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p-converse to a theorem of Gross–Zagier, Kolyvagin and Rubin

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Abstract Let E be a CM elliptic curve over the rationals and $p > 3$ a good ordinary prime for E. We show that

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1 \implies \mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}}) = 1$$

for the p^∞-Selmer group $\mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}})$ and the complex L-function $L(s, E_{/\mathbb{Q}})$. In particular, the Tate–Shafarevich group $Ш(E_{/\mathbb{Q}})$ is finite whenever $\mathrm{corank}_{\mathbb{Z}_p}$ $\mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1$. We also prove an analogous p-converse for CM abelian varieties arising from weight two elliptic CM modular forms with trivial central character. For non-CM elliptic curves over the rationals, first general results towards such a p-converse theorem are independently due to Skinner (A converse to a theorem of Gross, Zagier and Kolyvagin, arXiv:1405.7294, 2014) and Zhang (Camb J Math 2(2):191–253, 2014).

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Contents

1 Introduction

There seems to be a dichotomy in the arithmetic of elliptic curves over the rationals, the CM and non-CM elliptic curves. The CM action typically manifests itself and often relates the arithmetic of a CM elliptic curve to arithmetic of certain Hecke characters over the underlying CM field. In the CM case, we may hope to study arithmetic of $\mathrm{GL}_{2/\mathbb{Q}}$ via arithmetic of $\mathrm{GL}_{1/K}$ for the CM field K. In this article, we consider perhaps a new instance of such a phenomena in regards to the Birch and Swinnerton–Dyer (BSD) conjecture.

Elliptic curves of classical Diophantine interest are often CM. In regards to an arithmetic aspect, the case of CM elliptic curves typically precedes that of the non-CM elliptic curves. In this article, we consider an atypical instance with results being first obtained in the case of non-CM elliptic curves a few years ago.

Let E be an elliptic curve over the rationals. A fundamental arithmetic invariant is the Mordell–Weil rank given by the rank of the finitely generated abelian group $E(\mathbb{Q})$. As E varies, the rank is typically expected to be 0 or 1. The arithmetic complexity seems to deepen while moving from the former case to the latter. A mysterious structure governing the arithmetic of E is the conjecturally finite Tate–Shafarevich group $Ш(E_{/\mathbb{Q}})$. For a prime p, the p^∞-Selmer group $\mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}})$ encodes arithmetic of the elliptic curve via the exact sequence

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) \to Ш(E_{/\mathbb{Q}})[p^\infty] \to 0.$$

An underlying object on the analytic side is the complex L-function $L(s, E_{/\mathbb{Q}})$ corresponding to the elliptic curve $E_{/\mathbb{Q}}$ with $s \in \mathbb{C}$. A fundamental analytic invariant is the analytic rank given by the vanishing order $\mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}})$.

The BSD conjecture predicts a deep relation among the arithmetic and analytic facets. We consider the following typical instance.

Conjecture 1.1 *Let E be an elliptic curve over the rationals. For $r = 0, 1$, the following are equivalent.*

(1). $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = r$ *and* $Ш(E_{/\mathbb{Q}})$ *is finite.*
(2). $\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = r$ *for a prime* p.
(3). $\mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}}) = r$.

Part (2) evidently follows from part (1). On the other hand, the implication

$$\mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}}) = r \implies \mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = r, \#Ш(E_{/\mathbb{Q}}) < \infty$$

is a fundamental result on the BSD conjecture due to Coates–Wiles [16], Gross–Zagier [19], Kolyvagin [28], Rubin [37] and Kato [25]. This passage to the arithmetic facet from the analytic one goes back to mid 70's and mid 80's. In the case $r = 0$, the finiteness of the Mordell–Weil group for CM elliptic curves is due to Coates–Wiles around mid 70's. In the case $r \leq 1$, the implication is due to Gross–Zagier, Kolyvagin and Rubin for non-CM and CM elliptic curves, respectively around mid 80's. This is the theorem alluded to in the title. It is one of the rare instances where results were almost simultaneously obtained for non-CM and CM elliptic curves.

In this article, we refer to the implication

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = r \implies \mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}}) = r$$

as a p-converse theorem. Visibly, this is a p-adic criteria for an elliptic curve to have analytic rank r.

From now, we suppose that p is a good ordinary prime for $E_{/\mathbb{Q}}$.

In the case $r = 0$, p-converse theorem is well-known to follow from a divisibility in a p-adic Iwasawa main conjecture (IMC) for the elliptic curve. Here divisibility refers to a lower bound for an Iwasawa Selmer group associated to E along the $\mathbb{Z}_p$-cyclotomic extension of the rationals in terms of a p-adic L-function. For CM elliptic curves, the rank zero p-converse thus follows from a $\mathrm{GL}_{1/K}$-IMC due to Rubin [38] around early 90's for $p > 2$. Here K is the underlying CM field. For non-CM elliptic curves, the p-converse theorem follows from a divisibility in a $\mathrm{GL}_{2/\mathbb{Q}}$-IMC due to Skinner–Urban [42] around late 00's under certain hypotheses.

In the case $r = 1$, a p-converse theorem appeared out of reach until recently. For non-CM elliptic curves, a first general p-converse theorem is independently due to Skinner [43] and Zhang [50] a few years ago. The theorems were proven almost simultaneously under different hypotheses. The striking approaches due to Skinner and Zhang appear markedly distant at first. For now, we only mention that both crucially rely on an auxiliary IMC. The theorems and subsequent developments [14,48] exclude the case of CM elliptic curves.

The main result of the article is a p-converse theorem in the case of CM elliptic curves.

Theorem 1.2 *Let E be a CM elliptic curve over the rationals. Let $p > 3$ be a good ordinary prime for the elliptic curve $E_{/\mathbb{Q}}$.*

Then,

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1 \implies \mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}}) = 1.$$

In particular, $\mathrm{rank}_{\mathbb{Z}}E(\mathbb{Q}) = 1$ *and* $Ш(E_{/\mathbb{Q}})$ *is finite whenever* $\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1$.

Note that, "In particular" part follows from the work of Gross–Zagier, Kolyvagin and Rubin. We would like to emphasise that finiteness of the Tate–Shafarevich group $Ш(E_{/\mathbb{Q}})$ is not our hypothesis but in fact a consequence. For CM elliptic curves, the p-converse under finiteness of $Ш(E_{/\mathbb{Q}})[p^\infty]$ is indeed due to Rubin [40] around early 90's.

In the article, we prove a p-converse theorem for CM-abelian varieties arising from elliptic CM modular forms with weight two and trivial central character (Theorem 4.4).

We have a few consequences of the p-converse theorem.

Corollary 1.3 *Let E be a CM elliptic curve over the rationals. Let $p > 3$ be a good ordinary prime for $E_{/\mathbb{Q}}$. Suppose that* $\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1$.

Then, the p-part of full BSD conjecture holds for $E_{/\mathbb{Q}}$.

Proof From Theorem 1.2, the rank part of BSD holds. In particular,

$$\mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}}) = 1, \#Ш(E_{/\mathbb{Q}}) < \infty.$$

In view of the work of Perrin–Riou [35] and Rubin [40], the p-part of BSD formula thus holds (for example, [27, Cor. 1.4]). □

The following gives a mod p criteria for a CM elliptic curve to have analytic rank one and the p-part of underlying Tate–Shafarevich group to be trivial.

Corollary 1.4 *Let E be a CM elliptic curve over the rationals. Let $p > 3$ be a good ordinary prime for $E_{/\mathbb{Q}}$. Suppose that the following holds.*

(i). *The mod p Galois representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_p)$ arising from the p-torsion $E[p]$ is absolutely irreducible with $G_{\mathbb{Q}}$ an absolute Galois group of the rationals.*

(ii). *We have* $\mathrm{Sel}_p(E_{/\mathbb{Q}}) \simeq \mathbb{Z}/p\mathbb{Z}$ *for the p-Selmer group* $\mathrm{Sel}_p(E_{/\mathbb{Q}})$.

Then,

$$\mathrm{ord}_{s=1} L(s, E_{/\mathbb{Q}}) = 1, \text{Ш}(E_{/\mathbb{Q}})[p^\infty] = 0.$$

Proof In view of (i) and (ii),

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1.$$

Indeed, we note

$$\mathrm{Sel}_p(E_{/\mathbb{Q}}) = \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}})[p]$$

if $E(\mathbb{Q})[p] = 0$.

From Theorem 1.2, we now deduce

$$\mathrm{ord}_{s=1} L(s, E_{/\mathbb{Q}}) = 1, \#\text{Ш}(E_{/\mathbb{Q}}) < \infty.$$

From non-degeneracy of Cassels–Tate pairing on $\text{Ш}(E_{/\mathbb{Q}})[p^\infty]$ and (ii), we finally conclude

$$\text{Ш}(E_{/\mathbb{Q}})[p^\infty] = 0.$$

□

When $\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1$, the BSD conjecture predicts the existence of a non-torsion point in the Mordell–Weil group $E(\mathbb{Q})$. To approach a rank one p-converse theorem, one may typically begin with an auxiliary Heegner point as a candidate for being non-torsion and then attempt non-triviality based on a vertical/horizontal variation of certain Heegner points/ toric periods. We recall that auxiliary Heegner points are defined over certain imaginary quadratic fields.

Before describing our approach, we give a simplistic account of the approaches due to Skinner and Zhang for a rank one p-converse theorem in the non-CM case. In this paragraph alone, let E be a non-CM elliptic curve over the rationals with conductor N such that $\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1$ for an ordinary prime $p > 3$. Both begin with a choice of an auxiliary imaginary quadratic field K' satisfying the following Heegner hypothesis for $E_{/\mathbb{Q}}$.

(H) The number of primes dividing N which are inert or ramified in K' is even.

Accordingly, there exists a candidate for a desired non-torsion point $y_{K'} \in E(K')$ as a Heegner point. The non-triviality is then approached via different strategies. As in Gross–Zagier, Kolyvagin and Rubin, K' is more precsiely chosen so that

(i). Heegner hypothesis (H) holds for the pair (E, K'),
(ii). $L(1, E^{(K')}_{/\mathbb{Q}}) \neq 0$ for the quadratic twist $E^{(K')}$ and
(iii). p splits in K'.

From (ii) and Kolyvagin, note that $\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E_{/K'}) = 1$. In view of Gross–Zagier, the Heegner point $y_{K'} \in E(K')$ being non-torsion is equivalent to $\mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}}) = 1$. Skinner then resorts to p-adic Waldspurger formula due to Bertolini–Darmon–Prasanna [4,5] which expresses p-adic logarithm of $y_{K'}$ as a value of an anticyclotomic Rankin–Selberg p-adic L-function $L^{BDP}(E_{/K'})$ at the idenity Hecke character $1_{K'}$ over K' which is outside its interpolation range. Based on a variant of Galois descent, one notes that the implication

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E_{/K'}) = 1 \implies \widehat{1}_{K'}(L^{BDP}(E_{/K'})) \neq 0$$

follows from a divisibility in a Rankin–Selberg IMC for the p-adic L-function under a certain hypothesis. Here divisibility refers to a lower bound for an Iwasawa Selmer group of $E_{/K'}$ along the anticyclotomic $\mathbb{Z}_p$-extension of K' in terms of the p-adic L-function. Finally, such a divisibility is due to Wan [47]. To approach the non-triviality of the Heegner point $y_{K'}$, Zhang instead resorts to the technique of level raising and rank lowering along with Jochnowitz congruence due to Bertolini–Darmon, Jochnowitz and Vatsal ([46] and references therein). Under a certain hypothesis, Zhang in fact proves p-indivisibility of the Heegner point. For simplicity, let's suppose that $\mathrm{Sel}_p(E_{/K}) \simeq \mathbb{Z}/p$. Via Zhang's technique of level raising and rank lowering, one introduces a weight two elliptic newform g arising from a well-chosen level raising of the elliptic newform corresponding to the elliptic curve E such that $\mathrm{Sel}_p(g_{/K'}) = 0$. From the Jochnowitz congruence, the p-indivisibility of the Heegner point $y_{K'}$ is equivalent to the p-indivisibility of a noramlised central L-value $L^{\mathrm{alg}}(1, g_{/K'})$. Finally, the implication

$$\mathrm{Sel}_p(g_{/K'}) = 0 \implies p \nmid L^{\mathrm{alg}}(1, g_{/K'})$$

can be seen to follow from a divisibility in an IMC for g over K' due to Skinner–Urban [42].

These approaches seem to exclude the CM case in an essential manner. From now, let E be a CM elliptic curve over the rationals with CM by an order in an imaginary quadratic field K. A direct computation shows that the root number of E over K equals $+1$. In particular, the CM field K does not satisfy Heegner hypothesis (H) for $E_{/\mathbb{Q}}$. In the approaches of Skinner and Zhang, it may thus be tempting to choose an auxiliary quadratic field K' distinct from the CM field

K. However, Iwasawa theory in the CM case over imaginary quadratic fields distinct from the CM field seems to be in its early stages. As E has CM, it is not a semi-stable elliptic curve and the corresponding p-adic Galois image over K' is 'small'. On the other hand, being semi-stable or having 'large' Galois image seems indispensable for the Eisenstein congruence approach to obtain a lower bound for an Iwasawa Selmer group in an IMC. Consequently, neither Rankin–Selberg IMC divisibility for E over K' nor an IMC divisibility for an analogue of auxiliary newform g over K' is yet established.

