# ON FANO MANIFOLDS OF PICARD NUMBER ONE WITH BIG AUTOMORPHISM GROUPS 

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#### Abstract

Let $X$ be an $n$-dimensional smooth Fano complex variety of Picard number one. Assume that the VMRT at a general point of $X$ is smooth irreducible and non-degenerate (which holds if $X$ is covered by lines with index $>(n+2) / 2$ ). It is proven that $\operatorname{dim} \mathfrak{a u t}(X)>n(n+1) / 2$ if and only if $X$ is isomorphic to $\mathbb{P}^{n}, \mathbb{Q}^{n}$ or $\operatorname{Gr}(2,5)$. Furthermore, the equality $\operatorname{dim} \mathfrak{a u t}(X)=n(n+1) / 2$ holds only when $X$ is isomorphic to the 6 -dimensional Lagrangian Grassmannian $\operatorname{Lag}(6)$ or a general hyperplane section of $\operatorname{Gr}(2,5)$.


## 1. Introduction

For a smooth projective complex variety $X$, the Lie algebra $\mathfrak{a u t}(X)$ of its automorphism group $\operatorname{Aut}(X)$ is naturally identified with $H^{0}\left(X, T_{X}\right)$. A natural question is how big can this group be. In general, $\mathfrak{a u t}(X)$ can be very big with respect to its dimension. For example, when $X$ is the Hirzebruch surface $\mathbb{F}_{m}$, then $\operatorname{dim} \mathfrak{a u t}(X)=m+5$. On the other hand, in the case of Fano manifold of Picard number one, we have the following:

Conjecture 1.1 ([HM05], Conjecture 2). Let $X$ be an $n$-dimensional Fano manifold of Picard number one. Then $\operatorname{dim} \mathfrak{a u t}(X) \leq n^{2}+2 n$, with equality if and only if $X \simeq \mathbb{P}^{n}$.

In HM05 (Theorem 1.3.2), this conjecture is proven under the assumption that the variety of minimal rational tangents (VMRT for short, cf. Definition 3.1) at a general point of $X$ is smooth irreducible non-degenerate and linearly normal. The purpose of this note is to push further the ideas of [HM05], combined with the recent results in [FH12] and [FH18], to prove the following

Theorem 1.2. Let $X$ be an n-dimensional Fano manifold of Picard number one. Assume that the VMRT at a general point of $X$ is smooth irreducible and non-degenerate. Then we have
(a) $\operatorname{dim} \mathfrak{a u t}(X)>n(n+1) / 2$ if and only if $X$ is isomorphic to $\mathbb{P}^{n}, \mathbb{Q}^{n}$ or $\operatorname{Gr}(2,5)$.
(b) The equality $\operatorname{dim} \mathfrak{a u t}(X)=n(n+1) / 2$ holds only when $X$ is isomorphic to $\operatorname{Lag}(6)$ or a general hyperplane section of $\operatorname{Gr}(2,5)$.

Remark 1.3. As proved in Corollary 1.3.3 HM05, the assumption on the VMRT of $X$ is satisfied if there exists an embedding $X \subset \mathbb{P}^{N}$ such that $X$ is covered by lines with index $>\frac{n+2}{2}$.

Recall that $\operatorname{dim} \mathfrak{a u t}\left(\mathbb{P}^{n}\right)=n^{2}+2 n$ and $\operatorname{dim} \mathfrak{a u t}\left(\mathbb{Q}^{n}\right)=\operatorname{dim} \mathfrak{s o}_{n+2}=\frac{(n+1)(n+2)}{2}$. The previous theorem indicates that there may exist big gaps between the dimensions of automorphism groups of Fano manifolds of Picard number one.

To prove Theorem [1.2, we first show the following result, which could be of independent interest.

Theorem 1.4. Let $n \geq 2$ be an integer. Let $X \subsetneq \mathbb{P}^{n}$ be an irreducible and nondegenerate subvariety of codimension $c \geq 1$, which is not a cone. Let $G_{n}^{X}=\{g \in$ $\left.\mathrm{PGL}_{n+1}(\mathbb{C}) \mid g(X)=X\right\}$. Then
(a) $\operatorname{dim} G_{n}^{X} \leq \frac{n(n+1)}{2}-\frac{(c-1)(c+4)}{2}$.
(b) $\operatorname{dim} G_{n}^{X}=\frac{n(n+1)}{2}$ if and only if $X$ is a smooth quadratic hypersurface.
(c) if $X$ is smooth and is not a quadratic hypersurface, then $\operatorname{dim} G_{n}^{X} \leq \frac{n(n+1)}{2}-3$.

