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The multiplicity one conjecture for local theta correspondences

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Abstract Over a non-archimedean local field of characteristic zero, we prove multiplicity preservation of local theta correspondences for orthogonal-symplectic dual pairs. The same proof works for dual pairs of unitary groups.

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1 Introduction

Fix a non-archimedean local field k of characteristic zero. Let G be an orthogonal group $\mathrm{O}(m)$, and let G' be a symplectic group $\mathrm{Sp}(2n)$, both defined over k $(m, n > 0)$. Then as usual, they form a reductive dual pair in the larger symplectic group $\mathrm{Sp}(2mn)$. Denote by

$$1 \to \{\pm 1\} \to \widetilde{\mathrm{Sp}}(2mn) \to \mathrm{Sp}(2mn) \to 1 \qquad (1)$$

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the metaplectic cover of $\mathrm{Sp}(2mn)$. Write $\widetilde{G}$ and $\widetilde{G}'$ for the double covers of G and G', respectively, induced by the cover (1). They are centralizers of each other inside the group $\widetilde{\mathrm{Sp}}(2mn)$.

The purpose of this note is to prove the following

Theorem A *Let $\widetilde{G}$ and $\widetilde{G}'$ be as above. Let ω_ψ be a smooth oscillator representation of $\widetilde{\mathrm{Sp}}(2mn)$ associated to a nontrivial character ψ of* k. *Then for every genuine irreducible admissible smooth representation π of $\widetilde{G}$, and π' of $\widetilde{G}'$, the inequality*

$$\dim \mathrm{Hom}_{\widetilde{G}\times\widetilde{G}'}(\omega_\psi, \pi \otimes \pi') \leq 1$$

holds.

Recall that an irreducible admissible smooth representation of $\widetilde{G}$ is said to be genuine if it does not descend to a representation of G. The same terminology applies to all double covers of groups.

Theorem A is usually called multiplicity preservation of local theta correspondences. S. Rallis call it "the Multiplicity One Conjecture" and proves it in some cases [6]. It is complementary to the famous Local Howe Duality Conjecture. The archimedean analog of Theorem A is proved by R. Howe in [2]. When the residue characteristic of k is odd, Theorem A is proved by J.-L. Waldspurger in [10]. For applications to number theory, it is desirable to prove multiplicity preservation over all local fields.

Remarks (a) The same argument as in this note proves that multiplicity preservation of local theta correspondences also hold for dual pairs of unitary groups.

(b) For all reductive dual pairs in the archimedean case, and all type II dual pairs in the nonarchimedean case, both multiplicity preservation and the Local Howe Duality Conjecture are proved ([2, Theorem 1] and [3, Theorem 1]).

(c) Unfortunately, the method of this note does not work for quaternionic type I dual pairs.

2 Proof of Theorem A

To emphasize the symmetric roles played by orthogonal groups and symplectic groups, we introduce the following notation. Let $\epsilon = \pm 1$, and let $(E, \langle\,,\,\rangle_E)$ be a finite dimensional k-vector space with a non-degenerate ϵ-symmetric bilinear form. Denote by G the automorphism group of $(E, \langle\,,\,\rangle_E)$, which is an orthogonal group or a symplectic group, depending on $\epsilon = 1$ or -1. Following [4, Proposition 4.I.2], we extend G to a larger group $\breve{G}$, which contains G as a subgroup of index two, and consists of pairs $(g, \delta) \in \mathrm{GL}_{\mathrm{k}}(E) \times \{\pm 1\}$ such that either

$$\delta = 1 \quad \text{and} \quad g \in G,$$

or

$$\delta = -1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle v, u \rangle_E, \quad u, v \in E.$$

Write $\epsilon' = -\epsilon$, and let $(E', \langle\, ,\, \rangle_{E'})$ be a finite dimensional k-vector space with a non-degenerate ϵ'-symmetric bilinear form. Then

$$\mathbf{E} := E \otimes E'$$

is a symplectic space under the form

$$\langle u \otimes u', v \otimes v' \rangle_{\mathbf{E}} := \langle u, v \rangle_E \, \langle u', v' \rangle_{E'}.$$

Define the groups

$$\breve{G}' \supset G' \quad \text{and} \quad \breve{\mathrm{Sp}}(\mathbf{E}) \supset \mathrm{Sp}(\mathbf{E})$$

in the obvious way. Assume that both E and E' are nonzero. Then G and G' form a reductive dual pair in $\mathrm{Sp}(\mathbf{E})$.

