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Twisted Fermat curves over totally real fields

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1. Introduction

Let p be a prime number, F a totally real field such that $[F(\mu_p):F]=2$ and $[F:\mathbb{Q}]$ is odd. For $\delta \in F^{\times}$, let $[\ \delta\ ]$ denote its class in $F^{\times}/F^{\times p}$. In this paper, we show

Main Theorem. *There are infinitely many classes* $[\ \delta\ ] \in F^{\times}/F^{\times p}$ *such that the twisted affine Fermat curves*

$$W_\delta : \quad X^p + Y^p = \delta$$

have no F-rational points.

Remark. It is clear that if $[\ \delta\ ] = [\ \delta'\ ]$, then W_δ is isomorphic to $W_{\delta'}$ over F. For any $\delta \in F^{\times}$, W_δ/F has rational points locally everywhere.

To obtain this result, consider the smooth open affine curve:

$$C_\delta : V^p = U(\delta - U),$$

and the morphism:

$$\psi_\delta : W_\delta \longrightarrow C_\delta; \quad (x,y) \longmapsto (x^p, xy).$$

Let $C_\delta \to J_\delta$ be the Jacobian embedding of C_δ/F defined by the point $(0,0)$. We will show that:

(1) If $L(1, J_\delta/F) \neq 0$, then $J_\delta(F)$ is a finite group (cf. Theorem 2.1. of §2).

The proof is based on Zhang's extension of the Gross-Zagier formula to totally real fields and on Kolyvagin's technique of Euler systems. One might use techniques of congruence of modular forms to remove the restriction that the degree $[F:\mathbb{Q}]$ is odd.

(2) There are infinitely many classes $[\ \delta\ ]$ such that $L(1, J_\delta/F) \neq 0$ (cf. Theorem 3.1. of §3; see also 2.2.4.).

The proof is based on the theory of double Dirichlet series. The condition that $[F(\mu_p):F]=2$ is essential for the technique we use here.

Combining (1) and (2), one can see that the set

$$\Pi := \left\{ [\ \delta\ ] \in F^\times / F^{\times p} \ \middle|\ J_\delta(F) \text{ is torsion} \right\}$$

is infinite.

1.1. *Proof of the Main Theorem assuming* (1) *and* (2). For any $\delta \in F^\times$, consider the twisting isomorphism (defined over $F(\sqrt[p]{\delta})$):

$$\iota_\delta : C_\delta \longrightarrow C_1; \quad (u, v) \longmapsto (u/\delta, v/\sqrt[p]{\delta^2}).$$

Define $\eta_\delta : J_\delta \longrightarrow J_1$ to be the homomorphism associated to ι_δ.

Let Σ_δ denote the set $\iota_\delta\,(C_\delta(F))$. It is easy to see that:

(i) $\Sigma_\delta = \Sigma_{\delta'}$, if $[\ \delta\ ] = [\ \delta'\ ]$,

(ii) $\Sigma_\delta \cap \Sigma_{\delta'} = \{(0,0), (1,0)\}$, otherwise.

For any $\delta \in F^\times$ with $[\ \delta\ ] \in \Pi$, and $[\ \delta\ ] \neq 1$, the diagram

$$\begin{tikzcd} W_\delta(F) \arrow[r, "\psi_\delta"] & C_\delta(F) \arrow[r, hook] \arrow[d, "\iota_\delta"] & J_\delta(F) \arrow[d, "\eta_\delta"] \\ & C_1(F(\sqrt[p]{\delta})) \arrow[r, hook] & J_1(F(\sqrt[p]{\delta})) \end{tikzcd}$$

commutes.

Since the set

$$\bigcup_{\delta \in F^\times} J_1(F(\sqrt[p]{\delta}))_{\mathrm{tor}} \ \subset\ J_1(\overline{F})$$

is finite by the Northcott theorem, the set $\displaystyle\bigcup_{[\ \delta\ ] \in \Pi} \Sigma_\delta$ is finite. Thus, for all but finitely many $[\ \delta\ ] \in \Pi \setminus \{[1]\}$, $\quad \Sigma_\delta = \{(0,0), (1,0)\}$, and therefore W_δ has no F-rational points. $\square$

Remark. Our method is, in fact, effective: for any $[\ \delta\ ] \in F^\times / F^{\times p}$, let

$$\mathrm{Supp}^{(p)}\,([\ \delta\ ]) = \left\{ \mathfrak{p} \text{ prime of } F \ \middle|\ p \nmid v_{\mathfrak{p}}(\delta) \right\}.$$

Let L' be the Galois closure of $F(\mu_p)$, and let S be the set of places of F above $2D_{L'/\mathbb{Q}}$, where $D_{L'/\mathbb{Q}}$ is the discriminant of $L'/\mathbb{Q}$. If $\mathrm{Supp}^{(p)}\,([\ \delta\ ])$ is not contained in S and $L(1, J_\delta) \neq 0$, then the twisted Fermat curve W_δ has no F-rational points (see Proposition 2.2).

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2. Arithmetic methods

Fix $\delta \in F^\times \cap \mathcal{O}_F$ such that $(\delta, p) = 1$. Let $\zeta = \zeta_p$ be a primitive p-th root of unity. The abelian variety J_δ is absolutely simple, of dimension $g = \dfrac{p-1}{2}$, and has complex multiplication by $\mathbb{Z}[\zeta]$ over the field $F(\mu_p)$. In this section we show:

Theorem 2.1. *If $L(1, J_\delta/F) \neq 0$, then $J_\delta(F)$ is finite.*

Notation. In this section, for an abelian group M, set $\widehat{M} = M \otimes_{\mathbb{Z}} \prod_p \mathbb{Z}_p$ where p runs over all primes. For any ring R, let $R^\times$ denote the group of invertible elements. For any ideal $\mathfrak{a}$ of F, denote the norm $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{a})$ by $\mathrm{N}\mathfrak{a}$. Let $\mathbb{A}$ denote the adele ring of F, and $\mathbb{A}_f$ its finite part. Sometimes, we shall not distinguish a finite place from its corresponding prime ideal.

2.1. *The Hilbert newform associated to J_δ.* We first recall some facts about L-functions of twisted Fermat curves over arbitrary number fields (see [14], [32]). Let F be any number field, $L = F(\mu_p)$, $L_0 = \mathbb{Q}(\mu_p)$, and $F_0 = L_0 \cap F$.

For any place w of L, denote by w_0 and v its restrictions to $\mathbb{Q}(\mu_p)$ and F, respectively. Let χ_{w_0} and χ_w be the p-th power residue symbols on $L_0^\times$ and $L^\times$, respectively, given by class field theory. Then $\chi_w = \chi_{w_0} \circ \mathrm{N}_{L/\mathbb{Q}(\mu_p)}$. The Jacobi sum

$$j(\chi_w, \chi_w) = -\sum_{\substack{a \in \mathcal{O}_L/w \\ a \neq 0,1}} \chi_w(a)\chi_w(1-a)$$

is an integer in L_0 satisfying $j(\chi_w, \chi_w) = j(\chi_{w_0}, \chi_{w_0})^{i_{w/w_0}}$ and the Stickelberger relation:

$$(j(\chi_{w_0}, \chi_{w_0})) = \prod_{i=1}^{\frac{p-1}{2}} \sigma_i^{-1}(w_0)$$

as an ideal in L_0. Here, i_{w/w_0} is the inertial degree for w/w_0, and $\sigma_i \in \mathrm{Gal}(L_0/\mathbb{Q})$ is the image of i under the isomorphism $(\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathrm{Gal}(L_0/\mathbb{Q})$.

Since $\delta \in \mathcal{O}_F$ is coprime to p, C_δ has good reduction at w for any $w \nmid p\delta$. We know that the zeta-function of the reduction $\widetilde{C_\delta}$ of C_δ at a place v of F is

$$Z(\widetilde{C_\delta}, T) = \frac{P_v(T)}{(1-T)(1-\mathrm{N}vT)},$$

with

$$P_v(T) = \prod_{w|v} \prod_{\sigma} (1 - \chi_w(\delta^2)^\sigma j(\chi_w, \chi_w)^\sigma T^{f_v}),$$

where f_v is the order of $\mathrm{N}v$ modulo p, and σ runs over representatives in $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ of $\mathrm{Gal}(F_0/\mathbb{Q})$. Then the number of points on $\tilde{J}_\delta$ (the reduction of J_δ at v) is $P_v(1)$.

Now we give a bound on torsion points of $J_\delta(F)$. Let F' be the Galois closure of $F/\mathbb{Q}$, and assume that $F \cap L_0 = F' \cap L_0$. This assumption is satisfied if F is as in the main theorem, or F is Galois over $\mathbb{Q}$. Let $L' = F'(\mu_p)$, and let $q \nmid 2D_{L'/\mathbb{Q}}$ be a prime. Let ℓ be a prime for which there exists a place $w'|\ell$ of L' such that $\mathrm{Frob}_{L_0/F_0}(w'|_{L_0})$ is a generator of $\mathrm{Gal}(L_0/F_0)$, $\mathrm{Frob}_{F'/F_0}(w'|_{F'}) = 1$ and $\mathrm{Frob}_{\mathbb{Q}(\mu_q)/\mathbb{Q}}(w'|_{\mathbb{Q}(\mu_q)}) = 1$. Then, $\ell \equiv 1 \mod q$. Let v, w and w_0 be the places of F, L and L_0, respectively, below w'. Then, v is inert in L/F and $i_{w/w_0} = 1$. We have

$$P_v(1) = \prod_\sigma (1 - \chi_w(\delta^2)^\sigma j(\chi_w, \chi_w)^\sigma).$$

Since v is inert in L/F and $\delta \in F^\times$, we have $\chi_w(\delta^2) = 1$. Using the Stickelberger relation and the fact that $j(\chi_{w_0}, \chi_{w_0}) \equiv 1 \mod (1-\zeta_p)^2$, one can show that $j(\chi_w, \chi_w) = -\ell^f$, for $f = \frac{p-1}{2[F_0:\mathbb{Q}]}$. Then, $P_v(1) = (1+\ell^f)^{[F_0:\mathbb{Q}]} \equiv 2^{[F_0:\mathbb{Q}]} \mod q$. Consequently, there are no q-torsion points in $J_\delta(F)$.

Similarly, for the case $q|2D_{L'/\mathbb{Q}}$, let $c_q \geq 1$ be the smallest positive integer such that there is a $\sigma \in \mathrm{Gal}(L'(\mu_{q^{c_q}})/\mathbb{Q})$ for which $\sigma|_L$ is a generator of $\mathrm{Gal}(L/F)$, $\sigma|_{F'} = 1$, and the restriction of σ to $\mathrm{Gal}(\mathbb{Q}(\mu_{q^{c_q}})/\mathbb{Q})$ has order greater than $f = \frac{p-1}{2[F_0:\mathbb{Q}]}$. Then, $P_v(1) \not\equiv 0 \mod q^{c_q[F_0:\mathbb{Q}]}$. Let M be defined by $M := \prod_{q|2D_{L'/\mathbb{Q}}} q^{c_q[F_0:\mathbb{Q}]}$. It follows that $J_\delta(F)_{\mathrm{tor}} \subset J_\delta[M]$, the subgroup of M-torsion points of $J_\delta(\overline{F})$.

