NOTES ON TATE'S THESIS

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The aim of this short note is to explain Tate's thesis [Ta50] on the harmonic analysis on Adèles and Idèles, the functional equations of Dedekind Zeta functions and Hecke L-series. For general reference on adèles and idèles, we refer the reader to [We74].

1. Local Theory

1.1. Let k be a local field of characteristic 0, i.e. $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$. If k is p-adic, we denote by $\mathcal{O} \subset k$ the ring of integers in k, $\mathfrak{p} \subset \mathcal{O}$ the maximal ideal, and $\varpi \in \mathfrak{p}$ a uniformizer of $\mathcal{O}$. If $\mathfrak{a}$ is a fractional ideal of $\mathcal{O}$, we denote by $N\mathfrak{a} \in \mathbb{Q}$ the norm of $\mathfrak{a}$. So if $\mathfrak{a} \subset \mathcal{O}$ is an ideal, we have $N\mathfrak{a} = |\mathcal{O}/\mathfrak{a}|$. Let $|\cdot| : k \to \mathbb{R}_{\geq 0}$ be the normalized absolute value on k, i.e. for $x \in k$, we have

$$|x| = \begin{cases} |x|_{\mathbb{R}} & \text{if } k = \mathbb{R}; \\ |x|_{\mathbb{C}}^2 & \text{if } k = \mathbb{C}; \\ N(\mathfrak{p})^{-\mathrm{ord}_\varpi(x)} & \text{if } k \text{ is } p\text{-adic and } x = u\varpi^{\mathrm{ord}_\varpi(x)} \text{ with } u \in \mathcal{O}^\times. \end{cases}$$

We denote by k^+ the additive group of k. Consider the unitary character $\psi : k^+ \to \mathbb{C}^\times$ defined as follows:

$$\psi(x) = \begin{cases} e^{-2\pi i x} & \text{if } k = \mathbb{R}; \\ e^{-2\pi i(x+\bar{x})} & \text{if } k = \mathbb{C}; \\ e^{2\pi i \lambda(\mathrm{Tr}_{k/\mathbb{Q}_p}(x))} & \text{if } k \text{ is } p\text{-adic}, \end{cases} \tag{1.1.1}$$

where $\lambda(\cdot)$ means the decimal part of a p-adic number. For any $\xi \in k$, we note by ψ_ξ the additive character $x \mapsto \psi(x\xi)$ of k^+. Note that if k is non-archimedean, $\psi(x) = 1$ if and only if $x \in \mathfrak{d}^{-1}$, where $\mathfrak{d}$ is the different of k over $\mathbb{Q}_p$, i.e.

$$x \in \mathfrak{d}^{-1} \Leftrightarrow \mathrm{Tr}_{k/\mathbb{Q}_p}(xy) \in \mathbb{Z}_p \quad \forall y \in \mathcal{O}.$$

Proposition 1.2. *The map $\Psi : \xi \mapsto \psi_\xi$ defines an isomorphism of topological groups $k^+ \simeq \widehat{k^+}$, where $\widehat{k^+}$ denotes the group of unitary characters of k^+.*

Proof. If $k = \mathbb{R}$ or $\mathbb{C}$, this is well known in classical Fourier analysis. We assume here k is non-archimedean.

(1) It's clear that Ψ is a homomorphism of groups. We show first that Ψ is continuous (at 0). If $\xi \in \mathfrak{p}^m$, then ψ_ξ is trivial on $\mathfrak{d}^{-1}\mathfrak{p}^{-m}$. Since the subsets

$$U_m = \{\chi \in \widehat{k^+} \mid \chi \text{ is trivial on } \mathfrak{d}^{-1}\mathfrak{p}^{-m}\}$$

form a fundamental system of open neighborhoods of 0 in $\widehat{k^+}$, the continuity of Ψ follows immediately.

(2) Next, we show that $\Psi : k \to \Psi(k) \subset \widehat{k}$ is homoemorphism of k onto its image. We need to check that if $(x_n)_{n\geq 1} \in k$ is a sequence such that $\psi_{x_n} \to 1$ uniformly for all compact subsets of k, then x_n converges to 1 in k. Consider the compact open subgroup $\mathfrak{p}^{-m}$ for $m \in \mathbb{Z}$. Then for any $1/2 > \epsilon > 0$, there exists an integer $N > 0$ such that $|\psi(x_n z) - 1| < \epsilon$ for all $n > N$ and $z \in \mathfrak{p}^{-m}$. But $x_n\mathfrak{p}^{-m}$ is a subgroup and the open ball $B(1,\epsilon) \subset \mathbb{C}^\times$ contains no subgroup of S^1. Hence we have $\psi(x_n z) = 1$ for all $z \in \mathfrak{p}^{-m}$, so $x_n \in \mathfrak{d}^{-1}\mathfrak{p}^m$.

(3) The image of Ψ is dense in k. Let H be the image of Ψ, and $\bar{H} \subset \widehat{k}$ be its closure. Then we have

$$\begin{aligned}\bar{H}^\perp &= \{x \in \widehat{\widehat{k}} \simeq k \mid \chi(x) = 1,\ \forall \chi \in H\}\\ &= \{x \in k \mid \psi(x\xi) = 1,\ \forall \xi \in k\} = \{0\}\end{aligned}$$

Hence, we have $\bar{H} = \widehat{k}$.

(4) The proof of the Proposition will be complete by the Lemma 1.3 below. □

Lemma 1.3. *Let G be a locally compact topological group, $H \subset G$ be a locally compact subgroup. Then H is closed in G.*

Proof. Let h_n be a sequence in H that converges to $g \in G$. We need to prove that $g \in H$. Let $(U_r)_{r\geq 0}$ be a fundamental system of compact neighborhoods of 0. We have $\cap_{r\geq 0} U_r = \{0\}$. Then for any r, there exists an integer $N_r > 0$ such that $h_n \in g + U_r$ for all $n \geq N_r$. Up to modifying U_r, we may assume $h_n - h_m \in H \cap U_{r-1}$ for any $n, m \in N_r$. Note that $H \cap U_{r-1}$ is also compact by the local compactness of H. Up to replacing $\{U_r\}_{r\geq 0}$ by a subsequence, we may choose m_r for each integer r such that

$$h_{m_{r+1}} + U_r \cap H \subset h_{m_r} + U_{r-1} \cap H.$$

By compactness, the intersection

$$\bigcap_{r\geq 1} (h_{m_r} + U_{r-1} \cap H)$$

must contain an element $h \in H$. It's easy to see that $h = g$, since $\cap_{r\geq 0} U_r = \{0\}$.

□

1.4. Now we choose a Haar measure $\mathrm{d}x$ on k as follows. If $k = \mathbb{R}$, we take $\mathrm{d}x$ to be the usual Lebesgue measure on $\mathbb{R}$; if $k = \mathbb{C}$, we take $\mathrm{d}x$ to be twice of the usual Lebesgue measure on $\mathbb{C}$; and if k is non-archimedean, we normalize the measure by $\int_{\mathcal{O}} \mathrm{d}x = (N\mathfrak{d})^{-\frac{1}{2}}$. Let $L^1(k,\mathbb{C})$ be the space of complex valued absolutely integrable functions on k. For $f \in L^1(k,\mathbb{C})$, we define the Fourier transform of f to be

$$\hat{f}(\xi) = \int_k f(x)\psi(x\xi)\mathrm{d}x. \tag{1.4.1}$$

Let $\mathcal{S}(k)$ be the space of Schwartz functions on k, i.e.

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid \forall n, m \in \mathbb{N},\ |x^n \frac{d^m f}{dx^m}| \text{ is bounded}\};$$

we have a similar definition for $k = \mathbb{C}$; and if k is p-adic, $\mathcal{S}(k)$ consists of locally constant and compactly supported functions on k. In all these cases, the space $\mathcal{S}(k)$ is dense in $L^1(k, \mathbb{C})$.

Proposition 1.5. *The map $f \mapsto \hat{f}$ preserves $\mathcal{S}(k)$, and we have $\hat{\hat{f}}(x) = f(-x)$ for any $f \in \mathcal{S}(k)$.*

The following lemma will be useful in the sequels.

Lemma 1.6. *Assume k is non-archimedean. The local Fourier transform of $f = 1_{a+\mathfrak{p}^\ell}$, the characteristic function of the set $a + \mathfrak{p}^\ell$, is*

$$\hat{f}(x) = \psi(ax)(N\mathfrak{d})^{-\frac{1}{2}}(N\mathfrak{p})^{-\ell}1_{\mathfrak{d}^{-1}\mathfrak{p}^{-\ell}}. \tag{1.6.1}$$

In particular, we have $\hat{f} \in \mathcal{S}(k)$.

