\xi z_1$ and hence $x = \xi^{n-1}y + f^{n-c}z_1$ since ξ is a nonzero divisor. Therefore, we have $A_n[\xi] \subset \xi^{n-1}A_n + f^{n-c}A_n$ for all $n \geqslant c$. By assumption, N_n is flat over A_n. It follows that

$$N_n[\xi] = A_n[\xi] \otimes_{A_n} N_n = A_n[\xi] \otimes_A N.$$

The previous discussion on $A_n[\xi]$ then implies that $N_n[\xi] \subset \xi^{n-1}N_n + f^{n-c}N_n$ and hence $N[\xi] \subset \xi^{n-1}N + f^{n-c}N$ for all $n \geqslant c$. But as N is J-adically complete, it follows that

$$N[\xi] \subset \bigcap_{n \geqslant c} \left(\xi^{n-1}N + f^{n-c}N\right) = 0.$$

This proves the claim.

Now we can finish the proof of (3). The claim implies that $N \otimes^L_A A/\xi A = N/\xi N$, and hence that

$$N \otimes^L_A A/J = (N \otimes^L_A A/\xi A) \otimes^L_{A/\xi A} A/J = (N/\xi N) \otimes^L_{A/\xi A} A/J.$$

Then we conclude by applying the case of reduced prismatic type to $N/\xi N$. $\square$

Let $\mathbf{Mod}^\wedge_J(A)$ denote the subcategory of $\mathbf{Mod}(A)$ consisting of derived J-complete A-modules. Let M, N be objects of $\mathbf{Mod}^\wedge_J(A)$. Then $\mathrm{Hom}_A(M, N)$ is also derived J-complete by [11, Tag 0A6E]. We put

$$M \widehat{\otimes}_A N := H^0((M \otimes^L_A N)^\wedge). \tag{1.4.1}$$

Note that the functor $M \mapsto M\widehat{\otimes}_A N$ is right exact (cf. [11, Tag 0AAJ]) and we have (cf. [7, Appendix])

$$\mathrm{Hom}_A(L\widehat{\otimes}_A M, N) = \mathrm{Hom}_A(L, \mathrm{Hom}_A(M, N))$$

for all objects $L, M, N \in \mathbf{Mod}^\wedge_J(A)$. There is a canonical surjection from $M\widehat{\otimes}_A N$ to the classical J-adic completion of $M \otimes_A N$, which is in general not an isomorphism. Let $\mathbf{FMod}^\wedge_J(A)$ be the full subcategory of $\mathbf{Mod}^\wedge_J(A)$ consisting of J-completely flat A-modules.

Let $\phi : A \to B$ be a derived J-complete A-algebra. We denote by

$$\phi^* : \mathbf{Mod}^\wedge_J(A) \to \mathbf{Mod}^\wedge_{JB}(B) \tag{1.4.2}$$

the base change functor $M \mapsto M\widehat{\otimes}_A B$. Then ϕ^* is the left adjoint of the functor of restriction of scalars $\mathbf{Mod}^\wedge_{JB}(B) \to \mathbf{Mod}^\wedge_J(A)$.

Lemma 1.5. *Let $\phi : A \to B$ and $\psi : B \to C$ be morphisms of derived J-complete A-algebras. Then for any $M \in \mathbf{Mod}_J^\wedge(A)$, there is a canonical isomorphism*

$$\psi^*\phi^*(M) \simeq (\psi \circ \phi)^*(M).$$

Proof. Consider the distinguished triangle

$$\tau_{\leqslant -1}(M\otimes_A^L B)^\wedge \to (M\otimes_A^L B)^\wedge \to \phi^*(M) = H^0((M\otimes_A^L B)^\wedge) \to .$$

Taking derived tensor with C and derived J-completion, one gets

$$\big(\tau_{\leqslant -1}(M\otimes_A^L B)^\wedge \otimes_B^L C\big)^\wedge \to \big((M\otimes_A^L B)^\wedge \otimes_B^L C\big)^\wedge \to (\phi^*(M) \otimes_B^L C)^\wedge \to .$$

If we write $J = (f_1, \dots, f_r)$, then one has

$$\begin{aligned}\big((M\otimes_A^L B)^\wedge \otimes_B^L C\big)^\wedge &= R\lim_n \big((M \otimes_A^L B)^\wedge \otimes_B^L \mathrm{Kos}(C; f_1^n, \cdots, f_r^n)\big) \\ &\simeq R\lim_n \big((M\otimes_A^L \mathrm{Kos}(B; f_1^n, \dots, f_r^n) \otimes_B^L C\big) \\ &\simeq R\lim_n \big(M\otimes_A^L \mathrm{Kos}(C; f_1^n, \dots, f_r^n)\big) \\ &= (M \otimes_A^L C)^\wedge,\end{aligned}$$

where the second isomorphism uses Lemma 1.1. One deduces then

$$H^0\big(\tau_{\leqslant -1}(M\otimes_A^L B)^\wedge \otimes_B^L C\big)^\wedge \to (\psi \circ \phi)^*(M) \to \psi^*\phi^*(M) \to H^1\big(\tau_{\leqslant -1}(M\otimes_A^L B)^\wedge\big)$$

Since $\big(\tau_{\leqslant -1}(M\otimes_A^L B)^\wedge \otimes_B^L C\big)^\wedge$ is concentrated in degree $\leqslant -1$ by [11, Tag 0AAJ], the Lemma follows immediately. ∎

Lemma 1.6. *Let (A, J) be a pair of prismatic type (resp. of reduced prismatic type), and $\phi : A \to B$ be an A-algebra such that (B, JB) is also of prismatic type (resp. of reduced prismatic type).*

1. *If $M, N \in \mathbf{FMod}_J^\wedge(A)$, then $M\widehat{\otimes}_A N$ is J-completely flat and it coincides with the classical J-adic completion of $M \otimes_A N$.*
2. *The base change functor ϕ^* sends $\mathbf{FMod}_J^\wedge(A)$ to $\mathbf{FMod}_{JB}^\wedge(B)$.*

Proof. (1) By Proposition 1.4, we have

$$(M\otimes_A^L N)^\wedge = R\lim_n (M \otimes_A^L N \otimes_A^L A/J^n) = \lim_n (M \otimes_A N)/J^n(M \otimes_A N),$$

where the last equality uses the J-complete flatness of $M \otimes_A^L N$. The Lemma follows immediately.

(2) Let M be an object of $\mathbf{FMod}_J^\wedge(A)$. Then $(M \otimes_A^L B)^\wedge$ is JB-completely flat. By Proposition 1.4(2), we have $M\widehat{\otimes}_A B = (M \otimes_A^L B)^\wedge$. ∎

Let $B^{\otimes \bullet}$ be the Čech nerve of $\phi : A \to B$ in the category of J-completely flat A-algebras, i.e. $B^{\otimes\bullet}$ is the cosimplicial object of J-completely flat A-algebras with its n-component for $n \geqslant 0$ given by

$$B^{\otimes n} := \underbrace{B\widehat{\otimes}_A \cdots \widehat{\otimes}_A B}_{(n+1)\text{-fold}}.$$

In particular, for any integer i with $0 \leqslant i \leqslant n$, one has a map of A-algebras $\delta_i^n : B^{\otimes(n-1)} \to B^{\otimes n}$ given by

$$\delta_i^n : \quad b_0 \otimes \cdots \otimes b_{n-1} \mapsto b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{n-1}.$$

We have thus a diagram of morphisms of derived I-complete A-algebras:

$$B \underset{\delta_1^1}{\overset{\delta_0^1}{\rightrightarrows}} B\widehat{\otimes}_A B \overset{\delta_0^2}{\underset{\delta_2^2}{\overset{\delta_1^2}{\rightrightarrows\!\!\!\!\to}}} B\widehat{\otimes}_A B\widehat{\otimes}_A B,$$

Definition 1.7. A *descent pair relative to* $\phi : A \to B$ consists of

- a derived JB-complete B-module M,
- and an isomorphism of $B\widehat{\otimes}_A B$-modules

$$\varepsilon : \delta_0^{1,*}(M) = M\widehat{\otimes}_A B \xrightarrow{\sim} \delta_1^{1,*}(M) = B\widehat{\otimes}_A M,$$

called a descent datum on M, such that the usual cocyle condition

$$\delta_1^{2,*}(\varepsilon) = \delta_2^{2,*}(\varepsilon) \circ \delta_0^{2,*}(\varepsilon)$$

is statisfied.

A morphism of descent pairs $f : (M_1, \varepsilon_1) \to (M_2, \varepsilon_2)$ is a morphism of B-modules $f : M_1 \to M_2$ such that $\varepsilon_2 \circ \delta_1^{1,*}(f) = \delta_0^{1,*}(f) \circ \varepsilon_1$. We denote by $\mathbf{Desc}^{\wedge}_{B/A}$ the category of J-complete descent pair relative to ϕ.

For an object $N \in \mathbf{Mod}^{\wedge}_J(A)$, there is a canonical isomorphism

$$\varepsilon_N : \delta_0^{1,*}(N\widehat{\otimes}_A B) \simeq N\widehat{\otimes}_A (B\widehat{\otimes}_A B) \simeq \delta_1^{1,*}(N\widehat{\otimes}_A B)$$

which clearly satisfies the axiom for a descent pair. Thus we get thus a functor

$$\phi^{\natural} : \mathbf{Mod}^{\wedge}_J(A) \to \mathbf{Desc}^{\wedge}_{B/A}, \quad N \mapsto (N\widehat{\otimes}_A B, \varepsilon_N).$$

A descent pair relative to ϕ is called effective if it lies in the essential image of $\phi^{\natural}$.

Proposition 1.8. *Let (A, J) be a pair of (reduced) prismatic type, and $\phi : A \to B$ be a J-completely faithfully flat derived J-complete A-algebra.*

(1) *Let $M \in \mathbf{FMod}^{\wedge}_J(A)$, and $\underline{s}(M\widehat{\otimes}_A B^{\otimes\bullet})$ denote the complex associated to cosimplicial object $M\widehat{\otimes}_A B^{\otimes\bullet}$. Then the canonical map*

$$M \xrightarrow{\sim} \underline{s}(M\widehat{\otimes}_A B^{\otimes\bullet})$$

is a quasi-isomorphism. In particular, one has an exact sequence of sets

$$M \longrightarrow M\widehat{\otimes}_A B \rightrightarrows M\widehat{\otimes}_A B\widehat{\otimes}_A B.$$

(2) *Let $\mathbf{FDesc}^{\wedge}_{B/A}$ be the subcategory of $\mathbf{Desc}^{\wedge}_{B/A}$ consisting of objects (M, θ) with M an JB-completely flat B-module. Then the functor $\phi^{\natural}$ induces an equivalence of categories*

$$\mathbf{FMod}^{\wedge}_J(A) \xrightarrow{\sim} \mathbf{FDesc}^{\wedge}_{B/A}.$$

Proof. As usual, we treat here only the case of prismatic type, and that of reduced prismatic is similar and much simpler.

For (1), it suffices to prove the statement for $M = A$, the general case being obtained by applying $M \otimes^L_A \_$ and taking derived J-completion. By Lemma 1.2(2), it suffices to prove that $B \xrightarrow{\sim} \big(\underline{s}(B^{\otimes\bullet}) \otimes^L_A B\big)^{\wedge}$ is a quasi-isomorphism. But note that

$$\big(\underline{s}(B^{\otimes\bullet}) \otimes^L_A B\big)^{\wedge} \simeq \underline{s}(B^{\otimes\bullet}\widehat{\otimes}_A B),$$

where $B^{\otimes\bullet}\widehat{\otimes}_A B$ is nothing but the Čech nerve of $\delta_1^1 : B \to B\widehat{\otimes}_A B$, which admits a right inverse given by the multiplication map $B\widehat{\otimes}_A B \to B$. It follows from [11, Tag 019Z] that $B^{\otimes\bullet}\widehat{\otimes}_A B$ is homotopy equivalent to the constant cosimplicial object B. Hence the canonical map $B \xrightarrow{\sim} \underline{s}(B^{\otimes\bullet}\widehat{\otimes}_A B)$ is also a homotopy equivalence of complexes.

For (2), the full faithfulness of the restriction of $\phi^\natural$ to $\mathbf{FMod}_J^\wedge(A)$ follows easily from (1). It remains to prove the essential surjectivity. Let (M,ε) be an object of $\mathbf{FDesc}^\wedge_{B/A}$. For any integer $n \geqslant 1$, let $M_n := M/J^nB$ and

$$\varepsilon_n : \delta_0^{1,*}(M_n) = M_n \otimes_{A/J^n} B/J^nB \xrightarrow{\sim} \delta_1^{1,*}(M_n) = B/J^nB \otimes_{A/J^n} M_n$$

be the reduction modulo $J^n(B\widehat{\otimes}_A B)$ of ε. Then (M_n, ε_n) is a classical descent datum relative to $A/J^n \to B/J^nB$. By the classical fpqc-descent, the descent datum (M_n, θ_n) comes from a flat A/J^n-module N_n. It is clear that $N_{n+1} \otimes_{A/J^{n+1}} A/J^n \simeq N_n$. We put $N = \varprojlim_n N_n$. By Proposition 1.4(3), N is J-completely flat over A, and hence $(B\widehat{\otimes}^L_A N)^\wedge$ is JB-completely flat. By Proposition 1.4(2), we have

$$(N\widehat{\otimes}_A B) \otimes_B B/J^nB = (N\otimes^L_A B)^\wedge \otimes^L_B B/I^nB \simeq N_n \otimes_{A/J^n} B/J^nB \simeq M_n.$$

Passing to the limit, we get thus an isomorphism of descent data $N\widehat{\otimes}_A B \simeq M$, and θ coincides with the canonical isomorphism ε_N.

□

2. Prismatic Crystals and Finiteness Theorems

In this section, we fix a prime number p. All the rings are supposed to be $\mathbb{Z}_{(p)}$-algebras.