Let F be a totally real field as in the main theorem. We have:

PROPOSITION 2.2. *Let S be the set of places of F above $2D_{L'/\mathbb{Q}}$. If* $\mathrm{Supp}^{(p)}([\ \delta\ ])$ *is not contained in S and $L(1, J_\delta/F) \neq 0$, then the twisted Fermat curve W_δ has no F-rational points.*

Let F be as in the introduction. Then $F_0 = \mathbb{Q}(\mu_p)^+$ is the maximal totally real subfield of $L_0 = \mathbb{Q}(\mu_p)$. By the reciprocity law, one can see that $w \mapsto \chi_w(\delta^2)$ defines a Hecke character, which we denote by $\chi_{[\delta^2]}$. It depends only on the class of δ^2 and has conductor above δ. By Weil [32], the map $w \mapsto j(\chi_w, \chi_w)\mathrm{N}_{L/\mathbb{Q}}w^{-\frac{1}{2}}$ also defines a Hecke character on L, denoted by ψ, which has conductor above p. Thus, we have a (unitary) Hecke character on L,

$$\chi_{[\delta^2]}\psi : \mathbb{A}_L^\times \longrightarrow \mathbb{C}^\times,$$

which is not of the form $\phi \circ \mathrm{N}_{L/F}$, for any Hecke character ϕ over F. Then, there exists a unique holomorphic Hilbert newform f/F of pure weight 2 with trivial central character such that,

$$L_v(s, f/F) = \prod_{w|v} L_w(s - 1/2, \chi_{[\delta^2]}\psi),$$

for all places v of F. Actually, the field over $\mathbb{Q}$ generated by the Hecke eigenvalues attached to f is $F_0 = \mathbb{Q}(\mu_p)^+$, and for the CM abelian variety J_δ, we have

$$L(s, J_\delta/F) = \prod_{\sigma \in \mathrm{Gal}(L_0/\mathbb{Q}) \big/ \mathrm{Gal}(L_0/F_0)} L(s-1/2, \chi^\sigma_{[\delta^2]} \psi^\sigma)$$

$$= \prod_{\sigma : F_0 \hookrightarrow \mathbb{C}} L(s, f^\sigma/F).$$

Note that $L(s, J_\delta)$ only depends on the class $[\ \delta\ ]$ of δ, and the above equality holds for any local factor.

2.2. *A nonvanishing result.* Let π be the automorphic representation associated to f, and let N be its conductor. Let S_0 be any finite set of places of F, including all infinite places and the places dividing N. Choose a quadratic Hecke character ξ corresponding to a totally imaginary quadratic extension of F, unramified at N, where $\xi(N) \cdot (-1)^g = -1$ (since F is of odd degree, we have $(-1)^g = -1$); i.e., the epsilon factor of $L(s, \pi \otimes \xi)$ is -1. Let $\mathcal{D}(\xi; S_0)$ denote the set of quadratic characters χ of $F^\times/\mathbb{A}_F^\times$, for which $\chi_v = \xi_v$, for all $v \in S_0$. With the above notation and assumptions, by a theorem of Friedberg and Hoffstein [11], there exist infinitely many quadratic characters $\chi \in \mathcal{D}(\xi; S_0)$ such that $L(s, \pi \otimes \chi)$ has a simple zero at the center $s = 1/2$.

Choose such a χ, and let K be the totally imaginary quadratic extension of F associated to it. The conductor of χ is coprime to N, and the L-function $L(s, f/K) = L(s-1/2, \pi)L(s-1/2, \pi \otimes \chi)$ has a simple zero at $s = 1$. Let d denote the discriminant of K/F.

2.3. *Zhang's formula.*

2.3.1. *The* (N, K)*-type Shimura curves.* Let $\mathcal{O}$ be the subalgebra of $\mathbb{C}$ over $\mathbb{Z}$ generated by the eigenvalues of f under the Hecke operators. In our case, $\mathcal{O} = \mathbb{Z}[\zeta + \zeta^{-1}]$ is the ring of integers of F_0. In [33] (see also [5], [6]), Zhang constructs a Shimura curve X of (N, K)-type, and proves that there exists a unique abelian subvariety A of the Jacobian $\mathrm{Jac}(X)$ of dimension $[\mathcal{O} : \mathbb{Z}] = g$, such that

$$L_v(s, A) = \prod_{\sigma : \mathcal{O} \hookrightarrow \mathbb{C}} L_v(s, f^\sigma/F),$$

for all places v of F. By the construction of f, it follows that $L_v(s, A/F) = L_v(s, J_\delta/F)$ for all places v of F. Therefore, by the isogeny conjecture proved by Faltings, A is isogenous to J_δ over F. In particular, the complex multiplication by $\mathcal{O} \subset \mathbb{Q}(\mu_p)^+$ on A is defined over F.

Now, let us recall the constructions of X and A.

The L-function of $\pi \otimes \chi$ satisfies the functional equation

$$L(1-s, \pi \otimes \chi) = (-1)^{|\Sigma|} \mathrm{N}_{F/\mathbb{Q}}(Nd)^{2s-1} L(s, \pi \otimes \chi),$$

where $\Sigma = \Sigma(N, K)$ is the following set of places of F :

$$\Sigma(N, K) = \left\{ v \;\middle|\; v|\infty, \text{ or } \chi_v(N) = -1 \right\}.$$

Since the sign of the functional equation is -1, by our choice of K, the cardinality of Σ is odd. Let τ be any real place of F. Then, we have:

(1) Up to isomorphism, there exists a unique quaternion algebra B such that B is ramified at exactly the places in $\Sigma \backslash \{\tau\}$;

(2) There exist embeddings $\rho : K \hookrightarrow B$ over F.

From now on, we fix an embedding $\rho : K \to B$ over F.

Let G denote the algebraic group over F, which is an inner form of PGL_2 with $G(F) \cong B^\times / F^\times$. The group $G(F_\tau) \cong \mathrm{PGL}_2(\mathbb{R})$ acts on $\mathcal{H}^\pm = \mathbb{C} \backslash \mathbb{R}$. Now, for any open compact subgroup U of $G(\mathbb{A}_f)$, we have an analytic space

$$S_U(\mathbb{C}) = G(F)_+ \backslash \mathcal{H}^+ \times G(\mathbb{A}_f)/U,$$

where $G(F)_+$ denotes the subgroup of elements in $G(F)$ with positive determinant via τ.

Shimura has shown that $S_U(\mathbb{C})$ is the set of complex points of an algebraic curve S_U, which descends canonically to F (as a subfield of $\mathbb{C}$ via τ). The curve S_U over F is independent of the choice of τ.

There exists an order R_0 of B containing $\mathcal{O}_K$ with reduced discriminant N. One can choose R_0 as follows. Let $\mathcal{O}_B$ be a maximal order of B containing $\mathcal{O}_K$, and let $\mathcal{N}$ be an ideal of $\mathcal{O}_K$ such that

$$\mathrm{N}_{K/F}\mathcal{N} \cdot \mathrm{disc}_{B/F} = N,$$

where $\mathrm{disc}_{B/F}$ is the reduced discriminant of $\mathcal{O}_B$ over $\mathcal{O}_F$. Then, we take

$$R_0 = \mathcal{O}_K + \mathcal{N} \cdot \mathcal{O}_B.$$

Take $U = \prod_v R_v^\times / \mathcal{O}_v^\times$. The corresponding Shimura curve $X := S_U$ is compact.

Let $\xi \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ be the unique class whose degree is 1 on each connected component and such that,

$$\mathrm{T}_m \xi = \deg(\mathrm{T}_m)\xi,$$

for all integral ideals m of $\mathcal{O}_F$ coprime to Nd. Here, the T_m are the Hecke operators.

2.3.2. *Gross-Zagier-Zhang formula.* Now, we define the basic class in $\mathrm{Jac}(X)(K) \otimes \mathbb{Q}$, where $\mathrm{Jac}(X)$ is the connected component of $\mathrm{Pic}(X)$, from the CM-points on the curve X. The CM points corresponding to K on X form a set:

$$\mathcal{C} : \; G(F)_+ \backslash G(F)_+ \cdot h_0 \times G(\mathbb{A}_f)/U \cong T(F) \backslash G(\mathbb{A}_f)/U; \qquad [(h_0, g)] \leftrightarrow [g],$$

where $h_0 \in \mathcal{H}^+$ is the unique fixed point of the torus $T(F) = K^\times / F^\times$.

For a CM point $z = [g] \in \mathcal{C}$, represented by $g \in G(\mathbb{A}_f)$, let

$$\Phi_g : K \longrightarrow \widehat{B}, \qquad t \longmapsto g^{-1}\rho(t)g.$$

Then, $\mathrm{End}(z) := \Phi_g^{-1}(\widehat{R_0})$ is an order of K, say $\mathcal{O}_n = \mathcal{O}_F + n\mathcal{O}_K$, for a (unique) ideal n of F. The ideal n, called the conductor of z, is independent of the choice of the representative g. By Shimura's theory, every CM point of conductor n is defined over the abelian extension H'_n of K corresponding to $K^\times \setminus \widehat{K}^\times / \widehat{F}^\times \widehat{\mathcal{O}}_n^\times$ via class field theory.

Let P_1 be a CM point in X of conductor 1, which is defined over H'_1, the abelian extension of K corresponding to $K^\times \setminus \widehat{K}^\times / \widehat{F}^\times \widehat{\mathcal{O}}_K^\times$. The divisor $P = \mathrm{Gal}(H'_1/K) \cdot P_1$ together with the Hodge class defines a class

$$x := [P - \deg(P)\xi] \in \mathrm{Jac}(X)(K) \otimes \mathbb{Q},$$

where $\deg P$ is the multi-degree of P on the geometric components. Let x_f be the f-typical component of x. In [34], Zhang generalized the Gross-Zagier formula to the totally real field case, by proving that

$$L'(1, f/K) = \frac{2^{g+1}}{\sqrt{\mathrm{N}(d)}} \cdot \|f\|^2 \cdot \|x_f\|^2,$$

where $\|f\|^2$ is computed on the invariant measure on

$$\mathrm{PGL}_2(F) \setminus \mathcal{H}^g \times \mathrm{PGL}_2(\mathbb{A}_f)/U_0(N)$$

induced by $dxdy/y^2$ on $\mathcal{H}^g$, and where

$$U_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) \middle| c \in \widehat{N} \right\} \subset \mathrm{GL}_2(\widehat{F}),$$

and $\|x_f\|^2$ is the Neron-Tate pairing of x_f with itself.

2.3.3. *The equivalence of nonvanishing of L-factors.* For any $\sigma : F \hookrightarrow \mathbb{C}$, it is known by a result of Shimura that $L(1, f/F) \neq 0$ is equivalent to $L(1, f^\sigma/F) \neq 0$. One can also show this using Zhang's formula above. To see this, assume $L(1, f/F) \neq 0$. Then, $\|x_f\| \neq 0$, and therefore, $\|x_{f^\sigma}\| \neq 0$. It follows that $L'(1, f^\sigma/K) \neq 0$. Since $L(1, f/F) \neq 0$, the L-function $L(s, f^\sigma/F)$ has a positive sign in its functional equation. Thus, $L(1, f^\sigma/F) \neq 0$. In fact, to obtain our main theorem, we do not need this equivalence, but we may see that Theorem 3.1 is equivalent to statement (2) in the introduction.

2.4. *The Euler system of CM points.* We now assume that $L(1, \chi_{[\delta^2]}\psi) \neq 0$, or equivalently, $L(1, f/F) \neq 0$. Then by the equivalence of nonvanishing of $L(1, f^\sigma)$ for all embeddings $\sigma : F \hookrightarrow \mathbb{C}$, we have that $L(1, J_\delta/F) \neq 0$. By Zhang's formula, we also know that $\|x_f\| \neq 0$.

Let $\mathcal{N}$ be the set of square-free integral ideals of F whose prime divisors are inert in K and coprime to Nd. For any $n \in \mathcal{N}$, define

$$H_n = \prod_{\ell | n} H'_\ell \subset H'_n, \qquad H_1 = H'_1.$$

Let u_n denote the cardinality of $(\widehat{\mathcal{O}}_n^\times \cap K^\times \widehat{F}^\times)/\widehat{\mathcal{O}}_F^\times$. Then, H_ℓ/H_1 is a cyclic extension of degree $t(\ell) = \frac{\mathrm{N}(\ell)+1}{u_1/u_\ell}$.

For each $n \in \mathcal{N}$, let P_n be a CM point of order n such that P_n is contained in $\mathrm{T}_\ell P_m$ if $n = m\ell \in \mathcal{N}$ and ℓ is a prime ideal of F. Let $y_n = \mathrm{Tr}_{H'_n/H_n} \pi(P_n) \in A(H_n)$, where π is a morphism from X to $\mathrm{Jac}(X)$ defined by a multiple of the Hodge class.