Proof. By definition, we have

$$\hat{f}(x) = \int_{a+\mathfrak{p}^\ell} \psi(xy)\mathrm{d}y = \psi(ax)\int_{\mathfrak{p}^\ell} \psi(xy)\mathrm{d}y.$$

The lemma follows immediately from

$$\int_{\mathfrak{p}^\ell} \psi(xy)\mathrm{d}y = \begin{cases} (N\mathfrak{d})^{-\frac{1}{2}}(N\mathfrak{p})^{-\ell} & \text{if } x \in \mathfrak{d}^{-1}\mathfrak{p}^{-\ell} \\ 0 & \text{otherwise.} \end{cases}$$

□

Proof of 1.5. If k is archimedean, this is well-known in classical analysis. Consider here the non-archimedean case. Since any compactly supported locally constant function on k is a linear combination of functions $1_{a+\mathfrak{p}^\ell}$. We may assume thus $f = 1_{a+\mathfrak{p}^\ell}$. The first part of the proposition follows from the previous lemma. For the second part, we have

$$\begin{aligned}\hat{\hat{f}}(x) &= \int_k \hat{f}(y)\psi(xy)\mathrm{d}y = (N\mathfrak{d})^{-\frac{1}{2}}(N\mathfrak{p})^{-\ell}\int_{\mathfrak{d}^{-1}\mathfrak{p}^{-\ell}} \psi((x+a)y)\mathrm{d}y \\ &= (N\mathfrak{d})^{-\frac{1}{2}}(N\mathfrak{p})^{-\ell}(N\mathfrak{d})^{-\frac{1}{2}}(N\mathfrak{p})^{\mathrm{ord}_\varpi(\mathfrak{d})+\ell}1_{-a+\mathfrak{p}^\ell} \\ &= 1_{-a+\mathfrak{p}^\ell}.\end{aligned}$$

In the third equality above, we have used (1.6.1) with ℓ replaced by $-\mathrm{ord}_\varpi(\mathfrak{d}) - \ell$ and x replaced by $x + a$. Now it's clear that $\hat{\hat{f}}(x) = f(-x)$. □

1.7. Now consider the multiplicative group $k^\times$, and put

$$U = \{x \in k^\times \mid |x| = 1\}.$$

So $U = \{\pm 1\}$ if $k = \mathbb{R}$, $U = S^1$ is the group of unit circle if $k = \mathbb{C}$, and $U = \mathcal{O}^\times$ if k is p-adic. We have

$$k^\times/U = \begin{cases} \mathbb{R}_+^\times & \text{if } k = \mathbb{R}, \mathbb{C}; \\ \mathbb{Z} & \text{if } k \text{ is } p\text{-adic.} \end{cases}$$

Recall that a *quasi-character* of $k^\times$ is a continuous homomorphism $\chi : k^\times \to \mathbb{C}^\times$. We say χ is a (unitary) *character* if $|\chi(x)| = 1$ for all $x \in k^\times$, and χ is *unramified* if $\chi|_U$ is trivial. So χ is unramified if and only if there is $s \in \mathbb{C}$ such that $\chi(x) = |x|^s$. Note that such an s is determined by χ if $k = \mathbb{R}$ or $\mathbb{C}$, and determined up to $2\pi i/\log(N\mathfrak{p})$ if k is p-adic.

Lemma 1.8. *For any quasi-character χ of $k^\times$, there exists a unique unitary character χ_0 of $k^\times$ such that $\chi = \chi_0 |\cdot|^s$.*

Proof. For any $x \in k^\times$, one can write uniquely $x = \tilde{x}\rho$ where $\tilde{x} \in U$ and $\rho \in \mathbb{R}_+^\times$ if $k = \mathbb{R}$ or $\mathbb{C}$, and $\rho \in \varpi^{\mathbb{Z}}$ if k is non-archimedean. We define χ_0 as $\chi_0(x) = (\chi|_U)(\tilde{x})$. One checks easily that the quasi-character χ/χ_0 is unramified. □

Let χ be a quasi-character of $k^\times$, and $s \in \mathbb{C}$ be the number appearing in the Lemma above. Note that $\sigma(\chi) = \Re(s)$ is uniquely determined by χ, and we call it the *exponent* of χ. Let $\nu \in \mathbb{Z}_{\geq 0}$ be the minimal integer such that $\chi|_{1+\mathfrak{p}^\nu}$ is trivial. We call the ideal $\mathfrak{f}_\chi = \mathfrak{p}^\nu$ *conductor* of χ. So the conductor of χ is $\mathcal{O}$ if and only if χ is unramified.

1.9. We choose the Haar measure on $k^\times$ to be $\mathrm{d}^\times x = \delta(k)\mathrm{d}x/|x|$, where

$$\delta(k) = \begin{cases} 1 & \text{if } k = \mathbb{R}, \mathbb{C}; \\ \frac{N\mathfrak{p}}{N\mathfrak{p}-1} & \text{if } k \text{ is non-archimedean.} \end{cases} \tag{1.9.1}$$

If k is non-archimedean, the factor $\delta(k)$ is justified by the fact that

$$\int_U \mathrm{d}x = (N\mathfrak{d})^{-\frac{1}{2}}.$$

Definition 1.10. For $f \in \mathcal{S}(k)$, we put

$$\zeta(f, \chi) = \int_{k^\times} f(x)\chi(x)\,\mathrm{d}^\times x,$$

which converges for any quasi-character χ with $\sigma(\chi) > 0$. We call $\zeta(f, \chi)$ the *local zeta function* associated with f (in quasi-characters).

Proposition 1.11. *For any $f, g \in \mathcal{S}(k)$, we have*

$$\zeta(f, \chi)\zeta(\hat{g}, \hat{\chi}) = \zeta(\hat{f}, \hat{\chi})\zeta(g, \chi),$$

where $\hat{f}, \hat{g}$ are Fourier transforms of f and g, and $\hat{\chi} = |\cdot|\chi^{-1}$ for any quasi-character χ with $0 < \sigma(\chi) < 1$.

Proof.

$$\begin{aligned} \zeta(f, \chi)\zeta(\hat{g}, \hat{\chi}) &= \int_{k^\times} \left(\int_{k^\times} f(x)\hat{g}(xy)|x|\mathrm{d}^\times x \right) \chi(y^{-1})|y|\mathrm{d}^\times y \\ &= \delta(k) \int_{k^\times} \left(\int_k \int_k f(x)g(z)\psi(xyz)\mathrm{d}z\mathrm{d}x \right) \chi(y^{-1})|y|\mathrm{d}^\times y. \end{aligned}$$

To finish the proof of the Proposition, it suffices to note that the expression above is symmetric for f and g. □

We endow the set of quasi-characters with a structure of complex manifold such that for any fixed quasi-character χ the map $s \mapsto \chi|\cdot|^s$ induces an isomorphism of complex manifolds from $\mathbb{C}$ to a connected component of the set of quasi-characters.

Theorem 1.12. *For any $f \in \mathcal{S}(k)$, the function $\zeta(f,\chi)$ can be continued to a meromorphic function on the space of all quasi-characters. Moreover, it satisfies the functional equation*

$$\zeta(f,\chi) = \rho(\chi)\zeta(\hat{f},\hat{\chi}), \tag{1.12.1}$$

where $\rho(\chi)$ is a meromorphic function of χ independent of f given as follows:

(1) *If $k = \mathbb{R}$, then $\chi(x) = |x|^s$ or $\chi(x) = \mathrm{sgn}(x)|x|^s$ for some $s \in \mathbb{C}$. We have*
$$\rho(|\cdot|^s) = 2^{1-s}\pi^{-s}\cos(\frac{\pi s}{2})\Gamma(s), \quad \rho(\mathrm{sgn}|\cdot|^s) = i2^{1-s}\pi^{-s}\sin(\frac{\pi s}{2})\Gamma(s).$$

(2) *If $k = \mathbb{C}$, then there exists $n \in \mathbb{Z}$ and $s \in \mathbb{C}$ such that $\chi = \chi_n|\cdot|^s$ where χ_n is the unitary character $\chi_n(re^{i\theta}) = e^{in\theta}$. We have*
$$\rho(\chi_n|\cdot|^s) = i^{|n|}\frac{(2\pi)^{1-s}\Gamma(s+\frac{|n|}{2})}{(2\pi)^s\Gamma(1-s+\frac{|n|}{2})}.$$

(3) *Assume k is p-adic. If χ is unramified, then*
$$\rho(|\cdot|^s) = (N\mathfrak{d})^{s-\frac{1}{2}}\frac{1-(N\mathfrak{p})^{s-1}}{1-N\mathfrak{p}^{-s}}.$$
If $\chi = \chi_0|\cdot|^s$ is ramified, where χ_0 is unitary with $\chi_0(\varpi) = 1$ as in Lemma 1.8, then one has
$$\rho(\chi_0|\cdot|^s) = N(\mathfrak{d}\mathfrak{f}_\chi)^{s-\frac{1}{2}}\rho_0(\chi_0)$$
with
$$\rho_0(\chi_0) = N(\mathfrak{f}_\chi)^{-\frac{1}{2}}\sum_x \chi_0(-x)\psi(\frac{x}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}})$$
where x runs over a set of representatives of $\mathcal{O}^\times/(1+\mathfrak{f}_\chi)$.

