ON THE LEFSCHETZ TRACE FORMULA FOR LUBIN-TATE SPACES

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Abstract. We give a new proof of the Lefschetz trace formula for Lubin-Tate spaces. Our proof is based on the locally finite cell decompositions of these spaces and Mieda's theorem of Lefschetz trace formula for certain open adic spaces. This proof is rather different from those in the work of Strauch and Mieda, and is quite hopeful to be generalized to other Rapoport-Zink spaces as soon as there exist some suitable cell decompositions. For example, in another paper we have proved a Lefschetz trace formula for some unitary group Rapoport-Zink spaces by using similar ideas here.
MSC 2010: 14G35, 14G22. **Keywords**: Lubin-Tate spaces, Lefschetz trace formula, cell decomposition.

1. Introduction

We first recall some basic facts about Lubin-Tate spaces, see [18] for further details. Let p be a prime number, F be a finite extension of $\mathbb{Q}_p$, O be the ring of integers of F, and $\pi \in O$ be a uniformizer in O. We denote $\widehat{F}^{nr}$ as the completion of the maximal unramified extension of F, and $\widehat{O}^{nr}$ its ring of integers. For any integer $n \geq 1$, we consider the general linear group GL_n as well as its inner form $D^\times$ over F, where D is the central division algebra over F with invariant $\frac{1}{n}$ and $D^\times$ is the reductive group defined by invertible elements in D. Recall a formal O-module is a formal p-divisible group with an O-action over a base over O, such that the induced action on its Lie algebra is the canonical action of O. We consider the formal Lubin-Tate space $\widehat{\mathcal{M}} = \coprod_{i\in\mathbb{Z}} \widehat{\mathcal{M}}^i$ over $\widehat{O}^{nr}$: for any scheme $S \in \mathrm{Nilp}\widehat{O}^{nr}$, $\widehat{\mathcal{M}}(S) = \{(H,\rho)\}/\simeq$, where

- H is a formal O-module of dimension one over S,
- $\rho : \mathbb{H}_{\overline{S}} \to H_{\overline{S}}$ is a quasi-isogeny.

Here $\mathrm{Nilp}\widehat{O}^{nr}$ is the category of schemes over $\widehat{O}^{nr}$ on which π is locally nilpotent, $\mathbb{H}$ is the unique (up to isomorphism) one dimensional formal O-module over $\overline{\mathbb{F}}_p$ with O-height n, and $\overline{S}$ is the closed subscheme defined by π of $S \in \mathrm{Nilp}\widehat{O}^{nr}$. For $i \in \mathbb{Z}$, $\widehat{\mathcal{M}}^i$ is the open and closed subspace of $\widehat{\mathcal{M}}$ such that the quasi-isogenies ρ have O-height i. There is a natural (left) action of $D^\times$ on $\widehat{\mathcal{M}}$ by $\forall b \in D^\times, b : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}},\ (H,\rho) \mapsto (H, \rho\circ b^{-1})$. This action induces non-canonical isomorphisms

$$\widehat{\mathcal{M}}^i \simeq \widehat{\mathcal{M}}^0,$$

while one knows that there is a non-canonical isomorphism

$$\widehat{\mathcal{M}}^0 \simeq Spf(\widehat{O}^{nr}[[x_1,\dots,x_{n-1}]]).$$

Let $\mathcal{M} = \widehat{\mathcal{M}}^{an} = \coprod_{i\in\mathbb{Z}} \mathcal{M}^i$ be the Berkovich analytic fiber of $\widehat{\mathcal{M}}$ ([1]). By trivializing the local system over $\mathcal{M}$ defined by the Tate module of p-divisible group, we have the Lubin-Tate tower $(\mathcal{M}_K)_{K\subset GL_n(O)}$ over $\widehat{F}^{nr}$, and the group $GL_n(F)$ acts (on the right) on this tower through Hecke correspondences. In particular $\mathcal{M} = \mathcal{M}_{GL_n(O)}$. When $K = K_m := ker(GL_n(O) \to GL_n(O/\pi^m O))$ for some integer $m \geq 0$, there is a regular model $\widehat{\mathcal{M}}_m$ of $\mathcal{M}_{K_m}$ by introducing Drinfeld structures on formal O-modules. We will not use these models and we will work always on the Berkovich spaces $\mathcal{M}_K$. Note there is a natural action of $D^\times$ on each $\mathcal{M}_K$, which commutes with the Hecke action.

Fix a prime $l \neq p$, let $\overline{\mathbb{Q}}_l$ (resp. $\overline{\mathbb{Q}}_p$) be a fixed algebraic closure of $\mathbb{Q}_l$ (resp. $\mathbb{Q}_p$), and $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$. For each $i \geq 0$, we consider the cohomology with compact support

$$H^i_c(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \varinjlim_U \varprojlim_n H^i_c(U \times \mathbb{C}_p, \mathbb{Z}/l^n\mathbb{Z}) \otimes \overline{\mathbb{Q}}_l,$$

where the injective limit is taken over all locally compact open subsets $U \subset \mathcal{M}_K$, see [3] section 4 and [10]. We have

$$H^i_c(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \bigoplus_{j\in\mathbb{Z}} H^i_c(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l),$$

where

$$dim_{\overline{\mathbb{Q}}_l} H^i_c(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) < \infty$$

by theorem 3.3 in [10]. In fact we have also the usual l-adic cohomology groups $H^i(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$ which are Poincaré dual to those $H^i_c(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$, and (cf. [18] lemma 2.5.1)

$$H^i_c(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \neq 0 \Leftrightarrow n-1 \leq i \leq 2(n-1),$$

$$H^i(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \neq 0 \Leftrightarrow 0 \leq i \leq n-1.$$

The groups

$$\varinjlim_K H^i_c(\mathcal{M}_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$$

are natural smooth representations of $GL_n(F) \times D^\times \times W_F$ (W_F is the Weil group of F), and the local Langlands and Jacquet-Langlands correspondences between the three groups are realized in these cohomology groups, see [2], [3] and [8].

In [18] Strauch had proved a Lefschetz trace formula for regular elliptic elements action on the Lubin-Tate spaces. More precisely, we consider

$$H^*_c(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \sum_i (-1)^i H^i_c(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Let $\gamma = (g, b) \in GL_n(F) \times D^\times$ such that both g, b are regular elliptic elements, $gKg^{-1} = K$, $v_p(det(g)) + v_p(Nrd(b)) = 0$. Here and in the following $Nrd : D^\times \to F^\times$ is the reduced norm, v_p is the valuation on F such that $v_p(\pi) = 1$. Then we have an automorphism

$$\gamma : \mathcal{M}^j_K \to \mathcal{M}^j_K,$$

which induces morphisms on cohomology groups

$$\gamma : H^i_c(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \to H^i_c(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

We define

$$Tr(\gamma | H^*_c(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) := \sum_i (-1)^i Tr(\gamma | H^i_c(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)).$$

Strauch proved the following trace formula.