Write

$$\mathbf{H} := \mathbf{E} \times \mathrm{k}$$

for the Heisenberg group, where the group law is given by

$$(u, t)(u', t') := (u + u', t + t' + \langle u, u' \rangle_{\mathbf{E}}).$$

The group $\breve{\mathrm{Sp}}(\mathbf{E})$ acts on it as automorphisms by

$$(g, \delta).(u, t) := (gu, \delta t). \tag{2}$$

Denote the fiber product

$$\breve{\mathbf{G}} := \breve{G} \times_{\{\pm 1\}} \breve{G}' = \{(g, g', \delta) \mid (g, \delta) \in \breve{G},\ (g', \delta) \in \breve{G}'\},$$

which contains

$$\mathbf{G} := G \times G'$$

as a subgroup of index two. The action (2) and the homomorphism

$$\breve{\mathbf{G}} \to \breve{\mathrm{Sp}}(\mathbf{E}), \quad (g, g', \delta) \mapsto (g \otimes g', \delta) \tag{3}$$

induce an action of $\breve{\mathbf{G}}$ on $\mathbf{H}$. This defines a semidirect product

$$\breve{\mathbf{J}} := \breve{\mathbf{G}} \ltimes \mathbf{H},$$

which contains

$$\mathbf{J} := \mathbf{G} \ltimes \mathbf{H}$$

as a subgroup of index two.

Let the group

$$\{\pm 1\} \ltimes (\breve{\mathbf{G}} \times \breve{\mathbf{G}}) \tag{4}$$

act on $\breve{\mathbf{J}}$ by

$$(\delta, \breve{\mathbf{g}}_1, \breve{\mathbf{g}}_2).\breve{\mathbf{j}} := (\breve{\mathbf{g}}_1 \breve{\mathbf{j}} \breve{\mathbf{g}}_2^{-1})^{\delta}, \tag{5}$$

where the semidirect product in (4) is defined by the action

$$-1.(\breve{\mathbf{g}}_1, \breve{\mathbf{g}}_2) := (\breve{\mathbf{g}}_2, \breve{\mathbf{g}}_1).$$

Note that the fibre product

$$\breve{\mathbb{G}} := \{\pm 1\} \ltimes_{\{\pm 1\}} (\breve{\mathbf{G}} \times_{\{\pm 1\}} \breve{\mathbf{G}}) = \{(\delta, \breve{\mathbf{g}}_1, \breve{\mathbf{g}}_2) \mid \chi_{\mathbf{G}}(\breve{\mathbf{g}}_1) = \chi_{\mathbf{G}}(\breve{\mathbf{g}}_2) = \delta\}$$

is a subgroup of (4), where $\chi_{\mathbf{G}}$ is the quadratic character of $\breve{\mathbf{G}}$ with kernel $\mathbf{G}$. The group $\breve{\mathbb{G}}$ contains

$$\mathbb{G} := \mathbf{G} \times \mathbf{G}$$

as a subgroup of index two, and stabilizes $\mathbf{J}$ under the action (5).

We postpone the proof of the following proposition to the next section.

Proposition 2.1 *Every $\mathbb{G}$-orbit in $\mathbf{J}$ is $\breve{\mathbb{G}}$-stable.*

Note that $\breve{\mathbf{J}}$ is unimodular. Therefore by the localization principle of Bernstein and Zelevinsky [1, Theorem 6.9 and Theorem 6.15 A], Proposition 2.1 implies the following:

Corollary 2.2 *If a generalized function on $\mathbf{J}$ is $\mathbb{G}$-invariant, then it is also $\breve{\mathbb{G}}$-invariant.*

For the usual notion of generalized functions, see [7, Sect. 2], for example.

Recall the following version of the Gelfand-Kazhdan criterion (cf. [5]).