The idea of the proof of Theorem 1.2 is similar to that in HM05: the dimension of $\mathfrak{a u t}(X)$ is controlled by $n+\operatorname{dim} \mathfrak{a u t}\left(\hat{\mathcal{C}}_{x}\right)+\operatorname{dim} \mathfrak{a u t}\left(\hat{\mathcal{C}}_{x}\right)^{(1)}$, where $\mathcal{C}_{x}$ is the VMRT of $X$ at a general point, $\mathfrak{a u t}\left(\hat{\mathcal{C}}_{x}\right)$ is the Lie algebra of infinitesimal automorphisms of $\hat{\mathcal{C}_{x}}$ while $\mathfrak{a u t}\left(\hat{\mathcal{C}_{x}}\right)^{(1)}$ is the first prolongation of $\mathfrak{a u t}\left(\hat{\mathcal{C}}_{x}\right)$ (cf. Definition (3.4). By Theorem [1.4, we have an optimal bound for $\operatorname{dim} \mathfrak{a u t}\left(\hat{\mathcal{C}}_{x}\right)$, which gives the bound for $\operatorname{dim} \mathfrak{a u t}(X)$ in the case when $\mathfrak{a u t}\left(\hat{\mathcal{C}_{x}}\right)^{(1)}=0$. For the case when $\mathfrak{a u t}\left(\hat{\mathcal{C}_{x}}\right)^{(1)} \neq 0$, we have a complete classification of all such embeddings $\mathcal{C}_{x} \subset \mathbb{P} T_{x} X$ by [FH12] and [FH18]. Then a case-by-case check gives us the bound in Theorem 1.2, Finally we apply Cartan-Fubini extension theorem of Hwang-Mok ([HM01]) and the result of Mok ([Mok08]) to recover the variety $X$ from its VMRT.

Convention: For a projective variety $X$, we denote by $\mathfrak{a u t}(X)$ its Lie algebra of automorphism group, while for an embedded variety $S \subset \mathbb{P} V$, we denote by $\mathfrak{a u t}(\hat{S})$ the Lie algebra of infinitesimal automorphisms of $\hat{S}$, which is given by

$$
\mathfrak{a u t}(\hat{S}):=\left\{g \in \operatorname{End}(V) \mid g(\alpha) \in T_{\alpha}(\hat{S}), \text { for any smooth point } \alpha \in \hat{S}\right\}
$$

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## 2. Automorphism group of embedded varieties

For each positive integer $n$, we let

$$
G_{n}=\operatorname{Aut}\left(\mathbb{P}^{n}\right)=\operatorname{PGL}_{n+1}(\mathbb{C})
$$

If $X \subseteq \mathbb{P}^{n}$ is a subvariety, we denote by $G_{n}^{X} \subseteq G_{n}$ the subgroup of elements $g$ such that $g(X)=X$. Note that if $X \subset \mathbb{P}^{n}$ is non-degenerate, then $G_{n}^{X} \subset \operatorname{Aut}(X)$.

The goal of this section is to prove the following theorem, which is more general than Theorem 1.4 .

Theorem 2.1. Assume that $n \geq 2$. Let $X \subsetneq \mathbb{P}^{n}$ be an irreducible and non-degenerate subvariety of codimension $c \geq 1$. Then the set

$$
C_{X}:=\{x \in X \mid X \text { is a cone with vertex } x\}
$$

is a linear subspace. Set $r_{X}:=-1$ if $C_{X}=\emptyset$ and $r_{X}:=\operatorname{dim} C_{X}$ otherwise. Then we have

$$
\operatorname{dim} G_{n}^{X} \leq \frac{\left(n-r_{X}-1\right)\left(n-r_{X}\right)}{2}-\frac{(c-1)(c+4)}{2}+\left(r_{X}+1\right)(n+1)
$$

The idea of the proof is to cut $X$ by a general hyperplane, and then use induction on $n$ to conclude. To this end, we will first introduce the following notation. For a hyperplane $H$ in $\mathbb{P}^{n}$, we may choose a coordinates system $\left[x_{0}: x_{1}: \cdots: x_{n}\right]$ such that
$H$ is defined by $x_{0}=0$. For every $g \in G_{n}$, it has a representative $M \in \mathrm{GL}_{n+1}(\mathbb{C})$, such that its action on $\mathbb{P}^{n}$ is given by $g\left(\left[x_{0}: x_{1}: \cdots: x_{n}\right]\right)=\left[y_{0}: y_{1}: \cdots: y_{n}\right]$, where

$$
\left(\begin{array}{c}
y_{0} \\
y_{1} \\
\vdots \\
y_{n}
\end{array}\right)=M\left(\begin{array}{c}
x_{0} \\
x_{1} \\
\vdots \\
x_{n}
\end{array}\right)
$$