Our approach instead involves working over the CM field K itself via an auxiliary Rankin–Selberg setup which leads to a candidate for a desired non-torsion point as a Heegner point. We approach the non-triviality via Iwasawa theory of Heegner points, the study of which rests upon elliptic units and anticyclotomic CM Iwasawa theory over K. An essential role is also played by Gross–Zagier formula and its Iwasawa-analogue along with non-vanishing of arithmetic invariants which appear in the anticyclotomic CM Iwasawa theory. In the remaining introduction, we provide an impressionistic account. Some of the notation used here is not followed in the rest of the article.

Let $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ be complex and p-adic embeddings, respectively. Let λ be an arithmetic Hecke character over K associated to the CM elliptic curve E. As $\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1$, the root number of λ equals -1 from the parity conjecture. In this case, the parity conjecture is due to Nekovář [31].

The auxiliary Rankin–Selberg setup involves an anticyclotomic twist as follows. Let χ be a finite order Hecke character over K unramified at p such that

$$L\left(1, \lambda^* \cdot \frac{\chi}{\chi*}\right) \neq 0. \tag{1.1}$$

For a Hecke character ψ over K, here $\psi^* := \psi \circ c$ for $c \in \mathrm{Gal}(K/\mathbb{Q})$ the non-trivial element. Note that $\frac{\chi}{\chi*}$ is anticyclotomic Hecke character over K and the existence of χ is thus due to Rohrlich [36]. Let g be the CM modular form associated to the Hecke character $\lambda\chi^{-1}$. We have a key factorisation

$$L(s, g \times \chi) = L(s, \lambda) \cdot L\left(s, \lambda^* \cdot \frac{\chi}{\chi*}\right)$$

of complex L-functions for the Rankin-Selberg convolution $g \times \chi$ corresponding to the pair (g, χ). In particular, $\mathrm{ord}_{s=1}L(s, g \times \chi) = 1 \iff \mathrm{ord}_{s=1}L(s, E_{/\mathbb{Q}}) = 1$. On the other hand, the Rankin-Selberg convolution $g \times \chi$ satisfies generalised Heegner hypothesis in the sense of Yuan–Zhang–Zhang, namely

(GH1). $\omega \cdot \chi|_{\mathbb{A}^\times} = 1$ for ω the Neben-type of g and $\mathbb{A}$ the adeles over $\mathbb{Q}$;
(GH2). $\epsilon(g \times \chi) = -1$ for $\epsilon(g \times \chi)$ the global root number of the Rankin–Selberg convolution $g \times \chi$.

In other words, (g, χ) corresponds to a self-dual Rankin-Selberg convolution with root number -1. The generalised Heegner hypothesis holds in view of λ being self-dual[1] with root number -1 and (1.1). We thus have a Heegner point

$$P_{g,\chi} \in B(K) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for a CM abelian variety $B_{/K}$ corresponding to the Rankin–Selberg convolution $g \times \chi$ arising from a Shimura curve parametrisation. In view of generalisation of Gross–Zagier formula due to Yuan–Zhang–Zhang [49], we have

$$\mathrm{ord}_{s=1} L\big(s, g \times \chi\big) = 1 \iff P_{g,\chi} \neq 0.$$

In view of (1.1), note that

$$\mathrm{corank}_{\mathcal{O}_\wp} \mathrm{Sel}_{\wp^\infty}\left(\lambda^* \cdot \frac{\chi}{\chi^*}\right) = 0$$

due to Kolyvagin/ Rubin. Here $\wp | p$ is a prime in the Hecke field corresponding to the Rankin–Selberg convolution $g \times \chi$ determined via the p-adic embedding ι_p, $\mathcal{O}$ an order in the Hecke field and $\mathrm{Sel}_{\wp^\infty}(\lambda^* \cdot \frac{\chi}{\chi^*})$ the Bloch–Kato Selmer group corresponding to the Hecke character $\lambda^* \cdot \frac{\chi}{\chi^*}$. Summing up, p-converse theorem is equivalent to the implication

$$\mathrm{corank}_{\mathcal{O}_\wp} \mathrm{Sel}_{\wp^\infty}(B_{/K}) = 1 \implies P_{g,\chi} \neq 0. \tag{1.2}$$

We approach the implication based on anticyclotomic Iwasawa theory of Heegner points, namely a Heegner main conjecture (HMC) for the pair (g, χ) along the anticyclotomic $\mathbb{Z}_p$-extension K_∞^- of K. Let Λ be the anticyclotomic Iwasawa algebra and ι the involution of Λ arising from inversion on the Galois group $\mathrm{Gal}(K_\infty^-/K)$.

Let $\mathcal{S}$ be the anticyclotomic compact Selmer group associated to B given by $\mathcal{S} = \varprojlim_n \mathrm{Sel}_{\wp^\infty}(B_{/K_n^-})$ for K_n^- the n^{th}-layer in the Iwasawa extension K_∞^-/K. Let $\mathcal{X}$ be the anticyclotomic divisible Selmer group associated to B and $\mathcal{X}_{\mathrm{tor}}$ a maximal Λ-torsion submodule. Let $\kappa_0 \in \mathrm{Sel}_{\wp^\infty}(B_{/K})$ be the cohomology class arising from Kummer image of the Heegner point $P_{g,\chi}$. Variant

[1] Strictly speaking, λ is not self-dual as an automorphic representation of $\mathrm{GL}_{1/K}$. It is so after an automorphic induction to $\mathbb{Q}$. We follow this unconventional terminology throughout.

of the construction of the Heegner point $P_{g,\chi}$ along the anticyclotomic $\mathbb{Z}_p$-extension gives rise to a Λ-adic Heegner cohomology class $\kappa \in \mathcal{S}$ deforming κ_0. The HMC is essentially due to Perrin–Riou [35] and predicts the following.

(HMC1). The Heegner class $\kappa \in \mathcal{S}$ is Λ-non-torsion and $\mathrm{rank}_\Lambda \mathcal{S} = \mathrm{rank}_\Lambda \mathcal{X} = 1$.

(HMC2). $\mathrm{Char}_\Lambda \mathcal{S}/(\kappa) \cdot (\mathrm{Char}_\Lambda \mathcal{S}/(\kappa))^\iota = \mathrm{Char}_\Lambda \mathcal{X}_{\mathrm{tor}}$ for $\mathrm{Char}_\Lambda(\cdot)$ the characteristic ideal.

The desired implication (1.2) follows from HMC via Galois descent.

The Heegner cohomology class κ being Λ-non-torsion is nothing but Mazur's conjecture. It has been recently proven in [11]. The central character of g being non-trivial, the case seems to be excluded from the work of Cornut–Vatsal and subsequent developments [1,8–10,17,46].

We now turn towards our approach to the remaining parts of HMC. The motive corresponding to the Rankin–Selberg convolution associated to the pair (g, χ) being a direct sum of motives corresponding to the pair $(\lambda, \lambda^* \cdot \frac{\chi}{\chi^*})$ of self-dual Hecke characters over K with opposite parity is perhaps the key. The decomposition permeates through the approach. It is based on Λ-adic Gross–Zagier formula on the $\mathrm{GL}_{2/\mathbb{Q}}$-side [18] and anticyclotomic CM Iwasawa theory on the $\mathrm{GL}_{1/K}$-side [2,3,39]. An essential role is also played by a non-vanishing of an anticyclotomic regulator [7,11]. The Selmer groups $\mathcal{S}$ and $\mathcal{X}$ decompose in terms of the ones corresponding to the pair $(\lambda, \lambda^* \cdot \frac{\chi}{\chi^*})$. The assertion on ranks in HMC part (i) thus follows from the one for the Iwasawa Selmer groups corresponding to the pair $(\lambda, \lambda^* \cdot \frac{\chi}{\chi^*})$ due to Agboola–Howard [2] and Rubin [38]. The latter fundamentally relies on an Euler system of elliptic units. As for HMC part (ii), we commence with Disegni's Λ-adic Gross–Zagier formula which expresses the Heegner index corresponding to $\mathcal{S}/\kappa$ in terms of the cyclotomic derivative $L'_p(g \times \chi) \in \Lambda$ of an underlying two-variable Rankin–Selberg p-adic L-function $L_p(g \times \chi)$ over K up to an anticyclotomic regulator $\mathcal{R}_{Hg}$. Here the regulator corresponds to the Λ-module $\mathcal{S}$ with rank one (HMC part (i)). A factorisation

$$L'_p(g \times \chi) = L'_p(\lambda) \cdot L^-_p\left(\lambda^* \cdot \frac{\chi}{\chi *}\right)$$

of anticyclotomic p-adic L-functions initiates a passage to $\mathrm{GL}_{1/K}$. Here $L'_p(\lambda)$ is the cyclotomic derivative of a two-variable Katz p-adic L-function $L_p(\lambda)$ over K and $L^-_p(\lambda^* \cdot \frac{\chi}{\chi *})$ an anticyclotomic Katz p-adic L-function along K^-_∞/K. From the anticyclotomic CM Iwasawa theory, results of Agboola–Howard and Arnold (resp. Rubin) express $L'_p(\lambda)$ (resp. $L^-_p(\lambda^* \cdot \frac{\chi}{\chi *})$) in terms of characteristic ideal of Λ-module $\mathcal{X}(\lambda)_{\mathrm{tor}}$ up to an anticyclotomic regulator $\mathcal{R}_{El}$ (resp. $\mathcal{X}(\lambda^* \cdot \frac{\chi}{\chi *})$). Here $\mathcal{X}(\cdot)$ denotes an anticyclotomic divisible Selmer group and the regulator corresponds to the Λ-compact Selmer group associated to λ which turns out to have rank one. In view of the decomposition of anticyclotomic Selmer group $\mathcal{X}$, the discussion so far shows that HMC part (ii) would finally follow from the non-vanishing

$$\mathcal{R}_{Hg} = \mathcal{R}_{El} \neq 0$$

of the anticyclotomic regulator. Building on Hida's approach [22], the non-vanishing has been proven in [7]. In the case of imaginary quadratic fields K with class number one, the non-vanishing is in fact independently due to Rubin [2, App.]. In this manner, we end up proving HMC based on a Rankin–Selberg IMC in the sign -1 case along with non-vanishing of an anticyclotomic regulator.

The auxiliary twist thus leads to a Rankin–Selberg setup viable for Iwasawa theory. As if in a fugue, the manner in which various results from anticyclotomic CM Iwasawa theory complement each other is mysterious to us. It may be worth mentioning that unlike the approaches of Skinner and Zhang, a genuine $\mathrm{GL}_{2/\mathbb{Q}}$-IMC does not seem to be present in our approach to a p-converse theorem. The only $\mathrm{GL}_{2/\mathbb{Q}}$-ingredients seem to be Gross–Zagier formula, its Λ-adic analogue and Chai–Oort rigidity principle for self-products of modular curves in characteristic p [22]. Unlike Skinner's approach, our approach does not involve p-adic Waldspurger formula. The substitute seems to be Λ-adic Gross–Zagier formula perhaps at the expense of non-vanishing of an anticyclotomic regulator.

We would like to emphasise that generality of the Gross–Zagier formula [19] due to Yuan–Zhang–Zhang YZZ is foundational to the approach. In fact, their very formulation allows an access to a self-dual Rankin–Selberg setup for a pair (g, χ) with central character of the Hecke eigenform g or of the twist χ being possibly non-trivial. As indicated earlier, the generality is perhaps essential to us. The p-converse theorem seems to be one of the first results towards the BSD conjecture which crucially relies on the Yuan–Zhang–Zhang formalism with central character of the Hecke eigenform g or of the twist χ being non-trivial. We hope that the article initiates study of arithmetic aspects of Yuan–Zhang–Zhang formalism in its generality.

As is evident, the approach builds on anticyclotomic CM Iwasawa theory due to Rubin [38,39], Agboola–Howard [2] and Arnold [3]. The Euler system of elliptic units due to Rubin underlies the CM Iwasawa theory. An auxiliary Rankin–Selberg setup is perhaps our point of departure. It leads to Heegner points relevant to CM Iwasawa theory. Even though such Heegner points seem to be missing in the earlier works, we study Iwasawa theory of the Heegner points (HMC) partly based on [2] and [3]. A few developments following the earlier works contribute crucially to our study, namely Gross–Zagier formula due to Yuan–Zhang–Zhang [49] and its Λ-adic analogue due to Disegni [18] along with non-vanishing of the anticyclotomic regulator due to Rubin [2] and [7].