The points $\{y_n\}_{n \in \mathcal{N}}$ form an Euler system (see [29, Prop. 7.5], or [33, Lemma 7.2.2]) so that, for any $n = m\ell \in \mathcal{N}$ with ℓ a prime ideal of F,

(1) $\displaystyle {u_n}^{-1} \sum_{\sigma \in \mathrm{Gal}(H_n/H_m)} y_n^\sigma = {u_m}^{-1} a_\ell y_m;$

(2) For any prime ideal λ_m of H_m above ℓ, and for λ_n the unique prime above λ_m,
$$\mathrm{Frob}_{\lambda_m} y_m \equiv y_n \mod \lambda_n;$$

(3) The class x_f is equal to $y_K := \mathrm{tr}_{H_1/K} y_1$ in $\big(A(K) \otimes \mathbb{Q}\big)/\mathbb{Q}^\times$.

Theorem 2.1 follows with the nontrivial Euler system by Kolyvagin's standard argument (see [21], [23], [13], and [33, Th. A]).

3. Analytic methods

Let $r = 4$ or an odd prime, and let $L = F(\zeta_r)$, with $[L : F] = 2$. Let ψ be a unitary Hecke character of L. In this section, we show:

THEOREM 3.1. *There are infinitely many classes $\delta \in F^\times/F^{\times r}$ such that $L\left(\frac{1}{2}, \chi_{[\ \delta\ ]}\psi\right)$ does not vanish.*

Let ρ be a unitary Hecke character of F. The purpose of this section is to construct a perfect double Dirichlet series $Z(s, w; \psi; \rho)$ similar to an Asai-Flicker-Patterson type Rankin-Selberg convolution, which possesses meromorphic continuation to $\mathbb{C}^2$ and functional equations. Then, Theorem 3.1 will follow from the analytic properties of $Z(s, w; \psi; \rho)$ (when $r = 4$, see [7]). To do this, it is necessary to recall the Fisher-Friedberg symbol in [9].

3.1. *The r-th power residue symbol.* Let S' be a finite set of non-archimedean places of L containing all places dividing r, and such that the ring of S'-integers $\mathcal{O}_L^{S'}$ has class number one. We shall also assume that S' is closed under conjugation and that ψ and ρ are both unramified outside S'.

Let S_∞ denote the set of all archimedean places of L, and set $S = S' \cup S_\infty$. Let $I_L(S)$ (resp. $\mathcal{I}_L(S)$) denote the group of fractional ideals (resp. the set of all integral ideals) of $\mathcal{O}_L$ coprime to S'. In [9], Fisher and Friedberg have shown that the r-th order symbol χ_n can be extended to $I_L(S)$ i.e., $\chi_{\mathfrak{n}}(\mathfrak{m})$ is defined for $\mathfrak{m}$, $\mathfrak{n} \in I_L(S)$. Let us recall their construction.

For a non-archimedean place $v \in S'$, let $\mathfrak{P}_v$ denote the corresponding ideal of L. Define $\mathfrak{c} = \prod_{v \in S'} \mathfrak{P}_v^{r_v}$ with $r_v = 1$ if $\mathrm{ord}_v(r) = 0$, and r_v sufficiently large such that, for $a \in L_v$, $\mathrm{ord}_v(a-1) \geq r_v$ implies that $a \in (L_v^\times)^r$. Let $P_L(\mathfrak{c}) \subset I_L(S)$ be the subgroup of principal ideals (α) with $\alpha \equiv 1 \mod \mathfrak{c}$, and let $H_{\mathfrak{c}} = I_L(S)/P_L(\mathfrak{c})$ be the ray class group modulo $\mathfrak{c}$. Set $R_{\mathfrak{c}} = H_{\mathfrak{c}} \otimes \mathbb{Z}/r\mathbb{Z}$, and write the finite group $R_{\mathfrak{c}}$ as a direct product of cyclic groups. Choose a generator for each, and let $\mathfrak{E}_0$ be a set of ideals of $\mathcal{O}_L$, prime to S, which represent these generators. For each $\mathfrak{e}_0 \in \mathfrak{E}_0$, choose $m_{\mathfrak{e}_0} \in L^\times$ such that $\mathfrak{e}_0 \mathcal{O}_L^{S'} = m_{\mathfrak{e}_0} \mathcal{O}_L^{S'}$. Let $\mathfrak{E}$ be a full set of representatives for $R_{\mathfrak{c}}$ of the form $\prod_{\mathfrak{e}_0 \in \mathfrak{E}_0} \mathfrak{e}_0^{\lambda_{\mathfrak{e}_0}}$. Note that $\mathfrak{e}\mathcal{O}_L^{S'} = m_{\mathfrak{e}} \mathcal{O}_L^{S'}$ for all $\mathfrak{e} \in \mathfrak{E}$. Without loss, we suppose that $\mathcal{O}_L^{S'} \in \mathfrak{E}$ and $m_{\mathcal{O}_L^{S'}} = 1$.

Let $\mathfrak{m}, \mathfrak{n} \in I_L(S)$ be coprime. Write $\mathfrak{m} = (m)\mathfrak{e}\mathfrak{g}^r$ with $\mathfrak{e} \in \mathfrak{E}$, $m \in L^\times$, $m \equiv 1 \mod \mathfrak{c}$ and $\mathfrak{g} \in I_L(S)$, $(\mathfrak{g}, \mathfrak{n}) = 1$. Then the r-th power residue symbol $\left(\frac{m m_{\mathfrak{e}}}{\mathfrak{n}}\right)_r$ is defined. If $\mathfrak{m} = (m')\mathfrak{e}'\mathfrak{g}'^r$ is another such decomposition, then $\mathfrak{e}' = \mathfrak{e}$ and $\left(\frac{m' m_{\mathfrak{e}'}}{\mathfrak{n}}\right)_r = \left(\frac{m m_{\mathfrak{e}}}{\mathfrak{n}}\right)_r$.

In view of this, the r-th power residue symbol $\left(\frac{\mathfrak{m}}{\mathfrak{n}}\right)_r$ is defined to be $\left(\frac{m m_{\mathfrak{e}}}{\mathfrak{n}}\right)_r$, and the character $\chi_{\mathfrak{m}}$ is defined by $\chi_{\mathfrak{m}}(\mathfrak{n}) = \left(\frac{\mathfrak{m}}{\mathfrak{n}}\right)_r$. This extension of the r-th power residue symbol depends on the above choices. Let $S_{\mathfrak{m}}$ denote the support of the conductor of $\chi_{\mathfrak{m}}$. It can be easily checked that if $\mathfrak{m} = \mathfrak{m}'\mathfrak{a}^r$, then $\chi_{\mathfrak{m}}(\mathfrak{n}) = \chi_{\mathfrak{m}'}(\mathfrak{n})$ whenever both are defined. This allows one to extend $\chi_{\mathfrak{m}}$ to a character of all ideals of $I_L(S \cup S_{\mathfrak{m}})$.

The extended symbol possesses a reciprocity law: if $\mathfrak{m}, \mathfrak{n} \in I_L(S)$ are coprime, then $\alpha(\mathfrak{m}, \mathfrak{n}) = \chi_{\mathfrak{m}}(\mathfrak{n})\chi_{\mathfrak{n}}(\mathfrak{m})^{-1}$ depends only on the images of $\mathfrak{m}, \mathfrak{n}$ in $R_{\mathfrak{c}}$.

In our situation, we also need the following lemma:

Lemma 3.2. *The natural morphism*

$$I_F(S)/P_F(\mathfrak{c}) \longrightarrow I_L(S)/P_L(\mathfrak{c})$$

has kernel of order a power of 2.

Proof. If $[\mathfrak{n}]$ is in the kernel, i.e., $\mathfrak{n} = (\alpha)$ in $I_L(S)$ is a principal ideal with $\alpha \equiv 1 \mod \mathfrak{c}$, then $\alpha/\overline{\alpha}$ is a root of unity with $\alpha/\overline{\alpha} \equiv 1 \mod \mathfrak{c}$. Now let W be the set of roots of unity in L which are $\equiv 1 \mod \mathfrak{c}$. Let W_0 be the subset of W of elements of the form $u/\overline{u}$ for some unit u in $\mathcal{O}_L$ and $u \equiv 1 \mod \mathfrak{c}$. It is clear that $W_0 \supset W^2$. Then, the map

$$\mathrm{Ker}\,(I_F(S)/P_F(\mathfrak{c}) \to I_L(S)/P_L(\mathfrak{c})) \longrightarrow W/W_0; \qquad \mathfrak{n} \longmapsto \alpha/\overline{\alpha}$$

is obviously injective; i.e., the order of the kernel of the natural map in this lemma is a power of 2. □

Since r is odd, using the lemma, we may choose a suitable set $\mathfrak{E}_0$ of representatives since the beginning such that if $\mathfrak{m} \in I_F(S)$, then the decomposition $\mathfrak{m} = (m)\mathfrak{e}\mathfrak{g}^r$ is such that $m \in F^\times$, $\mathfrak{e}, \mathfrak{g} \in I_F(S)$.

Using the symbol $\chi_\mathfrak{n}$, we shall construct a perfect double Dirichlet series $Z(s, w; \psi; \rho)$ (i.e., possessing meromorphic continuation to $\mathbb{C}^2$) of type:

$$Z(s, w; \psi; \rho) \;=\; Z_S(s, w; \psi; \rho) \;\; = * \sum_{\mathfrak{n} \in \mathcal{I}_F(S)} L_S(s, \psi\, \chi_\mathfrak{n})\, \rho(\mathfrak{n})\, \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^{-w}, \tag{3.1}$$

where the sum is over the set of all integral ideals of $\mathcal{O}_F$ coprime to S', for $\mathfrak{n} \in \mathcal{I}_F(S)$ square-free, the function $L_S(s, \psi\, \chi_\mathfrak{n})$ is precisely the Hecke L-function attached to $\psi\, \chi_\mathfrak{n}$ with the Euler factors at all places in S removed, and where $*$ is a certain normalizing factor. For an arbitrary $\mathfrak{n} \in \mathcal{I}_F(S)$, write $\mathfrak{n} = \mathfrak{n}_1 \mathfrak{n}_2^r$ with $\mathfrak{n}_1$ r-th power free. If $L_S(s, \psi\, \chi_{\mathfrak{n}_1})$ denotes the Hecke L-series associated to $\psi\, \chi_{\mathfrak{n}_1}$ with the Euler factors at all places in S removed, then $L_S(s, \psi\, \chi_\mathfrak{n})$ is defined as $L_S(s, \psi\, \chi_{\mathfrak{n}_1})$ multiplied by a Dirichlet polynomial whose complexity grows with the divisibility of $\mathfrak{n}$ by powers (see (3.10), (3.12) and (3.13) for precise definitions).

Based on the analytic properties of $Z(s, w; \psi; \rho)$, we show the following result which is stronger than Theorem 3.1.

Theorem 3.3. 1) *There exist infinitely many* r-th *power free ideals* $\mathfrak{n}_1$ *in* $\mathcal{I}_F(S)$ *with trivial image in* $R_\mathfrak{c}$ *for which the special value* $L_S(\frac{1}{2}, \chi_\mathfrak{n}\psi)$ *does not vanish.*

2) *Let* $\kappa_\mathfrak{c}$ *denote the number of characters of* $R_\mathfrak{c}$ *whose restrictions to* F *are also characters of the ideal class group of* F, *and let* κ *be the residue of the Dedekind zeta function* $\zeta_F(s)$ *at* $s = 1$. *Then for* $x \to \infty$,

$$\sum_{\substack{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n}) < x \\ \mathfrak{n} \in \mathcal{I}_F(S) \\ \mathfrak{n} = (n) \\ [\mathfrak{n}] = 1}} L_S\left(\frac{1}{2}, \chi_\mathfrak{n}\psi\right) \;\sim\; \frac{\kappa \cdot \kappa_\mathfrak{c}}{h_F \cdot |R_\mathfrak{c}|}\, \frac{L_S(1, \psi)\, L_S(\frac{r}{2}, \psi^r)}{L_S(\frac{r}{2}+1, \psi^r)} \prod_{\substack{v \text{ in } F \\ v \in S'}} \left(1 - q_v^{-1}\right) \cdot x, \tag{3.2}$$

where $[\mathfrak{n}]$ *denotes the image of the ideal* $\mathfrak{n}$ *in* $R_\mathfrak{c}$.