2.1. **Prismatic site.** Let (A, I) be a bounded prism in the sense of [6, Def. 3.2], and X be a p-adic formal scheme over $\mathrm{Spf}(A/I)$. We recall first the prismatic site of X relative to A, denoted by $(X/A)_{\mathbb{\Delta}}$, introduced in [6, Def. 4.1]:

- The underlying category of $(X/A)_{\mathbb{\Delta}}$ is the opposite of bounded prisms (B, IB) over (A, I) together with a map of p-adic formal schemes $\mathrm{Spf}(B/IB) \to X$ over A/I; the notion of morphism in $(X/A)_{\mathbb{\Delta}}$ is the obvious one. We shall often denote such an object by
$$(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X);$$
or by $(B \to B/IB \leftarrow R)$ if $X = \mathrm{Spf}(R)$ is affine.
- A map
$$(\mathrm{Spf}(C) \leftarrow \mathrm{Spf}(C/IC) \to X) \to (\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$$
in $(X/A)_{\mathbb{\Delta}}$ is a flat cover if the underlying map of δ-A-algebras $B \to C$ is (p, I)-completely faithfully flat.

Following Grothendieck, we denote by $(X/A)^\sim_{\mathbb{\Delta}}$ the associated topos, and there is a canonical embedding $(X/A)_{\mathbb{\Delta}} \to (X/A)^\sim_{\mathbb{\Delta}}$.

Remark 2.2. Let $f : V := (\mathrm{Spf}(C) \leftarrow \mathrm{Spf}(C/IC) \to X) \to U := (\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$ and $g : W := (\mathrm{Spf}(D) \leftarrow \mathrm{Spf}(D/ID) \to X) \to U$ be two morphisms in $(X/A)_{\mathbb{\Delta}}$. Assume that one of f and g, say g, is (p, I)-completely flat (i.e. the underlying map of δ-A-algebras

$B \to D$ is (p, I)-completely flat). Then $E := (C \otimes_B^L D)^\wedge$ concentrated in degree 0 and it is a (p, I)-completely flat δ-C-algebra; moreover, there exists a canonical map

$$\mathrm{Spf}(E/IE) = \mathrm{Spf}(C/IC) \times_{\mathrm{Spf}(B/IB)} \mathrm{Spf}(D/ID) \to \mathrm{Spf}(B/IB) \to X.$$

Then the object $(\mathrm{Spf}(E) \leftarrow \mathrm{Spf}(E/IE) \to X)$ gives the fibre product $V \times_U W$.

We denote by $\mathcal{O}_{(X/A)_{\Delta}}$ (resp. $\overline{\mathcal{O}}_{\Delta} = \overline{\mathcal{O}}_{(X/A)_{\Delta}}$) the structural sheaf (resp. the reduced structural sheaf) on $(X/A)_{\Delta}$ which sends an object (B, J) in $(X/A)_{\Delta}$ to B (resp. to B/J). If no confusion arises, we will simply write $\mathcal{O}_{\Delta}$ for $\mathcal{O}_{(X/A)_{\Delta}}$ and $\overline{\mathcal{O}}_{\Delta}$ for $\overline{\mathcal{O}}_{(X/A)_{\Delta}}$.

Definition 2.3. An $\mathcal{O}_{\Delta}$-*crystal* (resp. an $\overline{\mathcal{O}}_{\Delta}$-crystal) on $(X/A)_{\Delta}$ is a sheaf of $\mathcal{O}_{\Delta}$-modules $\mathcal{F}$ on $(X/A)_{\Delta}$ such that

- for each object $(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$ of $(X/A)_{\Delta}$, the evaluation
$$\mathcal{F}_B := \mathcal{F}(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$$
is a derived (p, I)-complete and (p, I)-*completely flat* B-module (resp. a derived p-complete and p-completely flat B/I-module),
- for any morphism
$$(\mathrm{Spf}(C) \leftarrow \mathrm{Spf}(C/IC) \to X) \xrightarrow{f} (\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X),$$
in $(X/A)_{\Delta}$ the canonical linearized transition map
$$c_f(\mathcal{F}) : f^*(\mathcal{F}_B) := \mathcal{F}_B \widehat{\otimes}_B C \to \mathcal{F}_C$$
(resp. $c_{\bar{f}}(\mathcal{F}) : \bar{f}^*(\mathcal{F}_B) := \mathcal{F}_B \widehat{\otimes}_{B/IB} C/IC \to \mathcal{F}_C$) is an isomorphism, where $\widehat{\otimes}$ is the completed tensor product for the ideal (p, I) (resp. for the ideal (p)) defined in (1.4.1).

  An $\mathcal{O}_{\Delta}$-crystal (resp. an $\overline{\mathcal{O}}_{\Delta}$-crystal) $\mathcal{F}$ is called *locally free of finite rank* if $\mathcal{F}_B$ is a locally free B-module (resp. B/IB-module) of finite rank for each object $(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$.

We denote by $\mathbf{CR}((X/A)_{\Delta}, \mathcal{O}_{\Delta})$ (resp. $\mathbf{CR}((X/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta})$) the category of $\mathcal{O}_{\Delta}$-crystals (resp. $\overline{\mathcal{O}}_{\Delta}$-crystals).

Lemma 2.4. *The functor $\mathcal{F} \mapsto (\{\mathcal{F}_B\}, \{c_f(\mathcal{F})\})$ induces an equivalence of $\mathbf{CR}((X/A)_{\Delta}, \mathcal{O}_{\Delta})$ and the category of the data $(\{M_B\}, \{c_f\})$, where*

- *$\{M_B\}$ is the collection of derived (p, I)-complete and I-completely flat B-modules M_B corresponding to each object $(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$ of $(X/A)_{\Delta}$;*
- *$\{c_f\}$ is the collection of isomorphisms of C-modules*
$$c_f : C \widehat{\otimes}_B M_B \xrightarrow{\sim} M_C$$
for each morphism $f : (\mathrm{Spf}(C) \leftarrow \mathrm{Spf}(C/IC) \to X) \to (\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$ in $(X/A)_{\Delta}$ such that
  - *c_f is the identity map if f is the identity of an object of $(X/A)_{\Delta}$;*
  - *the cocycle condition*
$$c_{f \circ g} = c_g \circ g^*(c_f)$$
is satsified for a composition of morphisms.

Proof. It is sufficient to construct a quasi-inverse to the evaluation functor $\mathcal{F} \mapsto (\{\mathcal{F}_B\}, \{c_f(\mathcal{F})\})$. Given a datum $(\{M_B\}, \{c_f\})$ as above, the cocycle condition guarantees that

$$\mathcal{M} : (\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X) \mapsto M_B$$

together with transition map

$$\mathcal{M}_f : M_B \to f^*(M_B) = M_B \widehat{\otimes}_B C \xrightarrow{c_f} M_C$$

for each morphism $f : (\mathrm{Spf}(C) \leftarrow \mathrm{Spf}(C/IC) \to X) \to (\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$ in $(X/A)_{\mathbb{\Delta}}$ indeed defines a presheaf on $(X/A)_{\mathbb{\Delta}}$. Then Proposition 1.8(1) implies that $\mathcal{M}$ is indeed a sheaf. The fact that $\mathcal{M}$ is an $\mathcal{O}_{\mathbb{\Delta}}$-crystal follows immediately from the conditions on $\{c_f\}$. $\square$

Remark 2.5. There is an obvious analogue of this Lemma for $\overline{\mathcal{O}}_{\mathbb{\Delta}}$-crystals by requiring $c_{\bar{f}} : C/IC \widehat{\otimes}_{B/IB} M_B \to M_C$ to be an isomorphism for each morphism f as above.

Example 2.6. Assume that (A, I) is a perfect prism, and $X = \mathrm{Spf}(R)$ with R semiregular quasi-syntomic and $A/I \to R$ is surjective. Then the prismatic cohomology $\mathbb{\Delta}_{R/A} := R\Gamma((X/A)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$ is concentrated in degree 0 and $(\mathbb{\Delta}_{R/A}, I\mathbb{\Delta}_{R/A})$ becomes the final object in $(X/A)_{\mathbb{\Delta}}$. Therefore, $\mathbf{CR}((X/A)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$ is equivalent to the category of derived I-complete and completely I-flat $\mathbb{\Delta}_{R/A}$-modules.

2.7. **Functoriality.** Let $(A, I) \to (A', I')$ be a morphism of bounded prisms. Let

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \mathrm{Spf}(A'/I') \arrow[r] & \mathrm{Spf}(A/I) \end{tikzcd}$$

be a commutative diagram of p-adic formal schemes. Then f induces a morphism of sites

$$f_\sharp : (Y/A')_{\mathbb{\Delta}} \to (X/A)_{\mathbb{\Delta}}$$

given by

$$(\mathrm{Spf}(B') \leftarrow \mathrm{Spf}(B'/J') \to Y) \mapsto (\mathrm{Spf}(B') \leftarrow \mathrm{Spf}(B'/J') \to Y \xrightarrow{f} X).$$

It is easy to see that $f_\sharp$ is cocontinuous, hence it induces a morphism of topoi:

$$f_{\mathbb{\Delta}} = (f_{\mathbb{\Delta}}^{-1}, f_{\mathbb{\Delta},*}) : (Y/A')^{\sim}_{\mathbb{\Delta}} \to (X/A)^{\sim}_{\mathbb{\Delta}}.$$

For a sheaf $\mathcal{F}$ on $(X/A)_{\mathbb{\Delta}}$, its inverse image is given by

$$(f_{\mathbb{\Delta}}^{-1}\mathcal{F})(B', J') = \mathcal{F}(f_\sharp(B', J'))$$

For a sheaf $\mathcal{E}$ on $(Y/A')_{\mathbb{\Delta}}$, its direct image $f_{\mathbb{\Delta},*}(\mathcal{E})$ is described as follows. For an object (B, J) in $(X/A)_{\mathbb{\Delta}}$, let $Y_B = Y \times_X \mathrm{Spf}(B/J)$ and $B' = B \widehat{\otimes}_A A'$. Assume that (B', IB') is a bounded prism (which is the case if either A' or B is (p, I)-completely flat over A.) Then the direct image of a sheaf $\mathcal{E}$ on $(Y/A')_{\mathbb{\Delta}}$ is given by

$$f_{\mathbb{\Delta},*}(\mathcal{E})(B, J) = \Gamma((Y_B/B')_{\mathbb{\Delta}}, \mathcal{E}|_{(Y_B/B')_{\mathbb{\Delta}}})$$

where $\mathcal{E}|_{(Y_B/B')_{\mathbb{\Delta}}}$ is the pullback of $\mathcal{E}$ to $(Y_B/B')_{\mathbb{\Delta}}$.

For an $\mathcal{O}_{(X/A)_{\Delta}}$-module $\mathcal{F}$, we put

$$f_{\Delta}^{*}\mathcal{F} := f_{\Delta}^{-1}\mathcal{F}\widehat{\otimes}_{f_{\Delta}^{-1}\mathcal{O}_{(X/A)_{\Delta}}}\mathcal{O}_{(Y/A')_{\Delta}}.$$

Then the functor f_{Δ}^{*} induces the pullback map for prismatic crystals:

$$f_{\Delta}^{*}: \mathbf{CR}((X/A)_{\Delta}, \mathcal{O}_{(X/A)_{\Delta}}) \to \mathbf{CR}((Y/A')_{\Delta}, \mathcal{O}_{(Y/A')_{\Delta}}).$$

Similarly, the formula

$$\bar{f}_{\Delta}^{*}: \overline{\mathcal{F}} \mapsto f^{-1}\overline{\mathcal{F}} \otimes_{f_{\Delta}^{-1}\overline{\mathcal{O}}_{(X/A)_{\Delta}}} \overline{\mathcal{O}}_{(Y/A')_{\Delta}}$$

defines a pullback functor

$$\bar{f}_{\Delta}^{*}: \mathbf{CR}((X/A)_{\Delta}, \overline{\mathcal{O}}_{(X/A)_{\Delta}}) \to \mathbf{CR}((Y/A')_{\Delta}, \overline{\mathcal{O}}_{(Y/A')_{\Delta}}).$$

We can now state the main results of this article. Let $X_{\text{ét}}$ be the big étale site of X consisting of all p-adic formal schemes over X. Let $\nu_{X/A} : (X/A)_{\Delta}^{\sim} \to X_{\text{ét}}^{\sim}$ be the canonical projection. For any sheaf $\mathcal{F}$ on $(X/A)_{\Delta}$ and any object U of $X_{\text{ét}}$, we have

$$(\nu_{X/A,*}\mathcal{F})(U) = \Gamma((U/A)_{\Delta}, \mathcal{F}_U),$$

where $\mathcal{F}_U$ is the inverse image under the natural map of topos $(U/A)_{\Delta}^{\sim} \to (X/A)_{\Delta}^{\sim}$.

Theorem 2.8. *Let X be a smooth p-adic formal scheme over A/I of relative dimension n. Let $\mathcal{E}$ be an $\overline{\mathcal{O}}_{\Delta}$-crystal locally free of finite rank. Then the following statements hold:*

(1) *$R\nu_{X/A,*}(\mathcal{E})$ is a perfect complex of $\mathcal{O}_X$-modules with tor-amplitude in $[0, n]$.*
(2) *Let $(A, I) \to (A', I')$ be a morphism of p-torison free bounded prisms. Consider the cartesian diagram*

(2.8.1)
$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \mathrm{Spf}(A'/I') \arrow[r] & \mathrm{Spf}(A/I). \end{tikzcd}$$

The canonical base change map

$$f^{-1}R\nu_{X/A,*} * (\mathcal{E}) \otimes^{L}_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'} \xrightarrow{\sim} R\nu_{X'/A',*}(\bar{f}_{\Delta}^{*}\mathcal{E})$$

is an isomorphism.

The proof of this Theorem will be given in Section 5 after some local preparations. For the moment, one can deduce immediately from Theorem 2.8 the following finiteness result on the cohomology of an $\mathcal{O}_{\Delta}$-crystal.

Theorem 2.9. *Let X be a proper and smooth p-adic formal scheme over* $\mathrm{Spf}(A/I)$ *of relative dimension n. Let $\mathcal{F}$ be an $\mathcal{O}_{\Delta}$-crystal locally free of finite rank on $(X/A)_{\Delta}$. Then $R\Gamma((X/A)_{\Delta}, \mathcal{F})$ is a perfect complex of A-modules with tor-amplitude in $[0, 2n]$. Moreover, if $(A, I) \to (A', I')$ is a morphism of p-torsion free bounded prisms that induces the cartesian diagram* (2.8.1), *then the canonical base change map*

$$R\Gamma((X/A)_{\Delta}, \mathcal{F}) \otimes^{L}_{A} A' \xrightarrow{\sim} R\Gamma((X'/A')_{\Delta}, f_{\Delta}^{*}\mathcal{F})$$

is an isomorphism.