Non-vanishing of arithmetic invariants in anticyclotomic CM Iwasawa theory forms backbone of the approach. A search for auxiliary Rankin–Selberg setup over the CM field K invokes non-vanishing of central Hecke L-values due to Rohrlich [36]. Iwasawa theory of Heegner points (HMC part (i)) arising from the auxiliary setup invokes non-vanishing of the Heegner points in [11]. Finally, the proof of HMC part (ii) invokes non-vanishing of an anticyclotomic regulator. These non-vanishing results can in fact be proven uniformly based on Hida's approach to non-vanishing alluded to above.

On the whole, Iwasawa theory seems essential in our approach to the p-converse theorem.

In regards to the arithmetic of self-dual Hecke characters, the auxiliary setup seems to be rich. For example, it seems to lead to an anticyclotomic Euler system for a self-dual Hecke character over a CM field. This is in contrast to the fact that analogue of elliptic units over a general CM field being not yet known. The study of related topics will appear in the near future. We may ask if the current approach to p-converse works for all ordinary primes, for example $p = 2$.

For a class of congruent number CM elliptic curves, a p-converse theorem was established in [44] with $p = 2$. In this case, p is in fact a prime of bad supersingular reduction. In the non-CM supersingular case, we refer to Castella–Wan [15] for subsequent development regarding a p-converse theorem. This article is perhaps a follow up to the work of Castella [14], Skinner [43], Wan [48] and Zhang [50]. We refer to these articles for a general introduction.

The article is organised as follows. In Sect. 2, we describe Heegner main conjecture (HMC) for Rankin–Selberg convolution corresponding to a self-dual pair of a weight two elliptic newform and a Hecke character over an imaginary quadratic field with root number -1. In Sect. 2.1, we introduce the setup. In Sect. 2.2, we introduce the relevant Heegner points. In Sect. 2.3, we describe the conjecture. In Sect. 3, we consider Heegner main conjecture (HMC) for Rankin–Selberg convolution corresponding to a self-dual pair of a weight two elliptic CM modular form and a Hecke character over the same CM field with root number -1. In Sect. 3.1, we introduce the main result. In Sect. 3.2, we introduce underlying Selmer groups arising in our approach to HMC. In Sect. 3.3, we introduce underlying p-adic L-functions arising in the approach. In Sect. 3.4, we conclude with the proof. In Sect. 4, we prove a p-converse theorem in the CM case. In Sect. 4.1, we introduce an auxiliary twist of the underlying Hecke character. In Sect. 4.2, we conclude the proof based on a HMC which involves an auxiliary Rankin–Selberg setup arising from the twist.

Notation

We use the following notation unless otherwise stated.

Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$.

For a subfield $F \subset \overline{\mathbb{Q}}$, let $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. For a set of places Σ of F, let $F_\Sigma \subset \overline{\mathbb{Q}}$ denote the maximal extension of F unramified outside Σ and $G_{F,\Sigma} = \mathrm{Gal}(F_\Sigma/F)$. For $F = \mathbb{Q}$, we often drop the subscript 'F' from notation. For a place v of F, let $\overline{F}_v$ denote a fixed algebraic closure of F_v and $G_{F_v} = \mathrm{Gal}(\overline{F}_v/F_v)$. Let $I_v \subset G_{F_v}$ denote the inertia subgroup. In the case residue field of F_v being finite, let $\mathrm{Frob}_v \in G_{F_v}/I_{F_v}$ denote an arithmetic Frobenius. Typically, an F-linear embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{F}_v$ will be chosen which identifies G_{F_v} as a subgroup of G_F.

Let $\mathcal{O}_F$ be the corresponding integer ring and D_F the discriminant.

Let $\mathbb{A}_F$ denote the adeles over F. For a finite subset S of places in F, let $\mathbb{A}_F^{(S)}$ denote the adeles outside S and $\mathbb{A}_{F,S}$ the S-part. When F equals the rationals, we drop the subscript F. For a $\mathbb{Q}$-algebra C, let $C_{\mathbb{A}} = C \otimes_{\mathbb{Q}} \mathbb{A}$. Let $\widehat{C}^{(S)}$ (resp. C_S) denote the part outside S (resp. S-part) of $C_{\mathbb{A}}$.

For a place v of $\mathbb{Q}$ and a quadratic extension $K_v/\mathbb{Q}_v$, let η_v denote the corresponding quadratic character. For a quaternion algebra $B_v/\mathbb{Q}_v$, let $\epsilon(B_v)$ denote the corresponding invariant. In a non-standard manner, we take the invariant to be 1 (resp. -1) if the quaternion is split (resp. non-split).

For an imaginary quadratic extension $K/\mathbb{Q}$ and an integral ideal $\mathfrak{c}$ of $\mathbb{Q}$, let $H_{K,\mathfrak{c}}$ be the ring class field with conductor $\mathfrak{c}$ and $\mathrm{Pic}^{\mathfrak{c}}_{K/F}$ the relative ring class group with conductor $\mathfrak{c}$. Let h_K (resp. $h_{K,\mathfrak{c}}$) be the ideal class number of K (resp. $H_{K,\mathfrak{c}}$).

For a finite abelian group G, let $\widehat{G}$ denote the $\overline{\mathbb{Q}}^\times$-valued character group of G. For a $\mathbb{Z}$-algebra A, let $\widehat{A} = A \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ for $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n$. For a $\mathbb{Z}$-module M, let $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$.

2 Heegner main conjecture

In this section, we consider Heegner main conjecture (HMC) for Rankin–Selberg convolution corresponding to a self-dual pair of a weight two elliptic newform and a finite Hecke character over an imaginary quadratic field with root number -1. In Sect. 2.1, we introduce the setup. In Sect. 2.2, we introduce a norm compatible sequence of Heegner points. In Sect. 2.3, we describe the conjecture.

We emphasise that the root number of the newform over the imaginary quadratic field need not be -1. Such a conjecture is essentially due to Perrin–Riou [35].

2.1 Setup

In this subsection, we introduce the underlying objects and hypotheses.

2.1.1 *Embeddings*

Let p be an odd prime. We fix two embeddings $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \to \mathbb{C}_p$. Let v_p be the p-adic valuation induced via the embedding ι_p so that $v_p(p) = 1$.

2.1.2 *Imaginary quadratic field*

Let K be an imaginary quadratic field and $\mathcal{O}$ the ring of integers. We regard K as a subfield of $\mathbb{C}$ via the embedding ι_∞. Let η denote the quadratic character over $\mathbb{Q}$ corresponding to the extension $K/\mathbb{Q}$. Let c be the complex conjugation on $\mathbb{C}$ which induces the unique non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$ via ι_∞.

We assume the following:

(ord) p splits in K.

Let $\mathfrak{p}$ be the prime above p in K induced via the p-adic embedding ι_p. For a positive integer m, let H_m be the ring class field of K with conductor m and $\mathcal{O}_m = \mathbb{Z} + m\mathcal{O}$ the corresponding order. Let H be the Hilbert class field.

Let K_∞^- be the anticyclotomic $\mathbb{Z}_p$-extension of K and $\Gamma_K^- = \mathrm{Gal}(K_\infty^-/K)$. For each $n \geq 1$, let K_n^- be the subextension of K_∞^- with degree p^n over K. Let K_∞ be the $\mathbb{Z}_p^2$-extension of K and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$.

2.1.3 *Self-dual pair*

Let $g \in S_2(\Gamma_0(N), \omega)$ be a weight two elliptic newform with Neben-type ω and E_g the corresponding Hecke field. Let $A = A_g$ be an abelian variety over the rationals associated to g by Eichler–Shimura such that

$$L(s, A_{/\mathbb{Q}}) = \prod_{\sigma: E_g \to \mathbb{C}} L(s, g^\sigma).$$

We recall that the endomorphism ring for A contains an order $\mathcal{O}_g$ in the Hecke field E and the order is generated over $\mathbb{Z}$ by the Hecke eigenvalues corresponding to g.

Let χ be an arithmetic Hecke character over the imaginary quadratic field K. Let $E_{g,\chi} \subset \mathbb{C}$ be the subfield generated over $\mathbb{Q}$ by the Hecke eigenvalues of g and the image of χ on $\mathbb{A}_K^{(\infty),\times}$.

Let $\wp$ be a prime above p in $E_{g,\chi}$ determined via the embedding ι_p. Let L be the completion of $E_{g,\chi}$ at $\wp$ and $\mathcal{O}$ the corresponding integer ring. Let $\mathcal{O}_{g,\chi} \subset E_{g,\chi}$ be generated over $\mathcal{O}_g$ by the values of χ. Let $\wp_0$ be the prime of $\mathcal{O}_{g,\chi}$ given by $\wp \cap \mathcal{O}_{g,\chi}$. Let $\mathcal{O}_0$ be the localisation of $\mathcal{O}_{g,\chi}$ at $\wp_0$. By definition, $\mathcal{O}_0$ is a subring of $\mathcal{O}$.

Let

$$\Lambda^\circ = \mathcal{O}[[\Gamma_K^-]] \text{ and } \Lambda = \Lambda^\circ \otimes_{\mathcal{O}} L.$$

Let ι be the involution on Λ induced by the inverse map on the anticyclotomic Galois group Γ_K^-. For a Λ^0-module M, let M^ι denote the corresponding twist.

We adopt the following normalisation in regards to Rankin–Selberg convolution corresponding to the pair (g, χ). Let π be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by g and (ℓ_1, ℓ_2) the infinity type of χ. We then consider the Rankin–Selberg L-function $L(s, g \times \chi)$ given by

$$L(s, g \times \chi) = L\left(s - \frac{1+\ell_1+\ell_2}{2}, \pi \times \chi\right).$$

Then, $L(s, g \times \chi)$ satisfies a functional equation with center $\frac{2+\ell_1+\ell_2}{2}$ ([5, §4.1]).

Suppose that χ is with finite order and

(SD) $\omega \cdot \chi|_{\mathbb{A}^\times} = 1$

In particular, the Rankin–Selberg convolution $L(s, g \times \chi)$ corresponding to the pair (g, χ) is self-dual with functional equation around the center $s = 1$.

We have the following variant of Eichler–Shimura construction.

Definition 2.1 Let (g, χ) be a pair of an elliptic newform with weight two and a finite order Hecke character over an imaginary quadratic field K as above. Let $E_{g,\chi}$ be the Hecke field corresponding to the pair generated over $\mathbb{Q}$ by the Hecke eigenvalues of g and the image of χ. Let B be the Serre tensor $A \otimes \chi$ [45, p. 734] and [30, Def. 1.1]. It is an abelian variety defined over K such that

$$L(s, B_{/K}) = \prod_{\sigma: E_{g,\chi} \to \mathbb{C}} L(s, g^\sigma \times \chi^\sigma)$$

for $^\sigma$ being the Galois-conjugate.

In view of (SD), the dual abelian variety $B_{/K}^\vee$ is isogenous to $B_{/K}$. We fix such an isogeny.

2.1.4 Selmer groups

We now introduce Selmer groups associated to the abelian variety B over the anticyclotomic tower.

By definition, we have an embedding $\mathcal{O}_{g,\chi} \hookrightarrow \mathrm{End}(B)$ for $\mathcal{O}_{g,\chi}$ being generated over $\mathcal{O}_g$ by the values of χ as above. Recall $\wp | p$ denotes the prime of the Hecke field $E_{g,\chi}$ determined via the embedding ι_p as above and $\wp_0$ the corresponding prime of $\mathcal{O}_{g,\chi}$.

Let $\mathcal{S}(B)$ the anticyclotomic Selmer group given by

$$\mathcal{S}(B) := \varprojlim_n \varprojlim_m \mathrm{Sel}_{p^m}(B/K_n^-) \otimes_{\mathcal{O}_0} L.$$

Here $\mathrm{Sel}_{p^m}(B/K_n^-)$ denotes the usual Selmer group arising from p^m-torsion points of the abelian variety B over the anticyclotomic extension K_n^-. Let

$$\mathcal{X}(B) = \left(\varinjlim_n \varinjlim_m \mathrm{Sel}_{p^m}(B/K_n^-)\right)^\vee \otimes_{\mathcal{O}_0} L$$

be the anticyclotomic Selmer group for $(\cdot)^\vee$ being the Pontryagin dual. Note that these Selmer groups actually arise from $\wp_0^\infty$-torsion points on B.

We may analogously define Selmer groups $\mathcal{S}(B^\vee)$ and $\mathcal{X}(B^\vee)$ for the dual abelian variety $B^\vee_{/K}$.

2.1.5 p-ordinarity

In what follows, we suppose that the newform g is p-ordinary and $(p, \mathrm{cond}^{\mathrm{r}}(\chi)) = 1$ for $\mathrm{cond}^{\mathrm{r}}(\cdot)$ the conductor. In particular, the abelian variety $B_{/K}$ has ordinary reduction at the prime above p determined via the embedding ι_p.

By definition, $\mathcal{S}(B)$ and $\mathcal{X}(B)$ have a natural structure as a Λ-module. In the ordinary setup, $\mathcal{S}(B)$ and $\mathcal{X}(B)$ turn out to be finitely generated Λ-modules.

2.2 Heegner points

In this subsection, we introduce the underlying Heegner points.

For the pair (g, χ), we suppose the following generalised Heegner hypothesis.