Theorem 1.1 ([18], Theorem 3.3.1)**.** *Under the above assumptions and notations, we have*

$$Tr(\gamma | H^*_c(\mathcal{M}^j_K \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#Fix(\gamma | \mathcal{M}^j_K \times \mathbb{C}_p).$$

By applying the p-adic period mapping

$$\mathcal{M} \to \mathbf{P}^{n-1,an},$$

Strauch obtained a nice fixed points number formula for the quotient space $\mathcal{M}_K/\pi^{\mathbb{Z}}$ (theorem 2.6.8 in loc. cit.)

$$\#\mathrm{Fix}(\gamma|(\mathcal{M}_K/\pi^{\mathbb{Z}})(\mathbb{C}_p)) = n\#\{h \in GL_n(F)/\pi^{\mathbb{Z}}K | h^{-1}g_b h = g^{-1}\},$$

which can be rewritten as some suitable orbit integral, see [13] proposition 3.3. Here g_b is an element of $GL_n(F)$ which is stably conjugate to b, see [18] proposition 2.6.7. This Lefschetz trace formula and the above fixed points formula enable Strauch and Mieda to prove that the Jacquet-Langlands correspondence between smooth representations of $GL_n(F)$ and $D^\times$ is realized in the cohomology of the tower $(\mathcal{M}_K)_K$, not involving with Shimura varieties as in [8], see section 4 of [18] and [13].

There are two main ingredients in Strauch's proof of the above theorem. The first is some careful approximation theorems of Artin in this special (affine) case, and the second is Fujiwara's theorem of specialization of local terms ([7] proposition 1.7.1). In general case one has no sufficient approximation theorems, thus his method can be hardly generalized. In [12] Mieda proved a general Lefschetz trace formula for some open adic spaces by totally working in rigid analytic geometry. He verified that his conditions in the special case of Lubin-Tate spaces hold, thus he can reprove the above Lefschetz trace formula. Both Strauch and Mieda worked in the category of adic spaces, and studied the action of γ on some boundary stratas (outside the corresponding Berkovich space) of the analytic generic fiber of $\widehat{\mathcal{M}}_m$. Their boundary stratas are linked to the theory of generalized canonical subgroups (cf. [4] section 7) in this special case, hence their approachs can hardly be generalized.

In this note we work mainly with Berkovich spaces. We will consider Fargues's locally finite cell decomposition of Lubin-Tate spaces, cf. [5] chapter 1 and [6]. By studying the action of γ on these cells, we verify the conditions in Mieda's theorem of Lefschetz trace formula hold. Here we use the dictionary between the equivalent categories of Hausdorff strictly Berkovich k-analytic spaces and adic spaces which are taut and locally of finite type over $Spa(k, k^0)$. (k is a complete non-archimedean field and k^0 is its ring of integers.) Therefore we can reprove the above theorem, by a different method. The advantage of our method is that, once we know there exists a locally finite cell decomposition with the fundamental domain compact, then by studying the action on the cells we will easily verify Mieda's theorem applies. For example, we can treat the case of some unitary group Rapoport-Zink spaces in [16], and we will also treat the case of basic Rapoport-Zink spaces for GSp_4.

In next section we review the locally finite cell decomposition of $\mathcal{M}_K$, and in section 3 we study the action of γ on the cells and verify Mieda's theorem applies.

Acknowledgments. I would like to thank Laurent Fargues sincerely for sharing the idea of cell decomposition of Rapoport-Zink spaces, and the suggestion that this will suffice to prove the Lefschetz trace formula. I would like to thank also Bertrand Rémy sincerely for replying my question on buildings, and pointing me the article [15]. Yoichi Mieda had proposed me some useful questions after the first version of this article, so special thanks go to him as well. I should also thank the referee for careful reading and suggestions.

2. The locally finite cell decomposition of Lubin-Tate spaces

In [5] and [6] Fargues found some locally finite coverings of $\mathcal{M}_K$. We will call these locally finite coverings as locally finite cell decompositions, and the members of these coverings as cells. Therefore, although these cells are not disjoint, each cell has no empty intersection with only finitely many others. The parameter set of cells in [5] is the set of vertices of some Bruhat-Tits building. These cells for K varies form in fact a cell decomposition of the tower $(\mathcal{M}_K)_K$ but not of a fixed space $\mathcal{M}_K$. Therefore we will mainly follow the construction in [6], where the parameter set is essentially some set of Hecke correspondences. To consider the group actions on cells, we will relate the parameter set to a Bruhat-Tits building by borrowing some ideas from [5].

First consider the case without level structures. Fix a uniformizer $\Pi \in D^\times$, then it induces isomorphisms

$$\Pi^{-1} : \mathcal{M}^i \xrightarrow{\sim} \mathcal{M}^{i+1}.$$

Let $\mathcal{M}^{ss}$ be the semi-stable locus in $\mathcal{M}$, i.e. the locus where the associated p-divisible groups are semi-stable in the sense of [6] definition 4. It is a closed analytic domain in $\mathcal{M}$. Let $\mathcal{D} = \mathcal{M}^{ss,0} := \mathcal{M}^{ss} \cap \mathcal{M}^0$, then $\mathcal{M}^{ss} = \coprod_{i\in\mathbb{Z}} \Pi^{-i}\mathcal{D}$ and $\mathcal{D}$ is the compact fundamental domain of Gross-Hopkins, see [5] 1.5. The main results of [6] in our special case say that we have a locally finite covering

$$\mathcal{M} = \bigcup_{\substack{T\in GL_n(O)\backslash GL_n(F)/GL_n(O)\\ i=0,\ldots,n-1}} T\Pi^{-i}\mathcal{D},$$

where TA is the image under the Hecke correspondence T for a subset A, which is an analytic domain if A is. In the following we shall actually work with the component $\mathcal{M}^0$. We consider its induced cell decomposition

$$\mathcal{M}^0 = \bigcup_{\substack{T\in GL_n(O)\backslash GL_n(F)/GL_n(O)\\ i=0,\ldots,n-1}} ((T\Pi^{-i}\mathcal{D}) \cap \mathcal{M}^0).$$