Proof. Applying Theorem 2.8(1) to $\overline{\mathcal{F}} := \mathcal{F}/I\mathcal{F}$, we see that $R\nu_{X/A,*}(\overline{\mathcal{F}})$ is a perfect complex of $\mathcal{O}_X$-modules with perfect amplitude in $[0,n]$. Since X is assumed to be proper and smooth, it follows that

$$R\Gamma((X/A)_{\Delta}, \mathcal{F}) \otimes^L_A A/I \xrightarrow{\sim} R\Gamma((X/A)_{\Delta}, \overline{\mathcal{F}}) \cong R\Gamma(X_{\text{ét}}, R\nu_{X/A,*}(\overline{\mathcal{F}}))$$

is a perfect complex of A/I-modules with perfect amplitude $[0,2n]$. We then conclude by [11, Tag 07LU] that $R\Gamma((X/A)_{\Delta}, \mathcal{F})$ is a perfect complex with tor-amplitude in $[0,2n]$.

For the second part of the Theorem, according to the derived Nakayama Lemma [11, Tag 0G1U], it suffices to show that

$$R\Gamma((X/A)_{\Delta}, \mathcal{F}) \otimes^L_A A'/I' \xrightarrow{\sim} R\Gamma((X'/A')_{\Delta}, f^*_{\Delta}\mathcal{F}) \otimes^L_{A'} A'/I'$$

is an isomorphism. But this follows from the following sequence of canonical isomorphisms:

$$\begin{aligned} R\Gamma((X/A)_{\Delta}, \overline{\mathcal{F}}) \otimes^L_{A/I} A'/I' &\cong R\Gamma(X_{\text{ét}}, R\nu_{X/A,*}(\overline{\mathcal{F}})) \otimes^L_{A/I} A'/I' \\ &\cong R\Gamma(X'_{\text{ét}}, f^{-1}R\nu_{X/A,*}(\overline{\mathcal{F}}) \otimes^L_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'}) \\ &\cong R\Gamma(X'_{\text{ét}}, R\nu_{X'/A',*}(\bar{f}^*_{\Delta}\overline{\mathcal{F}})) \\ &\cong R\Gamma((X'/A')_{\Delta}, \bar{f}^*_{\Delta}\mathcal{F}). \end{aligned}$$

where the second isomorphism is the projection formula for coherent cohomology, and the third one is Theorem 2.8(2). □

3. Local description of prismatic crystals

In this section, we fix a bounded prism (A, I). Let $X = \mathrm{Spf}(R)$ be an affine smooth p-adic formal scheme over $\mathrm{Spf}(A/I)$ of relative dimension $n \geqslant 0$. We will make the following assumption:

Assumption 3.1. The A/I-algebra R admits a lift to a derived (p,I)-complete δ-A-algebra $\widetilde{R}$ that is formally smooth over A.

We fix such a lift $\widetilde{R}$, and let $\widetilde{X} := (\widetilde{R} \to \widetilde{R}/I\widetilde{R} \cong R)$ denote the resulting object in $(X/A)_{\Delta}$.

Lemma 3.2. *Under Assumption 3.1, for any object $(B \to B/IB \leftarrow R)$ in $(X/A)_{\Delta}$, the product of $(B \to B/IB \leftarrow R)$ and $\widetilde{X}$ in $(X/A)_{\Delta}$ exists. Moreover, if we denote this product by $(\widetilde{B} \to \widetilde{B}/I\widetilde{B} \leftarrow R)$, then $\widetilde{B}$ is (p,I)-completely faithfully flat. In particular, $\widetilde{X}$ is a cover of the final object of $(X/A)^{\sim}_{\Delta}$.*

Proof. Let $C := (B \otimes^L_A \widetilde{R})^\wedge$ be the derived (p,I)-completion of $B \otimes^L_A \widetilde{R}$. Then C is (p,I)-completely faithfully flat over B. Hence by Prop. 1.4(2) it is concentrated in degree 0 and coincides with the classical (p,I)-adic completion of $B \otimes_A \widetilde{R}$. By [6, Lemma 2.17], there is a unique δ-structure on C compatible with the natural product δ-structure on $B \otimes_A \widetilde{R}$. Consider the surjection $C \to B/IB \otimes_{A/I} R \to B/IB$, and denote its kernel by J. As R is formally smooth over A/I of relative dimension n, J is locally generated by I and a regular sequence of length n relative to B. Applying [6, Lemma 3.13], we get the prismatic envelope $\widetilde{B} := C\{\frac{J}{I}\}^\wedge$ which is (p,I)-completely faithfully flat over B, and commutes with base change in (B, IB). The fact that $(\widetilde{B} \twoheadrightarrow \widetilde{B}/I\widetilde{B} \leftarrow R)$ is the product of $(B \twoheadrightarrow B/IB \leftarrow R)$ and $\widetilde{X}$ in $(X/A)_{\Delta}$ follows easily from the universal property of the prismatic envelope.

The second part of the Lemma follows immediately from the following general fact: if $\mathcal{C}$ is a topos, an object $U \in \mathcal{C}$ is a cover of the final object of $\mathcal{C}$ if and only if for any object $V \in \mathcal{C}$ there exists a cover $W \to V$ such that W admits a morphism to U. $\square$

3.3. **Simplicial object.** For each integer $m \geqslant 0$, let $\widetilde{X}(m)$ be the $(m+1)$-fold self-product of the object $\widetilde{X}$ in $(X/A)_{\Delta}$. By the proof of Lemma 3.2, $\widetilde{X}(m)$ is explicitly given by as follows. Let $\widetilde{J}(m)$ be the kernel of the canonical surjection

$$\widetilde{R}^{\otimes(m+1)} := \underbrace{\widetilde{R}\widehat{\otimes}_A \dots \widehat{\otimes}_A \widetilde{R}}_{(m+1) \text{ copies}} \to R.$$

Then $\widetilde{J}(m)$ is generated by I together with a regular sequence since R is formally smooth over A/I. We denote by

$$\widetilde{R}(m) := \widetilde{R}^{\otimes(m+1)}\{\frac{\widetilde{J}(m)}{I}\}^{\wedge}$$

the prismatic envelope by the construction of [6, Lemma 3.13]. Then we have $\widetilde{X}(m) = (\widetilde{R}(m) \to \widetilde{R}(m)/I\widetilde{R} \leftarrow R)$. The $\widetilde{R}(m)$'s form naturally a cosimplicial δ-A-algebra. For an integer i with $0 \leqslant i \leqslant m$, let

$$\delta_i^m : \widetilde{R}(m-1) \to \widetilde{R}(m) \tag{3.3.1}$$

denote the map of δ-A-algebras corresponding to the strictly increasing map of simplexes $[m-1] := \{0, 1, \dots, m-1\} \to [m]$ that jumps at i, i.e. δ_i^m is induced by the map $\widetilde{R}^{\otimes m} \to \widetilde{R}^{\otimes(m+1)}$ given by

$$b_0 \otimes \cdots \otimes b_{m-1} \mapsto b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{m-1}.$$

In particular, we have a diagram of δ-A-algebras:

$$\begin{tikzcd} \widetilde{R} \arrow[r, shift left, "\delta_0^1"] \arrow[r, shift right, "\delta_1^1"'] & \widetilde{R}(1) \arrow[r, shift left=2, "\delta_0^2"] \arrow[r, "\delta_1^2" description] \arrow[r, shift right=2, "\delta_2^2"'] & \widetilde{R}(2). \end{tikzcd}$$

Definition 3.4. Let M be a derived (p, I)-complete and (p, I)-completely flat $\widetilde{R}$-module. A stratification of M over $\widetilde{R}(1)$ is an isomorphism of $\widetilde{R}(1)$-modules:

$$\epsilon : \delta_0^{1,*}(M) \xrightarrow{\sim} \delta_1^{1,*}(M)$$

where $\delta_i^{1,*}$ is the base change functor (1.4.2), such that the cocycle condition is satisfied:

$$\delta_1^{2,*}(\epsilon) = \delta_2^{2,*}(\epsilon) \circ \delta_0^{2,*}(\epsilon).$$

We denote by $\mathbf{Strat}(\widetilde{R}(1))$ the category of derived (p, I)-complete and (p, I)-completely flat $\widetilde{R}$-modules together with a stratification over $\widetilde{R}(1)$.

Remark 3.5. Let $\mu : \widetilde{R}(1) \to \widetilde{R}$ be the map induced by the multiplication $\widetilde{R}\widehat{\otimes}_A\widetilde{R} \to \widetilde{R}$ given by $a \otimes b \mapsto ab$. The definition of stratification in [8, Def. 3.9] requires further that

$\mu^*(\epsilon) = \mathrm{id}_M$. In fact, this is automatic. Indeed, by applying the functor $(\mathrm{id} \otimes \mu)^*$ to the cocycle condition $\delta_1^{2,*}(\epsilon) = \delta_2^{2,*}(\epsilon) \circ \delta_0^{2,*}(\epsilon)$, one gets

$$\epsilon = \epsilon \circ \delta_0^{1,*}(\mu^*(\epsilon))$$

and hence $\delta_0^{1,*}(\mu^*(\epsilon)) = \mathrm{id}_{\delta_0^{1,*}(M)}$. As $\delta_0^1 : \tilde{R} \to \tilde{R}(1)$ is (p, I)-completely faithfully flat, one deduces from Prop. 1.8 that $\mu^*(\epsilon) = \mathrm{id}_M$.

We have also a variant of Def. 3.4 and Prop. 3.7 for $\overline{\mathcal{O}}_{\Delta}$-crystals. In general, if f is morphism of objects over A (e.g. A-modules, A-algebras or A-formal schemes, ...), we denote by $\bar{f} : B/IB \to C/IC$ its reduction modulo I.

Definition 3.6. Let M be a derived p-complete and p-completely flat R-module. A stratification of M over $\widetilde{R}(1)/I\widetilde{R}(1)$ is an isomorphism of $\widetilde{R}(1)/I\widetilde{R}(1)$-modules

$$\epsilon : \bar{\delta}_0^{1,*}(M) \xrightarrow{\sim} \bar{\delta}_1^{1,*}(M)$$

such that $\bar{\delta}_1^{2,*}(\epsilon) = \bar{\delta}_2^{2,*}(\epsilon) \circ \bar{\delta}_0^{2,*}(\epsilon)$.

We denote by $\mathbf{Strat}(\widetilde{R}/I\widetilde{R}(1))$ the category of derived p-complete and p-completely flat R-modules together with a stratification over $\widetilde{R}(1)/I\widetilde{R}(1)$.

Let $\mathcal{F}$ be an $\mathcal{O}_{\Delta}$-crystal on $(X/A)_{\Delta}$, and $\mathcal{F}(\widetilde{X})$ be its value on $\widetilde{X}$. The crystal condition gives rise to a canonical isomorphism

$$\epsilon_{\mathcal{F}} : \delta_0^{1,*}(\mathcal{F}(\widetilde{X})) \xrightarrow{c_{\delta_0^1}(\mathcal{F})} \mathcal{F}(\widetilde{X}(1)) \xrightarrow{c_{\delta_1^1}(\mathcal{F})^{-1}} \delta_1^{1,*}\mathcal{F}(\widetilde{X})$$

which makes $(\mathcal{F}(\widetilde{X}), \epsilon_{\mathcal{F}})$ an object of $\mathbf{Strat}(\widetilde{R}(1))$. We get thus an evaluation functor

$$\mathrm{ev}_{\widetilde{X}} \colon \mathbf{CR}((X/A)_{\Delta}, \mathcal{O}_{\Delta}) \to \mathbf{Strat}(\widetilde{R}(1))$$

sending $\mathcal{F}$ to $(\mathcal{F}(\widetilde{X}), \epsilon_{\mathcal{F}})$. Similarly, we have also an evaluation functor

$$\overline{\mathrm{ev}}_{\widetilde{X}} \colon \mathbf{CR}((X/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta}) \to \mathbf{Strat}(\widetilde{R}(1)/I\widetilde{R}(1))$$

for $\overline{\mathcal{O}}_{\Delta}$-crystals.

Proposition 3.7. *Under the notation above, the functors* $\mathrm{ev}_{\widetilde{X}}$ *and* $\overline{\mathrm{ev}}_{\widetilde{X}}$ *are both equivalences of categories.*

Proof. We will prove only the statement for $\mathrm{ev}_{\widetilde{X}}$, and the case for $\overline{\mathrm{ev}}_{\widetilde{X}}$ is similar. We shall construct a functor quasi-inverse to $\mathrm{ev}_{\widetilde{X}}$. Let (M, ϵ) be an object of $\mathbf{Strat}(\widetilde{R}(1))$. We need to associate an $\mathcal{O}_{\Delta}$-crystal M_{Δ} to (M, ϵ).

Let $(B \to B/IB \leftarrow R)$ be an object in $(X/A)_{\Delta}$, and $(\widetilde{B} \to \widetilde{B}/I\tilde{B} \leftarrow R)$ be the product of $(B \to B/IB \leftarrow R)$ and $\widetilde{X}$ given by Lemma 3.2. By the universal property of prismatic envelopes, the canonical map of bounded prisms $p_B : (\widetilde{R}, I\widetilde{R}) \to (\widetilde{B}, I\widetilde{B})$ induces a commutative diagram of 2-truncated cosimplicial δ-A-algebras:

$$\begin{array}{ccccc}
\widetilde{R} & \rightrightarrows & \widetilde{R}(1) & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & \widetilde{R}(2) \\
\downarrow{\scriptstyle p_B} & & \downarrow{\scriptstyle p_{B(1)}} & & \downarrow{\scriptstyle p_{B(2)}} \\
\widetilde{B} & \rightrightarrows & \widetilde{B} \otimes_B \widetilde{B} & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & \widetilde{B} \otimes_B \widetilde{B} \otimes_B \widetilde{B}.
\end{array}$$

Applying the functor p_B^* to (M,ϵ), one gets a descent pair (Def. 1.7) $(p_B^*M, p_{B(1)}^*\epsilon)$ relative to the (p,I)-completely faithfully flat map $B\to\tilde{B}$ such that p_B^*M is (p,I)-completely flat over $\widetilde{B}$. By Prop. 1.8(2), there exists a derived (p,IB)-complete and (p,IB)-completely flat B-module M_B such that $M_B\widehat{\otimes}_B\widetilde{B}\simeq p_B^*(M)$.