(GH1). $\omega \cdot \chi|_{\mathbb{A}^\times} = 1$ and
(GH2). $\epsilon(g, \chi) = -1$ for the global root number $\epsilon(g, \chi)$ of the Rankin–Selberg convolution corresponding to the pair (g, χ).

2.2.1 Shimura curve

Following [49], we may introduce Heegner points on the abelian variety B over certain ring class fields K_m's. To do so, let D be the indefinite quaternion algebra over $\mathbb{Q}$ such that

$$\epsilon(g, \chi_v)\chi_v\eta_v(-1) = \epsilon(D_v)$$

for all finite places v. Here $\epsilon(g, \chi_v)$ denotes the local root number corresponding to the Rankin–Selberg convolution with χ_v the v-component of χ, η the quadratic character corresponding to the extension $K_{/\mathbb{Q}}$ and $\epsilon(D_v)$ the Hasse invariant.

From Tunnell [45] and Saito [41], the set of ramification places $\mathrm{ram}(D)$ satisfies

$$\mathrm{ram}(D) \subset \left\{ v|N\infty \middle| v \text{ non-split in } K \right\}$$

(for example, [13, Lem. 3.1]). In particular, the quaternion D is split at the prime p.

From construction of the quaternion algebra D, there exists a $\mathbb{Q}$-algebra embedding $\iota_K : K \hookrightarrow D$. We fix such an embedding for once and all.

We have the representation of $D_{\mathbb{A}}^{(\infty),\times}$ over the field $M := \mathrm{End}^0(A_{/\mathbb{Q}})$ arising from modular parametrisations of A given by

$$\pi = \varinjlim_{U \subset D_{\mathbb{A}}^{(\infty),\times}} \mathrm{Hom}^0_{\xi_U}(X_U, A).$$

Here $\xi = \varprojlim_U(\xi_U)$ is a Hodge class on the Shimura variety $X = \varprojlim_U X_U$ corresponding to the reductive group $D^\times$ as $U \subset D_{\mathbb{A}}^{(\infty),\times}$ varies over open compact subgroups. Let $\iota_\xi : X \to J$ be the corresponding quasi-embedding for $J = \varprojlim_U \mathrm{Alb}(X_U)$.[2]

2.2.2 Norm compatible Heegner points

The Heegner main conjecture (HMC) concerns Iwasawa theory of Heegner points on the abelian variety B along the anticyclotomic $\mathbb{Z}_p$-extension K_∞^-. We now briefly recall a norm compatible Heegner points along the anticyclotomic tower following [18, §10.1].

[2] We refer to [49] for details.

As the newform g is p-ordinary, note that the $D_{\mathbb{A}}^{(\infty),\times}$-representation π is $\wp$-ordinary [18, Def. 1.2.2]. Let α be the corresponding $\wp$-unit Hecke eigenvalue.

Test vectors. Let $f_\alpha^\circ \in \pi$ be a non-zero element satisfying the following.

(i). For the Hecke operator U_p,

$$U_p f_\alpha^\circ = \alpha f_\alpha^\circ$$

(ii). f_α° is a test vector away from p ([13, Def. 3.6]).[3]

For $n \geq 1$, let

$$s_n = \begin{pmatrix} p^n & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p) \simeq D^\times(\mathbb{Q}_p)$$

and

$$f_{\alpha,n} = |p|^{-n}\alpha^{-n} \cdot s_n f_\alpha^\circ.$$

Here we use $\mathrm{GL}_2(\mathbb{Q}_p)$ action on π. We then have

$$f_\alpha = (f_{\alpha,n})_n \in \pi^{(p)} \otimes \varprojlim_V \pi_p^V \tag{2.1}$$

for $\pi^{(p)}$ (resp. π_p) the component of π outside p (resp. at p), where V runs over open compact subgroups such that

$$\mathrm{Ker}(\omega_p) \subset V \subset K_p^\times$$

and ω_p the component of ω at p [18, Lem. 10.1.1].

Heegner points. As the imaginary quadratic field K embeds in the quaternion algebra D via the embedding ι_K, the torus $K^\times$ acts on the Shimura variety X. Let $P \in X^{K^\times}$ be a CM point.

In view of the definition of π, the choice of $f_{\alpha,n}$ gives rise to Heegner point

$$P(f_{\alpha,n}, \chi) = \int_{\mathrm{Gal}(K^{\mathrm{ab}}/K_n^-)} f_{\alpha,n}(\iota_\xi(P^\sigma)) \otimes \chi(\sigma) d\sigma \in B(K_n^-)_{\mathbb{Q}} \tag{2.2}$$

from the definition of abelian variety B (Definition 2.1).

For $n = 0$, let $P_{g,\chi}$ denote the corresponding Heegner point over K. For a character ν factoring through Γ_n^-, let $P_{g,\chi\nu}$ be the corresponding Heegner point arising from ν-component of $P(f_{\alpha,n}, \chi)$.

[3] We refer to [13, §3.3] for details.

In view of (2.1) and (2.2), we have

$$(P(f_{\alpha,n},\chi))_n \in \varprojlim_n B(K_n^-)_{\mathbb{Q}}$$

[18, (10.2.2)]. In view of the norm compatibility of the Heegner points $(P(f_{\alpha,n},\chi))_n$, the Kummer map

$$B(K_n^-)\otimes_{\mathcal{O}_0} L \to H^1(K_n^-, T_pB)\otimes_{\mathcal{O}_0} L$$

for the Tate module T_pB gives rise to Heegner cohomology class

$$\kappa \in \mathcal{S}(B).$$

As before, this corresponds to the $\wp_0$-component of the cohomology class arising from norm compatible Heegner points.

2.3 Formulation

In this subsection, we describe formulation of Heegner main conjecture (HMC).

After Kolyvagin [28] and Rubin [38], non-triviality of a Heegner point over a ring class field implies the Mordell–Weil rank of the underlying abelian variety over the ring class field being one and also finiteness of the corresponding Tate–Shafarevich group [34]. Moreover, index of the Heegner point in the Mordell-Weil group is closely related to the size of the Tate–Shafarevich group.

We have the following fundamental conjecture regarding Heegner points in Iwasawa-theoretic setup.

Conjecture 2.2 (Heegner Main Conjecture) *Let (g,χ) be a self-dual pair of an elliptic newform with weight two and a finite order Hecke character over an imaginary quadratic field K with root number -1. Let B be the corresponding abelian variety over K (Definition 2.1).*

Let p be an odd ordinary prime for the pair, K_∞^- the anticyclotomic $\mathbb{Z}_p$-extension of K, Λ the corresponding rational Iwasawa algebra and ι the involution of Λ as above. Let $\mathcal{S}(B)$ and $\mathcal{X}(B)$ be the rational version of Selmer groups associated to B along the anticyclotomic $\mathbb{Z}_p$-tower K_∞^-/K.

Let $\kappa \in \mathcal{S}(B)$ be the Heegner cohomology class as above. Then, the following holds.

(i). *The Heegner cohomology classes $\kappa \in \mathcal{S}(B)$ is Λ-non-torsion and*

$$\mathrm{rank}_\Lambda \mathcal{S}(B) = \mathrm{rank}_\Lambda \mathcal{X}(B) = 1.$$

Moreover, $\mathcal{S}(B)$ is Λ-torsion-free.

(ii).

$$\mathrm{Char}_\Lambda \mathcal{S}(B)/(\kappa) \cdot \mathrm{Char}_\Lambda (\mathcal{S}(B)/(\kappa))^\iota = \mathrm{Char}_\Lambda \mathcal{X}(B)_{\mathrm{tor}}$$

for $\mathrm{Char}_\Lambda(\cdot)$ *the characteristic ideal and* $(\cdot)_{\mathrm{tor}}$ *the* Λ*-torsion submodule.*

Mazur conjectured that the Heegner cohomology class κ is Λ-non-torsion. This has been essentially proven in [1,17,46] (also see [8,9] and [10]). Certain cases with the central character of g being non-trivial seem to be missing. The remaining part of HMC (i) has been essentially proven in [2,23] and [34] under certain hypothesis.

Remark 2.3 Heegner main conjecture does not seem to be stated in the literature for non-trivial χ. For our approach to a p-converse theorem, it is perhaps essential to consider the setup with a non-trivial χ. With an optimal choice of test vectors, we may formulate an integral version of the Heegner main conjecture and it would involve Tamagawa numbers [35]. We restrict to the rational version as it suffices for our study of a p-converse theorem.

3 Heegner main conjecture: CM case

In this section, we consider CM case of the Heegner main conjecture (HMC).

In §3.1, we describe the main result. In Sect. 3.2, we introduce underlying Selmer groups arising in our approach to HMC. In Sect. 3.3, we introduce underlying p-adic L-functions arising in the approach. In Sect. 3.4, we conclude with the proof.

3.1 Main result

In this section, we describe our main result towards HMC in the CM case.

We recall the following

Definition 3.1 A Hecke eigenform $g \in S_2(\Gamma_0(N), \epsilon)$ is said to be CM if there exists an imaginary quadratic field K and an arithmetic Hecke character λ over K such that g is the theta series $\theta(\lambda)$ associated to λ.

In this setup, we say that g has CM by K. Note that the infinity type of λ equals $(1, 0)$ or $(0, 1)$.

The main result of this section is the following case of Conjecture 2.2 in the CM case.

Theorem 3.2 *Let* K *be an imaginary quadratic field and* $p > 3$ *a prime split in* K. *Let* g *be a weight two CM modular form with CM by* K, *level* N_g *and* χ *a finite order Hecke character over* K. *Suppose that the Rankin–Selberg* *convolution corresponding to the pair* (g, χ) *is self-dual with root number* -1 *and* $p \nmid N_g \cdot \mathrm{cond}^{\mathrm{r}}(\chi)$.

Then, the Henger main conjecture (Conjecture 2.2) *holds for the pair* (g, χ).

Remark 3.3 (1). In view of (ord), the CM modular form g is p-ordinary.

(2). When g is a non-CM Hecke eigenform and χ trivial, the conjecture has been recently proven in [48, Thm. 1.2] and [14, Thm. 3.4] via an approach which builds on [23]. The results are under mild hypotheses as long as the image of mod p Galois representation associated to g is 'large' (also see [15]). The approach seems to exclude the CM case in an essential manner. To begin with, the results in [23] exclude the CM case.

3.2 Selmer groups

In this subsection, we describe generalities regarding Selmer groups arising in our approach to Heegner main conjecture in the CM case (Theorem 3.2).

Let the notation and assumptions be as in Sect. 2.1. In particular, p denotes an odd prime split in an imaginary quadratic field K. Moreover, $\mathfrak{p}$ denotes the prime of K above p induced via the embedding ι_p. Let $\mathfrak{p}^*$ be its conjugate.

3.2.1 Definitions

We introduce the definitions of underlying Selmer groups.

Let ψ be a Hecke character over K with infinity type $(1, 0)$ with respect to the embedding ι_∞. Let $g \in S_2(\Gamma_0(N_g), \omega)$ be the corresponding CM modular form. Recall that g is p-ordinary. Let χ be a finite order Hecke character over K such that $\omega \cdot \chi|_{\mathbb{A}_{\mathbb{Q}}^\times} = 1$. Let B be the abelian variety over K associated to the pair (g, χ) as in Definition 2.1. We suppose that $p \nmid \mathrm{cond}^{\mathrm{r}}(\chi)$ and thus B has ordinary reduction at primes above p.

Recall that $E_{g,\chi} \subset \mathbb{C}$ denotes the subfield generated over $\mathbb{Q}$ by the Hecke eigenvalues of g and the values of χ. Let L be the completion of $E_{g,\chi}$ at the prime above p induced via the embedding ι_p and $\mathcal{O}$ the corresponding ring of integers.

Galois representations. Let V_g be the p-adic Galois representation ρ_g : $G_{\mathbb{Q}} \to \mathrm{GL}_2(L)$ associated to the Hecke eigenform g. Let $L(\chi)$ be the one dimensional G_K-representation over L associated to the Hecke character χ. Then

$$V = V_g\big|_{G_K} \otimes_L L(\chi) \tag{3.1}$$

is a p-adic Galois representation of G_K ordinary at p.

The Galois representation admits an explicit description as follows. Let $\lambda = \psi\chi$. For a Hecke character ξ over K, let ξ^* denote the Hecke character

$\xi \circ c$ for $c \in \mathrm{Gal}(K/\mathbb{Q})$ the non-trivial element. We then have

$$V \cong L(\lambda) \oplus L(\psi^* \chi)$$

an isomorphism of $L[G_K]$-modules. Here we let ξ also denote p-adic avatar of ξ determined via the embedding. We now choose a G_K-stable lattice $T \subset V$ such that

$$T \cong \mathcal{O}(\lambda) \oplus \mathcal{O}(\psi^* \chi). \tag{3.2}$$

Let

$$W = V/T$$

and

$$W(\lambda) = L(\lambda)/\mathcal{O}(\lambda), \qquad W(\psi^* \chi) = L(\psi^* \chi)/\mathcal{O}(\psi^* \chi).$$

Selmer groups: $\mathrm{GL}_{2/\mathbb{Q}}$. We begin with underlying objects on the $\mathrm{GL}_{2/\mathbb{Q}}$-side.