For $T \in GL_n(O) \setminus GL_n(F)/GL_n(O), i = 0, \ldots, n-1$, we have

$$(T\Pi^{-i}\mathcal{D}) \cap \mathcal{M}^0 \neq \emptyset \Leftrightarrow -v_p(det(T)) + i = 0,$$

in which case we have the inclusion $T\Pi^{-i}\mathcal{D} \subset \mathcal{M}^0$. Here the composition $v_p \circ det$: $GL_n(F) \to \mathbb{Z}$ factors through $GL_n(O) \setminus GL_n(F)/GL_n(O) \to \mathbb{Z}$. Therefore we have the cell decomposition for $\mathcal{M}^0$

$$\mathcal{M}^0 = \bigcup_{\substack{T\in GL_n(O)\backslash GL_n(F)/GL_n(O)\\ i=0,\ldots,n-1\\ -v_p(det(T))+i=0}} T\Pi^{-i}\mathcal{D}.$$

Let $K \subset GL_n(O)$ be an open compact subgroup, $\pi_K : \mathcal{M}_K \to \mathcal{M}$ be the natural projection. We set

$$\mathcal{D}_K = \pi_K^{-1}(\mathcal{D}),$$

which is a compact analytic domain in $\mathcal{M}_K^0$. Since the group $GL_n(O)$ acts trivially on $\mathcal{M}$, any element in this group will stabilize $\mathcal{D}_K$. Therefore, for two Hecke correspondences $T_1, T_2 \in K \setminus GL_n(F)/K$ having the same image under the projection $K \setminus GL_n(F)/K \to GL_n(O) \setminus GL_n(F)/K$, we have

$$T_1\Pi^{-i}\mathcal{D}_K = T_2\Pi^{-i}\mathcal{D}_K.$$

Note $\Pi^{-i}\mathcal{D}_K = \pi_K^{-1}(\Pi^{-i}\mathcal{D})$ since π_K is $D^\times$-equivariant. We have the following locally finite cell decomposition at level K

$$\mathcal{M}_K = \bigcup_{\substack{T\in GL_n(O)\backslash GL_n(F)/K\\ i=0,\ldots,n-1}} T\Pi^{-i}\mathcal{D}_K.$$

We will denote the cells $T\Pi^{-i}\mathcal{D}_K$ by

$$\mathcal{D}_{T,i,K},$$

which are compact analytic domains. For any $T \in GL_n(O) \setminus GL_n(F)/K, i \in \mathbb{Z}$, we denote also $\mathcal{D}_{T,i,K} = T\Pi^{-i}\mathcal{D}_K$. Since the (right) action of $F^\times$ on $\mathcal{M}_K$ through the embedding $F^\times \to GL_n(F), z \mapsto z$ is the same as the (left) action of it on $\mathcal{M}_K$ through the embedding $F^\times \to D^\times, z \mapsto z$, the elements $(z, z^{-1}) \in GL_n(F) \times D^\times$ for $z \in F^\times$ act trivially on $\mathcal{M}_K$. For $z \in F^\times$, we have the equality

$$\mathcal{D}_{T,i,K} = \mathcal{D}_{Tz,i+nv_p(z),K}.$$

Now if $g \in GL_n(F)$ is an element such that $gKg^{-1} = K$, and $b \in D^\times$ is an arbitrary element, set

$$\gamma := (g, b).$$

Then automorphism $\gamma : \mathcal{M}_K \to \mathcal{M}_K$ naturally induces an action on the set of cells of $\mathcal{M}_K$:

$$\gamma(\mathcal{D}_{T,i,K}) = \mathcal{D}_{Tg,i-v_p(Nrd(b)),K}.$$

For the component $\mathcal{M}_K^0$, for $T \in GL_n(O) \setminus GL_n(F)/K, i = 0, \dots, n-1$, similarly as the case $K = GL_n(O)$, we have

$$(T\Pi^{-i}\mathcal{D}_K) \cap \mathcal{M}_K^0 \neq \emptyset \Leftrightarrow -v_p(det(T)) + i = 0,$$

in which case $T\Pi^{-i}\mathcal{D}_K \subset \mathcal{M}_K^0$. Thus we have a locally finite cell decomposition

$$\begin{aligned}
\mathcal{M}_K^0 &= \bigcup_{\substack{T \in GL_n(O)\setminus GL_n(F)/K \\ i=0,\dots,n-1}} ((T\Pi^{-i}\mathcal{D}_K) \cap \mathcal{M}_K^0) \\
&= \bigcup_{\substack{T \in GL_n(O)\setminus GL_n(F)/K \\ i=0,\dots,n-1 \\ -v_p(det(T))+i=0}} \mathcal{D}_{T,i,K}.
\end{aligned}$$

In fact for any $i \in \mathbb{Z}, T \in GL_n(O) \setminus GL_n(F)/K$ such that $-v_p(det(T)) + i = 0$, we have $\mathcal{D}_{T,i,K} \subset \mathcal{M}_K^0$ with the convention above. However, one can always by multiplying with some $z \in F^\times$ reduce to the cases $0 \le i \le n-1$. Now let $\gamma = (g, b) \in GL_n(F) \times D^\times$ be such that $gKg^{-1} = K, v_p(det(g)) + v_p(Nrd(b)) = 0$, then the action of γ on $\mathcal{M}_K$ induces $\gamma : \mathcal{M}_K^0 \to \mathcal{M}_K^0$. In this case γ acts on the set of cells of $\mathcal{M}_K^0$ in the same way as above.

To understand better the parameter set of cells of $\mathcal{M}_K^0$, we look at some ideas from [5]. Consider the embedding $\mathbb{G}_m \to GL_n \times D^\times, z \mapsto (z, z^{-1})$ of algebraic groups over F. Let $\mathcal{B}(GL_n \times D^\times, F)$ be the (extended) Bruhat-Tits building of $GL_n \times D^\times$ over F, and $\mathcal{B} = \mathcal{B}(GL_n \times D^\times, F)/F^\times$ be its quotient by the action of $F^\times$ through the above embedding. The set $\mathcal{B}^0$ of vertices of $\mathcal{B}$, which we define by the quotient of the vertices of $\mathcal{B}(GL_n \times D^\times, F)$, can be described as the set of equivalence classes

$$\{(\Lambda, M) | \Lambda \subset F^n \text{ is an } O\text{-lattice }, M \subset D \text{ is an } O_D^\times\text{-lattice}\}/\sim,$$

where

$$(\Lambda_1, M_1) \sim (\Lambda_2, M_2) \Leftrightarrow \exists i \in \mathbb{Z}, \Lambda_2 = \Lambda_1\pi^i, M_2 = \pi^{-i}M_1,$$