Remarks. i) By the above definition of the extended r-th power residue symbol, it is easy to see that the first part of this theorem is equivalent to Theorem 3.1.

ii) In fact, by a well-known result of Waldspurger [30], it will follow that $L_S(\frac{1}{2}, \chi_\mathfrak{n}\psi) \geq 0$, for $\mathfrak{n} \in \mathcal{I}_F(S)$, $\mathfrak{n} = (n)$ and trivial image in $R_\mathfrak{c}$. We will see this in the course of the proof of Theorem 3.3.

iii) Following [8], by a simple sieving process, one can prove the more familiar variant of the above asymptotic formula where the sum is restricted to square-free principal ideals.

3.2. *The series $Z_{\mathrm{aux}}(s, w; \psi; \rho)$ and metaplectic Eisenstein series.* To obtain the correct definition of $Z(s, w; \psi; \rho)$, let $G_0(\mathfrak{n}, \mathfrak{m})$, for $\mathfrak{m}$, $\mathfrak{n} \in \mathcal{I}_L(S)$, be given by

$$G_0(\mathfrak{n}, \mathfrak{m}) = \prod_{\substack{v \\ \mathrm{ord}_v(\mathfrak{n})=k \\ \mathrm{ord}_v(\mathfrak{m})=l}} G_0(\mathfrak{p}_v^k, \mathfrak{p}_v^l), \tag{3.3}$$

where, for k, $l \geq 0$,

$$G_0(\mathfrak{p}_v^k, \mathfrak{p}_v^l) = \begin{cases} 1 & \text{if } l = 0, \\ q_v^{\frac{k}{2}} & \text{if } k+1 = l;\ l \not\equiv 0 \pmod r, \\ -q_v^{\frac{k-1}{2}} & \text{if } k+1 = l;\ l > 0;\ l \equiv 0 \pmod r, \\ q_v^{\frac{l}{2}-1}(q_v - 1) & \text{if } k \geq l;\ l > 0;\ l \equiv 0 \pmod r, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Here q_v denotes the absolute value of the norm of v. Also, let $G(\chi^*_{\mathfrak{m}_1})$ (where $\mathfrak{m}_1$ denotes the r-th power free part of $\mathfrak{m}$ and $\chi^*_{\mathfrak{a}}(\mathfrak{b}) := \chi_{\mathfrak{b}}(\mathfrak{a})$) be the normalized Gauss sum appearing in the functional equation of the (primitive) Hecke L-function associated to $\chi^*_{\mathfrak{m}}$. If $\mathfrak{n}^*$ denotes the part of $\mathfrak{n}$ coprime to $\mathfrak{m}_1$, then set

$$G(\mathfrak{n}, \mathfrak{m}) := \overline{\chi^*_{\mathfrak{m}_1}(\mathfrak{n}^*)}\, G(\chi^*_{\mathfrak{m}_1})\, G_0(\mathfrak{n}, \mathfrak{m}).$$

Now, let ψ be as above. For $\mathfrak{n} \in \mathcal{I}_L(S)$ and $\mathrm{Re}(s) > 1$, let $\Psi_S(s, \mathfrak{n}, \psi)$ be the absolutely convergent Dirichlet series defined by

$$\Psi_S(s, \mathfrak{n}, \psi) = L_S\left(rs - \frac{r}{2} + 1, \psi^r\right) \sum_{\mathfrak{m} \in \mathcal{I}_L(S)} \frac{\psi(\mathfrak{m}) G(\mathfrak{n}, \mathfrak{m})}{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{m})^s}.$$

This series can be realized as a Fourier coefficient of a metaplectic Eisenstein series on the r-fold cover of GL(2) (see [18] and [24]). It follows as in Selberg [28], or alternatively, from Langlands' general theory of Eisenstein series [25] that $\Psi_S(s, \mathfrak{n}, \psi)$ has meromorphic continuation to $\mathbb{C}$ with only one possible (simple) pole at $s = \frac{1}{2} + \frac{1}{r}$. Moreover, this function is bounded when $|\mathrm{Im}(s)|$ is large in vertical strips, and satisfies a functional equation as $s \to 1 - s$ (see Kazhdan-Patterson [18, Cor. II.2.4]).

For $\mathrm{Re}(s)$, $\mathrm{Re}(w) > 1$, let $Z_{\mathrm{aux}}(s, w; \psi; \rho)$ be the auxiliary double Dirichlet series defined by

$$Z_{\mathrm{aux}}(s, w; \psi; \rho) = \sum_{\mathfrak{n} \in \mathcal{I}_F(S)} \frac{\Psi_S(s, \mathfrak{n}, \psi) \rho(\mathfrak{n})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^w}. \tag{3.5}$$

Let $\tilde{\rho}$ be the Hecke character of L given by $\tilde{\rho} = \rho \circ \mathrm{N}_{L/F}$. As we shall shortly see, $Z_{\mathrm{aux}}(s, w; \psi\,\tilde{\rho}; \overline{\rho})$ is the type of object that constitutes a building block in the process of constructing the perfect double Dirichlet series $Z(s, w; \psi; \rho)$. Set

$$\Gamma^*_{\mathrm{aux}}(s, \psi\,\tilde{\rho}) \;=\; \prod_{v \in S_\infty} \prod_{j=1}^{r-1} L_v\Big(s - \frac{1}{2} + \frac{j}{r}, \psi_v\,\tilde{\rho}_v\Big),$$

and let

$$\widehat{Z}_{\mathrm{aux}}(s, w; \psi\,\tilde{\rho}; \bar{\rho}) \;:=\; \Gamma^*_{\mathrm{aux}}(s, \psi\,\tilde{\rho}) \cdot Z_{\mathrm{aux}}(s, w; \psi\,\tilde{\rho}; \bar{\rho}).$$

Let $\mathcal{R}_1$ be the tube region in $\mathbb{C}^2$ whose base $\mathcal{B}_1$ is the convex region in $\mathbb{R}^2$ which lies strictly above the polygonal contour determined by $(0, 2)$, $(1, 1)$, and the rays $y = -2x + 2$ for $x \le 0$ and $y = 1$ for $x \ge 1$. As a simple consequence of the analytic properties of $\Psi_S(s, \mathfrak{n}, \psi)$ ($\mathfrak{n} \in \mathcal{I}_L(S)$), we have the following:

Proposition 3.4. *The double Dirichlet series $Z_{\mathrm{aux}}(s, w; \psi\,\tilde{\rho}, \bar{\rho})$ is holomorphic in $\mathcal{R}_1$, unless $\psi^r\,\tilde{\rho}^r = 1$ when it has only one simple pole at $s = \frac{1}{2} + \frac{1}{r}$. Furthermore, $\widehat{Z}_{\mathrm{aux}}(s, w; \psi\,\tilde{\rho}, \bar{\rho})$ satisfies the functional equation*

(3.6)
$$\widehat{Z}_{\mathrm{aux}}(s, w; \psi\,\tilde{\rho}, \bar{\rho}) \cdot \prod_{v \in S'} \Big(1 - (\psi\tilde{\rho})^{-r}(\pi_v)\, q_v^{rs - \frac{r}{2} - 1}\Big)$$
$$= \sum_{\eta,\,\tau} A^{(\psi,\,\rho)}_{\eta,\,\tau}(1 - s)\, \widehat{Z}_{\mathrm{aux}}(1 - s, 2s + w - 1; \psi^{-1}\,\tilde{\rho}^{-1}\eta, \psi\,\rho\,\tau),$$

where each $A^{(\psi,\,\rho)}_{\eta,\,\tau}(s)$ is a polynomial in the variables q_v^s, q_v^{-s} ($v \in S'$), and the sum is over a finite set of idéle class characters η and τ, unramified outside S and with orders dividing r.

3.3. *The double Dirichlet series $\widetilde{Z}(s, w; \psi; \rho)$.* It turns out that the function $Z_{\mathrm{aux}}(s, w; \psi\,\tilde{\rho}, \bar{\rho})$ possesses another functional equation. To describe it, we introduce a new double Dirichlet series $\widetilde{Z}(s, w; \psi; \rho)$ defined for $\mathrm{Re}(s)$, $\mathrm{Re}(w)$ > 1 by

(3.7)
$$\widetilde{Z}(s, w; \psi; \rho) \;=\; L_S(rs + rw + 1 - r, \psi^r\,\tilde{\rho}^r) \sum_{\substack{\mathfrak{m} \in \mathcal{I}_L(S) \\ \mathfrak{m}-\text{imaginary}}} \frac{\psi(\mathfrak{m})\, L_S(w,\, \chi^*_{\mathfrak{m}}\,\rho)}{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{m})^s}$$
$$\cdot \sum_{\mathfrak{h} \in \mathcal{I}_F(S)} \frac{(\psi\rho)(\mathfrak{h})\, \chi^*_{\mathfrak{m}}(\mathfrak{h}_1)}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{h})^{2s-1}\, \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{h})^w} \prod_{\substack{v \\ \mathrm{ord}_v(\mathfrak{h}_0) > 0}} \big[\, (\chi^*_{\mathfrak{m}}\,\rho)(\pi_v)\, q_v^{-w} \,-\, q_v^{-1}\,\big]$$
$$\cdot \prod_{\substack{v \\ \mathrm{ord}_v(\mathrm{N}_{L/F}(\mathfrak{m})) > 0 \\ \mathrm{ord}_v(\mathfrak{h}_2) > 0}} (1 - q_v^{-1}) \prod_{\substack{v-\text{split in } L \\ \mathrm{ord}_v(\mathrm{N}_{L/F}(\mathfrak{m})) = 0 \\ \mathrm{ord}_v(\mathfrak{h}_2) > 0}} \big[\, (\chi^*_{\mathfrak{m}}\,\rho)(\pi_v)\, q_v^{-w-1} \,+\, 1 \,-\, 2q_v^{-1}\,\big]$$
$$\cdot \prod_{\substack{v-\text{inert in } L \\ \mathrm{ord}_v(\mathfrak{h}_2) > 0}} \big[\, 1 \,-\, (\chi^*_{\mathfrak{m}}\,\rho)(\pi_v)\, q_v^{-w-1}\,\big]\,.$$

In the above formula, an ideal $\mathfrak{m} \in \mathcal{I}_L(S)$ is called *imaginary*, if it has no divisor in $\mathcal{I}_F(S)$, other than $\mathcal{O}_F$. The function $L_S(w,\, \chi^*_{\mathfrak{m}}\,\rho)$ represents the L-series defined over F (not necessarily primitive) associated to $\chi^*_{\mathfrak{m}}\,\rho$ with the Euler factors corresponding to places removed in S. Also, all the products are over places of F, π_v is the local parameter of F_v (F_v denoting the completion of F at v), and q_v is the absolute value of the norm in F of v.

Let $\mathcal{R}_2$ denote the tube region in $\mathbb{C}^2$ whose base $\mathcal{B}_2$ is the convex region in $\mathbb{R}^2$ which lies strictly above the polygonal contour determined by $(1,1)$, $(\frac{3}{2},0)$ and the rays $y=-x+\frac{3}{2}$ for $y\le 0$ and $x=1$ for $y\ge 1$. Recall that $L_S(w,\, \chi^*_{\mathfrak{m}}\,\rho)$ differs from a primitive L-series by only finitely many Euler factors (i.e., the factors corresponding to places in S and to places v for which $\mathrm{ord}_v(\mathrm{N}_{L/F}(\mathfrak{m}))\equiv 0 \pmod r$). Applying the functional equation of $L_S(w,\, \chi^*_{\mathfrak{m}}\,\rho)$ and some standard estimates, one can easily show that the function $\widetilde{Z}(s,w;\psi;\rho)$ is holomorphic in $\mathcal{R}_2$, unless $\rho=1$ where it has only one simple pole at $w=1$. The following proposition gives the functional equation connecting the double Dirichlet series $Z_{\mathrm{aux}}(s,w;\psi\,\tilde{\rho},\bar{\rho})$ and $\widetilde{Z}(s,w;\psi;\rho)$.