Lemma 2.3 *Let H be a t.d. group, i.e., a topological group which is Hausdorff, locally compact, secondly countable, and totally disconnected. Let S be a closed subgroup of H, and let σ be a continuous anti-automorphism of H.*

Assume that every bi-S-invariant generalized function on H is σ-invariant. Then for every irreducible admissible smooth representations π_H of H, one has that

$$\dim \mathrm{Hom}_S(\pi_H, \mathbb{C}) \cdot \dim \mathrm{Hom}_S(\pi_H^\vee, \mathbb{C}) \le 1.$$

Here and henceforth, we use "$^\vee$" to indicate the contragredient of an admissible smooth representation. A more general form of Lemma 2.3 is proved in [9, Theorem 2.3] for real reductive groups. The same proof works here and we omit the details.

Note that some elements of $\breve{\mathbb{G}} \setminus \mathbb{G}$ act on $\mathbf{J}$ as anti-automorphisms. Therefore Corollary 2.2 and Lemma 2.3 together imply the following

Lemma 2.4 *For every irreducible admissible smooth representations $\pi_{\mathbf{J}}$ of $\mathbf{J}$, one has that*

$$\dim \mathrm{Hom}_{\mathbf{G}}(\pi_{\mathbf{J}}, \mathbb{C}) \cdot \dim \mathrm{Hom}_{\mathbf{G}}(\pi_{\mathbf{J}}^\vee, \mathbb{C}) \le 1.$$

Now we come to the proof of Theorem A. Let $\widetilde{G}$, $\widetilde{G}'$ and $\widetilde{\mathrm{Sp}}(\mathbf{E})$ be double covers of G, G' and $\mathrm{Sp}(\mathbf{E})$, respectively, as in the Introduction. As in Theorem A, let π and π' be genuine irreducible admissible smooth representations of $\widetilde{G}$ and $\widetilde{G}'$, respectively. Recall that up to isomorphism, the oscillator representation ω_ψ is the unique genuine smooth representation of $\widetilde{\mathrm{Sp}}(\mathbf{E}) \ltimes \mathbf{H}$ which, as a representation of $\mathbf{H}$, is irreducible and has central character ψ.

It is easy to see that $\omega_\psi \otimes \pi^\vee \otimes \pi'^\vee$ descends to an irreducible admissible smooth representation of $\mathbf{J} = (G \times G') \ltimes \mathbf{H}$ (c.f. [7, Lemma 5.3]). Therefore Lemma 2.4 implies that

$$\dim \mathrm{Hom}_{\mathbf{G}}(\omega_\psi \otimes \pi^\vee \otimes \pi'^\vee, \mathbb{C}) \cdot \dim \mathrm{Hom}_{\mathbf{G}}(\omega_\psi^\vee \otimes \pi \otimes \pi', \mathbb{C}) \le 1. \quad (6)$$

By [8, Theorem 1.4], the two factors in the left hand side of (6) are equal to each other. Therefore

$$\dim \mathrm{Hom}_{\mathbf{G}}(\omega_\psi \otimes \pi^\vee \otimes \pi'^\vee, \mathbb{C}) \le 1,$$

and consequently,

$$\dim \mathrm{Hom}_{\widetilde{G} \times \widetilde{G}'}(\omega_\psi, \pi \otimes \pi') \le 1.$$

This finishes the proof of Theorem A.

3 Proof of Proposition 2.1

We continue to use the notation of the last section. Let $\breve{\mathbf{G}}$ act on $\mathbf{E}$ by

$$(g, g', \delta).(u \otimes u') := \delta g u \otimes g' u', \quad (7)$$

and act on

$$\mathbf{E}' := \mathrm{Hom}_{\mathrm{k}}(E, E')$$

by

$$((g, g', \delta).\phi)(u) := \delta\, g'(\phi(g^\tau u)),$$

where

$$g^\tau := \begin{cases} g^{-1}, & \text{if } \delta = 1, \\ \epsilon g^{-1}, & \text{if } \delta = -1. \end{cases}$$

Lemma 3.1 *As* k*-linear representations of* $\breve{\mathbf{G}}$, $\mathbf{E}$ *and* $\mathbf{E}'$ *are isomorphic to each other.*