Then $g \in G_{n}^{H}$ if and only if it can be represented by a matrix of the shape

$$
\left(\begin{array}{c|ccc}
a_{0} & 0 & \cdots & 0 \\
\hline a_{1} & & & \\
a_{2} & & A & \\
\vdots & & & \\
a_{n} & &
\end{array}\right)
$$

There is a natural morphism $r_{H}: G_{n}^{H} \rightarrow \operatorname{Aut}(H) \cong \mathrm{PGL}_{n}(\mathbb{C})$. Then an element $g$ is in the kernel $\operatorname{Ker} r_{H}$ if and only if it can be represented by a matrix of the shape

$$
\left(\begin{array}{c|ccc}
\lambda & 0 & \cdots & 0 \\
\hline a_{1} & & & \\
a_{2} & & \mathrm{Id}_{n} & \\
\vdots & & & \\
a_{n} & &
\end{array}\right)
$$

For such $g \in \operatorname{Ker} r_{H}$, we call $\lambda$ the special eigenvalue of $g$. The action of $g$ on the normal bundle of $H$ is then the multiplication by $\lambda$. We see that this is independant of the choice of representatives of $g$ in $\mathrm{GL}_{n+1}(\mathbb{C})$. We also note that if $g, h$ are two elements in $\operatorname{Ker} r_{H}$, with special eigenvalues $\lambda$ and $\mu$ respectively, then the special eigenvalue of $g h$ is equal to $\lambda \mu$. This gives a homomorphism $\chi_{H}: \operatorname{Ker} r_{H} \rightarrow \mathbb{C}^{*}$.

Before giving the proof of Theorem [1.4, we will first prove several lemmas.
Lemma 2.2. Let $H$ be a hyperplane in $\mathbb{P}^{n}$, and let $X \subseteq \mathbb{P}^{n}$ be any subvariety. Then

$$
\operatorname{dim} G_{n}^{X} \leq \operatorname{dim}\left(G_{n}^{H} \cap G_{n}^{X}\right)+n
$$

Proof. This lemma follows from the fact that $\operatorname{dim} G_{n}=\operatorname{dim} G_{n}^{H}+n$.
We also need the following Bertini type lemma.
Lemma 2.3. Assume that $n \geq 2$. Let $X \subseteq \mathbb{P}^{n}$ be a non-degenerate irreducible subvariety of positive dimension which is not a cone. Then for a general hyperplane $H$, the intersection $X \cap H$ is still non-degenerate in $H$ and is not a cone.

Proof. Since $X \subseteq \mathbb{P}^{n}$ is irreducible and non-degenerate, the intersection of $X$ and a general hyperplane $H$ is non-degenerate in $H$.

Let $V \subset X \times\left(\mathbb{P}^{n}\right)^{*}$ be the subset of pair $(x, H)$ such that $H \cap X$ is a cone with vertex $x$. Set $\pi_{1}: V \rightarrow X$ and $\pi_{2}: X \rightarrow\left(\mathbb{P}^{n}\right)^{*}$ the projections to the first and the second factors. If $\pi_{2}$ is not surjective, we concludes the proof. So we may assume that $\pi_{2}$ is surjective. Hence $\operatorname{dim} V \geq n$. Set $Y:=\pi_{1}(V)$.

We first assume that there is $x \in Y$ such that $\operatorname{dim} \pi_{1}^{-1}(x) \geq 1$. Since any non trivial complete one-dimensional family of hyperplanes in $\mathbb{P}^{n}$ covers the whole $\mathbb{P}^{n}$, this condition implies that for every point $x^{\prime} \in X \backslash\{x\}$, there is some hyperplane $H$ containing $x$ and $x^{\prime}$ such that $H \cap X$ is a cone with vertex $x$. Therefore, the line
joining $x$ and $x^{\prime}$ is contained in $H \cap X$ and hence in $X$. This shows that $X$ is a cone with vertex $x$. We obtain a contradiction.

So the morphism $\pi_{1}: V \rightarrow Y$ is finite. Then we get

$$
n \leq \operatorname{dim} V=\operatorname{dim} Y \leq \operatorname{dim} X<n
$$

which is a contradiction. This concludes the proof.
In the following lemmas, we will show that the kernel of $G_{n}^{X} \cap G_{n}^{H} \rightarrow G_{n-1}^{X \cap H}$ is a finite set. Note that this kernel is nothing but $G_{n}^{X} \cap \operatorname{Ker} r_{H}$, as $X \cap H$ is nondegenerate by Lemma 2.3, We will discuss according to the special eigenvalue of an element inside. We will first study the case when it is equal to 1 .