Let $f:(C\to C/IC\leftarrow R)\to(B\to B/IB\leftarrow R)$ be a morphism in $(X/A)_{\mathbb{\Delta}}$, and $(\widetilde{C}\to\widetilde{C}/I\widetilde{C}\leftarrow R)$ be the product of $(C\to C/IC\leftarrow R)$ with $\widetilde{X}$. We need to check that there exists a transition isomorphism

$$c_f: f^*(M_B)=M_B\widehat{\otimes}_BC\xrightarrow{\sim}M_C$$

satisfying the natural cocyle condition for a composition of morphisms as in Lemma 2.4. Denote by $\tilde{f}:(\widetilde{B},I\widetilde{B})\to(\widetilde{C},I\widetilde{C})$ the map of bounded prisms induced by f. Then one has $p_C=\tilde{f}\circ p_B$. By functoriality, one has a composition of isomorphisms

$$\tilde{c}_f\colon M_B\widehat{\otimes}_BC\widehat{\otimes}_C\widetilde{C}=M_B\widehat{\otimes}_B\widetilde{B}\widehat{\otimes}_{\widetilde{B}}\widetilde{C}\simeq\tilde{f}^*p_B^*(M)=p_C^*(M)\simeq M_C\widehat{\otimes}_C\widetilde{C};$$

moreover, the pullbacks of $\tilde{c}_f$ via the two canonical maps $\widetilde{C}\rightrightarrows\widetilde{C}\widehat{\otimes}_C\widetilde{C}$ coincide. Then the desired transition isomorphism c_f is obtained by descent. Given a composition of morphism

$$f\circ g:(D\to D/ID\leftarrow R)\xrightarrow{g}(C\to C/IC\leftarrow R)\xrightarrow{f}(B\to B/IB\leftarrow R)$$

in $(X/A)_{\mathbb{\Delta}}$, one has the cocycle condition $c_{f\circ g}=c_g\circ g^*(c_f)$. Indeed, this can be easily checked after base change to $\widetilde{D}$ by functoriality, and we conclude by faithfully flat descent (Prop. 1.8).

Now by Lemma 2.4, the data $(\{M_B\},\{c_f\})$ is equivalent to an $\mathcal{O}_{\mathbb{\Delta}}$-crystal $M_{\mathbb{\Delta}}$ on $(X/A)_{\mathbb{\Delta}}$. The construction $(M,\epsilon)\mapsto M_{\mathbb{\Delta}}$ is clearly functorial, which gives a functor that is easily checked to be a quasi-inverse of $\mathrm{ev}_{\widetilde{X}}$.

□

Remark 3.8. When (A,I) is a bounded prism over $(\mathbb{Z}_q[[q-1]],([p]_q))$ (with $\delta(q)=0$ and $[p]_q=\frac{q^p-1}{q-1}$), Prop. 3.7 was obtained by [8, Cor. 6.7] and [10, Chap. 3]. An analogue for q-crystalline crystals was proved in [7, Thm. 1.3.3].

We can use the simplicial object $\widetilde{X}(\bullet)$ to compute the cohomology of an $\mathcal{O}_{\mathbb{\Delta}}$-crystals or an $\overline{\mathcal{O}}_{\mathbb{\Delta}}$-crystal.

Proposition 3.9 (Čech–Alexandre complex)**.** *Let $\mathcal{F}$ be an $\mathcal{O}_{\mathbb{\Delta}}$-crystal or an $\overline{\mathcal{O}}_{\mathbb{\Delta}}$-crystal on $(X/A)_{\mathbb{\Delta}}$. Then under Assumption 3.1, $R\Gamma((X/A)_{\mathbb{\Delta}},\mathcal{F})$ is computed by the Čech–Alexandre complex $\Gamma(\widetilde{X}(\bullet),\mathcal{F})$, which is in turn isomorphic to*

$$\mathcal{F}(\widetilde{X})\to\mathcal{F}(\widetilde{X})\widehat{\otimes}_{\widetilde{R}}\widetilde{R}(1)\to\mathcal{F}(\widetilde{X})\widehat{\otimes}_{\widetilde{R}}\widetilde{R}(2)\to\cdots. \tag{3.9.1}$$

Here, we consider $\widetilde{R}(m)$ as a $\widetilde{R}$-algebra via the map corresponding to any morphism of simplexes $[0]\to[m]$.

Remark 3.10. Note that if $\mathcal{F}$ is an $\overline{\mathcal{O}}_{\mathbb{\Delta}}$-crystal, the Čech–Alexandre complex (3.9.1) is isomorphic to

$$\mathcal{F}(\tilde{X})\to\mathcal{F}(\tilde{X})\widehat{\otimes}_{\widetilde{R}/I\widetilde{R}}\widetilde{R}(1)/\widetilde{R}(1)\to\mathcal{F}(\tilde{X})\widehat{\otimes}_{\widetilde{R}/I\widetilde{R}}\widetilde{R}(2)/I\widetilde{R}(2)\to\cdots$$

For the proof of this proposition, we need the following

Lemma 3.11. *Let $\mathcal{F}$ be an $\mathcal{O}_{\Delta}$-crystal or $\overline{\mathcal{O}}_{\Delta}$-crystal on $(X/A)_{\Delta}$. Then for any object $U = (B \to B/IB \leftarrow R)$ of $(X/A)_{\Delta}$ and any integer $q > 0$, we have*

$$H^q(U, \mathcal{F}) = 0.$$

Proof. We will only treat the case when $\mathcal{F}$ is an $\mathcal{O}_{\Delta}$-crystal, the case of $\overline{\mathcal{O}}_{\Delta}$-crystal being similar. Let $V = (C \to C/IC \leftarrow R)$ be a cover of U in $(X/A)_{\Delta}$. Then by the crystal property of $\mathcal{F}$, the Čech complex of $\mathcal{F}$ for the cover $V \to U$ is identified with the complex associated to the cosimplicial B-module $\mathcal{F}(U)\widehat{\otimes}_B C^{\otimes \bullet}$. It follows from Prop. 1.8(1) that its higher cohomology groups vanish. Then we conclude by [2, Exposé V, Cor. 4.3] that $H^q(U, \mathcal{F}) = 0$ for all $q > 0$. $\square$

Proof of Prop. 3.9. Since $\widetilde{X}$ is a cover of the final object of the topos $(X/A)^{\sim}_{\Delta}$ (Lemma 3.2), we have a spectral sequence

$$E_1^{i,j} = H^j(\widetilde{X}(i), \mathcal{F}) \Longrightarrow H^{i+j}((X/A)_{\Delta}, \mathcal{F}).$$

By Lemma 3.11, we have $H^j(\widetilde{X}(i), \mathcal{F}) = 0$ for $j > 0$. Hence $R\Gamma((X/A)_{\Delta}, \mathcal{F})$ is computed by the Čech-Alexandre complex $\Gamma(\widetilde{X}(i), \mathcal{F})$, which is isomorphic to (3.9.1) by the crystal property of $\mathcal{F}$. $\square$

Corollary 3.12 (weak base change)**.** *Let X be a smooth p-adic formal scheme over* $\mathrm{Spf}(A/I)$ *(without assuming Assumption 3.1). Let $\mathcal{F}$ be an $\mathcal{O}_{\Delta}$-crystal (resp. an $\overline{\mathcal{O}}_{\Delta}$-crystal) on $(X/A)_{\Delta}$. Let $(A, I) \to (A', I')$ be a morphism of bounded prisms of finite tor-dimension, $X' = X \times_{\mathrm{Spf}(A/I)} \mathrm{Spf}(A'/I')$ and $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $(X'/A')_{\Delta}$. Then the natural completed base change map*

$$\begin{aligned} &(R\nu_{X/A,*}(\mathcal{F}) \otimes^L_A A')^\wedge \xrightarrow{\sim} R\nu_{X'/A',*}((X'/A')_{\Delta}, \mathcal{F}') \\ (\textit{resp.} \qquad &(R\nu_{X/A,*}(\mathcal{F}) \otimes^L_{A/I} A'/I')^\wedge \xrightarrow{\sim} R\nu_{X'/A',*}(\mathcal{F}')) \end{aligned}$$

is an isomorphism.

Proof. The problem is clearly local for the étale topology of X. Up to étale localization, we may impose thus Assumption 3.1. In this case, the statement follows immediately from Prop. 3.9 and the fact that the formation of Čech–Alexandre complex (3.9.1) commutes with the base change $A \to A'$. $\square$

Remark 3.13. After establishing Theorem 4.12, we will see that Corollary 3.12 holds without the assumption that $A \to A'$ is of finite tor-dimension.

4. Prismatic Crystals and Higgs Fields

In this section, we keep the notation of Section 3. We will restrict ourselves to the following special case of Assumption 3.1.

Situation 4.1. We assume that $X = \mathrm{Spf}(R)$ admits an (p, I)-completely étale map to $\mathrm{Spf}(A/I\langle \underline{T}_n\rangle)$, where $A/I\langle \underline{T}_n\rangle := A/I\langle T_1, \dots, T_n\rangle$ denotes the convergent power series ring over A/I in n variables (for the p-adic topology). Then by deformation theory, there exists a unique derived (p,I)-complete and (p,I)-completely étale $A\langle \underline{T}_n\rangle$-algebra $\widetilde{R}$ which makes the following diagram cocartesian:

$$\begin{tikzcd}
\widetilde{R} \arrow[r] & R \\
A\langle \underline{T}_n\rangle \arrow[u] \arrow[r] & A/I\langle \underline{T}_n\rangle \arrow[u]
\end{tikzcd}$$

We choose a δ-structure on $A\langle \underline{T}_n\rangle$ extending that on A. Then by [6, Lemma 2.18], it extends uniquely to a δ-structure on $\widetilde{R}$. In particular, Assumption 3.1 is satisfied. For technical reasons, we need to suppose that one of the following assumptions is satisfied:

(1) (A,I) is a crystalline prism, i.e. $I=(p)$;
(2) there exists a map of bounded prisms $(A',I')\to(A,I)$ such that
- the Frobenius map on A'/p is flat,
- A'/I' is p-torsion;
- the δ-structure on $A\langle \underline{T}_n\rangle$ descends to a δ-structure on $A'\langle \underline{T}_n\rangle$.

For instance, if $I=(d)$ is principal and we take the δ-structure with $\delta(T_i)=0$, then assumption (2) is satisfied with $(A',I')=(\mathbb{Z}_p\{d,\delta(d)^{-1}\}^\wedge,(d))$, the (d,p)-completed universal oriented prism.

We will give an explicit description of $\overline{\mathcal{O}}_{\Delta}$-crystals on $(X/A)_{\Delta}$ in terms of Higgs fields, and compute the cohomology of $\overline{\mathcal{O}}_{\Delta}$-crystals via the de Rham cohomology of its associated Higgs field. Recall that we have a cosimplicial object $\widetilde{R}(\bullet)$ defined in §3.3. We give first a more transparent description of $\widetilde{R}(1)$. Consider the canonical diagonal surjection

$$A\langle \underline{T}_n\rangle\widehat{\otimes}_A A\langle \underline{T}_n\rangle \to A/I\langle \underline{T}_n\rangle,$$

and denote its kernel by J_n. Note that we have an isomorphism

$$A\langle \underline{T}_n\rangle\widehat{\otimes}_A A\langle \underline{T}_n\rangle \simeq A\langle \underline{T}_n\rangle\langle \xi_1,\dots,\xi_n\rangle,$$

given by $T_i\otimes 1\mapsto T_i$ and $1\otimes T_i\mapsto T_i+\xi_i$ for $1\leqslant i\leqslant n$. Via this isomorphism, the ideal J_n corresponds to $(I,\xi_1,\dots,\xi_n)$. Recall that we have chosen a δ-structure on $A\langle \underline{T}_n\rangle$ compatible with that on A. We equip $A\langle \underline{T}_n\rangle\langle \xi_1,\dots,\xi_n\rangle$ with the δ-structure that corresponds to the canonical induced tensor δ-structure on $A\langle \underline{T}_n\rangle\widehat{\otimes}_A A\langle \underline{T}_n\rangle$: explicitly, if $\delta(T_i)=f_i(\underline{T})\in A\langle \underline{T}_n\rangle$, we have

$$\delta(\xi_i)=\sum_{j=1}^{p-1}\frac{1}{p}\binom{p}{j}\xi_i^jT_i^{p-j}+f_i(\underline{T}+\underline{\xi})-f_i(\underline{T})\in(\xi_1,\dots,\xi_n) \tag{4.1.1}$$

Applying the construction of [6, Lemma 3.13], we get the prismatic envelope

$$\big(A\langle \underline{T}_n\rangle\widehat{\otimes}_A A\langle \underline{T}_n\rangle\big)\{\frac{J_n}{I}\}^\wedge\simeq A\langle \underline{T}_n\rangle\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^\wedge,$$

which is (p,I)-completely faithfully flat over $A\langle \underline{T}_n\rangle$. The formation of $A\langle \underline{T}_n\rangle\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^\wedge$ commutes with base change in $(A\langle \underline{T}_n\rangle, IA\langle \underline{T}_n\rangle)$ in the following sense: If (C,IC) is a bounded prism over $(A\langle \underline{T}_n\rangle, IA\langle \underline{T}_n\rangle)$ and we extend the δ-structure on C to $C\langle \xi_1,\dots,\xi_n\rangle$ with $\delta(\xi_i)$ given by the image of (4.1.1) in $C\langle \xi_1,\dots,\xi_n\rangle$, then

$$C\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge} := C\widehat{\otimes}_{A\langle \underline{T}_n\rangle} A\langle \underline{T}_n\rangle\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}$$

is nothing but the prismatic envelope of $C\langle \xi_1,\dots,\xi_n\rangle$ with respect to the ideal $(I,\xi_1,\dots,\xi_n)$. Note that there exists a canonical map

$$C/IC \to C\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}/IC\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}$$

obtained by extension of scalars from the corresponding map for $C = A\langle \underline{T}_n\rangle$.

By the functoriality of prismatic envelope, there exists a canonical map of δ-A-algebras

$$A\langle \underline{T}_n\rangle\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge} \to \widetilde{R}(1).$$

Together with $\delta_1^1 : \widetilde{R} \to \widetilde{R}(1)$, it induces by pushout a map

$$\eta : \widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge} \to \widetilde{R}(1)$$

which induces a map of prisms over (A, I).