see [5] 1.5. We can understand $\mathcal{B}$ in the following way. The (extended) Bruhat-Tits building of GL_n over F is the product $\mathcal{B}(PGL_n, F) \times \mathbb{R}$ of the Bruhat-Tits building of PGL_n with $\mathbb{R}$, while the (extended) Bruhat-Tits building of $D^\times$ over F is $\mathcal{B}(D^\times, F) \simeq \mathbb{R}$. By construction

$$\mathcal{B} = \mathcal{B}(GL_n \times D^\times, F)/F^\times \simeq (\mathcal{B}(PGL_n, F) \times \mathbb{R} \times \mathbb{R})/\sim,$$

where the equivalence relation is defined by $(x, s, t) \sim (x', s', t') \Leftrightarrow x = x', s - s' = t' - t = nr$ for some $r \in \mathbb{Z}$. Therefore, any point $[x, s, t]$ in $\mathcal{B}$ can be written uniquely in the form $[x, s', t']$

for $x \in \mathcal{B}(PGL_n, F), s' \in \mathbb{R}, t' \in [0, n)$. The group $GL_n(F) \times D^\times$ acts on $\mathcal{B}$ by $\forall [x, s, t] \in \mathcal{B}$, $(g, b) \in GL_n(F) \times D^\times$,

$$(g, b)[x, s, t] = [g^{-1}x, s + v_p(det(g)), t + v_p(Nrd(b))].$$

If we consider the right action of $GL_n(F)$ on $\mathcal{B}(PGL_n, F)$ by $xg := g^{-1}x$, then we can also write $(g, b)[x, s, t] = [xg, s + v_p(det(g)), t + v_p(Nrd(b))]$.

On the other hand, consider the action of $F^\times$ on $GL_n(O) \setminus GL_n(F) \times D^\times / O_D^\times$ by $z(GL_n(O)g, dO_D^\times) = (GL_n(O)gz, zdO_D^\times)$, $\forall z \in F^\times$. Then the quotient set

$$(GL_n(O) \setminus GL_n(F) \times D^\times / O_D^\times)/F^\times$$

is naturally identified with the set $\mathcal{B}^0$ after fixing the vertex $[O^n, O_D] \in \mathcal{B}^0$. For an element $[GL_n(O)g, dO_D^\times]$, the associated point in $\mathcal{B}^0$ can be written as

$$[GL_n(O)F^\times g, v_p(det(g)), v_p(Nrd(d))].$$

Here $GL_n(O)F^\times g \in \mathcal{B}(PGL_n, F)$ by fixing the homothety class of O^n. Now let $K \subset GL_n(O)$ be an open compact subgroup, then the set

$$\mathcal{I}_K := (GL_n(O) \setminus GL_n(F)/K \times D^\times / O_D^\times)/F^\times$$

can be identified with the image of $\mathcal{B}^0$ in the quotient space $\mathcal{B}/K$:

$$\mathcal{B}^0/K.$$

If $\gamma = (g, b) \in GL_n(F) \times D^\times$ such that $gKg^{-1} = K$, then γ acts on the set $\mathcal{I}_K$ by $[T, d] \mapsto [Tg, bd]$. There is a map

$$\begin{aligned}\psi : GL_n(O) \setminus GL_n(F)/K \times D^\times / O_D^\times &\longrightarrow \mathbb{Z}\\ (T, d) &\mapsto -v_p(det(T)) - v_p(Nrd(d)).\end{aligned}$$

Since the action of $F^\times$ does not change the values of the above map, ψ factors through a map $\overline{\psi} : \mathcal{I}_K \to \mathbb{Z}$. In fact there is a well defined continuous map

$$\begin{aligned}\varphi : \mathcal{B} &\longrightarrow \mathbb{R}\\ [x, s, t] &\mapsto -s - t,\end{aligned}$$

with each fiber stable under the action of K. Hence the above map $\overline{\psi}$ is induced by φ. Let $(GL_n(O) \setminus GL_n(F)/K \times D^\times / O_D^\times)^0 := \psi^{-1}(0)$ be the inverse image of 0 under the map ψ. For the γ as above with further condition that $v_p(det(g)) + v_p(Nrd(b)) = 0$, it stabilizes the subset

$$\mathcal{I}_K^0 := \overline{\psi}^{-1}(0) = (GL_n(O) \setminus GL_n(F)/K \times D^\times / O_D^\times)^0/F^\times$$

for its above action on $\mathcal{I}_K$. For the map φ above, we see that $\mathcal{I}_K^0$ is identified with the quotient set $\varphi^{-1}(0)^0/K$, where $\varphi^{-1}(0)^0$ is the set of vertices in $\varphi^{-1}(0)$.

For any element $[T, d] \in \mathcal{I}_K$, the cell $[T, d]\mathcal{D}_K$ is well defined, which is what we denoted by $\mathcal{D}_{T, -v_p(Nrd(d)), K}$ above. As before we denote $[T, d]\mathcal{D}_K$ as

$$\mathcal{D}_{[T,d],K}.$$

Then we can rewrite the cell decompositions as

$$\mathcal{M}_K = \bigcup_{[T,d] \in \mathcal{I}_K} \mathcal{D}_{[T,d],K},$$

$$\mathcal{M}_K^0 = \bigcup_{[T,d] \in \mathcal{I}_K^0} \mathcal{D}_{[T,d],K}.$$

For $\gamma = (g, b) \in GL_n(F) \times D^\times$ as above, it acts on the cells in the way compatible with its action on $\mathcal{I}_K$:

$$\gamma(\mathcal{D}_{[T,d],K}) = \mathcal{D}_{[Tg,bd],K}.$$

Recall there is a metric $d(\cdot,\cdot)$ on $\mathcal{B}$, so that $(GL_n(F) \times D^\times)/F^\times$ acts on $\mathcal{B}$ by isometries. If $d'(\cdot,\cdot)$ is the metric on $\mathcal{B}(PGL_n, F)$, then for two points $[x,s,t],[x',s',t']$ with $x,x' \in \mathcal{B}(PGL_n, F), s,s' \in \mathbb{R}, t,t' \in [0,n)$ we have

$$d([x,s,t],[x',s',t']) = \sqrt{d'(x,x')^2 + (s-s')^2 + (t-t')^2}.$$

The group K acts on $\mathcal{B}$ through the natural morphisms $K \to GL_n(F) \times D^\times \to (GL_n(F) \times D^\times)/F^\times$. There is an induced metric $\overline{d}(\cdot,\cdot)$ on the quotient space $\mathcal{B}/K$:

$$\overline{d}(xK, yK) := \inf_{k,k' \in K} d(xk, yk') = \inf_{k \in K} d(xk, y) = \inf_{k \in K} d(x, yk),\ \forall\, xK, yK \in \mathcal{B}/K,$$

the last two equalities come from $d(xk, yk') = d(xk(k')^{-1}, y) = d(x, yk'k^{-1})$. Since K is compact, one checks it easily that this is indeed a metric on $\mathcal{B}/K$. With this metric, $\mathcal{I}_K, \mathcal{I}_K^0$ are both infinity discrete subsets of $\mathcal{B}/K$, and any closed ball in $\mathcal{B}/K$ contains only finitely many elements of $\mathcal{I}_K$ and $\mathcal{I}_K^0$. We will directly work with the induced metric space

$$\mathcal{I}_K = \mathcal{B}^0/K.$$

For $\gamma = (g,b) \in GL_n(F) \times D^\times$ with $gKg^{-1} = K$, one can check by definition of $\overline{d}$ that the above action of γ on $\mathcal{I}_K$ is isometric:

$$\overline{d}(\gamma x, \gamma x) = \overline{d}(x,x),\ \forall x \in \mathcal{I}_K.$$

Note that for $[T_1,d_1],[T_2,d_2] \in \mathcal{I}_K$, $\mathcal{D}_{[T_1,d_1],K} \cap \mathcal{D}_{[T_2,d_2],K} \neq \emptyset$ implies that $v_p(det((T_1))) + v_p(Nrd((d_1))) = v_p(det((T_2))) + v_p(Nrd((d_2)))$. We rewrite $[T_1,d_1] = [x_1K, s_1, t_1], [T_2,d_2] = [x_2K, s_2, t_2]$ with $x_1, x_2 \in \mathcal{B}(PGL_n, F), s_1, s_2 \in \mathbb{Z} \subset \mathbb{R}, t_1, t_2 \in [0,n) \cap \mathbb{Z}$. Recall that these equalities mean that $\exists r_1, r_2 \in \mathbb{Z}$, such that $v_p(det(T_i)) = s_i + nr_i, v_p(Nrd(d_i)) = t_i - nr_i, i = 1,2$. Then we have $s_1 + t_1 = s_2 + t_2, s_1 - s_2 = t_2 - t_1 \in [1-n, n-1]$. Therefore the distance

$$\overline{d}([T_1,d_1],[T_2,d_2]) = \inf_{k \in K} \sqrt{d'(x_1, x_2k)^2 + 2(s_1 - s_2)^2}$$

just depends on $\overline{d'}(x_1K, x_2K)$ for the induced metric $\overline{d'}$ on $\mathcal{B}(PGL_n, F)$, which is defined in the same way as $\overline{d}$. By the construction of the locally finite sell decomposition of $\mathcal{M}_K$, we have the following proposition.

Proposition 2.1. *There exists a constant $c > 0$, such that for any $[T_1,d_1],[T_2,d_2] \in \mathcal{I}_K$ with $\overline{d}([T_1,d_1],[T_2,d_2]) > c$, we have*

$$\mathcal{D}_{[T_1,d_1],K} \cap \mathcal{D}_{[T_2,d_2],K} = \emptyset.$$

Proof. We need to prove that, there exists a constant $c > 0$, such that for any $[T,d] \in \mathcal{I}_K$, and any $[T',d'] \in \{[T',d'] \in \mathcal{I}_K | \mathcal{D}_{[T',d'],K} \cap \mathcal{D}_{[T,d],K} \neq \emptyset\}$, we have $\overline{d}([T,d],[T',d']) \leq c$. This comes from the construction of the locally finite cell decomposition of $\mathcal{M}_K$, and the definition of $\overline{d}$. We just indicate some key points. First, for any fixed choice of fundamental domain V_K in $\mathcal{B}$ for the action of K, by definition $\forall x, y \in V_K, d(x,y) \geq \overline{d}(xK, yK)$. Next, by the proof of proposition 24 of [6], for any fixed Hecke correspondence $T \in GL_n(O) \setminus GL_n(F)/GL_n(O)$ and $i \in \mathbb{Z}$, the finite set

$$A_{[T,i]} := \{[T',j] \in (GL_n(O) \setminus GL_n(F)/GL_n(O) \times \mathbb{Z})/F^\times | T\Pi^{-i}\mathcal{D} \cap T'\Pi^{-j}\mathcal{D} \neq \emptyset\}$$

is such that $\forall [T',j] \in A_{[T,i]}$ we have $-v_p(det(T')) + j = -v_p(det(T)) + i$. By the Cartan decomposition $GL_n(O) \setminus GL_n(F)/GL_n(O) \simeq \mathbb{Z}_+^n = \{(a_1,\dots,a_n) \in \mathbb{Z}^n | a_1 \geq \cdots \geq a_n\}$, if T corresponds to the point $(a_1,\dots,a_n) \in \mathbb{Z}_+^n$, then for $j \in \mathbb{Z}/n\mathbb{Z}$ fixed, the set of $T' \in GL_n(O) \setminus GL_n(F)/GL_n(O)$ with $[T',j] \in A_{[T,i]}$, corresponds to the set of points $(a_1',\dots,a_n') \in \mathbb{Z}_+^n$

such that $\sum_{k=1}^{n} a'_k = \sum_{k=1}^{n} a_k - i + j \ (mod\ n\mathbb{Z}), |a_k - a'_k| \leq C$ for all $k = 1, \ldots, n$, where $C > 0$ is a constant which doesn't depend on $[T, i]$. From these two points one can easily deduce the proposition for $K = GL_n(O)$, and the general case will be obtained as soon as the case $K = GL_n(O)$ holds. □

In fact, by [4] corollary 9 the converse of the above proposition also holds. Thus there exists a constant $c > 0$, such that for any $[T, d], [T', d'] \in \mathcal{I}_K$, $\mathcal{D}_{[T',d'],K} \cap \mathcal{D}_{[T,d],K} \neq \emptyset$ if and only if $\overline{d}([T, d], [T', d']) \leq c$.