Proof. By Proposition 1.11, the function $\rho(\chi) = \frac{\zeta(f,\chi)}{\zeta(\hat{f},\hat{\chi})}$ is independent of f. This proves the functional equation (1.12.1). Note that $\zeta(f,\chi)$ is well defined if $\sigma(\chi) > 0$, and $\zeta(\hat{f},\hat{\chi})$ is well defined if $\sigma(\chi) < 1$. Therefore, once we show that $\rho(\chi)$ is meromorphic as in the statement, it will follow from the functional equation (1.12.1) that $\zeta(f,\chi)$ can be continued to a meromorphic function in χ. It remains to compute $\rho(\chi)$ by choosing special functions $f \in \mathcal{S}(k)$.

(1) Assume $k = \mathbb{R}$. If $\chi = |\cdot|^s$, we choose $f = e^{-\pi x^2}$. We have
$$\zeta(f,|\cdot|^s) = \int_{\mathbb{R}^\times} e^{-\pi x^2}|x|^s \mathrm{d}^\times x = 2\int_0^{+\infty} e^{-\pi x^2}x^{s-1}\mathrm{d}x = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}).$$
On the other hand,
$$\hat{f}(y) = \int_{\mathbb{R}} e^{-\pi(x^2+2ixy)}\mathrm{d}x = e^{-y^2}\int_{\mathbb{R}} e^{-\pi(x+yi)^2}\mathrm{d}x. \tag{1.12.2}$$

Using the well-known fact that

$$\int_{\mathbb{R}} e^{-\pi(x+yi)^2}\mathrm{d}x = \int_{\mathbb{R}} e^{-\pi x^2}\mathrm{d}x = 1,$$

we get $\hat{f} = f$. Hence, we have $\zeta(\hat{f}, |\cdot|^{1-s}) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})$ and

$$\rho(|\cdot|^s) = \frac{\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})}{\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})} = \pi^{-s}\sqrt{\pi}\frac{\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}$$

Now the formula for $\rho(|\cdot|^s)$ follows from the properties of Gamma functions

$$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = 2^{1-s}\sqrt{\pi}\Gamma(s), \quad \Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2}) = \frac{\pi}{\sin(\frac{\pi(1+s)}{2})}.$$

If $\chi = \mathrm{sgn}|\cdot|^s$, we take $f = xe^{-\pi x^2}$. A similar computation shows that

$$\zeta(f, \mathrm{sgn}|\cdot|^s) = \pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2}).$$

Taking derivatives with respect to y in (1.12.2), we get $\hat{f} = -if$. So we have

$$\zeta(\hat{f}, \mathrm{sgn}|\cdot|^{1-s}) = -i\pi^{\frac{s}{2}-1}\Gamma(1-\frac{s}{2}).$$

Therefore, we get

$$\begin{aligned}\rho(\mathrm{sgn}|\cdot|^s) &= \frac{\pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})}{-i\pi^{\frac{s}{2}-1}\Gamma(1-\frac{s}{2})} = i\pi^{-s}\sqrt{\pi}\frac{\Gamma(\frac{s+1}{2})\Gamma(\frac{s}{2})}{\Gamma(1-\frac{s}{2})\Gamma(\frac{s}{2})} \\ &= i2^{1-s}\pi^{-s}\sin(\frac{\pi s}{2})\Gamma(s).\end{aligned}$$

(2) Assume $k = \mathbb{C}$. If $\chi = |\cdot|^s$, we take $f(z) = e^{-\pi(z\bar{z})}$. The local zeta function associated with f is

$$\begin{aligned}\zeta(f, |\cdot|^s) &= \int_{\mathbb{C}^\times} e^{-\pi z\bar{z}}(z\bar{z})^s\mathrm{d}^\times z \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{+\infty} e^{-\pi r^2}r^{2s}\frac{2rdrd\theta}{r^2} \\ &= 4\pi\int_0^{+\infty} e^{-\pi r^2}r^{2s-1}dr \\ &= 4\pi\int_0^{+\infty} t^{s-\frac{1}{2}}e^{-\pi t}\frac{dt}{2\sqrt{t}} \quad (\text{set } t = r^2) \\ &= 2\pi^{1-s}\Gamma(s).\end{aligned}$$

The Fourier transform of f is

$$(1.12.3)\quad \begin{aligned}\hat{f}(z) &= \int_{\mathbb{C}} e^{-\pi w\bar{w}} e^{-2\pi i(zw+\bar{z}\bar{w})} \mathrm{d}w \\ &= 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\pi(u^2+v^2)} e^{-4\pi i(ux-vy)} dudv \quad (\text{put } z = x+iy, w = u+iv) \\ &= 2e^{-4\pi(x^2+y^2)} \int_{-\infty}^{+\infty} e^{-\pi(u+2ix)^2} du \int_{-\infty}^{+\infty} e^{-\pi(v-2iy)^2} dv \\ &= 2f(2z).\end{aligned}$$

Therefore, one has $\zeta(\hat{f}, |\cdot|^{1-s}) = 2^{2s-1}\zeta(f, |\cdot|^{1-s}) = 2^{2s}\pi^s\Gamma(1-s)$, thus

$$\rho(|\cdot|^s) = (2\pi)^{1-2s}\frac{\Gamma(s)}{\Gamma(1-s)}.$$

Let $n \geq 1$ and $\chi = \chi_{-n}|\cdot|^s$. We put $f_n = z^n e^{-\pi(z\bar{z})}$. We compute first the local zeta function of f_n:

$$(1.12.4)\quad \begin{aligned}\zeta(f_n, \chi_{-n}|\cdot|^s) &= \int_{\mathbb{C}^\times} z^n e^{-\pi(z\bar{z})}\chi_{-n}(z)(z\bar{z})^s \mathrm{d}^\times z \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{+\infty} e^{-\pi r^2} r^{2s+n}\frac{2rdrd\theta}{r^2} \\ &= 4\pi\int_0^{+\infty} e^{-\pi r^2} r^{2s+n-1} dr \\ &= 4\pi\int_0^{+\infty} e^{-\pi t} t^{s+\frac{n-1}{2}}\frac{dt}{2\sqrt{t}} \\ &= 2\pi^{1-(s+\frac{n}{2})}\Gamma(s+\frac{n}{2}).\end{aligned}$$

To find the Fourier transform of f_n, we consider the equality (1.12.3)

$$2e^{-4\pi(z\bar{z})} = \int_{\mathbb{C}} e^{-\pi(w\bar{w})} e^{-2\pi i(zw+\bar{z}\bar{w})} \mathrm{d}w.$$

Regarding z and $\bar{z}$ as independent variables and applying $\frac{\partial^n}{\partial z^n}$, we get

$$2(-2i\bar{z})^n e^{-4\pi z\bar{z}} = \int_{\mathbb{C}} w^n e^{-\pi(w\bar{w})} e^{-2\pi i(zw+\bar{z}\bar{w})} \mathrm{d}w,$$

that is, $\hat{f}_n(z) = 2\bar{f}_n(2iz)$. A similar computation as (1.12.4) shows that

$$\zeta(\hat{f}_n(z), \hat{\chi}) = \zeta(2\bar{f}_n(2iz), \chi_n|\cdot|^{1-s}) = (-i)^n 2^{2s}\pi^{s-\frac{n}{2}}\Gamma(s+\frac{n}{2}).$$

Therefore, we get

$$\rho(\chi_{-n}|\cdot|^s) = \frac{2\pi^{1-(s+\frac{n}{2})}\Gamma(s+\frac{n}{2})}{(-i)^n 2^{2s}\pi^{s-\frac{n}{2}}\Gamma(s+\frac{n}{2})} = i^n(2\pi)^{1-2s}\frac{\Gamma(s+\frac{n}{2})}{\Gamma(\frac{n}{2}+1-s)}.$$

The formulae for $\rho(\chi_n|\cdot|^s)$ can be proved in the same way by choosing $f = \bar{f}_n$.