Lemma 2.4. Assume that $n \geq 2$. Let $X \subseteq \mathbb{P}^{n}$ be an irreducible subvariety which is not a cone. Let $H$ be a hyperplane in $\mathbb{P}^{n}$ such that $X \nsubseteq H$. Let $g \in \operatorname{Ker} r_{H} \cap G_{n}^{X}$. If the special eigenvalue of $g$ is 1 , then $g$ is the identity in $G_{n}$. In other words, the map $\chi_{H}: \operatorname{Ker} r_{H} \cap G_{n}^{X} \rightarrow \mathbb{C}^{*}$ is injective.

Proof. Assume the opposite. We choose a homogeneous coordinates system $\left[x_{0}: x_{1}\right.$ : $\left.\cdots: x_{n}\right]$ such that $H$ is defined by $x_{0}=0$, and that $g\left(\left[x_{0}: x_{1}: \cdots: x_{n}\right]\right)=\left[y_{0}: y_{1}:\right.$ $\left.\cdots: y_{n}\right]$, where

$$
\left(\begin{array}{c}
y_{0} \\
y_{1} \\
y_{2} \\
\vdots \\
y_{n}
\end{array}\right)=\left(\begin{array}{c|lll}
1 & 0 & \cdots & 0 \\
\hline a_{1} & & & \\
a_{2} & & \mathrm{Id}_{n} & \\
\vdots & & & \\
a_{n} & &
\end{array}\right)\left(\begin{array}{c}
x_{0} \\
x_{1} \\
x_{2} \\
\vdots \\
x_{n}
\end{array}\right)=\left(\begin{array}{c}
x_{0} \\
x_{1}+a_{1} x_{0} \\
x_{2}+a_{2} x_{0} \\
\vdots \\
x_{n}+a_{n} x_{0}
\end{array}\right)
$$

By assumption, the $a_{i}$ are not all equal to zero. Let $p \in \mathbb{P}^{n}$ be the point with homogeneous coordinates $\left[0: a_{1}: \cdots: a_{n}\right]$. For any point $x \in X \backslash H$ with homogeneous coordinates $\left[1: x_{1}: \cdots: x_{n}\right]$, the point $g^{k}(x)$ has coordinates

$$
g^{k}(x)=\left[1: x_{1}+k a_{1}: \cdots: x_{n}+k a_{n}\right] .
$$

This shows that all $g^{k}(x)$ are on the unique line $L_{p, x}$ passing through $p$ and $x$. Since the $g^{k}(x)$ are pairwise different, this implies that $L_{p, x}$ has infinitely many intersection points with $X$. Therefore, $L_{p, x} \subseteq X$.

Since $X$ is irreducible, every point $y \in X \cap H$ is a limit of points in $X \backslash H$. Hence by continuity, for each point $x \in X \backslash\{p\}$, the line $L_{p, x}$ is contained in $X$. This implies that $X$ is a cone with vertex $p$. We obtain a contradiction.

Now we will look at the case when the special eigenvalue is different from 1.
Lemma 2.5. Assume that $n \geq 2$. Let $X \subseteq \mathbb{P}^{n}$ be a subvariety. Then there is a number $d(X)$ such that if a line $L$ intersects $X$ at more than $d(X)$ points, then $L \subseteq X$.

Proof. Assume that $X$ is defined as the common zero locus of homogeneous polynomials $P_{1}, \ldots, P_{k}$. Let $d(X)$ be the maximal degree of them. Assume that a line $L$ intersects $X$ at more than $d(X)$ points, then $L$ intersects the zero locus of each $P_{i}$ at more than $d(X)$ points. By degree assumption, this shows that $L$ is contained in the zero locus of each $P_{i}$. Hence $L \subseteq X$.

Lemma 2.6. Assume that $n \geq 2$. Let $X \subseteq \mathbb{P}^{n}$ be an irreducible subvariety which is not a cone. Let $H$ be a hyperplane in $\mathbb{P}^{n}$ such that $X \nsubseteq H$. Let $g \in \operatorname{Ker}\left(r_{H}\right) \cap G_{n}^{X}$.

Then the special eigenvalue $\lambda$ of $g$ is a root of unity. Moreover, its order is bounded by the number $d(X)$ from above.

Proof. We may assume that $\lambda$ is different from 1. Then $g$ is diagonalizable in this case. We may choose homogeneous coordinates $\left[x_{0}: x_{1}: \cdots: x_{n}\right]$ of $\mathbb{P}^{n}$ such that $H$ is defined by $x_{0}=0$ and that

$$
g\left(\left[x_{0}: x_{1}: \cdots: x_{n}\right]\right)=\left[\lambda x_{0}: x_{1}: \cdots: x_{n}\right] .
$$