We remark that, in [4] Fargues defined an $O_{D^\times}$-invariant continuous map of topological spaces

$$\mathcal{M}^0 \longrightarrow \mathcal{B}(PGL_n, F)/GL_n(O),$$

and identified the image of $\mathcal{D}$ under this map. However, this map depends quite on our special case. In general there is no such map from Rapoport-Zink spaces to Bruhat-Tits buildings. For any open compact subgroup $K \subset GL_n(O)$, there is also a continuous map $\mathcal{M}^0 \to \mathcal{B}(PGL_n, F)/K$, and we have a commutative diagram of continuous maps between topological spaces

$$\begin{array}{ccc} \mathcal{M}_K^0 & \longrightarrow & \mathcal{B}(PGL_n, F)/K \\ \downarrow & & \downarrow \\ \mathcal{M}^0 & \longrightarrow & \mathcal{B}(PGL_n, F)/GL_n(O). \end{array}$$

These maps are Hecke equivariant, thus compatible with the cell decomposition of $\mathcal{M}_K^0$.

3. Lefschetz trace formula for Lubin-Tate spaces

In this section $\gamma = (g, b) \in GL_n(F) \times D^\times$ is an element such that both g and b are regular elliptic, $gKg^{-1} = K$ and $v_p(det(g)) + v_p(Nrd(b)) = 0$. (Here we use the convention that an elliptic element is always semi-simple.) Since γ is regular elliptic, the set of γ-fixed vertices $(\mathcal{B}^0)^\gamma$ is non empty, cf. [17]. Let $\widehat{o}$ be a fixed choice of point in $(\mathcal{B}^0)^\gamma$, and $o \in \mathcal{I}_K$ be its image in the quotient space. One can take the above choice of $\widehat{o}$ so that $\widehat{o} \in \varphi^{-1}(0)^0, o \in \mathcal{I}_K^0$. Then $\gamma(o) = o$ by the action $\gamma : \mathcal{I}_K^0 \to \mathcal{I}_K^0$. For any real number $\rho > 0$, we consider the subset of $\mathcal{I}_K^0$

$$A_\rho = \{x \in \mathcal{I}_K^0 |\ \overline{d}(o, x) \leq \rho\},$$

which is a finite set for any fixed ρ. Moreover since $\gamma(o) = o$ and $\overline{d}$ is γ-isometric, we have $\gamma(A_\rho) = A_\rho$.

Definition 3.1. *For any finite set $A \subset \mathcal{I}_K^0$, we define two subspaces of $\mathcal{M}_K^0$*

$$V_A = \bigcup_{[T,d]\in A} \mathcal{D}_{[T,d],K},$$

$$U_A = \mathcal{M}_K^0 - \bigcup_{[T,d]\notin A} \mathcal{D}_{[T,d],K}.$$

Proposition 3.2. *U_A is an open subspace of $\mathcal{M}_K^0$, while V_A is a compact analytic domain, and $U_A \subset V_A$.*

Proof. Since $\mathcal{M}_K^0 - U_A = \bigcup_{[T,d]\notin A} \mathcal{D}_{[T,d],K}$, which is a locally finite union of closed subsets, therefore it is closed, and U_A is open. V_A is a finite union of compact analytic domains thus so is itself. The inclusion simply comes from the fact $\mathcal{M}_K^0 = V_A \cup (\mathcal{M}_K^0 - U_A)$. □

When $\rho \to \infty$, the finite sets A_ρ exhaust $\mathcal{I}_K^0$. For any $\rho \geq 0$, we denote $U_\rho = U_{A_\rho}, V_\rho = V_{A_\rho}$. Since U_ρ is relatively compact, we can compute the cohomology of $\mathcal{M}_K^0$ as

$$H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \varinjlim_{\rho} H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Moreover, for $\rho >> 0$ large enough, the cohomology group $H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$ is constant and bijective to $H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$, see proposition 3.5.

For the γ above, we consider the action $\gamma : \mathcal{M}_K^0 \to \mathcal{M}_K^0$. Since $\gamma(A_\rho) = A_\rho$, we have

$$\gamma(U_\rho) = U_\rho, \gamma(V_\rho) = V_\rho.$$

For the cells contained in V_ρ, γ acts as $\gamma(\mathcal{D}_{[T,d],K}) = \mathcal{D}_{[Tg,bd],K}$. Passing to the cohomology of U_ρ, γ induces an automorphism

$$\gamma : H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \to H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Consider

$$H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \sum_i (-1)^i H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)$$

as an element in some suitable Grothendieck group, and the trace of γ

$$Tr(\gamma | H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \sum_i (-1)^i Tr(\gamma | H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)).$$

Let $\mathrm{Fix}(\gamma | \mathcal{M}_K^0 \times \mathbb{C}_p)$ be the set of fixed points of γ on $\mathcal{M}_K^0 \times \mathbb{C}_p$, then each fixed point is simple since the p-adic period mapping is étale (cf. [18] theorem 2.6.8).

We will use our result of cell decomposition of $\mathcal{M}_K^0$, to verify that the action of γ satisfies the conditions of Mieda's theorem 3.13 [12], thus we will deduce a Lefschetz trace formula in our case. Recall that, if k is a complete non-archimedean field and k^0 is its ring of integers, then the category of Hausdorff strictly k-analytic spaces is equivalent to the category of adic spaces which are taut and locally of finite type over $spa(k, k^0)$, see [9] chapter 8. If X is a Hausdorff strictly k-analytic space, we denote by X^{ad} the associated adic space, which is taut and locally of finite type over $spa(k, k^0)$.

Theorem 3.3. *Let the notations and assumptions be as above. For $\gamma = (g, b) \in GL_n(F) \times D^\times$ such that both g and b are regular elliptic, $gKg^{-1} = K$ and $v_p(det(g)) + v_p(Nrd(b)) = 0$, there exist an open compact subgroup $K' \subset GL_n(O)$ and a real number ρ_0, such that for all open compact subgroup $K \subset K'$ which is normalized by g and all $\rho \geq \rho_0$, we have*

$$Tr(\gamma | H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \# Fix(\gamma | \mathcal{M}_K^0 \times \mathbb{C}_p).$$

For ρ sufficiently large, the left hand side is just $Tr(\gamma | H_c^*(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l))$.