Lemma 4.2. *The morphism η induces an isomorphisms of prisms*

$$(\widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}, I\widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}) \xrightarrow{\sim} (\widetilde{R}(1), I\widetilde{R}(1)).$$

Proof. We need to construct an inverse to η. Consider the following diagram

$$\begin{tikzcd}
A\langle \underline{T}_n\rangle \arrow[rr] \arrow[d, "T_i \mapsto T_i+\xi_i"'] & & \widetilde{R} \arrow[d] \arrow[dll, dashed, "i_2"'] \\
\widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge} \arrow[rr] & & \widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}/I\widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge},
\end{tikzcd}$$

where the right vertical map is the composed canonical map:

$$\widetilde{R} \to \widetilde{R}/I\widetilde{R} \to \widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}/I\widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}.$$

It is clear that the square of the above diagram is commutative. Since $A\langle \underline{T}_n\rangle \to \widetilde{R}$ is (p, I)-completely étale and $\widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}$ is I-adically complete, there exists a unique map $i_2 : \widetilde{R} \to \widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}$ as the dotted arrow that makes all triangles in the diagram commute. Moreover, since the left vertical arrow is a map of δ-A-algebras and the δ-structure on $\widetilde{R}$ is uniquely determined by its restriction to $A\langle \underline{T}_n\rangle$ (cf. [6, Lemma 2.18]), it follows that i_2 is a map of δ-A-algebras. Consider the morphism

$$f : \widetilde{R}\widehat{\otimes}_A\widetilde{R} \to \widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}$$

given by the tensor product of the natural inclusion and i_2. It is easy to see that f sends the ideal $\widetilde{J}(1) \subset \widetilde{R}\widehat{\otimes}_A\widetilde{R}$ to $I\widetilde{R}\langle \underline{T}_n\rangle\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}$. By the universal property of the prismatic envelope, it induces a morphism of prisms over (A, I):

$$\eta' : (\widetilde{R}(1), I\widetilde{R}(1)) \to (\widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}, I\widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{\wedge}).$$

Now it is routine to check that both η and η' are inverse of each other. $\square$

We now compare $\widetilde{R}(1)/I\widetilde{R}(1)$ with some divided power polynomial algebra. Let $\mathrm{Bl}_J(\mathbb{A}^n_{\widetilde{R}})$ denote the blow-up of $\mathrm{Spec}(\widetilde{R}[\xi_1,\dots,\xi_n])$ along the ideal $J=(I,\xi_1,\dots,\xi_n)$. Let U be the affine open subset of $\mathrm{Bl}_J(\mathbb{A}^n_{\widetilde{R}})$ where the inverse image of I generates the inverse image of $(I,\xi_1,\dots,\xi_n)$. The coordinate ring of U is isomorphic to the twisted polynomial algebra $\mathrm{Sym}_{\widetilde{R}}((I^{-1}\widetilde{R})^{\oplus n})$, and we write $\widetilde{R}\langle\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\rangle$ for its (p,I)-adic completion. If $I=(d)$ is generated by a distinguished element $d\in A$, then we have

$$\widetilde{R}\langle\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\rangle\cong\widetilde{R}\langle\xi_1,\dots,\xi_n,X_1,\dots,X_n\rangle/(\xi_i=dX_i:1\leqslant i\leqslant n)\cong\widetilde{R}\langle X_1,\dots,X_n\rangle.$$

Let $R\langle\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\rangle$ be the reduction modulo I of $\widetilde{R}\langle\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\rangle$, and $\widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{PD,\wedge}$ denote the (derived) (p,I)-completion of the PD-envelope of the canonical augmentation surjection

$$\widetilde{R}\langle\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\rangle\to\widetilde{R}.$$

If $I=(d)$, then $\widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{PD,\wedge}$ is nothing but the derived (p,I)-completion of the divided power polynomial ring in n-variables over R. Let $R^{PD}(1):=R\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{PD,\wedge}$ denote the reduction modulo I of $\widetilde{R}\{\frac{\xi_1}{I},\dots,\frac{\xi_n}{I}\}^{PD,\wedge}$.

Proposition 4.3. *Suppose that we are in Situation 4.1. Under the above notation, there exists an isomorphism of derived p-complete A/I-algebras:*

$$\psi:R^{PD}(1)\xrightarrow{\sim}\widetilde{R}(1)/I\widetilde{R}(1).$$

Moreover, via this isomorphism, the two maps $\bar{\delta}^1_i:R\to\widetilde{R}(1)/I\widetilde{R}(1)$ with $i=0,1$ are both identified with the natural map $R\to R^{PD}(1)$.

Proof. We assume first that $I=(d)$ is principal with $d\in A$ a distinguished element. By Lemma 4.2, we have an isomorphism of δ-A-algebras

$$\widetilde{R}(1)\cong\widetilde{R}\{\frac{\xi_1}{d},\dots,\frac{\xi_n}{d}\}^{\wedge}.$$

Recall that the convergent power series ring $\widetilde{R}\langle\xi_1,\dots,\xi_n\rangle$ is equipped with the δ-structure compatible with that on $\widetilde{R}$ and with $\delta(\xi_i)$ given by (4.1.1). By definition, we have

$$\widetilde{R}\{\frac{\xi_1}{d},\dots,\frac{\xi_n}{d}\}^{\wedge}=\Big(\widetilde{R}\langle\xi_1,\dots,\xi_n\rangle\{X_1,\cdots,X_n\}/(\xi_i-dX_i:1\leqslant i\leqslant n)_\delta\Big)^{\wedge}$$

where $\widetilde{R}\langle\xi_1,\dots,\xi_n\rangle\{X_1,\cdots,X_n\}$ is the free δ-algebra over $\widetilde{R}\langle\xi_1,\dots,\xi_n\rangle$ in n-variables, and $(\xi_i-dX_i:1\leqslant i\leqslant n)_\delta$ is the ideal generated by $\delta^r(\xi_i-dX_i)$ for all $r\geqslant 0$ and $1\leqslant i\leqslant n$. We denote by x_i the image of X_i in $\widetilde{R}(1)\cong\widetilde{R}\{\frac{\xi_1}{d},\dots,\frac{\xi_n}{d}\}^{\wedge}$. We prove first the following technical Lemma.

Lemma 4.4. *Let ϕ denote the Frobenius structure on $\widetilde{R}(1)$. For an integer $r\geqslant 0$, let $\mathcal{J}_r\subseteq\widetilde{R}(1)$ denote the closed ideal generated by $d^{p^j}\delta^j(x_k)$ with $0\leqslant j\leqslant r$ and $1\leqslant k\leqslant n$. Then for all $r\geqslant 0$ and $1\leqslant i\leqslant n$, there exists a $b_{i,r+1}\in\widetilde{R}(1)^\times$ such that*

$$\delta^r(\phi(x_i))=\delta^r(x_i)^p+p\delta^{r+1}(x_i)\equiv b_{i,r+1}d^{p^{r+1}}\delta^{r+1}(x_i)\mod\mathcal{J}_r.$$

Proof. We proceed by induction on $r \geqslant 0$. For $1 \leqslant i \leqslant n$, we have

$$\delta(\xi_i) = \delta(dx_i) = d^p\delta(x_i) + \phi(x_i)\delta(d).$$

In view of (4.1.1) and $\xi_j = dx_j$, we have $\delta(\xi_i) \in \mathcal{J}_0$. As d is distinguished, we have $\delta(d) \in A^\times$ and

$$\phi(x_i) \equiv -\frac{1}{\delta(d)} d^p\delta(x_i) \mod \mathcal{J}_0.$$

This proves the statement for $r = 0$. Suppose now that the statement holds for all integers $\leqslant r$. Then there exist $b_{i,j}^{r,k} \in \widetilde{R}(1)$ and $b_{i,r+1} \in \widetilde{R}(1)^\times$ such that

$$\delta^r(\phi(x_i)) = \sum_{j=0}^{r}\sum_{k=1}^{n} b_{i,j}^{r,k} d^{p^j}\delta^j(x_k) + b_{i,r+1}d^{p^{r+1}}\delta^{r+1}(x_i).$$

Applying δ, we get

$$\begin{aligned}\delta^{r+1}(\phi(x_i)) &= \delta\Bigg(\sum_{j=0}^{r}\sum_{k=1}^{n} b_{i,j}^{r,k} d^{p^j}\delta^j(x_k) + b_{i,r+1}d^{p^{r+1}}\delta^{r+1}(x_i)\Bigg)\\ &\equiv \sum_{j=0}^{r}\sum_{k=1}^{n} \delta(b_{i,j}^{r,k} d^{p^j}\delta^j(x_k)) + \delta(b_{i,r+1}d^{p^{r+1}}\delta^{r+1}(x_i)) \mod \mathcal{J}_{r+1}.\end{aligned}$$

Note that

$$\delta(b_{i,j}^{r,k} d^{p^j}\delta^j(x_k)) = \delta(b_{i,j}^{r,k} d^{p^j})\phi(\delta^j(x_k)) + (b_{i,j}^{r,k})^p d^{p^{j+1}}\delta^{j+1}(x_k),$$

and $\phi(\delta^j(x_k)) = \delta^j(\phi(x_k)) \in \mathcal{J}_j \subset \mathcal{J}_{r+1}$ by induction hypothesis. We have also

$$\delta(b_{i,r+1}d^{p^{r+1}}\delta^{r+1}(x_k)) = \delta(b_{i,r+1}d^{p^{r+1}})\phi(\delta^{r+1}(x_i)) + b_{i,r+1}^p d^{p^{r+2}}\delta^{r+2}(x_i).$$

It follows that

$$(1 - \delta(b_{i,r+1}d^{p^{r+1}}))\delta^{r+1}(\phi(x_i)) \equiv b_{i,r+1}^p d^{p^{r+2}}\delta^{r+2}(x_i) \mod \mathcal{J}_{r+1}.$$

Now by induction on $n \geqslant 1$, it is easy to see that $\delta(bd^n) \in (d^p, p)^{n-1}$ for any $b \in \widetilde{R}(1)$. It follows that $\delta(b_{i,r+1}d^{p^{r+1}})$ is topologically nilpotent and $1 - \delta(b_{i,r+1}d^{p^{r+1}})$ is invertible as $\widetilde{R}(1)$ is (p, d)-complete. We conclude that

$$\delta^{r+1}(\phi(x_i)) \equiv b_{i,r+2}d^{p^{p+2}}\delta^{r+2}(x_i) \mod \mathcal{J}_{r+1}$$

with $b_{i,r+2} = (1 - \delta(b_{i,r+1}d^{p^{r+1}}))^{-1}b_{i,r+1}^p \in \widetilde{R}(1)^\times$. This finishes the proof of Lemma 4.4. $\square$

We return to the proof of Proposition 4.3. Let $\widetilde{\mathcal{K}} \subset \widetilde{R}(1)$ be the kernel of the canonical augmentation map

$$\widetilde{R}(1) = \widetilde{R}\{\frac{\xi_1}{d}, \dots, \frac{\xi_n}{d}\}^\wedge \to \widetilde{R},$$

i.e. $\widetilde{\mathcal{K}}$ is the closure of the ideal generated by $\delta^r(x_i)$ with $r \geqslant 0$ and $1 \leqslant i \leqslant n$. For any element $x \in \mathcal{K}$, we put $\gamma_p(x) := -\frac{\delta(x)}{(p-1)!} \in \mathcal{K}$. We claim that

(1) $p!\gamma_p(x) \equiv x^p \mod I\widetilde{R}(1)$ for $x \in \widetilde{\mathcal{K}}$;
(2) $\gamma_p(ax) \equiv a^p\gamma_p(x) \mod I\widetilde{R}(1)$ for any $a \in \widetilde{R}(1)$ and $x \in \widetilde{\mathcal{K}}$;
(3) $\gamma_p(x + y) \equiv \gamma_p(x) + \gamma_p(y) + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!}x^iy^{p-i} \mod I\widetilde{R}(1)$ for $x, y \in \widetilde{\mathcal{K}}$.

Property (3) is a direct consequence of the additive property of δ. To verify (1) and (2), we first note that Lemma 4.4 implies

$$\delta^r(\phi(x_i)) = \phi(\delta^r(x_i)) = \delta^r(x_i)^p + p\delta^{r+1}(x_i) \in I\widetilde{R}(1)$$

for all $r \geqslant 0$ and $1 \leqslant i \leqslant n$, and hence $\phi(x) \in I\widetilde{R}(1)$ for all $x \in \widetilde{\mathcal{K}}$. Then (1) and (2) follow by an easy computation.

Let $\mathcal{K}$ denote the image of $\widetilde{\mathcal{K}}$ in $\widetilde{R}(1)/I\widetilde{R}(1)$. For $\bar{x} \in \mathcal{K}$, we denote by $\gamma_p(\bar{x}) \in \mathcal{K}$ the image of $\gamma_p(x)$ for an lift $x \in \widetilde{\mathcal{K}}$ of $\bar{x}$. As $\widetilde{\mathcal{K}} \cap I\widetilde{R}(1) = I\widetilde{\mathcal{K}}$, it is easy to see that $\gamma_p(\bar{x})$ does not depend on the choice of x. By the properties of γ_p and [11, Tag 07GS], there exists a unique PD-structure on the ideal $\mathcal{K} \subseteq \widetilde{R}(1)/I\widetilde{R}(1)$ such that the p-th divided power function is given by $\gamma_p : \mathcal{K} \to \mathcal{K}$. By the universal property of the divided power polynomial ring, there exists a unique map of R-algebras

$$\psi : R^{PD}(1) = R\{\frac{\xi_1}{d}, \ldots, \frac{\xi_n}{d}\}^{PD,\wedge} \to \widetilde{R}(1)/I\widetilde{R}(1)$$

sending $\frac{\xi_i}{d}$ to x_i.

We have to show that ψ is an isomorphism. It is clear from the construction that ψ is surjective since the reduction of $\delta^r(x_i)$ with $r \geqslant 0$ and $1 \leqslant i \leqslant n$ lies in the image of ψ. It remains to see that ψ is injective.