PROPOSITION 3.5. *The function $\widetilde{Z}(s,w;\psi;\rho)$ is holomorphic in $\mathcal{R}_2$, unless ρ is the trivial character when it has a simple pole at $w=1$. Furthermore, for* $\mathrm{Re}(s)$, $\mathrm{Re}(w)>1$, *there exist the functional equations*

$$\begin{aligned}(3.8)\quad &\prod_{v\in S_\infty} L_v\,(1-w,\rho_v)\cdot\prod_{v\in S'}\left(1-\rho^{-r}(\pi_v)\,q_v^{-rw}\right)\cdot\widetilde{Z}(s+w-\tfrac{1}{2},1-w;\psi;\rho)\\ &\qquad=\prod_{v\in S_\infty} L_v\left(w,\rho_v^{-1}\right)\cdot\sum_\tau B_\tau^{(\rho)}(w)\,Z_{\mathrm{aux}}(s,w;\psi\tilde{\rho}\,\tau,\bar{\rho}),\end{aligned}$$

and

$$\begin{aligned}(3.9)\quad &\prod_{v\in S_\infty} L_v\left(w,\rho_v^{-1}\right)\cdot\prod_{v\in S'}\left(1-\rho^{r}(\pi_v)\,q_v^{rw-r}\right)\cdot Z_{\mathrm{aux}}(s,w;\psi\tilde{\rho},\bar{\rho})\\ &\qquad=\prod_{v\in S_\infty} L_v\big(1-w,\rho_v\big)\cdot\sum_\tau C_\tau^{(\rho)}(1-w)\,\widetilde{Z}(s+w-\tfrac{1}{2},1-w;\psi\,\tau;\rho),\end{aligned}$$

where, as before, $B_\tau^{(\rho)}(w)$, $C_\tau^{(\rho)}(w)$ are polynomials in the variables q_v^w, q_v^{-w} ($v\in S'$). The above products are over the places of k corresponding to those in S, and the sums are over a finite set of idéle class characters τ, unramified outside S and orders dividing r.

The proof of this proposition will be given in the next section.

Let α and β be the involutions on $\mathbb{C}^2$ given by

$$\alpha:(s,w)\to(1-s,2s+w-1)\quad\text{and}\quad\beta:(s,w)\to(s+w-\tfrac{1}{2},1-w).$$

It can be easily checked that these involutions generate the dihedral group D_8 of order 8. It follows directly from Propositions 3.2 and 3.3 that both

$\widetilde{Z}(s+w-\frac{1}{2}, 1-w; \psi; \rho)$ and $Z_{\mathrm{aux}}(s, w; \psi\tilde{\rho}, \bar{\rho})$ can be continued to $\mathcal{R}_1 \cup \mathcal{R}_2$. Clearly, this applies to $Z_{\mathrm{aux}}(s, w; \psi, \rho)$ (replace ψ by $\psi\tilde{\rho}^{-1}$ and ρ by $\bar{\rho}$). It follows from the functional equation (3.6) that $Z_{\mathrm{aux}}(s, w; \psi\tilde{\rho}, \bar{\rho})$ can be continued to $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \alpha(\mathcal{R}_2)$, and hence, by (3.8), the function $\widetilde{Z}(s+w-\frac{1}{2}, 1-w; \psi; \rho)$ continues to this region. The double Dirichlet series $Z_{\mathrm{aux}}(s, w; \psi\widetilde{\rho}, \overline{\rho})$ may have only one simple pole in $\mathcal{R}_2$, namely $w = 1$, and this pole occurs only if ρ is the trivial character. This fact follows easily by inspection of the proof of Proposition 3.3 (see §3.1). Then from the functional equation (3.6), one can see that $Z_{\mathrm{aux}}(s, w; \psi\widetilde{\rho}, \overline{\rho})$ may have a pole only at $w = 2-2s$ in $\alpha(\mathcal{R}_2)$, provided $\psi^r|_{\mathcal{O}_F} \cdot \rho^r$ is trivial. The last fact also applies to $\widetilde{Z}(s+w-\frac{1}{2}, 1-w; \psi, \rho)$, by the functional equation β in (3.8).

3.4. *The double Dirichlet series $Z(s, w; \psi; \rho)$.* To define the perfect double Dirichlet series $Z(s, w; \psi; \rho)$, let $L_S(s, \chi_{\mathfrak{n}}\psi)$, for $\mathfrak{n} \in \mathcal{I}_F(S)$, be given by

$$L_S(s,\, \chi_{\mathfrak{n}}\psi) \;:=\; L_S(s,\, \chi_{\mathfrak{n}_1}\psi) P_{\mathfrak{n}}(s,\, \psi),$$

where $\mathfrak{n}_1$ denotes the r-th power free part of $\mathfrak{n}$, and $P_{\mathfrak{n}}(s, \psi)$ is the Dirichlet polynomial defined by

(3.10)

$$\begin{aligned} P_{\mathfrak{n}}(s,\, \psi) = &\prod_{\substack{v \\ \mathrm{ord}_v(\mathfrak{n}_1)>0}} \left(1 + \psi(\pi_v)\, q_v^{1-2s} + \cdots + \psi(\pi_v)^{\mathrm{ord}_v(\mathfrak{n})-1} q_v^{(\mathrm{ord}_v(\mathfrak{n})-1)(1-2s)}\right) \\ &\cdot \prod_{\substack{v \\ \mathrm{ord}_v(\mathfrak{n})=r\mu \\ v-\text{inert in } L}} \Bigg(\left(1 - \psi(\pi_v)\, q_v^{-2s}\right) \big(1 + \psi(\pi_v)\, q_v^{1-2s} + \cdots \\ &\qquad + \psi(\pi_v)^{r\mu-1}\, q_v^{(r\mu-1)(1-2s)}\big) + \psi(\pi_v)^{r\mu}\, q_v^{r\mu(1-2s)} \left(1 + q_v^{-1}\right) \Bigg) \\ &\cdot \prod_{\substack{v \\ \mathrm{ord}_v(\mathfrak{n})=r\omega \\ v=v'\bar{v}' \text{ in } L}} \Bigg((1 - (\chi_{\mathfrak{n}_1}\psi)(\pi_{v'})\, q_v^{-s})(1 - (\chi_{\mathfrak{n}_1}\psi)(\pi_{\bar{v}'})\, q_v^{-s}) \big(1 + \psi(\pi_v)\, q_v^{1-2s} + \cdots \\ &\qquad + \psi(\pi_v)^{r\omega-1}\, q_v^{(r\omega-1)(1-2s)}\big) + \psi(\pi_v)^{r\omega}\, q_v^{r\omega(1-2s)} \left(1 - q_v^{-1}\right) \Bigg). \end{aligned}$$

Here the products are over places v of F, and π_v denotes the local parameter of F_v. It can be seen that these polynomials satisfy a functional equation as $s \to 1-s$, and that we have the estimate

$$P_{\mathfrak{n}}(s,\, \psi) \ll_{\varepsilon} \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^{\varepsilon} \qquad (\varepsilon > 0,\ \mathrm{Re}(s) \geq \tfrac{1}{2}). \tag{3.11}$$

Furthermore, if $\psi(\overline{\mathfrak{m}}) = \overline{\psi(\mathfrak{m})}$, for $\mathfrak{m} \in \mathcal{I}_L(S)$, then $P_{\mathfrak{n}}(s,\, \psi) \geq 0$, for $s \in \mathbb{R}$. Later, we shall specialize ψ to be (essentially) a normalized Jacobi sum, which obviously satisfies this property.

For $\mathrm{Re}(s)$, $\mathrm{Re}(w) > 1$, we define $Z(s,w;\psi;\rho)$ as

$$
\begin{aligned}
(3.12)\qquad Z(s,w;\psi;\rho) &= Z_S(s,w;\psi;\rho) \\
&= L_S(rs+rw+1-r,\ \psi^r\tilde{\rho}^r) \sum_{\mathfrak{n}\in\mathcal{I}_F(S)} \frac{L_S(s,\ \chi_{\mathfrak{n}}\psi)\rho(\mathfrak{n})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^w}.
\end{aligned}
$$

Applying the functional equation and the convexity bound of $L_S(s,\chi_{\mathfrak{n}}\psi)$ ($\mathfrak{n}\in\mathcal{I}_F(S)$), we see that $Z(s,w;\psi;\rho)$ is holomorphic in $\mathcal{R}_1$, if the character ψ^r is nontrivial. Representing the normalizing factor $L_S(rs+rw+1-r,\psi^r\tilde{\rho}^r)$ by its Dirichlet series, then after multiplying and reorganizing, we can write $Z(s,w;\psi;\rho)$ as

$$
(3.13)\qquad Z(s,w;\psi;\rho) \;=\; \sum_{\mathfrak{n}\in\mathcal{I}_F(S)} \frac{L_S(s,\ \chi_{\mathfrak{n}_1}\psi)\, Q_{\mathfrak{n}}(s,\ \psi)\, \rho(\mathfrak{n})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^w},
$$

where $Q_{\mathfrak{n}}(s,\psi)$, for $\mathfrak{n}\in\mathcal{I}_F(S)$, is a new set of Dirichlet polynomials which can be easily expressed in terms of $P_{\mathfrak{n}}(s,\psi)$.

Referring to the definition of $\widetilde{Z}(s,w;\psi;\rho)$ given in (3.7), replace $L_S(w,\chi^*_{\mathfrak{m}}\rho)$ by its Dirichlet series, the sum being over $\mathfrak{n}$, say. For fixed $\mathfrak{m}\in\mathcal{I}_L(S)$ imaginary, and $\mathfrak{n}\in\mathcal{I}_F(S)$, collect the terms contributing to $(\chi^*_{\mathfrak{m}}\rho)(\mathfrak{n})\,\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^{-w}$. Switching the order of summation, we obtain:

Proposition 3.6. *For* $\mathrm{Re}(s)$, $\mathrm{Re}(w) > 1$,

$$
(3.14)\qquad Z(s,w;\psi;\rho) \;=\; L_S(2s,\psi)\widetilde{Z}(s,w;\psi;\rho),
$$

where the L-function is defined over F.

Assuming both ψ^r and $\psi^r\tilde{\rho}^r$ to be nontrivial, we see from Proposition 3.4 that

$$
L_S(2s+2w-1,\psi)\widetilde{Z}(s+w-\tfrac{1}{2},1-w;\psi;\rho)
$$

continues to $\beta(\mathcal{R}_1)$, and hence, from the above discussion, it continues to $\mathcal{R}_1\cup\beta(\mathcal{R}_1)\cup\mathcal{R}_2\cup\alpha(\mathcal{R}_2)$. Note that the convex closure of this tube region is $\mathbb{C}^2$. As $\psi^r\tilde{\rho}^r\neq 1$, and therefore, by Propositions 3.2 and 3.3, the function $\widetilde{Z}(s+w-\tfrac{1}{2},1-w;\psi;\rho)$ does not have a pole at $s=\frac{1}{2}+\frac{1}{r}$, one can easily check that the only possible poles of $L_S(2s+2w-1,\psi)\widetilde{Z}(s+w-\tfrac{1}{2},1-w;\psi;\rho)$ are the hyperplanes $w=0$ and $w=2-2s$. Clearly, both are simple poles, and they may occur only if ρ and $\psi^r|_{\mathcal{O}_F}\cdot\rho^r$ are both trivial.

Consequently, by the convexity theorem for holomorphic functions of several complex variables (see [16]) and by Proposition 3.4, we have the following:

Theorem 3.7. *When ψ^r and $\psi^r\tilde{\rho}^r$ are nontrivial, the function*

$$
(w-1)(2s+w-2)Z(s,w;\psi;\rho)
$$

has analytic continuation to $\mathbb{C}^2$, and for any fixed s, it is (*as a function of the variable w*) *of order one.*

The fact that, for any fixed s, the above function is of order one follows as in [8, Prop. 3.11].