Assume by contradiction that the order of $\lambda$ is greater than $d(X)$ (by convention, if $\lambda$ is not a root of unity, then its order is $+\infty)$. Let $p$ be the point with homogeneous coordinates $[1: 0: \cdots: 0]$. For any point $x \in X \backslash(H \cup\{p\})$ with homogeneous coordinates $\left[1: x_{1}: \cdots: x_{n}\right]$, the point $g^{k}(x)$ has coordinates

$$
g^{k}(x)=\left[\lambda^{k}: x_{1}: \cdots: x_{n}\right] .
$$

This shows that all of the $g^{k}(x)$ are on the unique line $L_{p, x}$ passing through $p$ and $x$. Moreover, we note that the cardinality of

$$
\left\{g^{k}(x) \mid k \in \mathbb{Z}\right\}
$$

is exactly the order of $\lambda$. By Lemma 2.5, we obtain that the line $L_{p, x}$ is contained in $X$. By the same continuity argument as in the proof of Lemma 2.4, this implies that for any point $x \in X \backslash\{p\}$, the line $L_{p, x}$ is contained in $X$. Hence $X$ is a cone, and we obtain a contradiction.

Lemma 2.7. Assume that $n \geq 2$. Let $X \subseteq \mathbb{P}^{n}$ be an irreducible subvariety which is not a cone. Let $H$ be a hyperplane in $\mathbb{P}^{n}$ such that $X \nsubseteq H$. Then $\operatorname{Ker}\left(r_{H}\right) \cap G_{n}^{X}$ is a finite set. As a consequence, we have

$$
\operatorname{dim} G_{n}^{X} \leq \operatorname{dim} G_{n-1}^{X \cap H}+n
$$

Proof. By Lemma 2.4, the map $\chi_{H}: \operatorname{Ker} r_{H} \cap G_{n}^{X} \rightarrow \mathbb{C}^{*}$ is injective, while Lemma 2.6 implies that its image has bounded order, hence $\operatorname{Ker}\left(r_{H}\right) \cap G_{n}^{X}$ is finite. By Lemma 2.2. we obtain that

$$
\operatorname{dim} G_{n}^{X} \leq \operatorname{dim} G_{n-1}^{X \cap H}+n
$$

Now we can conclude Theorem 1.4 .
Proof of Theorem 1.4. By Lemma 2.3, we can repeatedly apply Lemma 2.7 to get
$\operatorname{dim} G_{n}^{X} \leq \operatorname{dim} G_{n-1}^{X \cap H}+n \leq \cdots \leq \operatorname{dim} G_{c+1}^{X \cap H^{n-c-1}}+n+(n-1)+\cdots+(c+2)$
Let $C:=X \cap H^{n-c-1} \subset \mathbb{P}^{c+1}$ be a general curve section of $X$. As $C \subset \mathbb{P}^{c+1}$ is non-degenerate by Lemma [2.3, we have an inclusion $G_{c+1}^{C} \subset \operatorname{Aut}(C)$, while the latter has dimension at most 3 . This gives that

$$
\operatorname{dim} G_{n}^{X} \leq 3+\sum_{j=c+2}^{n} j=\frac{n(n+1)}{2}-\frac{(c-1)(c+4)}{2}
$$

which proves (a).
For (b), if $\operatorname{dim} G_{n}^{X}=\frac{n(n+1)}{2}$, then $c=1$ by (a), i.e. $X \subset \mathbb{P}^{n}$ is a hypersurface. As $X$ is not a cone, it must be smooth if it is quadratic. Therefore it remains to show that if $X$ is a hypersurface of degree at least 3 , then $\operatorname{dim} G_{n}^{X} \leq \frac{n(n+1)}{2}-1$. We prove it by induction on the dimension of $X$. When $\operatorname{dim} X=1$, pick a general line $H$ in $\mathbb{P}^{2}$. By Lemma 2.7, we have

$$
\operatorname{dim} G_{2}^{X} \leq \operatorname{dim} G_{1}^{X \cap H}+2
$$

In this case $X \cap H$ is a set of $\operatorname{deg} X \geq 3$ points, hence $\operatorname{dim} G_{1}^{X \cap H}=0$, it follows that $\operatorname{dim} G_{2}^{X} \leq 2=\frac{2(2+1)}{2}-1$. When $\operatorname{dim} X=n-1 \geq 2$, pick a general hyperplane $H$ in $\mathbb{P}^{n}$. By Lemma 2.7, we have

$$
\operatorname{dim} G_{n}^{X} \leq \operatorname{dim} G_{n-1}^{X \cap H}+n
$$

Then by induction hypothesis we have

$$
\operatorname{dim} G_{n}^{X} \leq \operatorname{dim} G_{n-1}^{X \cap H}+n \leq \frac{n(n-1)}{2}-1+n=\frac{n(n+1)}{2}-1
$$

For (c), if we assume further that $X$ is a smooth hypersurface of degree greater than 2, then $\operatorname{dim} G_{n}^{X}=0$ (see for example Theorem 1.2 [Poo05]). As a consequence, if $X$ is smooth and is not a quadratic hypersurface, then $\operatorname{dim} G_{n}^{X} \leq \frac{n(n+1)}{2}-3$. This completes the proof of the theorem.