We consider first the special case that $(d) = (p)$, i.e. $(A, I) = (A, (p))$ is a crystalline prism. In this case, one considers the following Frobenius structure on $\widetilde{R}\langle\frac{\xi_1}{p}, \ldots, \frac{\xi_n}{p}\rangle$ given by

$$\phi(\frac{\xi_i}{p}) = \frac{\phi(\xi_i)}{p} = \frac{\xi_i^p}{p} + \delta(\xi_i) = p^{p-1}(\frac{\xi_i}{p})^p + \delta(\xi_i),$$

with $\delta(\xi_i)$ given by (4.1.1). Note that $\delta(\xi_i)$ is divisible by p in $\widetilde{R}\langle\frac{\xi_1}{p}, \ldots, \frac{\xi_n}{p}\rangle$, hence $\phi(\frac{\xi_i}{p}) \in p\widetilde{R}\langle\frac{\xi_1}{p}, \ldots, \frac{\xi_n}{p}\rangle$. It follows that such a lift of Frobenius, or equivalently such a δ-structure, extends uniquely to $\widetilde{R}\{\frac{\xi_1}{p}, \ldots, \frac{\xi_n}{p}\}^{PD,\wedge}$: Indeed, if $\phi(\frac{\xi_i}{p}) = py_i$, then we put

$$\phi((\frac{\xi_i}{p})^{[n]}) = \frac{p^n}{n!}y_i^n,$$

which determines a unique Frobenius structure on $\widetilde{R}\{\frac{\xi_1}{p}, \ldots, \frac{\xi_n}{p}\}^{PD,\wedge}$. Here, $(\frac{\xi_i}{p})^{[n]}$ denotes the n-th divided power of $\frac{\xi_i}{p}$. By the universal property of the prismatic envelope $\widetilde{R}(1) \cong \widetilde{R}\{\frac{\xi_1}{p}, \cdots, \frac{\xi_n}{p}\}^{\wedge}$, there exists a unique map of δ-$\widetilde{R}$-algebras $\widetilde{R}(1) \to \widetilde{R}\{\frac{\xi_1}{p}, \ldots, \frac{\xi_n}{p}\}^{PD,\wedge}$. Reducing modulo p, one gets a map pf R-algebras

$$\psi^{-1} : \widetilde{R}(1)/p\widetilde{R}(1) \to R^{PD}(1) = R\{\frac{\xi_1}{d}, \ldots, \frac{\xi_n}{d}\}^{PD,\wedge}$$

which is easily seen to be the inverse of ψ.

We consider now the general case of d. First, we note that the formation of ψ commutes with the étale localization in R. Therefore, one can reduce to the case $R = A\langle\underline{T}_n\rangle$. By our assumptions in Situation 4.1, there exists a map of bounded prisms $(A', I') \to (A, I)$ such that the δ-structure on $A\langle\underline{T}_n\rangle$ descends to $A'\langle\underline{T}_n\rangle$ and

(a) the Frobenius map on A'/p is flat;
(b) A'/I' is p-torsion free.

Since the formation of ψ commutes with base change in (A, I), it suffices to prove the statement for (A', I'). Up to changing notation, we may assume that (A, I) satisfies conditions (a) and (b).

Consider the ring $B = A\{\frac{\phi(d)}{p}\}^\wedge$, the p-completed δ-A-algebra obtained by freely adjoining $\frac{\phi(d)}{p}$ to A. By [6, Cor. 2.38], B is identified with the p-completed PD-envelope of A with respect to (d). Let $\alpha : A \to B$ denote the composite of the canonical map $A \to B$ with the Frobenius $\phi : A \to A$. Then α induces a map of bounded prisms $(A, I) \to (B, (p))$ since $\alpha(I) \subset (p)$. Modulo p, we get a factorization

$$\bar{\alpha}\colon A/p \to A/(p, I) \xrightarrow{\bar{\phi}} A/(p, I^p) \xrightarrow{\iota} B/p.$$

where the first arrow is the canonical reduction, $\bar{\phi}$ is the Frobenius, and ι is induced by the canonical map $A \to B$. Note that the second and the third map are both faithfully flat. Denote by $\bar{\beta} : A/(p, I) \to B/p$ the composite of the second and third map above. Let $\bar{\psi}$ the reduction of ψ modulo p. Then one has a commutative diagram:

$$\begin{tikzcd}
R/p\{\frac{\xi_1}{d}, \dots, \frac{\xi_n}{d}\} \arrow[r, "\bar{\psi}"] \arrow[d] & \widetilde{R}(1)/(p, I)\widetilde{R}(1) \arrow[d] \\
(R/p \otimes_{A/(p,I),\bar{\beta}} B/p)\{\frac{\xi_1}{d}, \dots, \frac{\xi_n}{d}\} \arrow[r, "\bar{\psi}\otimes 1"] & \big(\widetilde{R}(1)/(p, I)\widetilde{R}(1)\big)\otimes_{A/(p,I),\bar{\beta}} B/p,
\end{tikzcd}$$

where the vertical maps are natural inclusions induced by $\bar{\beta}$, and the bottom map $\bar{\psi} \otimes 1$ is nothing but the base change of ψ via $\alpha : A \to B$, which we denote by ψ_B. As the formation of ψ commutes with base change in (A, I), the previous discussion in the case of $(d) = (p)$ implies that ψ_B is injective (hence an isomorphism), it follows that $\bar{\psi}$ is also injective. Let M denote the kernel of ψ. Since both the source and the target of ψ are flat over A/I, it follows that $M/pM = \ker(\bar{\psi}) = (0)$. As M is separate for the p-adic topology, it follows $M = (0)$. This finishes the construction of the isomorphism ψ when $I = (d)$ is principal.

To emphasize the dependence of ψ on the choice of d, we write it as ψ_d. If $d' = du$ with $u \in A^\times$ is another generator of I, then there exists a commutative diagram

$$\begin{tikzcd}
\widetilde{R}\{\frac{\xi_1}{d}, \dots, \frac{\xi_n}{d}\}^{PD,\wedge} \arrow[rr, "c_{d,d'}"] \arrow[dr, "\simeq", "\psi_d"'] & & \widetilde{R}\{\frac{\xi_1}{d'}, \dots, \frac{\xi_n}{d'}\}^{PD,\wedge} \arrow[dl, "\simeq"', "\psi_{d'}"] \\
& \widetilde{R}(1)/I\widetilde{R}(1), &
\end{tikzcd}$$

where $c_{d,d'}$ is the isomorphism sending $(\frac{\xi_i}{d})^{[n]}$ to $u^n(\frac{\xi_i}{d'})^{[n]}$.

Consider now the general case where I is not necessarily principal. Then there exists a (p, I)-completely flat δ-A-algebra A' such that $I' = IA'$ is principal. Put $\widetilde{R}' = \widetilde{R}\widetilde{\otimes}_A A'$, and $R' = \widetilde{R}'/I\widetilde{R}'$. After making a choice of a generator d' of I', one has an isomorphism $\psi_{d'} : R'\{\frac{\xi_1}{d'}, \dots, \frac{\xi_n}{d'}\}^{PD,\wedge} \xrightarrow{\sim} \widetilde{R}'(1)/I\widetilde{R}'(1)$. Then the compatibility of the isomorphism $\psi_{d'}$ for different choices of generators d' allows us to see that $\psi_{d'}$ descends to an isomorphism

$$\psi : R^{PD}(1) = R\{\frac{\xi_1}{I}, \dots, \frac{\xi_n}{I}\}^{PD,\wedge} \xrightarrow{\sim} \widetilde{R}(1)/I\widetilde{R}(1).$$

For the moreover part of Proposition 4.3, it is clear that $\bar{\delta}_1^1 : R \to \widetilde{R}(1)/I\widetilde{R}(1)$ is identified with the canonical map $R \to R^{PD}(1)$. To finish the proof, it suffices to see that $\bar{\delta}_0^1$ coincides with $\bar{\delta}_1^1$. In fact, via the isomorphism

$$\widetilde{R}\{\frac{\xi_1}{I}, \dots, \frac{\xi_n}{I}\}^\wedge \xrightarrow{\sim} \widetilde{R}(1)$$

in Lemma 4.2, $\delta_0^1 : \widetilde{R} \to \widetilde{R}(1)$ is identified with the unique map $i_2 : \widetilde{R} \to \widetilde{R}\{\frac{\xi_1}{I}, \dots, \frac{\xi_n}{I}\}^\wedge$ such that $T_i \mapsto T_i + \xi_i$. Since the image of ξ_i in $\widetilde{R}\{\frac{\xi_1}{I}, \dots, \frac{\xi_n}{I}\}^\wedge$ lies in $I\widetilde{R}\{\frac{\xi_1}{I}, \dots, \frac{\xi_n}{I}\}^\wedge$, it follows that $\bar{\delta}_0^1 = \bar{\delta}_1^1$. This finishes the proof of Proposition 4.3.

$\square$

Corollary 4.5. *For any integer $m \geqslant 1$, there exists an isomorphism of R-algebras:*

$$\widetilde{R}(m)/I\widetilde{R}(m) \cong R^{PD}(m) := R\{\frac{\xi_{i,j}}{I} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}^{PD,\wedge}$$

such that for $0 \leqslant k \leqslant m$ the map $\bar{\delta}_i^m : \widetilde{R}(m-1)/I\widetilde{R}(m-1) \to \widetilde{R}(m)/I\widetilde{R}(m)$ is compatible with the PD-structure and determined by

$$\bar{\delta}_k^m : \frac{\xi_{i,j}}{d} \mapsto \begin{cases} \frac{\xi_{i,j+1}}{d} & \text{if } k < j, \\ \frac{\xi_{i,j}}{d} + \frac{\xi_{i,j+1}}{d} & \text{if } k = j, \\ \frac{\xi_{i,j}}{d} & \text{if } k > j, \end{cases}$$

if d is a (local) generator I.

Proof. This follows immediately from Proposition 4.3 and the fact that (cf. Remark 2.2)

$$\widetilde{R}(m) = \underbrace{\widetilde{R}(1)\widehat{\otimes}_{\widetilde{R}} \cdots \widehat{\otimes}_{\widetilde{R}}\widetilde{R}(1)}_{m \text{ copies}}.$$

$\square$

4.6. **Higgs modules.** Let $\mathcal{K} \subset R^{PD}(1)$ denote the kernel of the canonical sujection

$$R^{PD}(1) \to R.$$

Then $\mathcal{K}$ is equipped with a PD-structure. Let $\mathcal{K}^{[2]} \subset \mathcal{K}$ be the closed ideal generated by all $x^{[n]}$ with $x \in \mathcal{K}$ and $n \geqslant 2$. Let Ω_R^1 be the module of continuous 1-differential form relative to A/I. In general, for an A/I-module N and an integer $i \geqslant 0$, we put $N\{-i\} = N \otimes_R (I/I^2)^{\vee, \otimes i}$. Then we have a canonical isomorphism

$$\mathcal{K}/\mathcal{K}^{[2]} \cong \Omega_R^1\{-1\},$$

such that if $d \in I$ is a generator then the class of $\frac{\xi_i}{d}$ in $\mathcal{K}/\mathcal{K}^{[2]}$ corresponds to $\mathrm{d}T_i \otimes d^{-1}$.

Definition 4.7. A Higgs module over R is a p-completely flat and derived p-complete R-module M together with a R-linear map

$$\theta : M \to M \otimes_R \Omega_R^1\{-1\}$$

such that the induced map $\theta \wedge \theta : M \to M \otimes \Omega_R^2\{-2\}$ vanishes.

Let (M,θ) be a Higgs module over R. Consider first the case where $I=(d)$ is principal. Since $\Omega^1_R\{-1\}=\oplus_{i=1}^n R\mathrm{d}T_i\otimes d^{-1}$, we can write $\theta=\sum_{i=1}^n\theta_i\mathrm{d}T_i\otimes d^{-1}$ with $\theta_i\in\mathrm{End}_R(M)$. Then the condition $\theta\wedge\theta=0$ is equivalent to $\theta_i\theta_j=\theta_j\theta_i$ for all i,j. For any n-tuple $\underline{m}=(m_1,\cdots,m_n)\in\mathbb{N}^n$, we put

$$\theta^{\underline{m}}=\prod_{i=1}^n\theta_i^{m_i}\in\mathrm{End}_R(M).$$

We say that (M,θ) is *topologically quasi-nilpotent*, if $\theta^{\underline{m}}(x)$ tends to 0 for all $x\in M$ as $|\underline{m}|:=\sum_{i=1}^n m_i$ tends to infinity. It is clear that such a definition does not depend on the choice of the generator $d\in I$. In the general case when I is not necessarily principal, we say that (M,θ) is *topologically quasi-nilpotent*, if for a (hence for all) (p,I)-completely faithfully flat δ-A-algebra with $I':=IA'$ principal, the induced Higgs module $(M\widehat{\otimes}_{A/I}A'/I',\theta')$ over $R\widehat{\otimes}_{A/I}A'/I'$ by base change is topologically quasi-nilpotent.

Let $\mathbf{Higgs}^{\wedge}(R)$ denote the category of topologically quasi-nilpotent Higgs modules over R.

4.8. **Stratification and Higgs modules.** Let (M,ϵ) be an object of $\mathbf{Strat}(\widetilde{R}(1)/I\widetilde{R}(1))$ (Definition 3.6). Note that we have an ismorphism:

$$\begin{aligned}\mathrm{Hom}_{\widetilde{R}(1)/I\widetilde{R}(1)}(M\widehat{\otimes}_R\widetilde{R}(1)/I\widetilde{R}(1),M\widehat{\otimes}_R\widetilde{R}(1)/I\widetilde{R}(1))&\cong\mathrm{Hom}_R(M,M\widehat{\otimes}_R\widetilde{R}(1)/I\widetilde{R}(1))\\&\cong\mathrm{Hom}_R(M,M\widehat{\otimes}_R R^{PD}(1))\end{aligned}$$

where the last isomorphism is Prop. 4.3. When no confusions arise, we will still denote by ϵ the image of ϵ in $\mathrm{Hom}_R(M,M\widehat{\otimes}_R R^{PD}(1))$. Let

$$\iota:M\to M\widehat{\otimes}_R R^{PD}(1)$$

be the natural inclusion induced by the structural map $R\to R^{PD}(1)$. By the same argument as Remark 3.5, we have $(\epsilon-\iota)(M)\subset M\widehat{\otimes}_R\mathcal{K}$. We denote by

$$\theta_\epsilon:M\to M\otimes_R\mathcal{K}/\mathcal{K}^{[2]}\cong M\otimes_R\Omega^1_R\{-1\}$$

the reduction of $\epsilon-\iota$ modulo $\mathcal{K}^{[2]}$.