By Proposition 3.4 and (3.7), one finds that, for $\mathrm{Re}(s) > \frac{1}{2}$,

(3.15)

$$\begin{aligned}
\operatorname*{Res}_{w=1} Z(s,w;\psi;1) &= L_S(2s,\psi)\, L_S(rs+1,\psi^r) \\
&\cdot \prod_{\substack{v \text{ in } F\\ v\in S'}} \Bigg[\left(1-q_v^{-1}\right) \sum_{\substack{\mathfrak{m}\in\mathcal{I}_L(S)\\ \mathfrak{m}-\text{imaginary}}} \Bigg(\frac{\kappa\, \psi(\mathfrak{m})^r \prod_{v'|\mathfrak{m}} \left(1-q_{v'}^{-1}\right)}{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{m})^{rs}} \sum_{\mathfrak{h}\in\mathcal{I}_F(S)} \frac{\psi(\mathfrak{h})^r}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{h})^{2rs}} \\
&\cdot \prod_{\substack{v\\ \mathrm{ord}_v(\mathrm{N}_{L/F}(\mathfrak{m}))>0\\ \mathrm{ord}_v(\mathfrak{h})>0}} \left(1-q_v^{-1}\right) \prod_{\substack{v-\text{split in } L\\ \mathrm{ord}_v(\mathrm{N}_{L/F}(\mathfrak{m}))=0\\ \mathrm{ord}_v(\mathfrak{h})>0}} \left(1-q_v^{-1}\right)^2 \prod_{\substack{v-\text{inert in } L\\ \mathrm{ord}_v(\mathfrak{h})>0}} \left(1-q_v^{-2}\right) \Bigg) \Bigg] \\
&= \kappa L_S(2s,\psi)\, L_S(rs,\psi^r) \prod_{\substack{v \text{ in } F\\ v\in S'}} \left(1-q_v^{-1}\right),
\end{aligned}$$

where κ denotes the residue at $w=1$ of the Dedekind zeta-function $\zeta_F(w)$.

We are now in the position to give the proof of Theorem 3.3.

Proof of Theorem 3.3. As before, let $\rho = \prod \rho_v$ be a unitary Hecke character of F unramified outside S. We further assume that ρ is of finite order. For $\mathrm{Re}(s)$, $\mathrm{Re}(w) > 1$, consider the double Dirichlet series $Z_1(s,w;\psi;\rho)$ defined by

$$Z_1(s,w;\psi;\rho) = \sum_{\substack{\mathfrak{n}\in\mathcal{I}_F(S)\\ \mathfrak{n}=(n)\\ [\mathfrak{n}]=1}} \frac{L_S(s,\, \chi_{\mathfrak{n}_1}\psi)\, Q_{\mathfrak{n}}(s,\, \psi)\, \rho(\mathfrak{n})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^w}. \tag{3.16}$$

By expressing this function as

$$Z_1(s,w;\psi;\rho) = \frac{1}{h_F\cdot |R_{\mathfrak{c}}|} \sum_{\rho_1,\,\rho_2} Z(s,w;\psi;\rho\rho_1\widehat{\rho}_2),$$

where ρ_1 ranges over the characters of the ideal class group of F, ρ_2 ranges over the characters of $R_{\mathfrak{c}}$, and $\widehat{\rho}_2$ is the restriction of ρ_2 to F, it follows from Theorem 3.5 that $Z_1(s,w;\psi;\rho)$ is holomorphic on $\mathbb{C}^2$, except for $w=1$ and $w=2-2s$, where it might have simple poles. Furthermore,

$$\lim_{w\to 1}(w-1)^2\, Z_1(\tfrac{1}{2},w;\psi;\rho) = \lim_{(s,w)\to(\frac{1}{2},1)} (w-1)(2s+w-2) Z_1(s,w;\psi;\rho) = 0,$$

and, therefore, $Z_1(\frac{1}{2},w;\psi;1)$ has at most a simple pole at $w=1$. To compute its residue, recall the functional equation satisfied by $L(s,\chi_{\mathfrak{n}_1}\psi)$ with $\mathfrak{n}_1\in\mathcal{I}_F(S)$ r-th power free (see [31, Ch. VII, §7]). Combining this with the functional equation of the polynomial $Q_{\mathfrak{n}}(s,\psi)$ ($\mathfrak{n}\in\mathcal{I}_F(S)$), we find that

$$L_S(s,\,\chi_{\mathfrak{n}_1}\psi)\,Q_{\mathfrak{n}}(s,\,\psi)=\varepsilon(s,\,\chi_{\mathfrak{n}_1}\psi)\cdot L_S(1-s,\,\chi_{\mathfrak{n}_1}\psi)\,Q_{\mathfrak{n}}(1-s,\,\psi)\cdot\prod_{v\in S_\infty}\frac{L_v(1-s,\,\psi_v)}{L_v(s,\,\psi_v)}\cdot\prod_{v\in S'}\frac{L_v\,(1-s,\,(\chi_{\mathfrak{n}_1}\psi)_v)}{L_v\,(s,\,(\chi_{\mathfrak{n}_1}\psi)_v)}.$$

A simple local computation shows that $\varepsilon(\frac{1}{2},\chi_{\mathfrak{n}_1}\psi)=\psi(\mathfrak{n})\varepsilon(\frac{1}{2},\psi)$. It immediately follows that $Z_1(s,w;\psi;1)$ satisfies the functional equation

(3.17)
$$\prod_{v\in S_\infty}L_v(s,\psi_v)\cdot\prod_{v\in S'}\left(1-\psi^r(\pi_v)\,q_v^{rs-r}\right)\cdot Z_1(s,w;\psi;1)=\prod_{v\in S_\infty}L_v(1-s,\psi_v)\cdot\sum_{\rho}D_\rho^{(\psi)}(1-s)\,Z_1(1-s,2s+w-1;\psi;\rho),$$

where $D_\rho^{(\psi)}(s)$ are polynomials in the variables $q_v^s,\,q_v^{-s},\,v\in S'$, and the sum is over a finite set of idéle class characters ρ, unramified outside S and orders dividing r. As r is odd, and ψ, restricted to the group of principal ideals of F, is quadratic and nontrivial, it follows that $Z_1(s,w;\psi;1)$ does not have a pole at $w=2-2s$. Then (3.15) yields

$$\operatorname*{Res}_{w=1}Z_1\left(\frac{1}{2},w;\psi;1\right)=\frac{\kappa\cdot\kappa_{\mathfrak{c}}}{h_F\cdot|R_{\mathfrak{c}}|}L_S(1,\psi)L_S\left(\frac{r}{2},\psi^r\right)\prod_{\substack{v\text{ in }F\\ v\in S'}}(1-q_v^{-1}),\tag{3.18}$$

where $\kappa_{\mathfrak{c}}$ denotes the number of characters of $R_{\mathfrak{c}}$ whose restrictions to F are also characters of the ideal class group of F.

To complete the proof, we define the double Dirichlet series $Z_0(s,w;\psi;\rho)$ by simply replacing in (3.16) the polynomial $Q_{\mathfrak{n}}(s,\psi)$ by $P_{\mathfrak{n}}(s,\psi)$ defined in (3.10). Note that

$$Z_0(s,w;\psi;\rho)=\frac{1}{h_F\cdot|R_{\mathfrak{c}}|}\sum_{\rho_1,\rho_2}\frac{Z(s,w;\psi;\rho\rho_1\rho_2)}{L_S(rs+rw+1-r,\psi^r\widetilde{\rho}^r\widetilde{\rho}_1^r)},$$

and therefore, $Z_0(s,w;\psi;\rho)$ may have additional poles at the zeros of the incomplete L-functions $L_S(rs+rw+1-r,\psi^r\widetilde{\rho}^r\widetilde{\rho}_1^r)$. It is well-known that these zeros occur in the region $\mathrm{Re}(s+w)<1$. In particular, the function $Z_0(\frac{1}{2},w;\psi;1)$ is holomorphic for $\mathrm{Re}(w)>\frac{1}{2}$, except for $w=1$, where it has a simple pole. Using (3.18), we can compute its residue as

(3.19)
$$\operatorname*{Res}_{w=1}Z_0\left(\frac{1}{2},w;\psi;1\right)=\frac{\kappa\cdot\kappa_{\mathfrak{c}}}{h_F\cdot|R_{\mathfrak{c}}|}\,\frac{L_S(1,\psi)\,L_S(\frac{r}{2},\psi^r)}{L_S(\frac{r}{2}+1,\psi^r)}\prod_{\substack{v\text{ in }F\\ v\in S'}}\left(1-q_v^{-1}\right)>0.$$

This implies that $L_S(\frac{1}{2},\chi_{\mathfrak{n}_1}\psi)\neq 0$ for infinitely many r-th power free ideals $\mathfrak{n}_1$ in $\mathcal{I}_F(S)$ with trivial image in $R_{\mathfrak{c}}$, which is the first assertion of Theorem 3.3.

For the remaining part, one needs to apply a Tauberian theorem. To keep the argument as simple as possible, note first that, as $\psi(\overline{\mathfrak{m}}) = \overline{\psi(\mathfrak{m})}$, for $\mathfrak{m} \in \mathcal{I}_L(S)$, we have $P_{\mathfrak{n}}(s, \psi) \geq 0$, for $s \in \mathbb{R}$. On the other hand, by the comment made right after Lemma 3.2, any r-th power free ideal $\mathfrak{n}_1$ in $\mathcal{I}_F(S)$ with trivial image in $R_{\mathfrak{c}}$ can be decomposed as $\mathfrak{n}_1 = (n_1)\mathfrak{g}^r$ with $n_1 \in F^\times$, $n_1 \equiv 1 \mod \mathfrak{c}$ and $\mathfrak{g} \in I_F(S)$. By definition, the character $\chi_{\mathfrak{n}_1}$ coincides with the classical r-th power residue symbol χ_{n_1} given by class field theory. It follows that the incomplete L-series $L_S(s, \chi_{\mathfrak{n}_1}\psi)$ differs from the complete Hecke L-series associated to $L(s, \chi_{n_1}\psi)$ by only finitely many local factors. Recall that the latter is the L-series associated to a Hilbert modular form. As the set S' is closed under conjugation, it follows from a well-known result of Waldspurger [31] that $L_S(\frac{1}{2}, \chi_{\mathfrak{n}}\psi) \geq 0$, for $\mathfrak{n} \in \mathcal{I}_F(S)$, $\mathfrak{n} = (n)$ and trivial image in $R_{\mathfrak{c}}$. Hence, the function $Z_0(\frac{1}{2}, w; \psi; 1)$, for $\Re(w) > 1$, is given by a Dirichlet series with nonnegative coefficients. The second part of Theorem 3.3 now follows from the Wiener-Ikehara Tauberian theorem. $\square$

Remark. With some additional effort, one can exhibit an error term on the order of $O(x^\theta)$ with $\theta < 1$ in the asymptotic formula (3.2). Also, the remark following Theorem 3.3 implies that the Hecke L-series $L_S(\frac{1}{2}, \chi_{\mathfrak{n}_1}\psi) \neq 0$ for infinitely many square-free principal ideals (n) in $\mathcal{I}_F(S)$ with trivial image in $R_{\mathfrak{c}}$. Any such ideal has a generator $n \in F$ with $n \equiv 1 \mod \mathfrak{c}$.

3.5. *Proof of Proposition* 3.3. Recall that for $\mathfrak{a} \in \mathcal{I}_L(S)$, we defined $\chi_{\mathfrak{a}}^*$ by $\chi_{\mathfrak{a}}^*(\mathfrak{b}) := \chi_{\mathfrak{b}}(\mathfrak{a})$ $(\mathfrak{b} \in \mathcal{I}_L(S))$. Note that every ideal $\mathfrak{m}$ of $\mathcal{O}_L$ can be uniquely decomposed as $\mathfrak{m} = \mathfrak{m}'\mathfrak{h}$, where $\mathfrak{m}'$ is an imaginary ideal of $\mathcal{O}_L$, and $\mathfrak{h}$ is a real ideal; that is, $\mathfrak{h} \in \mathcal{O}_F$. For $\mathfrak{m} \in \mathcal{I}_L(S)$ imaginary and r-th power free, let $\varepsilon(w, (\chi_{\mathfrak{m}}^*\rho)^{-1})$ denote the epsilon-factor in the functional equation of $L(w, (\chi_{\mathfrak{m}}^*\rho)^{-1})$ (as a Hecke L-function of F). Also, for $\mathfrak{m} \in \mathcal{I}_L(S)$ imaginary and $\mathfrak{h} \in \mathcal{I}_F(S)$, coprime and r-th power free, let $G(\chi_{\mathfrak{m}\mathfrak{h}}^*)$ be the normalized Gauss sum in the functional equation of the Hecke L-function (of the field L) associated to $\chi_{\mathfrak{m}\mathfrak{h}}^*$, i.e., $\varepsilon(\frac{1}{2}, \chi_{\mathfrak{m}\mathfrak{h}}^*)$. We set $\mathfrak{m}_0$ and $\mathfrak{h}_0$ to be the product of all distinct prime ideals dividing $\mathfrak{m}$ and $\mathfrak{h}$, respectively.