Proof. Since $g \in GL_n(F)$ is elliptic, we first prove the following claim: for any sufficiently small open compact subgroup $K \subset GL_n(O)$ such that $gKg^{-1} = K$, we have

$$\overline{d}(x, \gamma x) \to \infty, \text{ when } x \in \mathcal{I}_K^0, \overline{d}(o, x) \to \infty.$$

In fact, since $o, x \in \mathcal{I}_K^0$, write $o = [o'K, -s, s], x = [x'K, -t, t]$ with $o', x' \in \mathcal{B}(PGL_n, F)^0$, $s, t \in [0, n-1] \cap \mathbb{Z}$, then $\gamma(x) = [x'gK, v_p(det(g)) - t, v_p(Nrd(b)) + t] = [x'gK, -t', t']$ for some unique $t' = v_p(det(g)) - t + nr \in [0, n-1] \cap \mathbb{Z}$. If we denote the metric on $\mathcal{B}' = \mathcal{B}(PGL_n, F)$ by $d'(\cdot, \cdot)$ and the induced metric on $\mathcal{B}'/K$ by $\overline{d'}$ as before, then we just need to prove that

$$\overline{d'}(x'K, x'gK) \to \infty, \text{ when } x'K \in (\mathcal{B}')^0/K, \overline{d'}(o'K, x'K) \to \infty.$$

To prove this statement, we first work with $\mathcal{B}'$ itself. Since g is elliptic, the fixed points set $(\mathcal{B}')^g$ is nonempty and compact. Moreover, for K sufficiently small, $(\mathcal{B}')^g = (\mathcal{B}')^{g'}$ for any $g' \in gK$ (cf. the proof of lemma 12 in [17]). For $o' \in (\mathcal{B}')^g$ fixed, a simple triangle inequality shows that $d'(x',(\mathcal{B}')^g)\to\infty$ when $d'(x',o')\to\infty$, since $(\mathcal{B}')^g$ is compact. On the other hand, for any automorphism σ of $\mathcal{B}'$ with $(\mathcal{B}')^\sigma\neq\emptyset$, there exists a constant $0<\theta\leq\pi$ which just depends on $\mathcal{B}'$ and σ, such that

$$d'(x',\sigma x')\geq 2d'(x',(\mathcal{B}')^\sigma)\sin(\frac{\theta}{2}),$$

see [15] proposition 2.3. In particular, $d'(x',x'g')\to\infty$ when $d'(o',x')\to\infty$ for any $g'\in gK$. As K is compact this deduces the above statement.

To use the above claim, we will study the action of γ around the boundary points of U_ρ and V_ρ for ρ sufficiently large. We have

$$\mathcal{M}_K^0-U_\rho=\bigcup_{[T,d]\in\mathcal{I}_K^0-A_\rho}\mathcal{D}_{[T,d],K}$$
$$V_\rho-U_\rho=\bigcup_{[T,d]\in A_\rho-A_{\rho-c}}F_{[T,d]},$$

where for $[T,d]\in A_\rho$,

$$F_{[T,d]}=\mathcal{D}_{[T,d],K}\cap(\mathcal{M}_K^0-U_\rho),$$

which is nonempty if and only if $[T,d]\in A_\rho-A_{\rho-c}$ by the above proposition 2.1 (c is the constant constructed in this proposition), in which case $F_{[T,d]}$ is a compact analytic domain in $\mathcal{D}_{[T,d],K}\subset V_\rho$. For K sufficiently small, ρ sufficiently large and $[T,d]\in\mathcal{I}_K^0-A_{\rho-c}$, by the above claim $\overline{d}([T,d],\gamma([T,d]))>c$, in particular

$$\mathcal{D}_{[T,d],K}\cap\gamma(\mathcal{D}_{[T,d],K})=\emptyset,\ F_{[T,d]}\cap\gamma(F_{[T,d]})=\emptyset,\text{ for }[T,d]\in A_\rho-A_{\rho-c}.$$

To apply Mieda's theorem, we pass to adic spaces. We have the locally finite cell decomposition of the adic space $(\mathcal{M}_K^0)^{ad}$:

$$(\mathcal{M}_K^0)^{ad}=\bigcup_{[T,d]\in\mathcal{I}_K^0}\mathcal{D}_{[T,d],K}^{ad}$$

where each cell $\mathcal{D}_{[T,d],K}^{ad}$ is an open quasi-compact subspace, $\mathcal{D}_{[T_1,d_1],K}^{ad}\cap\mathcal{D}_{[T_2,d_2],K}^{ad}\neq\emptyset\Leftrightarrow\mathcal{D}_{[T_1,d_1],K}\cap\mathcal{D}_{[T_2,d_2],K}\neq\emptyset$, and the action of γ on $(\mathcal{M}_K^0)^{ad}$ induces an action on the cells in the same way as the case of Berkovich analytic spaces. By [9] 8.2, U_ρ^{ad} is an open subspace of $(\mathcal{M}_K^0)^{ad}$, which is separated, smooth, partially proper. On the other hand, $V_\rho^{ad}=\bigcup_{[T,d]\in A_\rho}\mathcal{D}_{[T,d],K}^{ad}$ is a quasi-compact open subspace. Consider the closure $\overline{V_\rho^{ad}}=\bigcup_{[T,d]\in A_\rho}\overline{\mathcal{D}_{[T,d],K}^{ad}}$ of V_ρ^{ad} in $(\mathcal{M}_K^0)^{ad}$, which is a proper pseudo-adic space. We know that $\overline{V_\rho^{ad}}$ (resp. $\overline{\mathcal{D}_{[T,d],K}^{ad}}$) is the set of all the specializations of points in V_ρ^{ad} (resp. $\mathcal{D}_{[T,d],K}^{ad}$). Moreover γ induces automorphisms $\gamma:\overline{V_\rho^{ad}}\to\overline{V_\rho^{ad}},V_\rho^{ad}\to V_\rho^{ad},U_\rho^{ad}\to U_\rho^{ad}$. Since $V_{\rho-c}^{ad}\subset U_\rho^{ad}\subset V_\rho^{ad}$, U_ρ^{ad} is closed under specializations, we have

$$\overline{V_\rho^{ad}}-V_\rho^{ad}=\bigcup_{[T,d]\in A_\rho-A_{\rho-c}}\overline{\mathcal{D}_{[T,d],K}^{ad}}-\bigcup_{[T,d]\in A_\rho-A_{\rho-c}}\mathcal{D}_{[T,d],K}^{ad}\subset\bigcup_{[T,d]\in A_\rho-A_{\rho-c}}(\overline{\mathcal{D}_{[T,d],K}^{ad}}-\mathcal{D}_{[T,d],K}^{ad}).$$

Note

$$\mathcal{D}_{[T_1,d_1],K}^{ad}\cap\mathcal{D}_{[T_2,d_2],K}^{ad}\neq\emptyset\Leftrightarrow\overline{\mathcal{D}_{[T_1,d_1],K}^{ad}}\cap\overline{\mathcal{D}_{[T_2,d_2],K}^{ad}}\neq\emptyset.$$