(3) Assume k is p-adic. Consider first the case $\chi = |\cdot|^s$. We take $f = 1_{\mathcal{O}}$. In the proof of Proposition 1.5, we have seen that $\hat{f} = (N\mathfrak{d})^{-\frac{1}{2}} 1_{\mathfrak{d}^{-1}}$. We have

$$\zeta(f,\chi) = \int_{\mathcal{O}-\{0\}} |x|^s \mathrm{d}^\times x.$$

As $\mathcal{O} - \{0\} = \coprod_{n=0}^{+\infty} \varpi^n \mathcal{O}^\times$, it follows that

$$\zeta(f,\chi) = \sum_{n=0}^{+\infty} (N\mathfrak{p})^{-ns} \int_{\mathcal{O}^\times} \mathrm{d}^\times x = (N\mathfrak{d})^{-\frac{1}{2}} \frac{1}{1-(N\mathfrak{p})^{-s}}$$

Similarly, using $\mathfrak{d}^{-1} - \{0\} = \coprod_{n=-\mathrm{ord}_\varpi(\mathfrak{d})}^{+\infty} \varpi^n \mathcal{O}^\times$, one obtains

$$\begin{aligned}
\zeta(\hat{f},\hat{\chi}) &= (N\mathfrak{d})^{-\frac{1}{2}} \int_{\mathfrak{d}^{-1}-\{0\}} |x|^{1-s} \mathrm{d}^\times x \\
&= (N\mathfrak{d})^{-\frac{1}{2}} \sum_{n=-\mathrm{ord}_\varpi(\mathfrak{d})}^{+\infty} (N\mathfrak{p})^{n(s-1)} \int_{\mathcal{O}^\times} \mathrm{d}^\times x \\
&= (N\mathfrak{d})^{-1} (N\mathfrak{p})^{\mathrm{ord}_\varpi(\mathfrak{d})(1-s)} \sum_{n=0}^{+\infty} N\mathfrak{p}^{n(s-1)} \\
&= (N\mathfrak{d})^{-s} \frac{1}{1-N\mathfrak{p}^{s-1}}.
\end{aligned}$$

The formula for $\rho(|\cdot|^s)$ follows immediately.

Now consider the case $\chi = \chi_0 |\cdot|^s$ with χ_0 ramified, unitary and $\chi_0(\varpi) = 1$. We take

$$f(x) = \psi(\frac{x}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}}) 1_{\mathcal{O}}.$$

The local zeta function of f is

$$\begin{aligned}
\zeta(f,\chi) &= \int_{\mathcal{O}-\{0\}} \psi(\frac{x}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}}) \chi_0(x) |x|^s \mathrm{d}^\times x \\
&= \sum_{n=0}^{+\infty} (N\mathfrak{p})^{-ns} \int_{\mathcal{O}^\times} \psi(\frac{x\varpi^n}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}}) \chi_0(x) \mathrm{d}^\times x
\end{aligned}$$

We claim that

$$\int_{\mathcal{O}^\times} \psi(\frac{x\varpi^n}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}}) \chi_0(x) \mathrm{d}^\times x = 0 \quad \text{for } n \geq 1. \tag{1.12.5}$$

Consider first the case $n \geq \mathrm{ord}_\varpi(f_\chi)$. We have

$$\psi(\frac{x\varpi^n}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}}) = 1 \quad \text{as} \quad \frac{x\varpi^n}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}} \in \mathfrak{d}^{-1}.$$

If S is a set of representatives of $\mathcal{O}^\times/(1+\mathfrak{f}_\chi)$, the integral above is equal to

$$\int_{\mathcal{O}^\times} \chi_0(x) \mathrm{d}^\times x = (\int_{1+\mathfrak{f}_\chi} \mathrm{d}^\times x) \sum_{x\in S} \chi_0(x) = 0.$$

Assume $0 \le n \le \mathrm{ord}_\varpi(\mathfrak{f}_\chi) - 1$. For any $y \in 1 + \mathfrak{p}^{-n}\mathfrak{f}_\chi$, we have

$$\psi(\frac{xy\varpi^n}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}}) = \psi(\frac{x\varpi^n}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}}).$$

Therefore, if $S_n \subset S$ denotes a subset of representatives of $\mathcal{O}^\times/(1+\mathfrak{p}^{-n}\mathfrak{f}_\chi)$, we get

$$\begin{aligned}\int_{\mathcal{O}^\times} \psi(\frac{x\varpi^n}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}})\chi_0(x)\mathrm{d}^\times x &= \Big(\int_{1+\mathfrak{f}_\chi} \mathrm{d}^\times x\Big)\sum_{x\in S}\chi_0(x)\psi(\frac{x\varpi^n}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}}) \\ &= \Big(\int_{1+\mathfrak{f}_\chi} \mathrm{d}^\times x\Big)\sum_{x\in S_n}\chi_0(x)\psi(\frac{x\varpi^n}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}})\sum_y \chi_0(y),\end{aligned}$$

where y runs over a set of representatives of $(1+\mathfrak{p}^{-n}\mathfrak{f}_\chi)/(1+\mathfrak{f}_\chi)$. Note that

$$\sum_y \chi_0(y) = \begin{cases} 0 & \text{if } 1 \le n \le \mathrm{ord}_\varpi(\mathfrak{f}_\chi), \\ 1 & \text{if } n = 0. \end{cases}$$

This proves the claim. It follows that
(1.12.6)

$$\zeta(f,\chi) = \Big(\int_{1+\mathfrak{f}_\chi} \mathrm{d}^\times x\Big)\sum_{x\in S}\chi_0(x)\psi(\frac{x}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}}) = \chi_0(-1)\Big(\int_{1+\mathfrak{f}_\chi} \mathrm{d}^\times x\Big)(N\mathfrak{f}_\chi)^{\frac{1}{2}}\rho_0(\chi_0),$$

where we have used the definition of ρ_0 in the last step. As in the proof of 1.5, the Fourier transform of f is

$$\begin{aligned}\hat{f}(x) &= \int_{\mathcal{O}} \psi(\frac{y}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}})\psi(xy)\mathrm{d}y \\ &= \int_{\mathcal{O}} \psi(y(x + \frac{1}{\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}}))\mathrm{d}y \\ &= (N\mathfrak{d})^{-\frac{1}{2}} 1_{-\varpi^{-\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}+\mathfrak{d}^{-1}}.\end{aligned}$$

We get the local zeta function of $\hat{f}$

$$\begin{aligned}\zeta(\hat{f},\hat{\chi}) &= (N\mathfrak{d})^{-\frac{1}{2}}\int_{-\varpi^{-\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}+\mathfrak{d}^{-1}} |x|^{1-s}\chi_0^{-1}(x)\mathrm{d}^\times x \\ &= (N\mathfrak{d})^{-\frac{1}{2}}(N\mathfrak{p})^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)(1-s)}\int_{-\varpi^{-\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}(1+\mathfrak{f}_\chi)} \chi_0^{-1}(x)\mathrm{d}^\times x\end{aligned}$$

Since $\chi_0^{-1}(-\varpi^{\mathrm{ord}_\varpi(\mathfrak{d}\mathfrak{f}_\chi)}(1+y)) = \chi_0(-1)$ for any $y \in \mathfrak{f}_\chi$, we get

$$\zeta(\hat{f},\hat{\chi}) = \chi_0(-1)(N\mathfrak{d})^{-\frac{1}{2}}N(\mathfrak{d}\mathfrak{f}_\chi)^{1-s}\Big(\int_{1+\mathfrak{f}_\chi} \mathrm{d}^\times x\Big).$$

It thus follows that

$$\rho(\chi_0|\cdot|^s) = \frac{\zeta(f,\chi_0|\cdot|^s)}{\zeta(\hat{f},\chi_0^{-1}|\cdot|^{1-s})} = N(\mathfrak{d}\mathfrak{f}_\chi)^{s-\frac{1}{2}}\rho_0(\chi_0).$$

□

Remark 1.13. The number $\rho_0(\chi_0)$ in (3) is a generalization of (normalized) Gauss sum. By the same method as the classical case, we can show that $|\rho_0(\chi_0)| = 1$. In general, it's an interesting and difficult problem to find the exact argument of $\rho_0(\chi_0)$.