Proposition 4.9. *The functor $(M,\epsilon)\mapsto(M,\theta_\epsilon)$ establishes an equivalence of categories*

$$\mathbf{Strat}(\widetilde{R}(1)/I\widetilde{R}(1))\xrightarrow{\sim}\mathbf{Higgs}^{\wedge}(R).$$

Proof. We prove first that, for any object (M,ϵ) of $\mathbf{Strat}(\widetilde{R}/I\widetilde{R})$, the attached object (M,θ_ϵ) is indeed a topologically quasi-nilpotent Higgs module over R. Up to base change to a (p,I)-completely faithfully flat δ-A-algebra, we may assume that $I=(d)$ is principal for a distinguished element d. We write as usual that

$$\theta_\epsilon=\sum_{i=1}^n\theta_{\epsilon,i}\,\mathrm{d}T_i\otimes d^{-1}.$$

Note that

$$M\widehat{\otimes}_R R^{PD}(1)=\left\{\sum_{\underline{m}\in\mathbb{N}^n}x_{\underline{m}}\left(\frac{\xi_\bullet}{d}\right)^{[\underline{m}]}:x_{\underline{m}}\in M,x_{\underline{m}}\to 0\text{ as }|\underline{m}|\to\infty\right\}$$

where we put

$$\left(\frac{\xi_\bullet}{d}\right)^{[\underline{m}]} = \prod_{i=1}^n \left(\frac{\xi_i}{d}\right)^{[m_i]}.$$

For $x \in M$, we write

$$\epsilon(x) = \sum_{\underline{m}\in\mathbb{N}^n} \Theta_{\underline{m}}(x)\left(\frac{\xi_\bullet}{d}\right)^{[\underline{m}]} \in M\widehat{\otimes}_R R^{PD}(1)$$

for some $\Theta_{\underline{m}} \in \mathrm{End}_R(M)$ with $\Theta_{\underline{m}} = \mathrm{id}_M$ if $\underline{m} = (0, \ldots, 0)$. By the definition of θ_ϵ, it is clear that $\theta_{\epsilon,i} = \Theta_{\underline{e}_i}$ with $\underline{e}_i \in \mathbb{N}^n$ the element with i-component equal to 1 and others components equal to 0. To see that (M, θ_ϵ) is an object of $\mathbf{Higgs}^\wedge(R)$, it suffices to show that

$$\theta_{\epsilon,i}\theta_{\epsilon,j} = \theta_{\epsilon,j}\theta_{\epsilon,i}, \quad \Theta_{\underline{m}} = \prod_{i=1}^n \theta_{\epsilon,i}^{m_i} = \theta_\epsilon^{\underline{m}}. \tag{4.9.1}$$

According to Corollary 4.5, we have $\widetilde{R}(2)/I\widetilde{R}(2) \cong R^{PD}(2) = R\{\frac{\xi_{i,j}}{d} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant 2\}$ and the maps $\bar{\delta}_i^2 : \widetilde{R}(1)/I\widetilde{R}(1) \to \widetilde{R}(2)/I\widetilde{R}(2)$ with $i = 0, 1, 2$ are given by

$$\bar{\delta}_0^2(\frac{\xi_i}{d}) = \frac{\xi_{i,2}}{d}, \qquad \bar{\delta}_1^2(\frac{\xi_i}{d}) = \frac{\xi_{i,1}}{d} + \frac{\xi_{i,2}}{d}, \qquad \bar{\delta}_2^2(\frac{\xi_i}{d}) = \frac{\xi_{i,1}}{d}.$$

for all $1 \leqslant i \leqslant n$. Then for all $x \in M$, we have

$$\begin{aligned} \bar{\delta}_1^{2,*}(\epsilon(x)) &= \sum_{\underline{m}\in\mathbb{N}^n} \Theta_{\underline{m}}(x)\left(\frac{\xi_{\bullet,1}}{d} + \frac{\xi_{\bullet,2}}{d}\right)^{[\underline{m}]} \\ &= \sum_{\underline{m}_1\in\mathbb{N}^n}\sum_{\underline{m}_2\in\mathbb{N}^n} \Theta_{\underline{m}_1+\underline{m}_2}(x)\left(\frac{\xi_{\bullet,1}}{d}\right)^{[\underline{m}_1]}\left(\frac{\xi_{\bullet,2}}{d}\right)^{[\underline{m}_2]}; \\ \bar{\delta}_2^{2,*}(\epsilon)\big(\bar{\delta}_0^{2,*}(\epsilon(x))\big) &= \sum_{\underline{m}_1\in\mathbb{N}^n}\sum_{\underline{m}_2\in\mathbb{N}^n} \Theta_{\underline{m}_1}(\Theta_{\underline{m}_2}(x))\left(\frac{\xi_{\bullet,1}}{d}\right)^{[\underline{m}_1]}\left(\frac{\xi_{\bullet,2}}{d}\right)^{[\underline{m}_2]}. \end{aligned} \tag{4.9.2}$$

Then the cocycle condition $\bar{\delta}_1^{2,*}(\epsilon) = \bar{\delta}_2^{2,*}(\epsilon) \circ \bar{\delta}_0^{2,*}(\epsilon)$ is equivalent to saying that

$$\Theta_{\underline{m}_1+\underline{m}_2} = \Theta_{\underline{m}_1} \circ \Theta_{\underline{m}_2},$$

for all $\underline{m}_1, \underline{m}_2 \in \mathbb{N}^n$, from which (4.9.1) follows immediately.

To finish the proof of the Proposition, it suffices to construct a functor quasi-inverse to $(M, \epsilon) \mapsto (M, \theta_\epsilon)$. Let (M, θ) be an object of $\mathbf{Higgs}^\wedge(R)$. We need to construct a stratification ϵ on M over $\widetilde{R}(1)/I\widetilde{R}(1)$. By the usual descent argument, we may reduce the problem to the case when $I = (d)$ is principal. For any $x \in M$, we put

$$\epsilon(x) := \sum_{\underline{m}\in\mathbb{N}^n} \theta^{\underline{m}}(x)\left(\frac{\xi_\bullet}{d}\right)^{[\underline{m}]} \in M\widehat{\otimes}_R R^{PD}(1) \cong M\widehat{\otimes}_R\widetilde{R}(1)/I\widetilde{R}(1),$$

which is well defined by the topological quasi-nilpotence of (M, θ). This defines an element

$$\epsilon \in \mathrm{Hom}_R(M, M\widehat{\otimes}_R\widetilde{R}(1)/I\widetilde{R}(1)) \cong \mathrm{End}_{\widetilde{R}(1)/I\widetilde{R}(1)}\big(M\widehat{\otimes}_R\widetilde{R}(1)/I\widetilde{R}(1)\big).$$

We claim first that ϵ is an isomorphism of $M\widehat{\otimes}_R\widetilde{R}(1)/I\widetilde{R}(1)$. Indeed, if ϵ' is the endomorphism of $M\widehat{\otimes}_R\widetilde{R}(1)/I\widetilde{R}(1)$ defined by

$$\epsilon'(x) = \sum_{\underline{m}\in\mathbb{N}^n} (-1)^{|\underline{m}|}\theta^{\underline{m}}(x)\left(\frac{\xi_\bullet}{d}\right)^{[\underline{m}]} \quad \forall x \in M,$$

then one verifies easily that ϵ' is the inverse of ϵ. It remains to check that ϵ verifies the cocycle condition $\bar{\delta}_1^{2,*}(\epsilon) = \bar{\delta}_2^{2,*}(\epsilon)\circ\bar{\delta}_0^{2,*}(\epsilon)$. By the computation (4.9.2), this follows immediately from the obvious fact that $\theta^{\underline{m}_1+\underline{m}_2} = \theta^{\underline{m}_1}\circ\theta^{\underline{m}_2}$. This construction gives a functor $\mathbf{Higgs}^\wedge(R) \to \mathbf{Strat}(\widetilde{R}/I\widetilde{R})$, which is easily seen to be a quasi-inverse to $(M,\epsilon)\mapsto(M,\theta_\epsilon)$. $\square$

Combining now Proposition 3.7 and 4.9, we obtain the following

Theorem 4.10. *Suppose that we are in Situation 4.1. Then we have a sequence of equivalence of categories*

$$\mathbf{CR}((X/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta}) \xrightarrow{\overline{\mathrm{ev}}_{\tilde{X}}} \mathbf{Strat}(\widetilde{R}(1)/I\widetilde{R}(1)) \xrightarrow{\text{Prop. 4.9}} \mathbf{Higgs}^\wedge(R).$$

Remark 4.11. (1) It is natural to expect that there still exists an equivalence of categories between $\mathbf{CR}((X/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta})$ and $\mathbf{Higgs}^\wedge(R)$ under the more general assumption 3.1.

(2) Gros–Le Strum–Quirós [8, Cor. 6.6] and Morrow–Tsuji [10, Thm. 3.2] give similar descriptions of $\mathcal{O}_{\Delta}$-crystals when (A,I) is a bounded prism over $(\mathbb{Z}_p[[q-1]], ([p]_q))$ with $[p]_q = \frac{q^p-1}{q-1}$. In these special cases, Theorem 4.10 is compatible with their results after modulo I.

We now describe the cohomology of an $\overline{\mathcal{O}}_{\Delta}$-crystal in terms of the de Rham complex of its associated Higgs module.

Theorem 4.12. *Suppose that we are in Situation 4.1. Let $\mathcal{E}$ be an $\overline{\mathcal{O}}_{\Delta}$-crystal, and $(\mathcal{E}_{\tilde{X}},\theta)$ be its associated object of* $\mathbf{Higgs}^\wedge(R)$ *via Theorem 4.10. Then $R\nu_{X/A,*}(\mathcal{E})$ is computed by the de Rham complex of $(\mathcal{E}_{\tilde{X}},\theta)$:*

$$\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}},\theta) := \mathcal{E}_{\tilde{X}} \xrightarrow{\theta} \mathcal{E}_{\tilde{X}}\otimes_R\Omega_R^1\{-1\} \xrightarrow{\theta} \cdots \xrightarrow{\theta} \mathcal{E}_{\tilde{X}}\otimes_R\Omega_R^n\{-n\}$$

We prove Theorem 4.12 by following the strategy of [3]. Recall the simiplicial object $\widetilde{X}(\bullet)$ in $(X/A)_{\Delta}$ defined in §3.3. For any integers $i,j\geqslant 0$, let $\mathcal{E}_{\tilde{X}(i)} \cong \mathcal{E}_{\tilde{X}}\widehat{\otimes}_R\widetilde{R}(i)/I\widetilde{R}(i)$ be the evaluation of $\mathcal{E}$ on $\widetilde{X}(i)$, and $\Omega^j_{\widetilde{R}(i)/I\widetilde{R}(i)}$ be the continuous differential i-forms of $\widetilde{R}(i)/I\widetilde{R}(i)$ relative to A/I. Note that any morphism $\widetilde{X}(i)\to\widetilde{X}$ induces an isomorphism $\mathcal{E}_{\tilde{X}(i)} \cong \mathcal{E}_{\tilde{X}}\widehat{\otimes}_R\widetilde{R}(i)/I\widetilde{R}(i)$. We consider the following double complex

$$\mathscr{C}^{\bullet,\bullet} = (\mathscr{C}^{i,j} = \mathcal{E}_{\tilde{X}(i)}\otimes_{\widetilde{R}(i)/I\widetilde{R}(i)}\Omega^j_{\widetilde{R}(i)/I\widetilde{R}(i)}\{-j\} : \quad i,j\geqslant 0)$$

such that for each integer $j\geqslant 0$ $(d_1^{\bullet,j} : \mathscr{C}^{\bullet,j})$ is the Čech complex induced by the simplicial object $\widetilde{X}(\bullet)$, and

$$d_2^{i,j}\colon \mathscr{C}^{i,j} = \mathcal{E}_{\tilde{X}}\widehat{\otimes}_R\Omega^j_{\widetilde{R}(i)/I\widetilde{R}(i)}\{-j\} \to \mathscr{C}^{i,j+1} = \mathcal{E}_{\tilde{X}}\widehat{\otimes}_R\Omega^{j+1}_{\widetilde{R}(i)/I\widetilde{R}(i)}\{-j-1\}$$

is given by

$$d_2^{i,j}(x\otimes\omega) = \theta(x)\wedge\omega + x\otimes \mathrm{d}\omega.$$

Lemma 4.13. *For an integer $i \geqslant 0$, any morphism of objects $\widetilde{X}(i) \to \widetilde{X}$ in $(X/A)_{\triangle}$ induces a quasi-isomorphism of complexes*

$$\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}}, \theta) \xrightarrow{\sim} (\mathscr{C}^{i,\bullet}, d_2^{i,\bullet}).$$

Proof. By Corollary 4.5, we have an isomorphism

$$\widetilde{R}(i)/I\widetilde{R}(i) \cong R^{PD}(i) = R\{\frac{\xi_{k,l}}{I} : 1 \leqslant k \leqslant n, 1 \leqslant l \leqslant i\}^{PD,\wedge}.$$

Note that there is a short exact sequence

$$0 \to \Omega^1_R\{-1\} \to \Omega^1_{R^{PD}(i)}\{-1\} \to \Omega^1_{R^{PD}(i)/R}\{-1\} \to 0,$$

where $\Omega^1_{R^{PD}(i)/R}\{-1\}$ is a locally free $R^{PD}(i)$-module with local basis $\frac{\mathrm{d}\xi_{k,l}}{d}$ with $1 \leqslant k \leqslant n$ and $1 \leqslant l \leqslant i$ if d is a local generator of I. This induces a filtration $(\mathrm{Fil}^m\Omega^j_{R^{PD}(i)}\{-j\} : m \geqslant 0)$ with graded piece

$$\mathrm{gr}^m\, \Omega^j_{R^{PD}(i)} \cong \Omega^m_R\{-m\} \otimes_R \Omega^{j-m}_{R^{PD}(i)/R}\{-j+m\}.$$

The natural inclusion $\Omega^j_R\{-j\} \subset \Omega^j_{R^{PD}(i)}\{-j\}$ induces an inclusion of complexes

$$\iota^i : \mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}}, \theta) \to \mathscr{C}^{i,\bullet}.$$