We introduce the local Bloch–Kato subgroups corresponding to the pair (g, χ).

(BK1). For $? = V, T$ or W and $w \nmid p$, we consider

$$H^1_f(K_w, ?) := \ker \big(H^1(K_w, V) \to H^1(I_w, V)\big).$$

Here I_w denotes the corresponding inertia subgroup.

(BK2). For $w|p$, we consider

$$H^1_f(K_w, V) := \ker(H^1(K_w, V) \longrightarrow H^1(K_w, V^-)).$$

Here V^- is the maximal unramified quotient of $V|_{G_{K_w}}$.[4] For $? = T, W$, we analogously define $H^1_f(K_w, ?)$.

These local conditions give rise to the Bloch–Kato Selmer group

$$H^1_f(K, ?) = \ker \left\{ H^1(K, ?) \to \prod_w \frac{H^1(K_w, ?)}{H^1_f(K_w, ?)} \right\}$$

[4] The existence follows from g being p-ordinary.

and the Tate–Shafarevich group

$$Ш_f(W_{/K}) = \frac{H^1_f(K, W)}{H^1_f(K, W)_{\mathrm{div}}}$$

for $(\cdot)_{\mathrm{div}}$ the maximal divisible subgroup.

Iwasaw version. Recall that $\Lambda^\circ = \mathcal{O}[[\Gamma_K^-]]$ for the anticyclotomic Galois group Γ_K^- and $\Lambda := \Lambda^\circ \otimes_{\mathcal{O}} L$ its rational version. We introduce the underlying Λ-adic Selmer groups.

(BK'1). For $w \nmid p$, we consider

$$H^1_f(K_w, V \otimes_L \Lambda) := \ker\left(H^1(K_w, V \otimes_L \Lambda) \to H^1(I_w, V \otimes_L \Lambda)\right).$$

Here I_w denotes the corresponding inertia subgroup.

(BK'2). For $w|p$, we consider

$$H^1_f(K_w, T \otimes_L \Lambda) := \ker(H^1(K_w, T \otimes_L \Lambda) \longrightarrow H^1(K_w, T^- \otimes_L \Lambda)).$$

Here T^- is the maximal unramified quotient of $T|_{G_{K_w}}$. For $? = V, W$, we analogously define $H^1_f(K_w, ? \otimes_L \Lambda)$.

Definition 3.4 The Λ-adic Selmer group $S(V \otimes_L \Lambda)$ corresponding to the pair (g, χ) is given by

$$S(V \otimes_L \Lambda) := \ker\left\{H^1(K, V \otimes_{L_0} \Lambda) \to \prod_w H^1(K_w, V \otimes_{L_0} \Lambda)/ H^1_f(K_w, V \otimes_{L_0} \Lambda)\right\}.$$

With the ordinary condition (BK'2), we also have the discrete Selmer group

$$\mathrm{Sel}(K, W \otimes_{\mathcal{O}} \Lambda^\circ)$$

and its dual

$$X(W) = \mathrm{Hom}_{\mathcal{O}}(\mathrm{Sel}(K, W \otimes_{\mathcal{O}} \Lambda^\circ), L/\mathcal{O}).$$

We define the analogous notions for the dual Galois representations.

Remark 3.5 The Λ-modules $S(V \otimes_L \Lambda)$ and $X(W)_L := X(W) \otimes_{\mathcal{O}} L$ are nothing but the Λ-modules $\mathcal{S}(B)$ and $\mathcal{X}(B)$ associated to the abelian variety B in Sect. 2.1, respectively.

Selmer groups: $\mathrm{GL}_{1/K}$. We now introduce underlying objects on the $\mathrm{GL}_{1/K}$-side.

We begin with the local Bloch–Kato subgroups corresponding to the underlying Hecke characters. For a finite place v of K, the underlying local Bloch–Kato Selmer groups are given by

$$H^1_f(K_w, L(\lambda)) = \begin{cases} H^1_{\mathrm{ur}}(K_w, L(\lambda)), & w \nmid p, \\ H^1(K_w, L(\lambda)) & w|\mathfrak{p}, \\ 0 & w|\mathfrak{p}^*. \end{cases}$$

and

$$H^1_f(K_w, L(\psi^*\chi)) = \begin{cases} H^1_{\mathrm{ur}}(K_w, L(\psi^*\chi)), & w \nmid p, \\ H^1(K_w, L(\psi^*\chi)) & w|\mathfrak{p}^*, \\ 0 & w|\mathfrak{p}. \end{cases}$$

Here $H^1_{\mathrm{ur}}(K_w, \cdot)$ denotes the unramified local Galois cohomology given by

$$H^1_{\mathrm{ur}}(F_w, \cdot) = \ker\left(H^1(F_w, \cdot) \to H^1(I_w, \cdot)\right).$$

The above description of the subgroups relies on the fact that the Hecke character λ (resp. $\psi^*\chi$) is with infinity type $(1, 0)$ (resp. $(0, 1)$).[5]

These local conditions give rise to the Bloch–Kato Selmer group

$$H^1_f(K, L(\cdot)) = \ker\left\{H^1(G_K, L(\cdot)) \to \prod_w \frac{H^1(K_w, L(\cdot))}{H^1_f(K_w, L(\cdot))}\right\}$$

We analogously define the Bloch–Kato Selmer group $H^1_f(K, L(\cdot)^\vee)$ for $(\cdot)^\vee$ being the dual, a discrete version $H^1_f(K, W(\cdot))$ and the Tate–Sharevich groups $Ш_f(W(\cdot)_{/K})$.

Iwasawa version. As in the $\mathrm{GL}_{2/\mathbb{Q}}$-case, we analogously introduce Iwasawa-version of the local conditions.

Definition 3.6 Let $\cdot$ denote the Hecke character λ or $\psi^*\chi$. The Λ-adic Selmer groups $S(L(\cdot) \otimes_L \Lambda)$ corresponding to χ is given by

$$S(L(\cdot) \otimes_L \Lambda) := \ker\left\{H^1(K, L(\cdot) \otimes_L \Lambda) \to \prod_w H^1(K_w, L(\cdot) \otimes_L \Lambda)/ H^1_f(K_w, L(\cdot) \otimes_L \Lambda)\right\}.$$

[5] We refer to [3, §1.2].

With the conditions, we have the discrete Selmer groups

$$\mathrm{Sel}(K, W(\lambda) \otimes_{\mathcal{O}} \Lambda^\circ), \qquad \mathrm{Sel}(K, W(\psi^*\chi) \otimes_{\mathcal{O}} \Lambda^\circ),$$

and also their duals

$$X(\lambda), \qquad X(\psi^*\chi).$$

3.2.2 Decomposition

We describe decomposition of the Selmer groups on the $\mathrm{GL}_{2/\mathbb{Q}}$-side.

We have the following relation among the Selmer groups on the $\mathrm{GL}_{2/\mathbb{Q}}$ and $\mathrm{GL}_{1/K}$-sides.

Lemma 3.7 *Let ψ (resp. χ) be a Hecke character over an imaginary quadratic field K with infinity type $(1, 0)$ (resp. finite order, unramified at p) and $\lambda = \psi\chi$. Let g be the CM modular form associated to ψ and V the p-adic Galois representation of G_K corresponding to the pair (g, χ) as above (3.1). Let $S(\cdot)$, $\mathrm{Sel}(\cdot)$ and $X(\cdot)$ be the anticyclotomic Selmer groups as above.*

Then, we have an isomorphism

$$S(V \otimes_L \Lambda) \cong S(L(\lambda) \otimes_L \Lambda) \oplus S(L(\psi^*\chi) \otimes_L \Lambda)$$

of Λ-modules and isomorphisms

$$\mathrm{Sel}(K, W \otimes_{\mathcal{O}} \Lambda^\circ) \cong \mathrm{Sel}(K, W(\lambda) \otimes_{\mathcal{O}} \Lambda^\circ) \oplus \mathrm{Sel}(K, W(\psi^*\chi) \otimes_{\mathcal{O}} \Lambda^\circ)$$

and

$$X(W) \cong X(\lambda) \oplus X(\psi^*\chi)$$

of Λ°-modules.

Proof We indicate the decomposition for $S(\cdot)$. The decomposition for $\mathrm{Sel}(\cdot)$ and $X(\cdot)$ can be proven analogously.

As noted earlier, we have decomposition of the p-adic Galois representation V as a direct sum of the ones corresponding to the Hecke characters λ and $\psi^*\chi$ (3.2). For a finite place v of K, we have the decomposition

$$H^1_f(K_v, V \otimes_L \Lambda) \cong H^1_f(K_v, L(\lambda) \otimes_L \Lambda) \oplus H^1_f(K_v, L(\psi^*\chi) \otimes_L \Lambda)$$

of the local Bloch–Kato Selmer groups.

This finishes the proof. $\square$

The decomposition does not rely on self-duality of the underlying Hecke characters. In particular, it is independent of the root numbers.

We have the following immediate

Corollary 3.8 *Let the notation and hypotheses be as in Lemma 3.7.*
Then,

$$\mathrm{Char}_{\Lambda} X(W)_{\mathrm{tor}} = \mathrm{Char}_{\Lambda} X(\lambda)_{\mathrm{tor}} \cdot \mathrm{Char}_{\Lambda} X(\psi^{*}\chi)_{\mathrm{tor}}.$$

Here Char_{Λ} *denotes the characteristic ideal and* $(\cdot)_{\mathrm{tor}}$ *the* Λ*-torsion submodule.*

3.2.3 Ranks

We describe ranks of the underlying Selmer groups under Heegner hypothesis.

For the pair (g, χ), we now suppose that

$$\epsilon(g, \chi) = -1.$$

Note that the Hecke characters λ and $\psi^{*}\chi$ are self-dual. Here we say a Hecke character ν over K to be self-dual if

$$\nu|_{\mathbb{A}^{\times}} = \eta|\cdot|_{\mathbb{A}^{\times}}.$$

Without loss of generality, we suppose that

$$\epsilon\left(\frac{1}{2}, \lambda\right) = -1, \quad \epsilon\left(\frac{1}{2}, \psi^{*}\chi\right) = +1.$$

We have the following proposition regarding the ranks of anticyclotomic Selmer groups.

Proposition 3.9 *Let* ψ *(resp.* χ*) be a Hecke character over an imaginary quadratic field* K *with infinity type* $(1, 0)$ *(resp. finite order) and* $\lambda = \psi\chi$*. Let* g *be the CM modular form associated to* ψ *with level* N_g *and* V *the* p*-adic Galois representation of* G_K *corresponding to the pair* (g, χ) *as above* (3.1)*. Suppose that* λ *(resp.* $\psi^{*}\chi$*) has root number* -1 *(resp.* 1*) and* $p \nmid 6N_g \cdot \mathrm{cond}^{\mathrm{r}}(\chi)$*. Let* $S(\cdot)$ *and* $X(\cdot)$ *denote the anticyclotomic Selmer groups as above.*
Then, we have

$$\mathrm{rank}_{\Lambda} S(V \otimes_L \Lambda) = 1$$

and

$$\mathrm{rank}_{\Lambda} S(L(\lambda) \otimes_L \Lambda) = 1, \qquad \mathrm{rank}_{\Lambda} S(L(\psi^{*}\chi) \otimes_L \Lambda) = 0.$$

Moreover, the analogous result holds for the divisible Selmer groups $X(\cdot)$.

Proof In view of Lemma 3.7, the latter assertion implies the former.

From our hypotheses on the root numbers, the latter assertion is nothing but [3, Thm. 2.1 and Thm. 2.2] (also see [2,38]). The approach is based on an underlying Euler system of elliptic units and its non-triviality. □

3.2.4 Regulators

We end this subsection with generalities regarding the underlying anticyclotomic regulators.

Let ν be a finite order character of the anticyclotomic Galois group Γ_K^- and $L(\nu)$ the extension of L obtained by adjoining values of ν. We may regard it as a $L[G_K]$-module.

Being in a p-ordinary setup, there exists an $L(\nu)$-valued canonical p-adic Height pairing

$$\langle\ ,\ \rangle_\nu : H^1_f(K, V \otimes_L L(\nu)) \times H^1_f(K, (V \otimes_L L(\nu))^*(1)) \longrightarrow L(\nu).$$

Here $H^1_f(K,\cdot)$ denotes the Bloch–Kato Selmer group corresponding to the ν-twist and $(\cdot)^*$ the dual [32, Thm. 4.2].

There exists a Λ-adic height pairing

$$\langle\ ,\ \rangle : \qquad S(V \otimes_L \Lambda) \otimes_\Lambda S((V \otimes_L \Lambda)^*(1)) \longrightarrow \Lambda$$

interpolating the p-adic height pairings $\langle\ ,\ \rangle_\nu$ as ν varies over finite order characters of Γ_K^- [33, Prop. 11.1.9].[6] In view of self-duality (SD), the pairing can be viewed as

$$\langle\ ,\ \rangle : \qquad S(V \otimes_L \Lambda) \otimes_\Lambda S(V \otimes_L \Lambda)^\iota \longrightarrow \Lambda. \tag{3.3}$$

In other words, here we utilise the isogeny between $B_{/K}$ and $B^\vee_{/K}$.