2. Global Theory

Let F be a number field, $\mathcal{O}_F$ be its ring of integers. Let Σ be the set of all places of F, and $\Sigma_f \subset \Sigma$ (resp. $\Sigma_\infty \subset \Sigma$) be the subset of non-archimedean (resp. archimedean) places. For $v \in \Sigma$, we denote by F_v the completion of F at v. Let $\mathrm{d}x_v$ be the self-dual Haar measure on F_v defined in 1.4. If v is finite, we denote by $\mathcal{O}_v$ the ring of integers of F_v, by $\mathfrak{p}_v$ the maximal ideal of $\mathcal{O}_v$, and we fix a uniformizer $\varpi_v \in \mathfrak{p}_v$. Let $\mathbb{A}_F$ be the adèle ring of F, i.e. the subring of $\prod_{v\in\Sigma} F_v$ consisting of elements $x = (x_v)_v$ with $x_v \in \mathcal{O}_v$ for almost all v, and $\mathbb{A}_{F,f}$ be the ring of finite adèles. We choose the Haar measure on $\mathbb{A}_F$ as $\mathrm{d}x = \prod_v \mathrm{d}x_v$. It induces a quotient Haar measure on $\mathbb{A}_F/F$.

Lemma 2.1. *Under the notation above, we have* $\int_{\mathbb{A}_F/F} \mathrm{d}x = 1$.

Proof. By Chinese reminders theorem, we have $\mathbb{A}_F = F + \prod_{v\in\Sigma_f} \mathcal{O}_v \times \prod_{v\in\Sigma_\infty} F_v$. We get thus an isomorphism

$$\mathbb{A}_F/F \simeq (\prod_{v\in\Sigma_f} \mathcal{O}_v \times \prod_{v\in\Sigma_\infty} F_v)/\mathcal{O}_F.$$

Hence we have

$$\begin{aligned}\int_{\mathbb{A}_F/F} \mathrm{d}x &= \prod_{v\in\Sigma_f} \int_{\mathcal{O}_v} \mathrm{d}x_v \times \int_{(\prod_{v\in\Sigma_\infty} F_v)/\mathcal{O}_F} \prod_{v\in\Sigma_\infty} \mathrm{d}x_v \\ &= \prod_{v\in\Sigma_f} (N\mathfrak{d}_v)^{-\frac{1}{2}} |\Delta_F|^{1/2},\end{aligned}$$

where $\mathfrak{d}_v$ denotes the different of F_v and Δ_F is the discriminant of F. If $\mathfrak{d}$ denotes the different of $F/\mathbb{Q}$, then the lemma follows easily from the product formula:

$$|\Delta_F| = N\mathfrak{d} = \prod_{v\in\Sigma_f} N\mathfrak{d}_v.$$

□

For $v \in \Sigma$, let ψ_v be the additive character of the local field F_v defined in (1.1.1). It's easy to check that $\psi = \prod_{v\in\Sigma} \psi_v$ is trivial on additive group F, therefore it defines a character of the quotient $\mathbb{A}_F/F$. We call it the basic character of $\mathbb{A}_F/F$ (or $\mathbb{A}_F$). For any $\xi \in \mathbb{A}_F$, let $\psi_\xi : \mathbb{A}_F \to \mathbb{C}^\times$ be the character given by $x \mapsto \psi(x\xi)$.

Proposition 2.2. *The map* $\Psi : \xi \mapsto \psi_\xi$ *defines an isomorphism between* $\mathbb{A}_F$ *and its topological dual* $\widehat{\mathbb{A}}_F$. *Moreover* ψ_ξ *is a character of* $\mathbb{A}_F/F$ *if and only if* $\xi \in F$, *i.e.* $\xi \mapsto \psi_\xi$ *gives rise to an isomorphism of topological groups* $F \simeq \widehat{\mathbb{A}_F/F}$.

Proof. The proof is similar to that of Proposition 1.2. One checks easily that Ψ is continuous and injective, and Ψ induces a homeomorphism of $\mathbb{A}_F$ onto its image. Conversely, let $\psi' : \mathbb{A}_F \to \mathbb{C}^\times$ be a continuous character. The restriction $\psi'_v = \psi'|_{F_v}$ to the v-th local component defines a continuous character of F_v. By Proposition 1.2, there exists $\xi_v \in F_v$ such that $\psi'_v = \psi_v(\xi_v \cdot \_)$. Since ψ' is continuous, there exists an open neighborhood $\prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{O}_v$ of 0 such that its image under ψ' lies in $B(1, 1/2) \subset \mathbb{C}^\times$. As $B(1, 1/2)$ contains no non-trivial subgroups of S^1, we see that for any $v \notin S$, we have $\xi_v \in \mathcal{O}_v$. This shows that $\xi = (\xi_v)_{v\in\Sigma} \in \mathbb{A}_F$, and $\psi' = \psi_\xi$. This shows that $\Psi : \mathbb{A}_F \to \widehat{\mathbb{A}}_F$ is a bijective continuous homomorphism of topological groups. To conclude that Ψ is an isomorphism, we need to show that if $\xi_n \in \mathbb{A}_F$ is a sequence such that $\psi_{\xi_n} \to 1$ in $\widehat{\mathbb{A}}_F$, we have $\xi_n \to 0$ in $\mathbb{A}_F$ as $n \to +\infty$. Actually, for any compact subset $U_v \subset F_v$ with $U_v = \mathcal{O}_v$ for almost all v and any $\epsilon > 0$, we have $|\psi_{\xi_n} - 1|_{\prod_v U_v} < \epsilon$ for n sufficiently large. By Proposition 1.2, for any finite subset $S \subset \Sigma$ containing Σ_∞, we can take $(U_v)_{v\in S}$ sufficiently large and $U_v = \mathcal{O}_v$ for $v \notin S$ such that $|\xi_n|_v < \epsilon$ for $v \in S$ and $\xi_n \in \mathcal{O}_v$ for $v \notin S$. This means that $\xi_n \to 0$ in $\mathbb{A}_F$.

For the second part, let $\Gamma \subset \mathbb{A}_F$ be the subgroup such that $\Psi(\Gamma) \subset \widehat{\mathbb{A}}_F$ consists of all characters trivial on F. It's clear that $F \subset \Gamma$ since ψ is trivial on F. To show that $\Gamma = F$, we consider first the case $F = \mathbb{Q}$. Let $\gamma \in \Gamma$. Since $\mathbb{A}_\mathbb{Q} = \mathbb{Q} + (-\frac{1}{2}, \frac{1}{2}] \times \prod_p \mathbb{Z}_p$, we can write $\gamma = b + c$, where $b \in \mathbb{Q}$, $c_\infty \in (-1/2, 1/2]$ and $c_p \in \mathbb{Z}_p$ for all primes p. Then we have

$$1 = \psi_\gamma(1) = \psi(\gamma) = \psi(b + c) = \psi(c) = e^{-2\pi i c_\infty}.$$

Hence we have $c_\infty = 0$. Moreover, for any prime p and any integer $n \geq 0$, we deduce from

$$1 = \psi_\gamma(\frac{1}{p^n}) = \psi(\frac{1}{p^n}(b + c)) = e^{2\pi i \lambda(\frac{c_p}{p^n})}$$

that $c_p \in p^n\mathbb{Z}_p$, i.e. we have $c_p = 0$. This shows $\gamma = b$, and hence $\Gamma = \mathbb{Q}$. In the general case, we note that the basic character of $\mathbb{A}_F$ is the composition of that on $\mathbb{A}_\mathbb{Q}$ with the trace map $\mathrm{Tr}_{F/\mathbb{Q}} : \mathbb{A}_F \to \mathbb{A}_\mathbb{Q}$. The following lemma will conclude the proof. □

Lemma 2.3. *Let $x = (x_v)_{v\in\Sigma} \in \mathbb{A}_F$ such that $\mathrm{Tr}_{F/\mathbb{Q}}(xy) \in \mathbb{Q} \subset \mathbb{A}_\mathbb{Q}$ for all $y \in F$. Then we have $x \in F$.*

Proof. Let $(e_i)_{1\leq i\leq d}$ be a basis of $F/\mathbb{Q}$, and $(e_i^*)_{1\leq i\leq d}$ be the dual basis with respect to the perfect pairing $F \times F \to \mathbb{Q}$ given by $(x, y) \mapsto \mathrm{Tr}_{F/\mathbb{Q}}(xy)$. For any place $p \leq \infty$ of $\mathbb{Q}$, we have a canonical isomorphism of $\mathbb{Q}_p$-algebras

$$F \otimes \mathbb{Q}_p \simeq \prod_{v|p} F_v.$$

We put $x_p = (x_v)_{v|p} \in \prod_{v|p} F_v$. Then we can write $x_p = \sum_{i=1}^d a_{p,i} e_i$ with $a_{p,i} \in \mathbb{Q}_p$. As $\mathrm{Tr}_{F/\mathbb{Q}}(xe_i^*) \in \mathbb{Q} \subset \mathbb{A}_\mathbb{Q}$ for any i, we deduce that $a_{p,i} \in \mathbb{Q}$ and it's independent of p. This shows that $x \in F$. □