Proof of Theorem [2.1. If $C_{X}=\emptyset$, then we conclude the proof by Theorem 1.4. Now assume that $C_{X} \neq \emptyset$. By Proposition 1.3.3 [Rus16], it is a linear subspace. For simplicity, we set $r=r_{X}=\operatorname{dim} C_{X}$.

Pick a coordinates system of $\mathbb{P}^{n}$ such that $C_{X}$ is defined by $x_{0}=\cdots=x_{n-r-1}=0$. Let $V$ be the subspace of $\mathbb{P}^{n}$ defined by $x_{n-r}=\cdots=x_{n}=0$ and we identify it with $\mathbb{P}^{n-r-1}$. We let $\pi: \mathbb{P}^{n} \backslash C_{X} \rightarrow V \cong \mathbb{P}^{n-r-1}$ be the projection

$$
\pi:\left[x_{0}: \cdots: x_{n}\right] \mapsto\left[x_{0}: \cdots: x_{n-r-1}: 0: \cdots: 0\right]
$$

Denote the image $\pi\left(X \backslash C_{X}\right)$ by $Y$. Then we have $X \backslash C_{X}=\pi^{-1}(Y)$, and $Y=X \cap V$. Moreover, $Y$ is not a cone and it is non-degenerate in $\mathbb{P}^{n-r-1}$. Since $C_{X}$ is preserved by $G_{n}^{X}$, we see that $G_{n}^{X} \subseteq G_{n}^{C_{X}}$.

Each element $g$ in $G_{n}^{C_{X}}$ can be represented by a matrix of the shape

$$
\left(\begin{array}{c|c}
A & 0 \\
\hline B & C
\end{array}\right)
$$

such that $A$ and $C$ are square matrices of dimension $n-r$ and $r+1$ respectively. We may now define an action of $G_{n}^{C_{x}}$ on $V$ as follows. For each

$$
y=\left[a_{0}: \cdots: a_{n-r-1}: 0: \cdots: 0\right] \in V,
$$

the new action $g * y$ of $g$ on $y$ is defined as

$$
g * y=\pi(g . y),
$$

where $g . y$ represents the standard action of $G_{n}$ on $\mathbb{P}^{n}$. With the representative above, this action is just defined as

$$
g *\left[a_{0}: \cdots: a_{n-r-1}: 0: \cdots: 0\right]=\left[b_{0}: \cdots: b_{n-r-1}: 0: \cdots: 0\right]
$$

where

$$
\left(\begin{array}{c}
b_{0} \\
b_{1} \\
\vdots \\
b_{n-r-1}
\end{array}\right)=A\left(\begin{array}{c}
a_{0} \\
a_{1} \\
\vdots \\
a_{n-r-1}
\end{array}\right)
$$

Thanks to this action, and by identifying $V$ with $\mathbb{P}^{n-r-1}$, we obtain a group morphism $\rho: G_{n}^{C_{X}} \rightarrow G_{n-r-1}$.

ON FANO MANIFOLDS OF PICARD NUMBER ONE WITH BIG AUTOMORPHISM GROUPS 7
On the one hand, we note that an element $g \in G_{n}^{C_{X}}$ belongs to $G_{n}^{X}$ if and only if $\rho(g) \in G_{n-r-1}^{Y}$. On the other hand, an element $g \in G_{n}^{C_{X}}$ belongs to $\operatorname{Ker} \rho$ if and only if it can be represented by a matrix of the shape

$$
\left(\begin{array}{c|c}
\mathrm{Id} & 0 \\
\hline B & C
\end{array}\right)
$$

Hence $\operatorname{dim} \operatorname{Ker} \rho=(r+1)(n+1)$. As we can see that $\operatorname{Ker} \rho \subseteq G_{n}^{X}$, we obtain that

$$
\operatorname{dim} G_{n}^{X}=\operatorname{dim} G_{n-r-1}^{Y}+(r+1)(n+1)
$$

Finally, by Theorem 1.4, we get

$$
\operatorname{dim} G_{n}^{X} \leq \frac{(n-r-1)(n-r)}{2}-\frac{(c-1)(c+4)}{2}+(r+1)(n+1)
$$