Such a Λ-adic height pairing also exists for Selmer groups arising from arithmetic Hecke characters over K, in particular for the Selmer group $S(L(\lambda) \otimes_L \Lambda)$ [32, Thm. 4.2] and [33, Prop. 11.1.9].[7]

Recall that the regulator of a Λ-adic height pairing is nothing but the characteristic ideal of its cokernel viewed as a Λ-module [2, Def. 3.1.3], also [3, p.77].

6 We also refer to [33, §11.1].

7 We also refer to [3, §4.2] and references therein.

Definition 3.10 Let $R(g \times \chi)$ (resp. $R(\lambda)$) be the regulator corresponding to the Λ-adic height pairing (3.3) on the Selmer group $S(V \otimes_L \Lambda)$ (resp. $S(L(\lambda) \otimes_L \Lambda)$).

We have the following relation among the regulators.

Corollary 3.11 *Let ψ (resp. χ) be a Hecke character over an imaginary quadratic field K with infinity type $(1, 0)$ (resp. finite order) and $\lambda = \psi\chi$. Let g be the CM modular form associated to ψ with level N_g and V the p-adic Galois representation of G_K corresponding to the pair (g, χ) as above. Suppose that λ (resp. $\psi^*\chi$) has root number -1 (resp. 1) and $p \nmid N_g \cdot \mathrm{cond}^{\mathrm{r}}(\chi)$. Let $R(\cdot)$ denote the anticyclotomic regulator as above.*

Then,

$$R(g \times \chi) = R(\lambda).$$

Proof As the Λ-module $S(L(\psi^*\chi) \otimes_L \Lambda)$ is torsion (Proposition 3.9), the assertion follows from Lemma 3.7 and definition of the Λ-adic height pairing. $\square$

3.3 p-adic L-functions

In this subsection, we describe generalities regarding p-adic L-functions arising in our approach to Heegner main conjecture (HMC) in the CM case (Theorem 3.2). Even though HMC does not explicitly involve p-adic L-functions in the formulation, they seem to appear inevitably in the approach.

3.3.1 Definitions

We introduce interpolation property for the underlying p-adic L-functions.

Let the notation and assumptions be as in Sect. 3.2. In particular, $g \in S_2(\Gamma_0(N_g), \omega)$ denotes CM modular form associated to a Hecke character ψ over K with infinity type $(1, 0)$ and χ a finite order Hecke character over K such that $\omega \cdot \chi|_{\mathbb{A}_{\mathbb{Q}}^{\times}} = 1$. Moreover, we suppose that $\epsilon(g, \chi) = -1$. Note that $\omega = \eta_K \cdot \psi|_{\mathbb{A}_{\mathbb{Q}}^{\times}}$.

Let $\lambda = \psi\chi$. Without loss of generality, we suppose that $\epsilon(\frac{1}{2}, \lambda) = -1$. Note that the CM modular form f associated to the Hecke character λ is of weight two and trivial central character.

In view of the decomposition of Galois representation V corresponding to the pair (g, χ) in terms of Hecke characters (3.2) and Artin formalism, we have a factorisation

$$L(s, g \times \chi) = L(s, \lambda) \cdot L(s, \psi^*\chi)$$

of complex L-functions. In what follows, we consider an Iwasawa analogue of the above factorisation.

p-adic L-functions: $\mathrm{GL}_{2/\mathbb{Q}}$. We begin with the $\mathrm{GL}_{2/\mathbb{Q}}$-side.

We have a two-variable Rankin–Selberg p-adic L-function $L_p(g \times \chi) \in \mathcal{O}[[\Gamma_K]] \otimes_{\mathcal{O}} L$. To recall the interpolation, let (g, χ) be a general pair as in Sect. 2.1 with g not necessarily a CM modular form. Recall that g is p-ordinary with U_p-eigenvalue α and $p \nmid \mathrm{cond}^{\mathrm{r}}(\chi)$.

Definition 3.12 Let (g, χ) be a self-dual pair of an elliptic newform with weight two and finite order Hecke character over an imaginary quadratic field as above. Let

$$L_p(g \times \chi) \in \mathcal{O}[[\Gamma_K]] \otimes_{\mathcal{O}} L$$

be the Rankin–Selberg p-adic L-function characterised by the interpolation property

$$\widehat{\chi'}(L_p(g \times \chi)) = \frac{e_p(g \times (\chi\chi')^{-1})}{\alpha^{v(\mathrm{cond}^{\mathrm{r}}(\chi'))}} \cdot \frac{L^{(p)}(1, g \times (\chi\chi')^{-1})}{\Omega_g}$$

for all sufficiently p-ramified finite order characters $\chi' \colon \Gamma_K \to \mathbb{C}_p^{\times}$. Here

- $\Omega_g := L(1, \mathrm{ad}(g))$ for $\mathrm{ad}(g)$ the adjoint,
- $e_p(g \times (\chi\chi')^{-1}) = \varepsilon(0, \chi_{\mathfrak{p}}\chi'_{\mathfrak{p}}) \cdot \varepsilon(0, \chi_{\mathfrak{p}^*}\chi'_{\mathfrak{p}^*})$ for the local ε-factor $\varepsilon(0, \cdot)$ and
- $L^{(p)}(\cdot)$ the L-function with Euler factors at primes above p removed

[18, Thm. A].

We refer to [18, Intro.] for "sufficiently ramified" (also see [11, Proof of Thm. 2.3]). The local epsilon factors as above are with respect to some uniform choice of additive characters of $K_{\mathfrak{q}}$ of level one for the primes $\mathfrak{q}|p$.

p-adic L-functions: $\mathrm{GL}_{1/K}$. We now turn towards the $\mathrm{GL}_{1/K}$-side.

We have a two-variable Katz p-adic L-functions $L_\Sigma(\lambda), L_{\Sigma^*}(\psi^*\chi) \in W[[\Gamma_K]]$. Here W denotes a finite flat extension of the Witt ring $W(\mathbb{F})$ for an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$.

Let $\pi : \Gamma_K^{\sharp} \twoheadrightarrow \Gamma_K$ be a finite cover arising from a finite extension of K_∞ contained in K^{ab}. This corresponds to fixing a tame level prime to p and we exclude the tame level from the notation for simplicity. Let Σ be a p-adic CM type of K as above, in other words the complex embedding ι_∞.

By [20, Thm. II] (also see [26]), there exists an element

$$L_\Sigma \in W[[\Gamma^{\sharp}]],$$

uniquely characterised by an interpolation property. The domain of interpolation consists of arithmetic Hecke characters $\lambda' \colon \Gamma_K^\sharp \to \mathbb{C}_p^\times$ with infinity type $(k+\kappa, -\kappa)$ for $k, \kappa \in \mathbb{Z}$, such that

(i). $k \geq 1$ and $\kappa \geq 0$ or
(ii). $k \leq 1$ and $k + \kappa > 0$.

The interpolation property is then the following.

There exist p-adic CM periods $\Omega_{\Sigma,p} \in \mathbb{C}_p^\times$ and complex CM periods $\Omega_{\Sigma,\infty} \in \mathbb{C}^\times$ such that for any character $\lambda' \colon \Gamma_K^\sharp \to \mathbb{C}_p^\times$ in the domain we have[8]

$$\frac{L_\Sigma(\lambda')}{\Omega_{\Sigma,p}^{k+2\kappa}} = e_p((\lambda')^{-1}) \cdot \frac{L^{(p)}(0, ((\lambda')^{-1}))}{\Omega_{\Sigma_E,\infty,}^{k+2\kappa}} \cdot \frac{\pi^\kappa \Gamma(k+\kappa)}{(\mathrm{Im}\theta)^\kappa} \cdot [O_K^\times : \mathbb{Z}^\times]$$

Here $L^{(p)}(\cdot)$ denotes the L-function with Euler factors at primes above p removed, Γ the usual Γ-function and $\theta \in K$ as in [24, §3.1].

If $\chi = (\lambda')^{-1}$ is ramified at the primes dividing p, the p-Euler factor is given by

$$e_p(\chi) = \frac{L(0, \chi_\mathfrak{p})}{\varepsilon(0, \chi_\mathfrak{p}) L(1, \chi_\mathfrak{p}^{-1})}.$$

For our consideration, we choose a certain tame level and consider the restriction of L_Σ to certain open subsets of $\Gamma_K^\sharp$.

Definition 3.13 Let λ_0 be a p-adic Hecke character over the imaginary quadratic field K with values in L and Σ is a p-adic CM type as above. Let

$$L_\Sigma(\lambda_0) \in W[[\Gamma_K]] \otimes L$$

be the Katz p-adic L-function given by

$$\widehat{\chi'}(L_\Sigma(\lambda_0)) := L_\Sigma(\lambda_0 \chi').$$

The interpolation property of L_Σ gives rise to an analogous interpolation for the Katz p-adic L-function $L_\Sigma(\lambda_0)$.

[8] Note that we are ignoring interpolation factors at places away from p appearing elsewhere in the literature, since those, while non-integral, can be interpolated by polynomial functions on $W[[\Gamma_K^\sharp]]$.

3.3.2 Factorisation

We describe factorisation of the Rankin–Selberg p-adic L-function on the $\mathrm{GL}_{2/\mathbb{Q}}$-side.

We have the following relation among the p-adic L-functions on $\mathrm{GL}_{2/\mathbb{Q}}$ and $\mathrm{GL}_{1/K}$-sides.

Lemma 3.14 *Let ψ (resp. χ) be a Hecke character over an imaginary quadratic field K with infinity type $(1, 0)$ (resp. finite order) and $\lambda = \psi\chi$. Let g be the CM modular form associated to ψ. Let $L_p(g \times \chi)$, $L_\Sigma(\lambda)$ and $L_{\Sigma^*}(\psi^*\chi)$ be the p-adic L-functions as in Definitions 3.12 and 3.13. Here Σ denotes a CM type of K and Σ^* the conjugate CM type Σ.*

Then,

$$L_p(g \times \chi) \doteq L_\Sigma(\lambda) \cdot L_{\Sigma^*}(\psi^*\chi).$$

Here '$\doteq$' denotes equality up to a constant in $\overline{\mathbb{Q}}_p^\times$ [11, § 2.4].

The factorisation does not rely on self-duality of the underlying Hecke characters. In particular, it is independent of the underlying root numbers.

Remark 3.15 The above factorisation is compatible with the decomposition in Lemma 3.7.

3.3.3 Cyclotomic derivative

We consider cyclotomic derivative of the underlying p-adic L-functions.

Recall that $\epsilon(\frac{1}{2}, g \times \chi) = -1$. Moreover, λ and $\psi^*\chi$ are self-dual Hecke characters over the imaginary quadratic field K. We may then suppose that

$$\epsilon\left(\frac{1}{2}, g \times \chi\right) = \epsilon\left(\frac{1}{2}, \lambda\right) = -1, \qquad \epsilon\left(\frac{1}{2}, \psi^*\chi\right) = 1.$$

Accordingly, we are led to consider cyclotomic derivatives of Rankin–Selberg and Katz p-adic L-functions.

As in Lemma 3.14, we have an analogous factorisation for cyclotomic derivatives of the p-adic L-functions.

Corollary 3.16 *Let ψ (resp. χ) be a Hecke character over an imaginary quadratic field K with infinity type $(1, 0)$ (resp. finite order) and $\lambda = \psi\chi$. Let g be the CM modular form associated to ψ. Let $L_p(g \times \chi)$, $L_\Sigma(\lambda)$ and $L_{\Sigma^*}(\psi^*\chi)$ be the p-adic L-functions as above. Here Σ denotes a CM type of K and Σ^* the conjugate CM type.*

Then,

$$L'_p(g \times \chi) \doteq L'_\Sigma(\lambda) \cdot L^-_{\Sigma^*}(\psi^* \chi).$$

Here

- *'$\doteq$' denotes equality up to a constant in* $\overline{\mathbb{Q}}_p^\times$,
- $L'_\square(\cdot) = \frac{d}{dS} L_\square(\cdot)\Big|_{S=0}$ *the cyclotomic derivative with* S *being the cyclotomic variable and*
- $L^-_\square(\cdot) = L_\square(\cdot)\Big|_{S=0}$ *the anticyclotomic projection.*

Proof In view of the hypothesis

$$\epsilon(g \times \chi) = \epsilon\left(\frac{1}{2}, \lambda\right) = -1,$$

we have

$$L^-_p(g \times \chi) = L^-_\Sigma(\lambda) = 0.$$

as the interpolated central L-values vanish identically by the functional equation of the underlying Rankin–Selberg L-functions. Here $L^-_p(g \times \chi)$ and $L^-_\Sigma(\lambda)$ denote the anticyclotomic projections.