For $[T,d]\in A_\rho-A_{\rho-c}$, let $W_{[T,d]}=\overline{\mathcal{D}_{[T,d],K}^{ad}}-\mathcal{D}_{[T,d],K}^{ad}$. By the paragraph above, for $\rho>>0$ we have $\gamma(W_{[T,d]})\cap W_{[T,d]}=\emptyset$. Since we can write $\overline{V_\rho^{ad}}-V_\rho^{ad}=\bigcup_{[T,d]\in A_\rho-A_{\rho-c}}W'_{[T,d]}$

where $W'_{[T,d]} \subset W_{[T,d]}$ is some locally closed subset for each $[T,d] \in A_\rho - A_{\rho-c}$, one sees the conditions of theorem 3.13 of [12] hold for V_ρ^{ad} and its compactification $\overline{V_\rho^{ad}}$, i.e.

$$Tr(\gamma|H_c^*(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#\mathrm{Fix}(\gamma|V_\rho^{ad} \times \mathbb{C}_p) = \#\mathrm{Fix}(\gamma|V_\rho \times \mathbb{C}_p).$$

Here and in the following $V_\rho^{ad} \times \mathbb{C}_p := V_\rho^{ad} \times spa(\mathbb{C}_p, O_{\mathbb{C}_p})$, and similar notations for other adic spaces. By [10] proposition 2.6 (i) and lemma 3.4, we have

$$Tr(\gamma|H_c^*(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = Tr(\gamma|H_c^*(U_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) + Tr(\gamma|H_c^*((V_\rho^{ad} - U_\rho^{ad}) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)).$$

Since $V_\rho^{ad} - U_\rho^{ad} = \bigcup_{[T,d]\in A_\rho - A_{\rho-c}} F'_{[T,d]}$, where $F'_{[T,d]} \subset F^{ad}_{[T,d]}$ is some locally closed subset for each $[T,d] \in A_\rho - A_{\rho-c}$, by the paragraph above, $F'_{[T,d]} \cap \gamma(F'_{[T,d]}) = \emptyset$. As $A_\rho - A_{\rho-c}$ is a finite set, there are only finitely many orbits for the action of γ. For the union of the subspaces $F'_{[T,d]}$ over an orbit of γ, the trace of γ on its cohomology is 0. Then one can repeat by the above argument to get that $Tr(\gamma|H_c^*((V_\rho^{ad} - U_\rho^{ad}) \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = 0$. We can conclude by Huber's comparison theorem for compactly support cohomology of Berkovich spaces and adic spaces (cf. proposition 8.3.6 of [9]),

$$Tr(\gamma|H_c^*(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = Tr(\gamma|H_c^*(U_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = Tr(\gamma|H_c^*(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l)) = \#\mathrm{Fix}(\gamma|V_\rho \times \mathbb{C}_p).$$

But by the reason above, for $\rho >> 0$ there is no fixed points of γ outside $V_\rho \times \mathbb{C}_p$,

$$\#\mathrm{Fix}(\gamma|V_\rho \times \mathbb{C}_p) = \#\mathrm{Fix}(\gamma|\mathcal{M}_K^0 \times \mathbb{C}_p).$$

The last statement in the theorem comes from the following proposition 3.5. □

Remark 3.4. *In fact we can use V_ρ to compute the cohomology of $\mathcal{M}_K^0$ directly when passing to adic spaces:*

$$H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq H_c^i((\mathcal{M}_K^0)^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) = \varinjlim_\rho H_c^i(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l), \ \forall\, i \geq 0,$$

here the second equality comes from proposition 2.1 (iv) of [10]. *We prefer to transfer back the results to Berkovich spaces, so we insist on working with the open subspaces U_ρ.*

In fact, the formal models $\widehat{\mathcal{M}}_K^0$ are algebraizable: they are the formal completions at closed points of some Shimura varieties as in [8], or one can find the algebraization directly as in theorem 2.3.1 in [18]. So we have for all integer $i \geq 0$

$$H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq (\varprojlim_r H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \mathbb{Z}/l^r\mathbb{Z})) \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l,$$

and similarly for the cohomology without compact support. We conclude this paper by proving the following proposition.

Proposition 3.5. *Let the notations be as above. Then for $\rho >> 0$ and all integer $i \geq 0$, we have bijections*

$$H_c^i(\mathcal{M}_K^0 \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq H_c^i(V_\rho^{ad} \times \mathbb{C}_p, \overline{\mathbb{Q}}_l) \simeq H_c^i(U_\rho \times \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

Proof. These two bijections come from the description of V_ρ and Huber's theorem 2.9 in [11]. Recall the fundamental domain $\mathcal{D} \subset \mathcal{M}^0$ is associated to an admissible open subset $\mathcal{D}^{rig} \subset (\mathcal{M}^0)^{rig}$. On the rigid analytic space $(\mathcal{M}^0)^{rig}$ there is a natural coordinate system $x_1, \ldots, x_{n-1}$, such that for $x = (x_1, \ldots, x_{n-1}) \in (\mathcal{M}^0)^{rig}$, the Newton polygon of the group $H_x[\pi]$ is given by the convex envelope of the points $(q^i, v(x_i))_{0\leq i\leq n}$, where $x_0 = 0, x_n = 1, q = \#(O/\pi)$, H_x is the p-divisible group associated to x, cf. [5] 1.1.5. Under this coordinate system

$$\mathcal{D}^{rig} = \{x = (x_1, \ldots, x_{n-1}) \in (\mathcal{M}^0)^{rig} | v(x_i) \geq 1 - \frac{i}{n}, i = 1, \ldots, n-1\},$$

cf. loc. cit. 1.4. Thus after base change to $\mathbb{C}_p$ it is isomorphic to a closed ball. In [4] section 5 Fargues had described the Newton polygons of the points in a Hecke orbit. In particular at level $K = GL_n(O)$ the admissible open subsets $V_\rho^{rig} \times \mathbb{C}_p$ are locally described by closed balls. Then this is also the case for any level K. Now pass to adic spaces, $V_\rho^{ad} \times \mathbb{C}_p$ is a quasi-compact open subset and locally described by $\mathbb{B}_{\epsilon_\rho} = \{z \in (\mathcal{M}^0)^{ad} \times \mathbb{C}_p ||x_i(z)| \leq \epsilon_\rho\}$. Since $U_\rho^{ad} \times \mathbb{C}_p, (\mathcal{M}^0)^{ad} \times \mathbb{C}_p$ can be described as unions of ascending chains of quasi-compact open subsets locally in the above forms, by theorem 2.9 of [11] and proposition 8.3.6 of [9] one can conclude. □

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