## 3. Proof of the main theorem

Definition 3.1. Let $X$ be a uniruled projective manifold. An irreducible component $\mathcal{K}$ of the space of rational curves on $X$ is called a minimal rational component if the subscheme $\mathcal{K}_{x}$ of $\mathcal{K}$ parameterizing curves passing through a general point $x \in X$ is non-empty and proper. Curves parameterized by $\mathcal{K}$ will be called minimal rational curves. Let $\rho: \mathcal{U} \rightarrow \mathcal{K}$ be the universal family and $\mu: \mathcal{U} \rightarrow X$ the evaluation map. The tangent map $\tau: \mathcal{U} \rightarrow \mathbb{P} T(X)$ is defined by $\tau(u)=\left[T_{\mu(u)}\left(\mu\left(\rho^{-1} \rho(u)\right)\right)\right] \in$ $\mathbb{P} T_{\mu(u)}(X)$. The closure $\mathcal{C} \subset \mathbb{P} T(X)$ of its image is the VMRT-structure on $X$. The natural projection $\mathcal{C} \rightarrow X$ is a proper surjective morphism and a general fiber $\mathcal{C}_{x} \subset$ $\mathbb{P} T_{x}(X)$ is called the VMRT at the point $x \in X$. The VMRT-structure $\mathcal{C}$ is locally flat if for a general $x \in X$, there exists an analytical open subset $U$ of $X$ containing $x$ with an open immersion $\phi: U \rightarrow \mathbb{C}^{n}, n=\operatorname{dim} X$, and a projective subvariety $Y \subset \mathbb{P}^{n-1}$ with $\operatorname{dim} Y=\operatorname{dim} \mathcal{C}_{x}$ such that $\phi_{*}: \mathbb{P} T(U) \rightarrow \mathbb{P} T\left(\mathbb{C}^{n}\right)$ maps $\left.\mathcal{C}\right|_{U}$ into the trivial fiber subbundle $\mathbb{C}^{n} \times Y$ of the trivial projective bundle $\mathbb{P} T\left(\mathbb{C}^{n}\right)=\mathbb{C}^{n} \times \mathbb{P}^{n-1}$.

Examples 3.2. An irreducible Hermitian symmetric space of compact type is a homogeneous space $M=G / P$ with a simple Lie group $G$ and a maximal parabolic subgroup $P$ such that the isotropy representation of $P$ on $T_{x}(M)$ at a base point $x \in M$ is irreducible. The highest weight orbit of the isotropy action on $\mathbb{P} T_{x}(M)$ is exactly the VMRT at $x$. The following table (e.g. Section 3.1 [FH12]) collects basic information on these varieties.

| Type | I.H.S.S. $M$ | VMRT $S$ | $S \subset \mathbb{P} T_{x}(M)$ | $\operatorname{dim} \mathfrak{a u t}(M)$ | $\operatorname{dim} \mathfrak{a u t}(S)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| I | $\operatorname{Gr}(a, a+b)$ | $\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$ | Segre | $(a+b)^{2}-1$ | $a^{2}+b^{2}-2$ |
| II | $\mathbb{S}_{r}$ | $\operatorname{Gr}(2, r)$ | Plücker | $r(2 r-1)$ | $r^{2}-1$ |
| III | $\operatorname{Lag}(2 r)$ | $\mathbb{P}^{r-1}$ | Veronese | $r(2 r+1)$ | $r^{2}-1$ |
| IV | $\mathbb{Q}^{r}$ | $\mathbb{Q}^{r-2}$ | Hyperquadric | $(r+1)(r+2) / 2$ | $(r-1) r / 2$ |
| V | $\mathbb{O P}^{2}$ | $\mathbb{S}_{5}$ | Spinor | 78 | 45 |
| VI | $E_{7} /\left(E_{6} \times U(1)\right)$ | $\mathbb{O P}^{2}$ | Severi | 133 | 78 |

Lemma 3.3. Let $M$ be an IHSS of dimension $n$ different from projective spaces and $S \subset \mathbb{P}^{n-1}$ its $V M R T$ at a general point. Then
(1) $\operatorname{dimaut}(M) \leq \frac{n(n+1)}{2}$ unless $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{2}$ or the natural embedding of $\mathbb{Q}^{n-2} \subset \mathbb{P}^{n-1}$.
(2) The equality holds if and only if $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to the second Veronese embedding of $\mathbb{P}^{2}$.

Proof. For Type (I), we have $M=\operatorname{Gr}(a, a+b), n=\operatorname{dim} M=a b$ and $\operatorname{dim} \mathfrak{a u t}(M)=$ $(a+b)^{2}-1$. As $M$ is not a projective space, we may assume $b \geq a \geq 2$. Then the inequality $(a+b)^{2}-1 \geq \frac{a b(a b+1)}{2}$ is equivalent to $3 a b+2 \geq\left(a^{2}-2\right)\left(b^{2}-2\right)$, which holds if and only if $(a, b)=(2,2)$ or $(2,3)$. In both cases, the inequality is strict.