As for the vanishing, note that the anticyclotomic line is a self-dual line for the Rankin–Selberg convolution and the Hecke character. Moreover, it lies in the interpolation region of the p-adic L-functions $L_p(g \times \chi)$ and $L_\Sigma(\lambda)$.

From Lemma 3.14, this finishes the proof. □

3.4 Heegner main conjecture

In this subsection, we conclude the proof of Heegner main conjecture (HMC) for a a self-dual pair of a weight two CM form and a Hecke character over the same CM field with root number -1 (Theorem 3.2).

Let the notation and assumptions be as in Sect. 3.2. In particular, $g \in S_2(\Gamma_0(N_g), \omega)$ denotes CM modular form associated to a Hecke character ψ over K with infinity type $(1, 0)$ and χ a finite order Hecke character over K such that $\omega \cdot \chi|_{\mathbb{A}_\mathbb{Q}^\times} = 1$. Moreover, we suppose that $\epsilon(g, \chi) = -1$. Without loss of generality, we suppose that $\epsilon(\frac{1}{2}, \psi\chi) = -1$. As before, let $\lambda = \psi\chi$. We consider HMC (Conjecture 2.2) for the pair (g, χ) when p is an odd prime split in the imaginary quadratic field K.

Approach. Our approach to Theorem 3.2 seems to consist of the following key ingredients.

- Λ-adic Gross–Zagier formula due to Disegni.
- Anticyclotomic IMC in the sign -1 case due to Agboola–Howard and Arnold along with anticyclotomic IMC in the sign $+1$ case due to Rubin.
- Non-vanishing of a Λ-adic regulator in the sign -1 case.

Roughly speaking, Conjecture 2.2 concerns Heegner cohomology classe $\kappa \in \mathcal{S}(B)$ with genesis on the $\mathrm{GL}_{2/\mathbb{Q}}$-side. Here B denotes the abelian variety corresponding to the pair (g, χ) as in Definition 2.1. The Λ-adic Gross–Zagier formula expresses the Λ-adic height pairing of the Heegner class κ [33, Ch. 11] in terms of the cyclotomic derivative $L'_p(g \times \chi)$ of the Rankin–Selberg p-adic L-function $L_p(g \times \chi)$ up to an anticyclotomic regulator.

The CM setup leads to decompositions of Selmer groups and factorisations of p-adic L-functions as in Lemma 3.7 and Corollary 3.16. In this manner, Conjecture 2.2 enters an Iwasawa-theoretic setup on the $\mathrm{GL}_{1/K}$-side. The anticyclotomic IMC's due to Agboola–Howard and Arnold (resp. Rubin) express the cyclotomic derivative $L_\Sigma(\lambda)$ (resp. anticyclotomic Katz p-adic L-function $L^-_{\Sigma^*}(\psi^*\chi)$) of Katz p-adic L-function in terms of anticyclotomic Selmer groups up to an anticyclotomic regulator (resp. without a regulator).

We finish the proof via non-vanishing of the anticyclotomic regulators based on Hida's approach to non-vanishing.

Proof of Theorem 3.2 We build upon the consideration in Sects. 3.2 and 3.3. HMC (i). *Non-triviality*. For a finite order character ν of the anticyclotomic Galois group Γ^-_K, we have a factorsiation

$$L(s, g \times \chi\nu) = L(s, \lambda\nu) \cdot L\big(s, \psi^*\chi\nu\big)$$

of complex L-functions.

As

$$\epsilon\left(\frac{1}{2}, \lambda\right) = -1, \qquad \epsilon\left(\frac{1}{2}, \psi^*\chi\right) = 1,$$

we have

$$\mathrm{ord}_{s=1} L(s, \lambda\nu) = 1, \qquad \mathrm{ord}_{s=1} L(s, \psi^*\chi\nu) = 0$$

for all but finitely many ν due to Rohrlich [36, p. 384].

In view of the Gross–Zagier formula due to Yuan–Zhang–Zhang [49, Thm. 1.2], we have

$$\mathrm{ord}_{s=1} L(s, g \times \chi\nu) = 1 \iff P_{g,\chi\nu} \neq 0$$

for the Heegner point $P_{g,\chi\nu}$ in Sect. 2.1.2.

From injectivity of the Kummer map on rational points, we conclude that the Heegner cohomology class κ is Λ-non-torsion.

Ranks. The rank part of HMC (i) is nothing but Proposition 3.9.

Torsion-free. In view of Lemma 3.7, it suffices to show that the $\mathrm{GL}_{1/K}$-Selmer groups $S(\lambda)$ and $S(\psi^*\chi)$ are Λ-torsion-free. The latter is a part of [3, Thm. 2.14].

HMC (ii). We begin with the $\mathrm{GL}_{2/\mathbb{Q}}$-side.

We have Λ°-adic Gross–Zagier formula

$$\langle \kappa, \kappa^\iota \rangle \doteq L'_p(g \times \chi)$$

due to Disegni [18, Thm. C]. Here $\langle\ ,\ \rangle : \mathcal{S}(B) \otimes \mathcal{S}(B)^\iota \longrightarrow \Lambda$ is the Λ-adic height pairing in (3.3) and $L'_p(g \times \chi)$ the cyclotomic derivative as in Sect. 3.3.3. Moreover, '$\doteq$' denotes up to a non-zero constant in $\overline{\mathbb{Q}}_p^\times$. In particular, the equality holds for ideals of Λ arising from both sides of the formula.

Recall that the Heegner class κ is Λ-non-torsion and

$$\mathrm{rank}_\Lambda \mathcal{S}(B) = 1$$

(HMC (i)). It then follows that

$$(L'_p(g \times \chi)) = (\langle \kappa, \kappa^\iota \rangle) = \mathrm{Char}_\Lambda \mathcal{S}(B)/(\kappa) \cdot \mathrm{Char}_\Lambda (\mathcal{S}(B)/(\kappa))^\iota \cdot R(g \times \chi). \tag{3.4}$$

Anticyclotomic IMC: $\mathrm{GL}_{1/K}$. We now turn towards underlying anticyclotomic Iwasawa theory on the $\mathrm{GL}_{1/K}$-side.

As λ is self-dual with $\epsilon(\frac{1}{2}, \lambda) = -1$, we have the following results towards anticyclotomic CM IMC due to Agboola–Howard [2, Thm. A] and Arnold [3, Thm. 2.14 & Thm. 4.17].

(IMC'1). The divisible Iwasawa module $X(\lambda)$ has Λ-rank one.
(IMC'2). Moreover,

$$(L'_\Sigma(\lambda)) = \mathrm{Char}_\Lambda X(\lambda)_{\mathrm{tor}} \cdot R(\lambda)$$

As $\psi^*\chi$ is self-dual with $\epsilon(\frac{1}{2}, \psi^*\chi) = 1$, we have the following results towards anticyclotomic CM IMC due to Rubin [38, Thm. 4.1], also see [3, Thm. 2.1].

(IMC1). The divisible Iwasawa module $X(\psi^*\chi)$ has Λ-rank zero.
(IMC2). Moreover,

$$(L^-_{\Sigma^*}(\psi^*\chi)) = \mathrm{Char}_\Lambda X(\psi^*\chi).$$

We thus have

$$(L'_{\Sigma}(\lambda) \cdot L^{-}_{\Sigma^*}(\psi^*\chi)) = \mathrm{Char}_{\Lambda} X(\lambda)_{\mathrm{tor}} \cdot \mathrm{Char}_{\Lambda} X(\psi^*\chi) \cdot R(\lambda). \quad (3.5)$$

Non-vanishing. In view of Lemma 3.7 and the non-vanishing of anticyclotomic regulators [2, App.], [7, Thm. A] and also see [11, Intro.], we have

$$R(g \times \chi) = R(\lambda) \neq 0. \quad (3.6)$$

Strictly speaking, the μ-invariant of the cyclotomic derivative $L'_{\Sigma}(\lambda)$ is determined in [7] under the hypothesis $p \nmid h_K$ for h_K the class number. However, the proof shows the non-vanishing of $L'_{\Sigma}(\lambda)$ even when $p \mid h_K$. For example, the argument in [7, §3] goes through verbatim for open subsets $b\Gamma'$ introduced in [24, §5.1].

From the anticyclotomic IMC due to Agboola–Howard and Arnold as above, non-vanishing of the anticyclotomic regulator $R(\lambda)$ follows from that of $L'_{\Sigma}(\lambda)$. From Lemma 3.7, we then deduce non-vanishing of the anticyclotomic regulator $R(g \times \chi)$.

HMC. We now relate the $\mathrm{GL}_{2/\mathbb{Q}}$ and $\mathrm{GL}_{1/K}$-sides.

From Corollary 3.16, Lemma 3.7 and Corollary 3.8, we conclude that

$$\mathrm{Char}_{\Lambda} \mathcal{S}(B)/(\kappa) \cdot \mathrm{Char}_{\Lambda} (\mathcal{S}(B)/(\kappa))^{\iota} \cdot R(g \times \chi) = \mathrm{Char}(\mathcal{X}(B)_{\mathrm{tor}}) \cdot R(\lambda).$$

In view of (3.4), (3.5) and (3.6), it thus follows that

$$\mathrm{Char}_{\Lambda} \mathcal{S}(B)/(\kappa) \cdot \mathrm{Char}_{\Lambda} (\mathcal{S}(B)/(\kappa))^{\iota} = \mathrm{Char}(\mathcal{X}(B)_{\mathrm{tor}}).$$

This finishes the proof. □

Remark 3.17 (1). The hypothesis $p \nmid 6N_g \cdot \mathrm{cond}^{\mathrm{r}}(\chi)$ arises only due to its occurence in the anticyclotomic IMC due to Agboola–Howard and Arnold.

(2). We may ask for a refinement of the approach so as to consider Λ°-adic version of the Heegner main conjecture. As mentioned earlier, it would involve Tamagawa numbers.

Remark 3.18 The approach seems rather different from the one in the non-CM case [12,14,15,48]. For example, the Euler system of Heegner points is not directly used for the 'Euler system' divisibility in Theorem 3.2 i.e. an upper bound for the Λ-torsion module $\mathcal{S}(B)/(\kappa)$. In fact, as the Galois representation has small image, such an argument would perhaps require an additional input. For the 'modular divisibility' i.e. a lower bound for the Λ-torsion module $\mathcal{S}(B)/(\kappa)$, we do not directly rely upon Eisenstein congruence on higher rank unitary group group $U(3,1)_{/K}$ (for example, [47]). In fact, as this is not a semistable setup and the Galois representation has small image, such an argument would again require an additional input. We circumvent these potential issues via passage to $\mathrm{GL}_{1/K}$. For example, Euler system of Heegner points is replaced by Euler system of elliptic units and the unitary group $U(3,1)_{/K}$ replaced with the unitary group $U(2,1)_{/K}$. The approach seems to utilise most of the known results regarding anticyclotomic CM Iwasawa theory in an essential manner. In particular, elliptic units are fundamental in the approach.

4 *p*-converse theorem

In this section, we prove a p-converse theorem in the CM rank one case. In Sect. 4.1, we introduce an auxiliary twist of the underlying Hecke character. In Sect. 4.2, we conclude with the p-converse theorem based on Heegner main conjecture arising from the twist.

4.1 Auxiliary twist

In this subsection, we introduce an auxiliary twist of a Hecke character such that non-vanishing of a certain central Hecke L-value holds.

4.1.1 Auxiliary twist, I

We introduce an auxiliary twist of a Hecke character in a general setup.

Let K be an imaginary quadratic field and λ a self-dual Hecke character over K. We consider a twist of λ of the following kind.

Proposition 4.1 *Let K be an imaginary quadratic field field. Let λ be a self-dual arithmetic Hecke character over K with infinity type* $(1,0)$. *Let T be a finite set of primes of K.*

Then, there exists a finite order Hecke character χ over K with $w \nmid \mathrm{cond}^{\mathrm{r}}(\chi)$ for $w \in T$ such that

$$L\left(1, \lambda \cdot \frac{\chi^*}{\chi}\right) \neq 0.$$

Here $\eta^* = \eta \circ c$ for a Hecke character η over K with $c \in \mathrm{Gal}(K/\mathbb{Q})$ the non-trivial element.

Proof This is based on the main result of [6].

Non-vanishing. Let $\theta(\lambda)$ be the CM modular form corresponding to the Hecke character λ.

From [6, pp. 543–544], there exists a weight two elliptic newform g arising from a quadratic twist unramified at T of the CM modular form $\theta(\lambda)$ such that

$$L(1, g) \neq 0. \tag{4.1}$$

Note that g is again a CM modular form with CM by K. Thus, there exists a Hecke character λ' over K with the same infinity type as λ such that $g = \theta(\lambda^{'})$. Note that the Hecke character λ'/λ is of finite order and also anticylotomic.

Desired form. From $L(s, \lambda^* \cdot \frac{\chi}{\chi^*}) = L(s, g)$, the non-vanishing (4.1) finishes the proof as a finite order anticyclotomic Hecke character has the form χ^*/χ for a finite order Hecke character χ over K [21, Lem. 5.31]. Moreover, the proof of [21, Lem. 5.31] shows that χ can be taken to be unramified at T. □

4.1.2 Auxiliary twist, II

We introduce an auxiliary twist of a Hecke character in a special setup.