We will show that this is a quasi-isomorphism. Consider the decreasing filtration of the complex $(\mathscr{C}^{i,\bullet}, d_2^{i,\bullet})$ given by

$$\mathrm{Fil}^m\mathscr{C}^{i,j} = \begin{cases} \mathscr{C}^{i,j} & m \leqslant 0; \\ \text{the image of } \mathcal{E}_{\tilde{X}}\widehat{\otimes}_R\mathrm{Fil}^m\Omega^j_R\{-j\} & m \leqslant j \end{cases}$$

with graded piece

$$\mathrm{gr}^m\,\mathscr{C}^{i,\bullet} = \mathcal{E}_{\tilde{X}}\widehat{\otimes}_R\Omega^m_R\{-m\} \otimes_R \Omega^{\bullet-m}_{R^{PD}(i)/R}\{-\bullet+m\}[-m].$$

The filtration $(\mathrm{Fil}^m\mathscr{C}^{i,\bullet}, m \in \mathbb{Z})$ induces via ι^i a filtration on $\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}}, \theta)$ with graded piece

$$\mathrm{gr}^m\,\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}}, \theta) \cong \mathcal{E}_{\tilde{X}}\widehat{\otimes}_R\Omega^m_R\{-m\}[-m].$$

Since $R^{PD}(i)$ is a divided power polynomial ring over R, the natural map

$$R \xrightarrow{\sim} \Omega^\bullet_{R^{PD}(i)/R}$$

is a quasi-isomorphism. It follows that ι^i induces a quasi-isomorphism on the graded pieces

$$\mathrm{gr}^m\,\iota^i\colon\ \mathrm{gr}^m\,\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}}, \theta) \xrightarrow{\sim} \mathrm{gr}^m\,\mathscr{C}^{i,\bullet},$$

and hence ι^i itself is also a quasi-isomorphism. □

End of the proof of Theorem 4.12. Note that $\mathscr{C}^{\bullet,0}$ is the Čech–Alexandre complex of the sheaf $\mathcal{E}$ as in Prop. 3.9, and $\mathscr{C}^{0,\bullet} = \mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}}, \theta)$. By Prop. 3.9, $R\nu_{X/A,*}(\mathcal{E})$ is computed by $\mathscr{C}^{\bullet,0}$. Therefore, to finish the proof, it suffices to show that both $\mathscr{C}^{\bullet,0}$ and $\mathscr{C}^{0,\bullet}$ are quasi-isomorphic to $\underline{s}(\mathscr{C}^{\bullet,\bullet})$, the simple complex associated to $\mathscr{C}^{\bullet,\bullet}$. By definition, the map of complexes

$$d_1^{i,\bullet}\colon \mathscr{C}^{i,\bullet} \to \mathscr{C}^{i+1,\bullet}$$

is given by $\sum_{k=0}^{i+1}(-1)^k\delta_k^{i+1,*}$, where $\delta_k^{i+1,*}$ is the map induced by the map

$$\delta_k^{i+1} : \widetilde{R}(i) \to \widetilde{R}(i+1)$$

defined in (3.3.1). By Lemma 4.13, $\mathscr{C}^{i,\bullet}$ and $\mathscr{C}^{i+1,\bullet}$ are both isomorphic to $\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}},\theta)$ in the derived category of R-modules and each $\delta_k^{i+1,*}$ corresponds to the identity map of $\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}},\theta)$. Therefore, via these quasi-isomorphisms, $d_1^{i,\bullet}$ is the identity map of $\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}},\theta)$ and the zero map if i is even. It follows that $\underline{s}(\mathscr{C}^{\bullet,\bullet})$ is quasi-isomorphic to

$$\underline{s}\big(\,\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}},\theta)\xrightarrow{0}\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}},\theta)\xrightarrow{\mathrm{id}}\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}},\theta)\xrightarrow{0}\cdots\big)\cong\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}},\theta)=\mathscr{C}^{0,\bullet}.$$

On the other hand, by [3, Lemma 2.17], the complex $\mathscr{C}^{\bullet,j}$ is homotopic to zero for any integer $j\geqslant 1$. Hence, the natural map

$$\mathscr{C}^{\bullet,0}\xrightarrow{\sim}\underline{s}(\mathscr{C}^{\bullet,\bullet})$$

is a quasi-isomorphism. This finishes the proof of Theorem 4.12.

□

Remark 4.14. It is clear that the trivial $\overline{\mathcal{O}}_{\mathbb{\Delta}}$-crystal $\mathcal{E}=\overline{\mathcal{O}}_{\mathbb{\Delta}}$ corresponds to the trivial Higgs module $(\mathcal{O}_X,0)$. Thus Theorem 4.12 implies that

$$R\nu_{X/A,*}(\overline{\mathcal{O}}_{\mathbb{\Delta}})=\bigoplus_{i=0}^{n}\Omega^1_R\{-i\}.$$

From this, it is easy to deduce Bhatt–Scholze's Hodge–Tate comparison theorem [6, Theorem 4.11] for general smooth p-adic formal scheme over $\mathrm{Spf}(A/I)$.

We can now finish the proof of Theorem 2.8.

4.15. **End of the proof of 2.8.** By Corollary 3.12 and Lemma 1.2, the statements are local for the flat topology in A. Therefore, up to replacing A by a faithfully flat cover, we may assume that I is pricipal. Moreover, the statements are also local for the étale topology of X. Up to étale localization, we may assume that $X=\mathrm{Spf}(R)$ is affine and satisfies the assumptions in Situation 4.1. Then statement 4.12(1) follows immediately from Theorem 4.12. Let $(A,I)\to(A',I')$ be a morphism of p-torsion free bounded prisms. Put $\widetilde{R}'=\widetilde{R}\widehat{\otimes}_A A'$, $R'=\widetilde{R}'/I'\widetilde{R}'$ and $X'=\mathrm{Spf}(R')$. If $\mathcal{E}$ corresponds to the Higgs module $(\mathcal{E}_{\tilde{X}},\theta)$, then $\bar{f}^*_{\mathbb{\Delta}}\mathcal{E}$ corresponds to $(\mathcal{E}_{\tilde{X}}\otimes_R R',\theta')$ with θ' obtained by θ by extension of scalars. By Theorem 4.12, the canonical base change map $f^{-1}R\nu_{X/A,*}(\mathcal{E})\otimes^L_{f^{-1}\mathcal{O}_X}\mathcal{O}_{X'}\to R\nu_{X'/A',*}(\bar{f}^*_{\mathbb{\Delta}}\mathcal{E})$ is identified in the derived category of R'-modules with the base change map

$$\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}},\theta)\otimes_R R'\xrightarrow{\sim}\mathrm{DR}^\bullet(\mathcal{E}_{\tilde{X}}\otimes_R R',\theta')$$

which is clearly an isomorphism of complexes.

5. Poincaré Duality

We fix a bounded prism (A,I), and a smooth p-adic formal scheme of relative dimension n over $\mathrm{Spf}(A/I)$.

5.1. Let $\mathbf{CR}((X/A)_{\triangle}, \overline{\mathcal{O}}_{\triangle})^{fr}$ be the category of $\overline{\mathcal{O}}_{\triangle}$-crystals locally free of finite rank. We have the natural notions of tensor product and internal hom in $\mathbf{CR}((X/A)_{\triangle}, \overline{\mathcal{O}}_{\triangle})^{fr}$: If $\mathcal{E}_1, \mathcal{E}_2$ are two objects of $\mathbf{CR}((X/A)_{\triangle}, \overline{\mathcal{O}}_{\triangle})^{fr}$, their tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$ is the $\overline{\mathcal{O}}_{\triangle}$-crystal such that

$$(\mathcal{E}_1 \otimes \mathcal{E}_2)(U) = \mathcal{E}_1(U) \otimes_B \mathcal{E}_2(U)$$

for every object $U = (\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \to X)$ of $(X/A)_{\triangle}$, and $\underline{\mathrm{hom}}(\mathcal{E}_1, \mathcal{E}_2)$ is the $\overline{\mathcal{O}}_{\triangle}$-crystal such that

$$\underline{\mathrm{hom}}(\mathcal{E}_1, \mathcal{E}_2)(U) = \mathrm{Hom}_B(\mathcal{E}_1(U), \mathcal{E}_2(U)).$$

Assume that $X = \mathrm{Spf}(R)$ satisfies the assumptions in Situation 4.1. Let (M_i, θ_i) with $i = 1, 2$ be the object of $\mathbf{Higgs}^{\wedge}(R)$ corresponding to $\mathcal{E}_i$ via the equivalence of categories in Theorem. 4.10. Then $\mathcal{E}_1 \otimes \mathcal{E}_2$ corresponds to the Higgs module $(M_1 \otimes_R M_2, \theta_1 \otimes 1 + 1 \otimes \theta_2)$, and $\underline{\mathrm{hom}}(\mathcal{E}_1, \mathcal{E}_2)$ corresponds to $(\mathrm{Hom}_R(M_1, M_2), \theta)$ such that

$$\theta(f)(x) = \theta_2(f(x)) - (f \otimes 1)(\theta_1(x)) \in M_2 \otimes_R \Omega^1_R\{-1\}$$

for all $x \in M_1$.

5.2. **Duality pairing.** Let $\mathcal{E}$ be an object of $\mathbf{CR}((X/A)_{\triangle}, \overline{\mathcal{O}}_{\triangle})^{fr}$. We put $\mathcal{E}^{\vee} := \underline{\mathrm{hom}}(\mathcal{E}, \underline{\mathcal{O}}_{\triangle})$ and

$$\mathcal{E}\{i\} := \mathcal{E} \otimes_{A/I} (I/I^2)^{\otimes i}$$

for all integers i. Then the cup product induces a pairing

$$\begin{aligned} R\nu_{X/A,*}(\mathcal{E}^{\vee}\{n\}) \otimes^L_{\mathcal{O}_X} R\nu_{X/A,*}(\mathcal{E}) &\to R\nu_{X/A,*}(\mathcal{E} \otimes \mathcal{E}^{\vee}\{n\}) \\ &\to R\nu_{X/A,*}(\overline{\mathcal{O}}_{\triangle}\{n\}) \to \Omega^n_X[-n], \end{aligned} \tag{5.2.1}$$

where the last map is induced by the Hodge–Tate comparison isomorphism [6, Theorem 4.11]

$$R^n\nu_{X/A,*}(\overline{\mathcal{O}}_{\triangle}) \cong \Omega^n_X\{-n\}.$$

If moreover X is proper over $\mathrm{Spf}(A/I)$, then one has similarly a pairing of perfect complexes of A/I-modules:

$$R\Gamma((X/A)_{\triangle}, \mathcal{E}^{\vee}\{n\}) \otimes^L_{A/I} R\Gamma((X/A)_{\triangle}, \mathcal{E}) \to R\Gamma((X/A)_{\triangle}\overline{\mathcal{O}}_{\triangle}\{n\}) \to A/I[-2n], \tag{5.2.2}$$

where the last map is induced by the isomorphism

$$H^{2n}((X/A)_{\triangle}, \overline{\mathcal{O}}_{\triangle}\{n\}) \cong H^n(X_{\text{ét}}, R^n\nu_{X/A,*}(\overline{\mathcal{O}}_{\triangle}\{n\})) = H^n(X_{\text{ét}}, \Omega^n_X) \cong A/I. \tag{5.2.3}$$

Theorem 5.3. *Under the notation above, the following statements hold:*

(1) *The pairing* (5.2.1) *induces an isomorphism of perfect complexes of $\mathcal{O}_X$-modules:*

$$R\nu_{X/A,*}(\mathcal{E}^{\vee}\{n\}) \xrightarrow{\sim} \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(R\nu_{X/A,*}(\mathcal{E}), \Omega^n_X)[-n].$$

(2) *If moreover X is proper over* $\mathrm{Spf}(A/I)$, *the pairing* (5.2.2) *induces an isomorphism of perfect complexes of (A/I)-modules*

$$R\Gamma((X/A)_{\triangle}, \mathcal{E}^{\vee}\{n\}) \cong R\,\mathrm{Hom}_{A/I}(R\Gamma((X/A)_{\triangle}, \mathcal{E}), A/I)[-2n]$$

Proof. It is clear that statement (2) is an immediate consequence of (1) and the classical Grothendieck–Serre duality. Since statement (1) is local for the étale topology of X, up to étale localization we may assume thus that $X = \mathrm{Spf}(R)$ satisfies the assumptions in Situation 4.1. Let (M, θ) be the object of $\mathbf{Higgs}^{\wedge}(R)$ corresponding to $\mathcal{E}$ by Theorem 4.10, and $(M^\vee\{n\}, \theta^\vee)$ be the object corresponding to $\mathcal{E}^\vee\{n\}$. Then we have $M^\vee\{n\} = \mathrm{Hom}_R(M, R\{n\})$ and $\theta^\vee$ is given by

$$\theta^\vee(f)(x) = -(f \otimes 1)(\theta(x)) \in \Omega^1_R\{n-1\},$$

for all $x \in M$ and $f \in M^\vee\{n\}$. By Theorem 4.12, the pairing (5.2.1) is represented by the pairing of complexes

$$\mathrm{DR}^\bullet(M^\vee\{n\}, \theta^\vee) \otimes_R \mathrm{DR}^\bullet(M, \theta) \to \Omega^n_R[-n]$$

given by

$$\langle f \otimes \omega_i, x \otimes \eta_j \rangle = \begin{cases} 0 & \text{if } i+j \neq n, \\ f(x)\omega_i \wedge \eta_j & \text{if } i+j = n, \end{cases}$$

for $x \in M$, $f \in M^\vee\{n\}$, $\omega_i \in \Omega^i_R\{-i\}$ and $\eta_j \in \Omega^j_R\{-j\}$. Now, it is straightforward to verify that, with the sign convention §0.2 , such a pairing induces an isomorphism of complexes

$$\mathrm{DR}^\bullet(M^\vee\{n\}, \theta^\vee) \cong \mathrm{Hom}(\mathrm{DR}^\bullet(M, \theta), \Omega^n_R)[-n].$$

This finishes the proof of (1). ☐

Remark 5.4. Assume that X is proper and smooth of relative dimension n over $\mathrm{Spf}(A/I)$. If one can construct a trace map

$$\mathrm{Tr}_X \colon H^{2n}((X/A)_{\Delta}, \mathcal{O}_{\Delta}\{n\}) \to A$$

which reduces to the isomorphism (5.2.3) when modulo I (so that Tr_X itself is an isomorphism), then Theorem 5.3 implies a similar duality for $\mathcal{O}_{\Delta}$-crystals.

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