The following lemma is a consequence of a standard local computation. The details will be omitted.

Lemma 3.8. *Let $\mathfrak{m}$ and $\mathfrak{h}$ be integral ideals as above. Assume that the images of $\mathfrak{m}\mathfrak{h}$ and $\mathfrak{m}$ in $R_{\mathfrak{c}}$ are $\mathfrak{e}$ and $\mathfrak{e}'$, respectively. Then,*

$$G(\chi_{\mathfrak{m}\mathfrak{h}}^*)\,\varepsilon\left(\frac{1}{2}, (\chi_{\mathfrak{m}}^*\rho)^{-1}\right) = C_{\mathfrak{e},\mathfrak{e}',\rho} \cdot \eta(\mathfrak{e})^{-1}\eta(\mathfrak{m}_1\mathfrak{h}_1)\,\tilde{\rho}(\mathfrak{m}_0)^{-1}\,\chi_{\mathfrak{m}}^*(\mathfrak{h}_0)\,\chi_{\mathfrak{h}}^*(\mathfrak{m}_0)\,\chi_{\mathfrak{m}}^*(\overline{\mathfrak{m}}_0)^{-1},$$

where $\tilde{\rho} = \rho \circ \mathrm{N}_{L/F}$, $C_{\mathfrak{e},\mathfrak{e}',\rho}$ is a constant depending on just $\mathfrak{e}$, $\mathfrak{e}'$ and ρ, and η is a Hecke character unramified outside S and order dividing r. Furthermore, *if $\mathfrak{e}'$ is replaced by $\mathfrak{e}''$ with $\mathfrak{e}'/\mathfrak{e}''$ a real ideal, then both $C_{\mathfrak{e},\mathfrak{e}',\rho}$ and η do not change.*

Proof of Proposition 3.3. Using (3.5), we have

(3.20)

$$\begin{aligned}
&Z_{\mathrm{aux}}(s,w;\psi\,\tilde{\rho},\bar{\rho})\\
&\quad= \sum_{\mathfrak{n}\in\mathcal{I}_F(S)} \frac{\Psi_S(s,\mathfrak{n},\psi\,\tilde{\rho})\,\overline{\rho(\mathfrak{n})}}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^w}\\
&\quad= L_S\left(rs-\frac{r}{2}+1,\psi^r\tilde{\rho}^r\right)\sum_{\substack{\mathfrak{m}\in\mathcal{I}_L(S)\\ \mathfrak{n}\in\mathcal{I}_F(S)}} \frac{(\psi\,\tilde{\rho})(\mathfrak{m})\,\overline{\rho(\mathfrak{n})}\,G(\mathfrak{n},\mathfrak{m})}{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{m})^s\,\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^w}\\
&\quad= L_S\left(rs-\frac{r}{2}+1,\psi^r\tilde{\rho}^r\right)\sum_{\substack{\mathfrak{m}\in\mathcal{I}_L(S)\\ \mathfrak{n}\in\mathcal{I}_F(S)}} \frac{(\psi\,\tilde{\rho})(\mathfrak{m})\,\overline{\rho(\mathfrak{n})}\;\overline{\chi^*_{\mathfrak{m}_1}(\mathfrak{n}^*)}\,G(\chi^*_{\mathfrak{m}_1})\,G_0(\mathfrak{n},\mathfrak{m})}{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{m})^s\,\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^w},
\end{aligned}$$

where $\mathfrak{n}^*$ denotes the part of $\mathfrak{n}$ coprime to $\mathfrak{m}_1$. In the last sum, replace $\mathfrak{m}$ by $\mathfrak{m}\mathfrak{h}$ with $\mathfrak{m}\in\mathcal{I}_L(S)$ imaginary and $\mathfrak{h}$ real. As we shall see, the only contribution to the sum comes from $\mathfrak{m}$ and $\mathfrak{h}$ for which their r-th power free parts $\mathfrak{m}_1$ and $\mathfrak{h}_1$ are coprime. Then, we have

(3.21)

$$\begin{aligned}
\sum_{\substack{\mathfrak{m}\in\mathcal{I}_L(S)\\ \mathfrak{n}\in\mathcal{I}_F(S)}} \frac{(\psi\,\tilde{\rho})(\mathfrak{m})\,\overline{\rho(\mathfrak{n})}\;\overline{\chi^*_{\mathfrak{m}_1}(\mathfrak{n}^*)}\,G(\chi^*_{\mathfrak{m}_1})\,G_0(\mathfrak{n},\mathfrak{m})}{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{m})^s\,\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^w} &= \sum_{\substack{\mathfrak{m}\in\mathcal{I}_L(S)\\ \mathfrak{m}-\text{imaginary}}} \frac{(\psi\,\tilde{\rho})(\mathfrak{m})}{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{m})^s}\\
\cdot \sum_{\substack{\mathfrak{h}\in\mathcal{I}_L(S)\\ \mathfrak{n}\in\mathcal{I}_F(S)\\ \mathfrak{h}-\text{real}}} &\frac{(\psi\,\tilde{\rho})(\mathfrak{h})\,\overline{\rho(\mathfrak{n})}\;\overline{\chi^*_{\mathfrak{m}_1\mathfrak{h}_1}(\mathfrak{n}^*)}\,G(\chi^*_{\mathfrak{m}_1\mathfrak{h}_1})\,G_0(\mathfrak{n},\mathfrak{m}\mathfrak{h})}{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{h})^s\,\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^w}.
\end{aligned}$$

Next, we separate the contribution of $\mathfrak{h}$ in the inner sum. To do so, let $\mathfrak{m}_1$ denote the r-th power free part of an ideal $\mathfrak{m}\in\mathcal{I}_L(S)$, and set $\mathfrak{m}_0$ to be the product of all distinct prime ideals dividing $\mathfrak{m}_1$, and

$$\mathfrak{m}_2 \quad := \prod_{\substack{v\\ \mathrm{ord}_v(\mathfrak{m})=re_v}} \mathfrak{p}_v^{re_v}.$$

For fixed $\mathfrak{m}$, $\mathfrak{n}$ and $\mathfrak{h}$ as above, let $\mathfrak{p}_v$ be a prime ideal of L dividing $\mathfrak{h}_0$. Upon replacing this prime ideal by its conjugate, we can assume that $\mathrm{ord}_v(\mathfrak{m})=0$. Recall that

$$G_0(\mathfrak{n},\,\mathfrak{m}) \quad = \prod_{\substack{v\\ \mathrm{ord}_v(\mathfrak{n})=k\\ \mathrm{ord}_v(\mathfrak{m})=l}} G_0(\mathfrak{p}_v^k,\mathfrak{p}_v^l),$$

where $G_0(\mathfrak{p}_v^k, \mathfrak{p}_v^l)$ is given by (3.4). As $\mathrm{ord}_v(\mathfrak{m}\mathfrak{h}) = \mathrm{ord}_v(\mathfrak{h}) \not\equiv 0 \pmod{r}$ (this condition implying that $\mathrm{ord}_v(\mathfrak{n}) = \mathrm{ord}_v(\mathfrak{h}) - 1$), and $\mathfrak{n} \in \mathcal{I}_F(S)$, we can decompose $\mathfrak{n} = (\mathfrak{h}/\mathfrak{h}_0\mathfrak{h}_2)\mathfrak{n}'$ with $\mathfrak{n}' \in \mathcal{I}_F(S)$ coprime to $\mathfrak{h}_1$. Also, we have

$$\begin{aligned}\mathrm{ord}_v(\mathfrak{n}) = \mathrm{ord}_{\bar{v}}(\mathfrak{n}) \;&\geq\; \mathrm{ord}_{\bar{v}}(\mathfrak{m}\mathfrak{h}) - 1\\ &= \mathrm{ord}_{\bar{v}}(\mathfrak{m}) + \mathrm{ord}_v(\mathfrak{h}) - 1 \;=\; \mathrm{ord}_{\bar{v}}(\mathfrak{m}) + \mathrm{ord}_v(\mathfrak{n}),\end{aligned}$$

which implies $\mathrm{ord}_{\bar{v}}(\mathfrak{m}) = 0$. It immediately follows that $\mathfrak{m}$ and $\mathfrak{h}_1$ are coprime. Then, by (3.4), we can write

$$\begin{aligned}G(\chi^*_{\mathfrak{m}_1\mathfrak{h}_1})\, G_0(\mathfrak{n}, \mathfrak{m}\mathfrak{h}) &= G(\chi^*_{\mathfrak{m}_1\mathfrak{h}_1})\, G_0\left(\frac{\mathfrak{h}}{\mathfrak{h}_0\mathfrak{h}_2}, \frac{\mathfrak{h}}{\mathfrak{h}_2}\right) G_0(\mathfrak{n}', \mathfrak{m}\mathfrak{h}_2)\\ &= G(\chi^*_{\mathfrak{m}_1\mathfrak{h}_1})\, \mathrm{N}_{L/\mathbb{Q}}\left(\frac{\mathfrak{h}}{\mathfrak{h}_0\mathfrak{h}_2}\right)^{\frac{1}{2}} G_0(\mathfrak{n}', \mathfrak{m}\mathfrak{h}_2).\end{aligned} \tag{3.22}$$

Furthermore, we have

$$\begin{aligned}G_0(\mathfrak{n}', \mathfrak{m}\mathfrak{h}_2) &= \prod_{\substack{v\\ \mathrm{ord}_v(\mathfrak{n}')=k_v\\ \mathrm{ord}_v(\mathfrak{m})=l_v\\ \mathrm{ord}_v(\mathfrak{h}_2)=re_v}} G_0(\mathfrak{p}_v^{k_v}, \mathfrak{p}_v^{l_v+re_v})\\
&= \prod_{\substack{v\\ l_v\not\equiv 0\,(r)\\ k_v+1=l_v+re_v}} G_0(\mathfrak{p}_v^{k_v}, \mathfrak{p}_v^{l_v+re_v}) \cdot \prod_{\substack{v\\ l_v\equiv 0\,(r)\\ k_v+1\geq l_v+re_v}} G_0(\mathfrak{p}_v^{k_v}, \mathfrak{p}_v^{l_v+re_v})\\
&= \prod_{\substack{v\\ l_v\not\equiv 0\,(r)\\ k_v+1=l_v+re_v}} q_v^{\frac{(l_v-1)+re_v}{2}} \cdot \prod_{\substack{v\\ l_v\equiv 0\,(r)\\ k_v+1=l_v+re_v>0}} - q_v^{\frac{l_v+re_v-2}{2}} \cdot \prod_{\substack{v\\ l_v\equiv 0\,(r)\\ k_v\geq l_v+re_v>0}} q_v^{\frac{l_v+re_v}{2}}(1-q_v^{-1})\\
&= \mathrm{N}_{L/\mathbb{Q}}\left(\frac{\mathfrak{m}\mathfrak{h}_2}{\mathfrak{m}_0}\right)^{\frac{1}{2}} \cdot \prod_{\substack{v\\ l_v\equiv 0\,(r)\\ k_v+1=l_v+re_v>0}} - q_v^{-1} \cdot \prod_{\substack{v\\ l_v\equiv 0\,(r)\\ k_v\geq l_v+re_v>0}} (1-q_v^{-1}).\end{aligned}$$