Proof It is routine to check that the k-linear isomorphism

$$\begin{aligned} \mathbf{E} &\to \mathbf{E}', \\ u \otimes u' &\mapsto (v \mapsto \langle v, u\rangle_E\, u') \end{aligned}$$

is $\breve{\mathbf{G}}$-intertwining. □

Denote by

$$\mathfrak{g} := \{x \in \mathrm{End}_{\mathrm{k}}(E) \mid \langle xu, v\rangle_E + \langle u, xv\rangle_E = 0\}$$

the Lie algebra of G, and put

$$\tilde{\mathfrak{g}} := \{(x, F) \mid x \in \mathfrak{g},\, F \text{ is a k-subspace of } E,\ x|_F = 0\}.$$

Let $\breve{G}$ act on $\tilde{\mathfrak{g}}$ by

$$(g, \delta).(x, F) := (\delta g x g^{-1}, gF).$$

The action of $\breve{\mathbf{G}}$ on $\mathbf{E}'$ induces an action of

$$\breve{G} = \breve{\mathbf{G}}/G'$$

on the quotient space $G'\backslash\mathbf{E}'$.

Lemma 3.2 *There exists a* $\breve{G}$*-equivariant embedding from* $G'\backslash\mathbf{E}'$ *into* $\tilde{\mathfrak{g}}$.

Proof Recall that the map

$$x \mapsto \langle x\,\cdot, \cdot\rangle_E$$

establishes a k-linear isomorphism form $\mathfrak{g}$ onto the space of ϵ'-symmetic forms on E. Define a map

$$\begin{aligned} \Xi : \mathbf{E}' = \mathrm{Hom}_{\mathrm{k}}(E, E') &\to \tilde{\mathfrak{g}}, \\ \phi &\mapsto (x, F), \end{aligned}$$

where F is the kernel of ϕ, and x is specified by the formula

$$\langle \phi(u), \phi(v) \rangle_{E'} = \langle xu, v \rangle_E, \quad u, v \in E.$$

Use Witt's Theorem, one finds that two elements of $\mathbf{E}'$ stay in the same G'-orbit precisely when they have the same image under the map Ξ. Therefore Ξ descends to an embedding

$$G' \backslash \mathbf{E}' \hookrightarrow \tilde{\mathfrak{g}}.$$

Finally, one checks easily that this embedding is $\breve{G}$-equivariant. □

The following lemma is stated in [4, Proposition 4.I.2]. We will not provide a proof here.

Lemma 3.3 *For every* $(x, F) \in \tilde{\mathfrak{g}}$, *there is an element* $(g, -1) \in \breve{G}$ *such that*

$$gxg^{-1} = -x \quad \text{and} \quad gF = F.$$

In other words, every element of $\tilde{\mathfrak{g}}$ is fixed by an element of $\breve{G} \setminus G$. Therefore every G-orbit in $\tilde{\mathfrak{g}}$ is $\breve{G}$-stable. Now Lemma 3.2 implies that every G-orbit in $G' \backslash \mathbf{E}'$ is $\breve{G}$-stable, or equivalently, every $\mathbf{G}$-orbit in $\mathbf{E}'$ is $\breve{\mathbf{G}}$-stable. Finally, by Lemma 3.1, we have

Lemma 3.4 *Every* $\mathbf{G}$*-orbit in* $\mathbf{E}$ *is* $\breve{\mathbf{G}}$*-stable.*

Now we are ready to prove Proposition 2.1. Note that every $\mathbb{G} = \mathbf{G} \times \mathbf{G}$-orbit in $\mathbf{J}$ intersect the subgroup $\mathbf{H}$, and the subgroup

$$\breve{\mathbf{G}} = \{\pm 1\} \times_{\{\pm 1\}} (\Delta(\breve{\mathbf{G}})) \quad \text{of} \quad \breve{\mathbb{G}} = \{\pm 1\} \ltimes_{\{\pm 1\}} (\breve{\mathbf{G}} \times_{\{\pm 1\}} \breve{\mathbf{G}})$$

stabilizes $\mathbf{H}$, where "Δ" stands for the diagonal group. Therefore in order to prove Proposition 2.1, it suffices to show that every $\mathbf{G}$-orbit in $\mathbf{H}$ is $\breve{\mathbf{G}}$-stable. Note that as a $\breve{\mathbf{G}}$-space,

$$\mathbf{H} = \mathbf{E} \times \mathrm{k},$$

where $\mathbf{E}$ carries the action (7), and k carries the trivial $\breve{\mathbf{G}}$-action. We finish the proof by Lemma 3.4.

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