In Sect. 4.2, we consider a special case of Proposition 4.1. As it can be approached in another manner, we describe it separately.

Let K be a p-ordinary imaginary quadratic field as in Sect. 2.1.1. Let λ be a self-dual Hecke character over K with infinity type $(1, 0)$. We consider an auxiliary twist of λ of the following kind.

Proposition 4.2 *Let K be a p-ordinary imaginary quadratic field for an odd prime p. Let λ be a self-dual arithmetic Hecke character over K with infinity type* $(1, 0)$.

Then, there exists a finite order Hecke character χ over K with $p \nmid \mathrm{cond}^{\mathrm{r}}(\chi)$ such that

$$L\left(1, \lambda \cdot \frac{\chi^*}{\chi}\right) \neq 0.$$

Here $\eta^* = \eta \circ c$ for a Hecke character η over K with $c \in \mathrm{Gal}(K/\mathbb{Q})$ the non-trivial element.

Proof This is based on the main result of [36].

Root number. First, there exists a finite order Hecke character χ_0 over K with $p \nmid \mathrm{cond}^{\mathrm{r}}(\chi_0)$ such that

$$\epsilon\left(\frac{1}{2}, \lambda \cdot \frac{\chi_0^*}{\chi_0}\right) = +1$$

[11, Lem. 2.5].

Non-vanishing. In view of [36, p. 384], there exists a finite order anticyclotomic Hecke character ψ over K with $p \nmid \mathrm{cond}^{\mathrm{r}}(\lambda \cdot \chi_0)$ such that

$$L\left(1, \lambda \cdot \frac{\chi_0^*}{\chi_0}\psi\right) \neq 0.$$

In fact, ψ can be chosen to have ℓ-power order conductor with ℓ an odd prime unramified in K such that $\ell \nmid p \cdot \mathrm{cond}^{\mathrm{r}}(\lambda\chi_0)$.

In view of [21, Lem. 5.31], this again finishes the proof.[9] □

Remark 4.3 The above approach has an anticyclotomic Iwasawa-theoretic flavour perhaps more in sync with the approach to a p-converse theorem (Sect. 4.2). We also note that the approach in [6] builds on a $\mathrm{GL}_{2/\mathbb{Q}}$-setup, whereas the one in [36] on a $\mathrm{GL}_{1/K}$-setup.

4.2 p-converse theorem

In this subsection, we prove a p-converse theorem in the CM rank one case. The proof is based on a Heegner main conjecture arising from an auxiliary twist in Sect. 4.1.

4.2.1 Setup

Let the notation be as before. In particular, we fix two embeddings $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \to \mathbb{C}_p$ as in Sect. 2. Moreover, $f \in S_2(\Gamma_0(N))$ denotes a weight two newform with trivial Neben-type and E_f the corresponding Hecke field.

Let $A = A_f$ be an abelian variety over $\mathbb{Q}$ associated to f by Eichler–Shimura such that

$$L(s, A_{/\mathbb{Q}}) = \prod_{\sigma : E_f \to \mathbb{C}} L(s, f^\sigma).$$

The endomorphism ring for A contains an order $\mathcal{O}_f$ in the Hecke field E_f and the order is generated over $\mathbb{Z}$ by the Hecke eigenvalues corresponding to g. Replacing A by an isogeny, we assume that A has real multiplication by the ring of integers in E_f. Let $\wp$ be a prime above p in E_f determined via the embedding ι_p. Let $\mathcal{O}$ the localisation of $\mathcal{O}_{E_f}$ at $\wp$.

Then the $\wp^\infty$-Selmer group $\mathrm{Sel}_{\wp^\infty}(A_{/\mathbb{Q}})$ associated to $A_{/\mathbb{Q}}$ is a co-finitely generated $\mathcal{O}_\wp$-module. As the case of elliptic curves in the introduction, we have a BSD conjecture in the setup (Conjecture 1.1) and an analogous result

[9] As the reader may note, hypothesis (ord) is inessential in this approach as well.

towards it due to Gross–Zagier [19, I.6]), Kolyvagin ([28] and Rubin [37], [38, §11].[10]

4.2.2 Main result

We consider a p-converse to the theorem of Gross–Zagier, Kolyvagin and Rubin.

The main result of the article is the following p-converse theorem in the CM case.

Theorem 4.4 *Let $f \in S_2(\Gamma_0(N))$ be an elliptic CM modular form of weight two, trivial neben-type with complex multiplication by an imaginary quadratic field. Let E_f be the corresponding Hecke field with integer ring $\mathcal{O}_{E_f}$. Let $A_{/\mathbb{Q}}$ be a corresponding GL_2-type abelian variety with $\mathcal{O}_{E_f} \subset \mathrm{End}(A)$. Let $p > 3$ be a good ordinary prime for A, $\wp$ a prime above p in E_f determined via the embedding ι_p and $\mathcal{O}$ the completion of $\mathcal{O}_{E_f}$ at $\wp$.*

Then,

$$\mathrm{corank}_{\mathcal{O}_\wp} \mathrm{Sel}_{\wp^\infty}(A_{/\mathbb{Q}}) = 1 \implies \mathrm{ord}_{s=1} L(s, A) = [E_f : \mathbb{Q}].$$

In particular, $\mathrm{rank}_{\mathcal{O}} A(\mathbb{Q}) = 1$ *and* $Ш(A_{/\mathbb{Q}})$ *is finite whenever* $\mathrm{corank}_{\mathcal{O}_\wp} \mathrm{Sel}_{\wp^\infty}(A_{/\mathbb{Q}}) = 1$.

Note that, "In particular" part follows from the work of Gross–Zagier, Kolyvagin and Rubin. We would like to emphasise that finiteness of the Tate–Shafarevich group $Ш(A_{/\mathbb{Q}})$ is not our hypothesis but indeed a consequence.

Corollary 4.5 *Let $f \in S_2(\Gamma_0(N))$ be an elliptic CM modular form of weight two, trivial neben-type with complex multiplication by an imaginary quadratic field. Let $A_{/\mathbb{Q}}$ be corresponding GL_2-type abelian variety and E_f the Hecke field with integer ring $\mathcal{O}_{E_f}$. Let $p > 3$ be a good ordinary prime for A, $\wp$ a prime above p in E_f determined via the embedding ι_p and $\mathcal{O}$ the completion of $\mathcal{O}_{E_f}$ at $\wp$. Suppose that* $\mathrm{corank}_{\mathcal{O}_\wp} \mathrm{Sel}_{\wp^\infty}(A_{/\mathbb{Q}}) = 1$.

Then, the p-part of full BSD conjecture holds for $A_{/\mathbb{Q}}$.

Proof From Theorem 4.4, the rank part of BSD holds. In particular,

$$\mathrm{ord}_{s=1} L(s, A_{/\mathbb{Q}}) = [E_f : \mathbb{Q}], \#Ш(A_{/\mathbb{Q}}) < \infty.$$

In view of the work of Perrin–Riou [35] and Rubin [40], the p-part of BSD formula thus holds (for example, [27, Cor. 1.4]).[11] □

Remark 4.6 In the setup, an analogue of Corollary 1.3 holds as well.

[10] We also refer to [27, Intro].

[11] We also refer to [29, Thm. 1.1].

4.2.3 Proof of the main result

The approach to Theorem 4.4 relies upon Heegner main conjecture (Theorem 3.2) for an auxiliary pair (g, χ) as in Sect. 2.1.

Proof of Theorem 4.4 Let K be the underlying CM field. As f is p-ordinary, the prime p splits in K.

Parity. Let λ be arithmetic Hecke character over K with infinity type $(1, 0)$ corresponding to the CM modular form f. Here the infinty type is with respect to the embedding ι_∞. Note that λ is self-dual.

As $\mathrm{corank}_{\mathcal{O}_\wp} \mathrm{Sel}_{\wp^\infty}(A_{/\mathbb{Q}}) = 1$, we deduce

$$\epsilon\left(\frac{1}{2}, \lambda\right) = -1 \tag{4.2}$$

from the parity conjecture due to Nekovář [31, Thm. A'].

Auxiliary twist. Let χ be a finite order Hecke character over K as in Proposition 4.2. In particular,

$$L\left(1, \lambda^* \cdot \frac{\chi}{\chi^*}\right) \neq 0. \tag{4.3}$$

Self-dual pair. Let g be the CM modular form associated to the Hecke character $\lambda\chi^{-1}$ over K with the same infinity type $(1, 0)$. In what follows, we consider the pair (g, χ) in the setup of Sect. 2.1.

We have a factorisation

$$L(s, g \times \chi) = L(s, \lambda) \cdot L\left(s, \lambda^* \cdot \frac{\chi}{\chi^*}\right)$$

of complex L-functions. By (4.3), we thus have

$$\mathrm{ord}_{s=1} L(s, \lambda) = 1 \iff \mathrm{ord}_{s=1} L(s, g \times \chi) = 1.$$

We first note that the pair satisfies the Heegner hypotheses (H) in Sect. 2.1. Here we rely upon (4.2) and (4.3).

In view of the Gross–Zagier formula due to Yuan–Zhang–Zhang [49, Thm. 1.2], it thus follows that

$$\mathrm{ord}_{s=1} L(s, \lambda) = 1 \iff P_{g,\chi} \neq 0.$$

Here $P_{g,\chi} \in B(K)_\mathbb{Q}$ is the Heegner point on the abelian variety $B_{/K}$ associated to the pair (g, χ) as in Sect. 2.1 (Definition 2.1).

Heegner main conjecture. We approach non-vanishing of the Heegner point via Heegner main conjecture in the setup (Theorem 3.2).

From Rubin ([38, Thm. 11.1], also see [3, Thm. 2.1]),

$$L\left(1, \lambda^* \cdot \frac{\chi}{\chi^*}\right) \neq 0 \implies \mathrm{corank}_L H^1_f\left(K, L\left(\lambda^* \cdot \frac{\chi}{\chi^*}\right)\right) = 0. \quad (4.4)$$

Here $H^1_f(K, L(\lambda^* \cdot \frac{\chi}{\chi^*}))$ denotes the corresponding Bloch–Kato Selmer group.[12]

In view of the Selmer decomposition in Lemma 3.7, we thus have

$$\mathrm{corank}_L(\mathrm{Sel}_{p^\infty}(B_{/K}) \otimes_{\mathcal{O}_0} L) = 1.$$

Here the notation $\mathcal{O}_0$ and L is as in Sect. 2.1.1.

We recall part (ii) of Theorem 3.2, namely

$$\mathrm{Char}_\Lambda \mathcal{S}(B)/(\kappa) \cdot \mathrm{Char}_\Lambda(\mathcal{S}(B)/(\kappa))^\iota = \mathrm{Char}_\Lambda \mathcal{X}(B)_{\mathrm{tor}} \quad (4.5)$$

an equality of ideals in Λ. The Iwasawa module $\mathcal{X}(B)$ has Λ-rank one (Proposition 3.9).

Descent. We now consider descent of the equality (4.5).

Let $\gamma^- \in \Gamma_K^-$ be a topological generator and $I = (\gamma^- - 1) \subset \Lambda$.

We have a natural morphism

$$\mathcal{X}(B)/I \cdot \mathcal{X}(B) \to \mathrm{Sel}_{p^\infty}(B_{/K})^\vee \otimes_{\mathcal{O}_0} L \quad (4.6)$$

with finite kernel and cokernel.[13] In view of Lemma 3.7, such a control for $\mathcal{X}(B)$ indeed follows from an analogous control for the $\mathrm{GL}_{1/K}$-Selmer groups $X(\lambda)$ and $X(\lambda^* \cdot \frac{\chi}{\chi^*})$. The latter control is nothing but [3, Prop. 4.3].

As $\mathrm{corank}_L(\mathrm{Sel}_{p^\infty}(B_{/K}) \otimes_{\mathcal{O}_0} L) = 1$, we first note that $\mathcal{X}(B)/I \cdot \mathcal{X}(B)$ is not a torsion L-module (4.6). In view of HMC (4.5), it now follows that the Heegner cohomology class corresponding to the Heegner point $P_{g,\chi}$ is non-trivial.

This finishes the proof. □

Remark 4.7 The construction of a desired non-trivial Heegner point $P_{g,\chi}$ perhaps crucially relies on the parity conjecture.

Remark 4.8 In view of the proof, the approach would generalise to elliptic CM modular forms with weight greater than two upon availability of complex and

[12] As $L(s, \lambda^* \cdot \frac{\chi}{\chi^*}) = L(s, \theta(\lambda^* \cdot \frac{\chi}{\chi^*}))$, the finiteness (4.4) also follows from [28].

[13] Usually referred as a 'control theorem'.

Λ°-adic Gross-Zagier formulae along with non-vanishing of complex Bloch–Beilinson height corresponding to a non-torsion Heegner cycle underlying the setup.

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