Let $\mathcal{S}(\mathbb{A}_F)$ be the space of Schwartz functions on $\mathbb{A}_F$, i.e. the space of finite linear combinations of functions on $\mathbb{A}_F$ of the form $f = \prod_v f_v$, where $f_v \in \mathcal{S}(F_v)$ and $f_v = 1_{\mathcal{O}_v}$ for almost all v. For any $f \in \mathcal{S}(\mathbb{A}_F)$, we define the Fourier transform of f to be

$$\hat{f}(\xi) = \int_{\mathbb{A}_F} f(x)\psi(x\xi)\mathrm{d}x. \qquad (2.3.1)$$

Proposition 2.4. (a) *The Fourier transform* $f \mapsto \hat{f}$ *preserves the space* $\mathcal{S}(\mathbb{A}_F)$*, and* $\hat{\hat{f}}(x) = f(-x)$*.*

(b) *If* $f = \otimes_v f_v$ *with* $f_v \in \mathcal{S}(F_v)$ *and* $f_v = 1_{\mathcal{O}_v}$ *for almost all* v*. Then* $\hat{f} = \otimes_v \hat{f}_v$*, where* $\hat{f}_v$ *is the local Fourier transform* (1.4.1) *of* f_v*.*

(c) *For any* $f \in \mathcal{S}(\mathbb{A})$*, the infinite sum* $\sum_{x\in F} |f(x)|$ *converges, and we have the Poisson formulae*

$$\sum_{x\in F} f(x) = \sum_{\xi\in F} \hat{f}(\xi). \tag{2.4.1}$$

Proof. Statement (a) is a direct consequence of (b), which in turn follows from the local computations in the proof of 1.5. Now we start to prove (c). We may assume $f = \otimes_v f_v$ with $f_v \in \mathcal{S}(F_v)$ and $f_v = 1_{\mathcal{O}_v}$ for almost all v. Then there exists an open compact subgroup $U \subset \mathbb{A}_f$ such that $\mathrm{Supp}(f) \subset U \times \prod_{v\in\Sigma_\infty} F_v$. Put $\mathcal{O}_U = F \cap (U \times \prod_{v\in\Sigma_\infty} F_v)$. This is a lattice in F. Each individual term in the summation $\sum_{x\in F} f(x)$ is non-zero only if $x \in \mathcal{O}_U$. Write $f = f^\infty f_\infty$, where $f^\infty = \otimes_{v\in\Sigma_f} f_v$ and $f_\infty = \otimes_{v\in\Sigma_\infty} f_v$. Then there exists a constant $C > 0$ such that $|f^\infty(x)| < C$ for all $x \in U$. Hence, we have

$$\sum_{x\in F} |f(x)| = \sum_{x\in\mathcal{O}_U} |f(x)| < C \sum_{x\in\mathcal{O}_U} |f_\infty(x)|.$$

By classical analysis, the sum on the right hand side is convergent. This proves the first part of (c). It remains to show Poisson's summation formula (2.4.1). Consider the function $g(x) = \sum_{y\in F} f(x+y)$, which converges for any $x \in \mathbb{A}_F$ by the first part of (c). As $g(x)$ is invariant under translation of F, we regard $g(x)$ as a function on $\mathbb{A}_F/F$. Its Fourier transform of $g(x)$ is

$$\begin{aligned}\hat{g}(\xi) &= \int_{\mathbb{A}_F/F} g(x)\psi(x\xi)\mathrm{d}x \qquad (\text{for } \xi \in F)\\ &= \int_{\mathbb{A}_F} f(x)\psi(x\xi)\mathrm{d}x = \hat{f}(\xi).\end{aligned}$$

By the Fourier inverse formulae (a), we have

$$g(x) = \sum_{\xi\in F} \hat{g}(\xi)\psi(-x\xi).$$

The formulae (2.4.1) follows by setting $x = 0$. □

2.5. Let $\mathbb{I}_F = \mathbb{A}_F^\times$ be the multiplicative group of idèles of F, i.e. the subgroup of $\prod_{v\in\Sigma} F_v^\times$ consisting of elements $x = (x_v)_v$ with $x_v \in \mathcal{O}_v^\times$ for almost all v, and $\mathbb{I}_F^1$ be the subgroup of $\mathbb{I}_F$ of idèles with norm 1. The diagonal embedding $F^\times \hookrightarrow \mathbb{I}_F^1$ identifies $F^\times$ with a discrete subgroup of $\mathbb{I}^1$ for the induced restricted product topology on $\mathbb{I}_F^1$. A fundamental theorem in the theory of idèles says that the quotient $\mathbb{I}_F^1/F^\times$ is compact [We74, IV §4 Thm.6]. We consider the Haar measure $\mathrm{d}^\times x = \prod_v \mathrm{d}^\times x_v$ on $\mathbb{I}_F$, where $\mathrm{d}^\times x_v$ is the local Haar measure on $F_v^\times$ considered in 1.9. We use the same notation for the induced Haar measures on $\mathbb{I}_F^1$ and $\mathbb{I}_F^1/F^\times$.

Proposition 2.6. *Under the notation above, we have*

$$\mathrm{Vol}(\mathbb{I}_F^1/F^\times) = \int_{\mathbb{I}_F^1/F^\times} \mathrm{d}^\times x = \frac{2^{r_1}(2\pi)^{r_2} hR}{|\Delta_F|^{1/2} w},$$

where r_1 (resp. r_2) is the number of real places (resp. complex places) of F, h is the class number of F, Δ_F is the discriminant, R is the regulator, and w denotes the number of roots of unity in F.

Proof. Note first that $\mathrm{Vol}(\mathbb{I}_F^1/F^\times)$ is finite, since $\mathbb{I}_F^1/F^\times$ is compact. For each $x = (x_v)_{v\in\Sigma} \in \mathbb{I}_F$, we denote by $\mathrm{Div}(x) = \prod_{v\in\Sigma_f} \mathfrak{p}_v^{\mathrm{ord}_v(x_v)}$ be the fractional ideal associated with x. Then Div induces a short exact sequence

$$0 \to (\prod_{v\in\Sigma_f} \mathcal{O}_v^\times \times (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}) \times F^\times \to \mathbb{I}_F \to \mathrm{Cl}_F \to 0,$$

where Cl_F denotes the class group of F. Let Ω be the subgroup of $(\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$ with product of absolute values $\prod_{i=1}^{r_1} |x_i| \times \prod_{i=1}^{r_2} |z_i|_{\mathbb{C}} = 1$. The the exact sequence above induces a similar exact sequence

$$0 \to (\prod_{v\in\Sigma_f} \mathcal{O}_v^\times \times \Omega) \times F^\times \to \mathbb{I}_F^1 \to \mathrm{Cl}_F \to 0.$$

Therefore, one gets

$$\int_{\mathbb{I}_F^1/F^\times} \mathrm{d}^\times x^\times = h \int_{(\prod_v \mathcal{O}_v^\times \times\Omega)/(\prod_v \mathcal{O}_v^\times \times\Omega)\cap F^\times} \mathrm{d}^\times x.$$

Let U_F denote the group of units of F. We have $(\prod_v \mathcal{O}_v^\times \times \Omega) \cap F^\times = U_F$, and hence

$$\int_{(\prod_v \mathcal{O}_v^\times \times\Omega)/F^\times\cap(\prod_v \mathcal{O}_v^\times \times\Omega)} = (\prod_{v\in\Sigma_f} \int_{\mathcal{O}_v^\times} \mathrm{d}x_v^\times) \times \int_{\Omega/U_F} \mathrm{d}^\times x = \prod_{v\in\Sigma_f} N\mathfrak{d}_v^{-\frac{1}{2}} \int_{\Omega/U_F} \mathrm{d}^\times x.$$

In view of the product formula $\prod_{v\in\Sigma_f} N\mathfrak{d}^{-\frac{1}{2}} = |\Delta_F|^{-\frac{1}{2}}$, to complete the proof, it suffices to prove that

$$\int_{\Omega/U_F} \mathrm{d}^\times x = \frac{2^{r_1}(2\pi)^{r_2} R}{w}. \tag{2.6.1}$$