One can decompose $\mathfrak{n}'$ as

$$\begin{aligned}\mathfrak{n}' = \mathfrak{n}_1 \cdot \mathrm{N}_{L/F}\left(\frac{\mathfrak{m}}{\mathfrak{m}_0}\right) \cdot \mathfrak{h}_2\\ \cdot \prod_{\substack{v-\text{complex}\\ l_v\equiv 0\,(r);\; l_{\bar{v}}=0\\ l_v+re_v>0\\ \alpha_v:=1+k_v-l_v-re_v\geq 0}} \mathrm{N}_{L/F}(\mathfrak{p}_v)^{\alpha_v-1} \cdot \prod_{\substack{v-\text{real}\\ e_v>0\\ \beta_v:=1+k_v-re_v\geq 0}} \mathfrak{q}_v^{\beta_v-1},\end{aligned}$$

with $\mathfrak{n}_1$ coprime to $\mathfrak{m}\mathfrak{h}$. Here, if v is complex such that $l_v = l_{\bar v} = 0$, then one chooses either v or $\bar v$, but not both. As $\mathfrak{n} = (\mathfrak{h}/\mathfrak{h}_0\mathfrak{h}_2)\mathfrak{n}'$, we also have

$$\mathfrak{n} = \mathfrak{n}_1 \cdot \mathrm{N}_{L/F}\left(\frac{\mathfrak{m}}{\mathfrak{m}_0}\right) \cdot \frac{\mathfrak{h}}{\mathfrak{h}_0} \cdot \prod_{\substack{v-\text{complex} \\ l_v \equiv 0\,(r);\; l_{\bar v}=0 \\ l_v + re_v > 0 \\ \alpha_v := 1+k_v-l_v-re_v \geq 0}} \mathrm{N}_{L/F}(\mathfrak{p}_v)^{\alpha_v - 1} \cdot \prod_{\substack{v-\text{real} \\ e_v>0 \\ \beta_v := 1+k_v-re_v\geq 0}} \mathfrak{q}_v^{\beta_v-1}.$$

Recall that $\mathfrak{n}^*$ denotes the part of $\mathfrak{n}$ coprime to $\mathfrak{m}_1\mathfrak{h}_1$. It follows that

$$\mathfrak{n}^* = \mathfrak{n}_1 \cdot \left(\frac{\overline{\mathfrak{m}}}{\overline{\mathfrak{m}}_0\overline{\mathfrak{m}}_2}\right) \cdot \mathrm{N}_{L/F}(\mathfrak{m}_2)\cdot\mathfrak{h}_2 \cdot \prod_{\substack{v-\text{complex} \\ l_v \equiv 0\,(r);\; l_{\bar v}=0 \\ l_v + re_v > 0 \\ \alpha_v := 1+k_v-l_v-re_v \geq 0}} \mathrm{N}_{L/F}(\mathfrak{p}_v)^{\alpha_v - 1} \cdot \prod_{\substack{v-\text{real} \\ e_v>0 \\ \beta_v := 1+k_v-re_v\geq 0}} \mathfrak{q}_v^{\beta_v-1}.$$

Combining all these with (4.26), we obtain

$$\sum_{\substack{\mathfrak{m}\in\mathcal{I}_L(S) \\ \mathfrak{m}-\text{imaginary}}} \frac{(\psi\tilde\rho)(\mathfrak{m})}{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{m})^s} \sum_{\substack{\mathfrak{h}\in\mathcal{I}_L(S) \\ \mathfrak{n}\in\mathcal{I}_F(S) \\ \mathfrak{h}-\text{real}}} \frac{(\psi\tilde\rho)(\mathfrak{h})\,\overline{\rho(\mathfrak{n})}\;\overline{\chi^*_{\mathfrak{m}_1\mathfrak{h}_1}(\mathfrak{n}^*)}\,G(\chi^*_{\mathfrak{m}_1\mathfrak{h}_1})\,G_0(\mathfrak{n},\mathfrak{m}\mathfrak{h})}{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{h})^s\,\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})^w}$$

$$= \sum_{\substack{\mathfrak{m}\in\mathcal{I}_L(S) \\ \mathfrak{m}-\text{imaginary}}} \frac{\psi(\mathfrak{m})\tilde\rho(\mathfrak{m}_0)\,\overline{\chi^*_{\mathfrak{m}_1}\left(\frac{\overline{\mathfrak{m}}}{\overline{\mathfrak{m}}_0}\right)}\,\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{m}_0)^{w-\frac{1}{2}}}{\mathrm{N}_{L/\mathbb{Q}}(\mathfrak{m})^{s+w-\frac{1}{2}}}$$

$$\cdot \sum_{\mathfrak{h}\in\mathcal{I}_F(S)} \frac{(\psi\rho)(\mathfrak{h})\,\rho(\mathfrak{h}_0)\,\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{h}_0)^{w-1}\,\chi^*_{\mathfrak{h}_1}(\mathfrak{m})\,\chi^*_{\mathfrak{h}_1}(\mathfrak{m}_0)^{-1}G(\chi^*_{\mathfrak{m}_1\mathfrak{h}_1})}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{h})^{2s+w-1}} \prod_{\substack{v \\ \mathrm{ord}_v(\mathrm{N}_{L/F}(\mathfrak{m}_1))>0 \\ \mathrm{ord}_v(\mathfrak{h}_2)>0}} (1-q_v^{-1})$$

$$\cdot \prod_{\substack{v \\ \mathrm{ord}_v(\mathrm{N}_{L/F}(\mathfrak{m}_2))>0 \\ \mathrm{ord}_v(\mathfrak{h})=0}} \Big[-\;(\chi^*_{\mathfrak{m}_1}\rho)(\pi_v)\,q_v^{w-1} + (1-q_v^{-1})\cdot\sum_{\alpha_v\geq 0}(\,(\chi^*_{\mathfrak{m}_1}\rho)^{-1}(\pi_v)\,q_v^{-w}\,)^{\alpha_v}\Big]$$

$$\cdot \prod_{\substack{v \\ \mathrm{ord}_v(\mathrm{N}_{L/F}(\mathfrak{m}_2))>0 \\ \mathrm{ord}_v(\mathfrak{h}_2)>0}} \Big[-\;(\chi^*_{\mathfrak{m}_1}\rho)(\pi_v)\,q_v^{w-1}(1-q_v^{-1}) + (1-q_v^{-1})^2\cdot\sum_{\alpha_v\geq 0}(\,(\chi^*_{\mathfrak{m}_1}\rho)^{-1}(\pi_v)\,q_v^{-w}\,)^{\alpha_v}\Big]$$

$$\cdot \prod_{\substack{v-\text{split in } L \\ \mathrm{ord}_v(\mathrm{N}_{L/F}(\mathfrak{m}))=0 \\ \mathrm{ord}_v(\mathfrak{h}_2)>0}} \Big[(\chi^*_{\mathfrak{m}_1}\rho)(\pi_v)\,q_v^{w-2} + (1-q_v^{-1})^2\cdot\sum_{\alpha_v\geq 0}(\,(\chi^*_{\mathfrak{m}_1}\rho)^{-1}(\pi_v)\,q_v^{-w}\,)^{\alpha_v}\Big]$$

$$\cdot \prod_{\substack{v-\text{inert in } L \\ \mathrm{ord}_v(\mathfrak{h}_2)>0}} \Big[-\;(\chi^*_{\mathfrak{m}_1}\rho)(\pi_v)\,q_v^{w-2} + (1-q_v^{-2})\cdot\sum_{\beta_v\geq 0}(\,(\chi^*_{\mathfrak{m}_1}\rho)^{-1}(\pi_v)\,q_v^{-w}\,)^{\beta_v}\Big]$$

$$\cdot \sum_{\substack{\mathfrak{n}_1\in\mathcal{I}_F(S) \\ (\mathfrak{n}_1,\mathfrak{m}\mathfrak{h})=1}} \frac{\overline{\rho(\mathfrak{n}_1)}\;\overline{\chi^*_{\mathfrak{m}_1}(\mathfrak{n}_1)}}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n}_1)^w}.$$

Note that the last sum represents an incomplete Hecke L-function. After evaluating the geometric series inside the last four products, the missing Euler factors corresponding to places of F dividing $\mathrm{N}_{L/F}(\mathfrak{m}_2)\mathfrak{h}_2$ can be incorporated. Also, multiply and divide by the Euler factors corresponding to places of F dividing $\mathfrak{h}_0$, forcing in this way $L_S(w,\, (\chi^*_{\mathfrak{m}_1}\,\rho)^{-1})$ to appear.

Let $R_{\mathfrak{c}}^+$ be the subgroup of $R_{\mathfrak{c}}$ generated by the images (in $R_{\mathfrak{c}}$) of all real fractional ideals of L coprime to S'. Let $\mathfrak{e}'$ be a fixed element of $R_{\mathfrak{c}}$ which is the image of an imaginary ideal $\mathfrak{m} \in \mathcal{I}_L(S)$. Replacing ψ by $\psi\tau_1\tau_2$ with τ_1 and τ_2 characters of $R_{\mathfrak{c}}$ and $R_{\mathfrak{c}}/R_{\mathfrak{c}}^+$, respectively, and making a standard linear combination, one can restrict the first two sums over ideals $\mathfrak{m}$ and $\mathfrak{h}$, for which the image of $\mathfrak{m}_1$ in $R_{\mathfrak{c}}$ is $\mathfrak{e}'$ modulo $R_{\mathfrak{c}}^+$ and the image of $\mathfrak{m}_1\mathfrak{h}_1$ is a fixed element $\mathfrak{e}$ of $R_{\mathfrak{c}}$.

Now, invoke the functional equation of $L(w,\, (\chi^*_{\mathfrak{m}_1}\,\rho)^{-1})$. It is well-known, see [31], that the incomplete Hecke L-function (defined over F)

$$L_S\left(w,\, (\chi^*_{\mathfrak{m}_1}\,\rho)^{-1}\right) \,=\, \prod_{v\notin S} L_v\left(w,\, (\chi^*_{\mathfrak{m}_1}\,\rho)_v^{-1}\right) \,=\, \prod_{v\notin S}\left[1 \,-\, (\chi^*_{\mathfrak{m}_1}\,\rho)_v^{-1}(\pi_v)\, q_v^{-w}\right]^{-1}$$

satisfies the functional equation

$$\begin{aligned} L_S\left(w,\, (\chi^*_{\mathfrak{m}_1}\,\rho)^{-1}\right) = \varepsilon\left(w,\, (\chi^*_{\mathfrak{m}_1}\,\rho)^{-1}\right)\cdot L_S\left(1-w,\, \chi^*_{\mathfrak{m}_1}\,\rho\right) \\ \cdot \prod_{v\in S_\infty} \frac{L_v\left(1-w,\, \rho_v\right)}{L_v\left(w,\, \rho_v^{-1}\right)} \cdot \prod_{v\in S'} \frac{L_v\left(1-w,\, (\chi^*_{\mathfrak{m}_1}\,\rho)_v\right)}{L_v\left(w,\, (\chi^*_{\mathfrak{m}_1}\,\rho)_v^{-1}\right)}. \end{aligned}$$

Replace ψ by $\psi\,\eta^{-1}$, and combine the above functional equation with Lemma 3.6. Here $\mathrm{Re}(s)$ is taken sufficiently large to ensure convergence. Using the Fisher-Friedberg extension of the reciprocity law [9], one can see that

$$\overline{\chi^*_{\mathfrak{m}_1}(\overline{\mathfrak{m}})}\,\chi^*_{\mathfrak{h}_1}(\mathfrak{m}) \,=\, C'_{\mathfrak{e},\,\widehat{\mathfrak{e}'}}\cdot \chi^*_{\mathfrak{m}}(\mathfrak{h}_1),$$

where $C'_{\mathfrak{e},\,\widehat{\mathfrak{e}'}}$ is a constant depending on just $\mathfrak{e}$ and the class $\widehat{\mathfrak{e}'}$ in $R_{\mathfrak{c}}/R_{\mathfrak{c}}^+$. Also, note that

$$\prod_{v\in S'}\left(1-\rho^{-r}(\pi_v)\, q_v^{-rw}\right)^{-1}\cdot\;\; \frac{L_v\left(1-w,\, (\chi^*_{\mathfrak{m}_1}\,\rho)_v\right)}{L_v\left(w,\, (\chi^*_{\mathfrak{m}_1}\,\rho)_v^{-1}\right)}$$

is the inverse of a polynomial in the variables q_v^w, q_v^{-w} corresponding to places $v \in S'$ of the totally real field F. The characters involved in its coefficients are trivial on real ideals. Now, the functional equation (3.8) immediately follows, after we replace ψ with $\psi\tau$, where τ ranges over a finite set of idéle class characters unramified outside S and orders dividing r, and make a combination such that the above product over $v \in S'$ disappears.

Starting from the definition of

$$\prod_{v\in S'}\left(1-\rho^{r}(\pi_v)\, q_v^{rw-r}\right)^{-1}\cdot\;\; \widetilde{Z}(s+w-\tfrac{1}{2}, 1-w; \psi; \rho),$$

one can easily check (3.9) by reversing the above argument. ∎

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