CELL DECOMPOSITION OF SOME UNITARY GROUP RAPOPORT-ZINK SPACES

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Let $p > 2$ be a fixed prime, $\mathbb{Q}_{p^2}|\mathbb{Q}_p$ be a quadratic unramified extension. Let $(V, \langle,\rangle)$ be a hermitian space over $\mathbb{Q}_{p^2}$, and $G = GU(V, \langle,\rangle)$ be the associated unitary similitude group over $\mathbb{Q}_p$. Denote by $n = dim_{\mathbb{Q}_{p^2}} V$, and assume there exists an autodual $\mathbb{Z}_{p^2}$-lattice in V. This implies that G is unramified. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. Then after fixing a basis of V we have an isomorphism $G_{\overline{\mathbb{Q}}_p} \simeq GL_{n\overline{\mathbb{Q}}_p} \times \mathbb{G}_{m\overline{\mathbb{Q}}_p}$. Consider the cocharacter $\mu : \mathbb{G}_{m\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$, such that under the above isomorphism it is given by $z \mapsto (diag(z, \cdots, z, 1), z)$. Let $b = b_0 \in B(G, \mu)$ (the Kottwitz set) be the basic element, J_b be the associated inner form of G. We remark that if n is odd, then $J_b \simeq G$, and if n is even J_b is up to isomorphism the unique non quasi-split inner form of G.

Let $W = W(\overline{\mathbb{F}}_p), L = W_{\mathbb{Q}}$. Consider the associated Rapoport-Zink space $\widehat{\mathcal{M}}$ over $SpfW$: for any $S \in \text{Nilp}W, \widehat{\mathcal{M}}(S) = \{(H, \iota, \lambda, \rho)\}/\simeq$, where H is a p-divisible group over S, ι is a $\mathbb{Z}_{p^2}$-action on H satisfying the determinant condition corresponding to μ, λ is a polarization which is compatible with ι, and $\rho : \mathbb{H}_{\overline{S}} \to H_{\overline{S}}$ is a quasi-isogeny (cf. [8] for more details). Here $\mathbb{H}$ is the standard unitary p-divisible group over $\overline{\mathbb{F}}_p$. We consider the Berkovich analytic generic fiber $\mathcal{M} = \widehat{\mathcal{M}}^{an}$ over L. As usual, there is in fact a tower of L-analytic spaces $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$, where the index set is the open compact subgroups K of $G(\mathbb{Z}_p)$ and $\mathcal{M}_{G(\mathbb{Z}_p)} = \mathcal{M}$. $J_b(\mathbb{Q}_p)$ acts naturally on each space $\mathcal{M}_K$ by modifying the quasi-isogeny, and moreover, $G(\mathbb{Q}_p)$ acts on the tower $(\mathcal{M}_K)_{K \subset G(\mathbb{Z}_p)}$ by Hecke correspondences. Note we have the decompositions (cf.[8])

$$\widehat{\mathcal{M}} = \coprod_{i \in \mathbb{Z},\, ni\ even} \widehat{\mathcal{M}}^i, \mathcal{M} = \coprod_{i \in \mathbb{Z},\, ni\ even} \mathcal{M}^i.$$

To state the theorem, we should fix some data. If n is even, fix an element $g_1 \in J_b(\mathbb{Q}_p)$ such that it induces an isomorphism $\mathcal{M}^0 \to \mathcal{M}^1$. We fix also a $\Lambda \in \mathcal{B}(J_b^{der}, \mathbb{Q}_p)$, the set of vertices of the Bruhat-Tits building of the derived subgroup J_b^{der} of J_b, such that $t(\Lambda)$ is maximal (cf. [8] for the precise meaning of the function t). Let $Stab(\Lambda)$ be the stabilizer of Λ in $J_b^{der}(\mathbb{Q}_p)$.

Theorem 1. *There exists a relatively compact analytic domain $\mathcal{D} \subset \mathcal{M}^0$, such that we have a locally finite covering*

$$\mathcal{M} = \bigcup_{\substack{T \in G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \\ g \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)}} T.g\mathcal{D}$$

if n is odd, and

$$\mathcal{M} = \bigcup_{\substack{T \in G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \\ g \in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda) \\ j=0,1}} T.gg_1^j\mathcal{D}$$

if n is even.

The proof of this theorem is based some ideas developed in [3] and [4]. In particular we use the theory of Harder-Narasimhan filtration of finite flat group schemes to study the p-analytic geometry of $\mathcal{M}$. The fundamental inequality between Harder-Narasimhan polygon and Newton polygon (Théorème 21 of [4]) can be easily generalized to our case. But we have to modify Fargues's algorithm in [4] a little to produce totally isotropic finite flat group schemes to be compatible with Hecke correspondences. The analytic domain $\mathcal{D}$ is defined as following. Let $\mathcal{M}^{ss}$ be the semi-stable locus in $\mathcal{M}$ (cf. Définition 4 of [4]). Consider

$$\mathcal{C} = \{x \in \mathcal{M} |\, \exists \text{some finite extension}\, K'|\mathcal{H}(x), \text{ and a finite flat}\, \mathbb{Z}_{p^2}-\text{subgroup} \\ \text{scheme}\, G \subset H_x[p]\, \text{over}\, O_{K'}, \text{such that}\, H_x/G\, \text{is semi-stable over}\, O_{K'}\}.$$

Then one can prove that $\mathcal{C}$ is a closed analytic domain of $\mathcal{M}$. Note $\mathcal{M}^{ss} \subset \mathcal{C}$. Let Λ be as above, and $\mathcal{M}_\Lambda \subset \mathcal{M}^0_{red}$ be the associated projective subvariety of the reduced special fiber of $\widehat{\mathcal{M}}^0$ defined by Vollaard-Wedhorn in [8]. Consider the specialization map $sp : \mathcal{M}^0 \to \mathcal{M}^0_{red}$, then $sp^{-1}(\mathcal{M}_\Lambda)$ is an open subspace of $\mathcal{M}^0$. The analytic domain $\mathcal{D}$ is defined by $\mathcal{D} := \mathcal{C} \bigcap sp^{-1}(\mathcal{M}_\Lambda)$. The relatively compactness of $\mathcal{D}$ is proved by introducing some special unitary Shimura varieties (cf. [1] and [8]), and the fact that their Harder-Narasimhan stratification and Newton stratification coincide (cf. [6] and [7]). We remark that our methods of proof of the above theorem in some other places are also different from that of [4].

This theorem has many useful applications. First, we have corresponding coverings of the associated p-adic period domain and Shimura varieties. Second, we have the locally finite coverings for all Rapoport-Zink spaces $\mathcal{M}_K$ for any open compact subgroup $K \subset G(\mathbb{Z}_p)$. By studying the action of regular semi-simple elliptic elements on the coverings of the later, we can verify easily that the conditions of Theorem 3.13 in [5] hold. Thus we can establish a Lefschetz trace formula for some sufficiently large subspaces. For more details, see section 11 of [7]. This formula should be useful for proving the realization of local Jacquet-Langlands correspondence in our case.

References

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