Consider the map

$$\begin{aligned} \mathrm{Log} : (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} &\to \mathbb{R}^{r_1+r_2} \\ ((x_i)_{1\le i\le r_1}, (z_j)_{1\le j\le r_2}) &\mapsto ((\log|x_i|)_{1\le i\le r_1}, (\log|z_j|^2)_{1\le j\le r_2}). \end{aligned}$$

Let S^1 be the unit circle subgroup of $\mathbb{C}^\times$, and V be the subspace of $\mathbb{R}^{r_1+r_2}$ defined by the linear equation $\sum_{i=1}^{r_1} x_i + \sum_{j=1}^{r_2} y_j = 0$. Then the map Log induces a short exact sequence of abelian groups

$$0 \to \{\pm 1\}^{r_1} \times (S^1)^{r_2} \to \Omega \xrightarrow{\mathrm{Log}} V \to 0.$$

If μ_F denotes the group of roots of unity in F, we have $(\{\pm 1\}^{r_1} \times (S^1)^{r_2}) \cap U_F = \mu_F$. Therefore, one obtains

$$\int_{\Omega/U_F} \mathrm{d}^\times x = \Big(\int_{\{\pm 1\}^{r_1}\times (S^1)^{r_2}/\mu_F} \mathrm{d}^\times x\Big) \times \Big(\int_{V/\mathrm{Log}(U_F)} \mathrm{d}^\times x\Big).$$

By the definition of the Haar measure on $\mathbb{I}_F$, the induced measure on $\Omega \subset (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$ is determined as follows. On each copy of $\mathbb{R}^\times$, the measure is given by $\mathrm{d}^\times x = \frac{dx}{|x|}$, where dx is the usual Lebesgue measure on $\mathbb{R}$; on each copy of $\mathbb{C}^\times$, the measure is given by

$$\mathrm{d}^\times z = 2\frac{dx\wedge dy}{|z|^2} = \frac{d(r^2)}{r^2}\wedge d\theta,$$

where $z = x+iy = re^{i\theta}$. Therefore, the Haar measure on $\mathbb{I}_F$ induces the usual Lebesgue measure on $\mathrm{Log}(\Omega) = V$, and the measure $\prod_{j=1}^{r_2} d\theta_j$ on $(S^1)^{r_2}$.[1] It follows that

$$\int_{\{\pm 1\}^{r_1}\times (S^1)^{r_2}/\mu_F} \mathrm{d}^\times x = \frac{2^{r_1}(2\pi)^{r_2}}{w}.$$

By the definition of the regulator, we have $R = \int_{V/\mathrm{Log}(U_F)} dx$. Now the formula (2.6.1) follows immediately. This finished the proof. □

2.7. A *Hecke character (or Grössencharacter)* of F is a continuous homomorphism $\chi : \mathbb{I}_F/F^\times \to \mathbb{C}^\times$. We say χ is unramified if there exists a complex number $s\in\mathbb{C}$ such that $\chi(x) = |x|^s$. We denote by X the set of Hecke characters of F. We equip X with a structure of Riemann surface such that for each fixed character χ, the map $s\mapsto \chi|\cdot|^s$ is a local isomorphism of $\mathbb{C}$ into X.

Now we choose a splitting $\mathbb{I}_F/F^\times = \mathbb{I}_F^1/F^\times \times \mathbb{R}_+^\times$ of the norm map $|\cdot| : \mathbb{I}_F/F^\times \to \mathbb{R}_+^\times$. For every Hecke character χ, we put $\chi_0 = \chi|_{\mathbb{I}^1_F/F^\times}$ and denote still by χ_0 its extension to $\mathbb{I}_F/F^\times$ by requiring χ_0 is trivial on the chosen complement $\mathbb{R}_+^\times$ of $\mathbb{I}_F^1/F^\times$. Note that χ_0 is necessarily unitary since $\mathbb{I}_F^1/F^\times$ is compact and χ/χ_0 is unramified, i.e. $\chi = \chi_0|\cdot|^s$ with $s\in\mathbb{C}$. We put $\sigma(\chi) = \Re(s)$, which is independent of the choice of the splitting. For $v\in\Sigma$, we put $\chi_v = \chi|_{F_v^\times}$. The local component χ_v is unramified for almost all v.

Definition 2.8. Let $f\in\mathcal{S}(\mathbb{A}_F)$ and χ be a Hecke character of F. We define the zeta function of f at χ to be

$$\zeta(f,\chi) = \int_{\mathbb{I}_F} f(x)\chi(x)\mathrm{d}^\times x.$$

Lemma 2.9. *Let $f\in\mathcal{S}(\mathbb{A}_F)$ and $\chi\in X$. Then the zeta function $\zeta(f,\chi)$ converges absolutely for $\sigma(\chi)>1$.*

---

[1] Note that the finiteness of $\mathrm{Vol}(\mathbb{I}^1/F^\times)$ implies that $\int_{V/\mathrm{Log}(U_F)} \mathrm{d}^\times x$ is finite, and hence $\mathrm{Log}(U_F)\subset V$ is a lattice. This actually gives another proof of Dirichlet's theorem that U_F has rank r_1+r_2-1.

Proof. We may assume $f = \otimes_{v\in\Sigma} f_v$ with $f_v = 1_{\mathcal{O}_v}$ for almost all $v \in \Sigma_f$, and $\chi = \chi_0|\cdot|^s$ where $\chi_0 : \mathbb{I}_F^1/F^\times \to S^1$ unitary. By definition, we have an Euler product

$$\zeta(f,\chi) = \prod_{v\in\Sigma} \zeta(f_v, \chi_v),$$

where $\zeta(f_v,\chi_v)$ is the local zeta function defined in 1.10. We have seen in the proof of Theorem 1.12(3) that

$$\zeta(1_{\mathcal{O}_v}, |\cdot|_v^s) = (N\mathfrak{d}_v)^{-\frac{1}{2}} \frac{1}{1-N\mathfrak{p}_v^{-s}}.$$

Thus there exists a finite subset S of places such that

$$|\zeta(f,\chi_0|\cdot|^s)| \le \prod_{v\in S} |\zeta(f_v,\chi_{0,v}|\cdot|_v^s)| \prod_{v\notin S} \frac{1}{1-N\mathfrak{p}_v^{-\sigma}}.$$

Since each $\zeta(f_v,\chi_{0,v}|\cdot|_v^s)$ converges for $\Re(s) > 0$, we are reduced to showing that the product $\prod_{v\notin S}\frac{1}{1-N\mathfrak{p}_v^{-\sigma}}$ converges absolutely for $\sigma > 1$. If $F = \mathbb{Q}$, this is a well-known theorem of Euler. In the general case, we have

$$\prod_{v\notin S} \frac{1}{1-N\mathfrak{p}_v^{-\sigma}} \le \prod_p \prod_{v|p} \frac{1}{1-N\mathfrak{p}_v^{-\sigma}} \le (\prod_p \frac{1}{1-p^{-\sigma}})^{[F:\mathbb{Q}]}.$$

□

Theorem 2.10 (Tate)**.** *Let $f \in \mathcal{S}(\mathbb{A}_F)$. The zeta function $\zeta(f,\chi)$ can be analytically continued to a meromorphic function on the whole complex manifold X. It satisfies the functional equation*

$$\zeta(f,\chi) = \zeta(\hat{f},\hat{\chi}), \tag{2.10.1}$$

where $\hat{f}$ is the Fourier transform of f (2.3.1), and $\hat{\chi} = |\cdot|\chi^{-1}$. Moreover, $\zeta(f,\chi)$ is holomorphic on the complex manifold X except for two simple poles at $\chi = 1$ and $\chi = |\cdot|$, with residues $-f(0)\mathrm{Vol}(\mathbb{I}_F^1/F^\times)$ at $\chi = 1$ and $\hat{f}(0)\mathrm{Vol}(\mathbb{I}_F^1/F^\times)$ at $\chi = |\cdot|$, where

$$\mathrm{Vol}(\mathbb{I}_F^1/F^\times) = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|\Delta_F|}}.$$

Proof. Let $\mathbb{I}_F^{\ge 1}$ (resp. $\mathbb{I}_F^{\le 1}$) be the subset of $\mathbb{I}_F$ with norm ≥ 1 (resp. ≤ 1). Since $\mathbb{I}_F^1 = \mathbb{I}_F^{\ge 1} \cap \mathbb{I}_F^{\le 1}$ has Haar measure 0 in $\mathbb{I}_F$, we have

$$\zeta(f,\chi) = \int_{\mathbb{I}_F} f(x)\chi(x)\mathrm{d}^\times x = \int_{\mathbb{I}_F^{\ge 1}} f(x)\chi(x)\mathrm{d}^\times x + \int_{\mathbb{I}_F^{\le 1}} f(x)\chi(x)\mathrm{d}^\times x.$$

Note that f is well-behaved when $|x| \to \infty$, the first integral $\int_{\mathbb{I}_F^{\ge 1}} f(x)\chi(x)\mathrm{d}^\times x$ converges absolutely for all $\chi \in X$, thus defines a holomorphic function on the whole complex manifold X. For the second integral, we have

$$\int_{\mathbb{I}_F^{\le 1}} f(x)\chi(x)\mathrm{d}^\times x = \int_{\mathbb{I}_F^{\le 1}/F^\times} (\sum_{\xi\in F^\times} f(\xi x))\chi(x)\mathrm{d}^\times x$$

by the triviality of χ on $F^\times$. It's easy to check that the Fourier transform of $f(x\cdot\_)$ is $|x|^{-1}\hat{f}(\frac{\cdot}{x})$. It follows from the Poisson formulae (2.4.1) that

$$\sum_{\xi\in F^\times} f(x\xi)=\sum_{\xi\in F^\times}\frac{1}{|x|}\hat{f}(\frac{\xi}{x})+\frac{1}{|x|}\hat{f}(0)-f(0).$$

Therefore, we get

$$\int_{\mathbb{I}_F^{\leq 1}/F^\times} f(x)\chi(x)\mathrm{d}^\times x=\int_{\mathbb{I}_F^{\leq 1}/F^\times}(\sum_{\xi\in F^\times}\frac{1}{|x|}\hat{f}(\frac{\xi}{x}))\chi(x)\mathrm{d}^\times x+\int_{\mathbb{I}_F^{\leq 1}/F^\times}(\frac{1}{|x|}\hat{f}(0)-f(0))\chi(x)\mathrm{d}^\times x.$$

Making the change of variable $y=\frac{1}{x}$, the first term on the right hand side above becomes

$$\begin{aligned}\int_{\mathbb{I}_F^{\leq 1}/F^\times}(\sum_{\xi\in F^\times}\hat{f}(\frac{\xi}{x}))\chi(x)\mathrm{d}^\times &=\int_{\mathbb{I}_F^{\geq 1}/F^\times}(\sum_{\xi\in F^\times}\hat{f}(y\xi))\hat{\chi}(y)\mathrm{d}^\times y\\ &=\int_{\mathbb{I}_F^{\geq 1}}\hat{f}(y)\hat{\chi}(y)\mathrm{d}^\times y.\end{aligned}$$

We choose a splitting $\mathbb{I}_F/F^\times=\mathbb{I}_F^1/F^\times\times\mathbb{R}_+^\times$ as in 2.7, and write that $\chi=\chi_0|\cdot|^s$, with $\chi_0:\mathbb{I}_F^1/F^\times\to\mathbb{C}^\times$ is unitary and $s\in\mathbb{C}$. We have

$$\int_{\mathbb{I}_F^{\leq 1}/F^\times}(\frac{\hat{f}(0)}{|x|}-f(0))\chi(x)\mathrm{d}^\times x=(\int_{\mathbb{I}_F^1/F^\times}\chi_0(x)\mathrm{d}^\times x)(\int_{t=0}^1(\frac{\hat{f}(0)}{t}-f(0))t^{s-1}dt.$$

We have

$$\int_{\mathbb{I}_F^1/F^\times}\chi_0(x)\mathrm{d}^\times x=\mathrm{Vol}(\mathbb{I}_F^1/F^\times)\delta_{\chi_0,1}=\begin{cases}0 & \text{if } \chi_0 \text{ is non-trivial;}\\ \mathrm{Vol}(\mathbb{I}_F^1/F^\times) & \text{if } \chi_0 \text{ is trivial.}\end{cases}$$

For the second term, we have

$$\int_{t=0}^1(\frac{\hat{f}(0)}{t}-f(0))t^{s-1}dt=\frac{\hat{f}(0)}{s-1}-\frac{f(0)}{s}.$$

Combining all the computations above, we get

$$\zeta(f,\chi)=\int_{\mathbb{I}_F^{\geq 1}}f(x)\chi(x)\mathrm{d}^\times x+\int_{\mathbb{I}_F^{\geq 1}}\hat{f}(x)\hat{\chi}(x)\mathrm{d}^\times x+\mathrm{Vol}(\mathbb{I}_F^1/F^\times)(\frac{\hat{f}(0)}{s-1}-\frac{f(0)}{s})\delta_{\chi_0,1}.$$

Now it's clear that the right hand side of the equation above is invariant with f replaced by $\hat{f}$ and χ replaced by $\hat{\chi}$. Thus (2.10.1) follows immediately. The moreover part follows from the fact that the first two integrals above define holomorphic functions on X. □

2.11. We indicate how to apply Tate's general theory to recover the classical results on the Dedekind Zeta function of a number field. Recall that Dedekind's zeta function is defined to be

$$\zeta_F(s)=\prod_{v\in\Sigma}\frac{1}{1-N\mathfrak{p}_v^{-s}}=\sum_{\mathfrak{a}\subset\mathcal{O}_F}\frac{1}{(N\mathfrak{a})^s},$$

which converges absolutely for $\Re(s)>1$. In the classical theory of Dedekind's zeta function, we have

Theorem 2.12. *Let F be a number field with r_1 real places and r_2 complex places. We put*

$$Z_F(s) = G_1(s)^{r_1} G_2(s)^{r_2} \zeta_F(s),$$

where $G_1(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$, $G_2(s) = (2\pi)^{1-s}\Gamma(s)$. Then $Z_F(s)$ is a meromorphic function in the s-plan, holomorphic except for simple zeros at $s = 0$ and $s = 1$, and satisfies the functional equation

$$Z_F(s) = |\Delta_F|^{\frac{1}{2}-s} Z_F(1-s).$$

Its residues at $s = 0$ and $s = 1$ are respectively $-\sqrt{|\Delta_F|}\mathrm{Vol}(\mathbb{I}_F^1/F^\times)$ and $\mathrm{Vol}(\mathbb{I}_F^1/F^\times)$.

Proof. We apply Tate's theorem 2.10 to $\chi = |\cdot|^s$, and $f = \otimes f_v$ with

$$f_v = \begin{cases} e^{-\pi x_v^2} & \text{if } v \text{ is real;} \\ e^{-\pi x_v \bar{x}_v} & \text{if } v \text{ is complex;} \\ 1_{\mathcal{O}_v} & \text{if } v \text{ is non-archimedean.} \end{cases}$$

By the local computations in 1.12, we have

$$\zeta(f_v, |\cdot|^s) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & \text{if } v \text{ is real;} \\ 2\pi^{1-s}\Gamma(s) & \text{if } v \text{ is complex;} \\ (N\mathfrak{d}_v)^{-\frac{1}{2}} \frac{1}{1-N\mathfrak{p}^{-s}} & \text{if } v \text{ is non-archimedean.} \end{cases}$$

Therefore, we get

$$\zeta(f, \chi) = \prod_{v\in\Sigma} \zeta(f_v, |\cdot|^s) = 2^{r_2 s} |\Delta_F|^{-\frac{1}{2}} Z_F(s).$$

On the other hand, we have $\hat{f} = \otimes_v \hat{f}_v$ with $\hat{f}_v = f_v$ if v is real, $\hat{f}_v(z) = 2f_v(2z)$ if v is complex, and $\hat{f}_v = (N\mathfrak{d}_v)^{-\frac{1}{2}} 1_{\mathfrak{d}_v^{-1}}$ if v is non-archimedean. The local zeta functions are

$$\zeta(\hat{f}_v, |\cdot|^{1-s}) = \begin{cases} \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2}) & \text{if } v \text{ is real;} \\ 2^s(2\pi)^s\Gamma(1-s) & \text{if } v \text{ is complex;} \\ (N\mathfrak{d}_v)^{-s} \frac{1}{1-N\mathfrak{p}^{s-1}} & \text{if } v \text{ is non-archimedean.} \end{cases}$$

Hence, we obtain

$$\zeta(\hat{f}, |\cdot|^s) = \prod_{v\in\Sigma} \zeta(\hat{f}_v, |\cdot|^s) = 2^{r_2 s} |\Delta_F|^{-s} Z_F(1-s).$$

The functional equation of $Z_F(s)$ follows immediately from $\zeta(f, |\cdot|^s) = \zeta(\hat{f}, |\cdot|^{1-s})$. The resides of $Z_F(s)$ follows from the residues of $\zeta(f, |\cdot|^s)$ and the fact that $f(0) = 1$ and $\hat{f}(0) = 2^{r_2}|\Delta_F|^{-\frac{1}{2}}$. □

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