For Type (II), we have $M=\mathbb{S}_{r}, n=\operatorname{dim} M=r(r-1) / 2$ and $\operatorname{dim} \mathfrak{a u t}(M)=r(2 r-$ 1). We may assume $r \geq 5$ as $\mathbb{S}_{4} \simeq \mathbb{Q}^{6}$. Then one checks that $\operatorname{dim} \mathfrak{a u t}(M)<\frac{n(n+1)}{2}$.

For Type (III), we have $M=\operatorname{Lag}(2 \mathrm{r}), n=\operatorname{dim} M=r(r+1) / 2$ and $\operatorname{dim} \mathfrak{a u t}(M)=$ $r(2 r+1)$. We may assume $r \geq 3$ as $\operatorname{Lag}(4) \simeq \mathbb{Q}^{3}$. Then one checks that $\operatorname{dim} \mathfrak{a u t}(M) \leq$ $\frac{n(n+1)}{2}$, with equality if and only if $r=3$. In this case, $S \subset \mathbb{P}^{5}$ is the second Veronese embedding of $\mathbb{P}^{2}$.

For type (IV), we have $M=\mathbb{Q}^{r}$ and $\operatorname{dim} \mathfrak{a u t}(M)=(r+1)(r+2) / 2$, which does not satisfy $\operatorname{dim} \mathfrak{a u t}(M) \leq \frac{r(r+1)}{2}$.

For types (V) and (VI), it is obvious that dim $\mathfrak{a u t}(M) \leq \frac{n(n+1)}{2}$.
Definition 3.4. Let $V$ be a complex vector space and $\mathfrak{g} \subset \operatorname{End}(V)$ a Lie subalgebra. The $k$-th prolongation (denoted by $\mathfrak{g}^{(k)}$ ) of $\mathfrak{g}$ is the space of symmetric multi-linear homomorphisms $A: \operatorname{Sym}^{k+1} V \rightarrow V$ such that for any fixed $v_{1}, \cdots, v_{k} \in V$, the endomorphism $A_{v_{1}, \ldots, v_{k}}: V \rightarrow V$ defined by

$$
v \in V \mapsto A_{v_{1}, \ldots, v_{k}, v}:=A\left(v, v_{1}, \cdots, v_{k}\right) \in V
$$

is in $\mathfrak{g}$. In other words, $\mathfrak{g}^{(k)}=\operatorname{Hom}\left(\operatorname{Sym}^{k+1} V, V\right) \cap \operatorname{Hom}\left(\operatorname{Sym}^{k} V, \mathfrak{g}\right)$.
It is shown in HM05 that for a smooth non-degenerate variety $C \subsetneq \mathbb{P}^{n-1}$, the second prolongation satisfies $\mathfrak{a u t}(\hat{C})^{(2)}=0$.

Examples 3.5. Fix two integers $k \geq 2, m \geq 1$. Let $\Sigma$ be an $(m+2 k)$-dimensional vector space endowed with a skew-symmetric 2 -form $\omega$ of maximal rank. The symplectic Grassmannian $M=\operatorname{Gr}_{\omega}(k, \Sigma)$ is the variety of all $k$-dimensional isotropic subspaces of $\Sigma$, which is not homogeneous if $m$ is odd. Let $W$ and $Q$ be vector spaces of dimensions $k \geq 2$ and $m$ respectively. Let $\mathbf{t}$ be the tautological line bundle over $\mathbb{P} W$. The VMRT $\mathcal{C}_{x} \subset \mathbb{P} T_{x}(M)$ of $\operatorname{Gr}_{\omega}(k, \Sigma)$ at a general point is isomorphic to the projective bundle $\mathbb{P}\left((Q \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2}\right)$ over $\mathbb{P} W$ with the projective embedding given by the complete linear system

$$
H^{0}\left(\mathbb{P} W,\left(Q \otimes \mathbf{t}^{*}\right) \oplus\left(\mathbf{t}^{*}\right)^{\otimes 2}\right)=(W \otimes Q)^{*} \oplus \operatorname{Sym}^{2} W^{*}
$$

By Proposition 3.8 [FH12], we have $\mathfrak{a u t}\left(\hat{\mathcal{C}}_{x}\right) \simeq\left(W^{*} \otimes Q\right) \rtimes(\mathfrak{g l l}(W) \oplus \mathfrak{g l}(Q))$ and $\mathfrak{a u t}\left(\hat{\mathcal{C}}_{x}\right)^{(1)} \simeq \operatorname{Sym}^{2} W^{*}$. This gives that

$$
\operatorname{dim} \mathfrak{a u t}\left(\hat{\mathcal{C}}_{x}\right)=m^{2}+k^{2}+k m \text { and } \operatorname{dim} \mathfrak{a u t}\left(\hat{\mathcal{C}}_{x}\right)^{(1)}=k(k+1) / 2 .
$$

We denote by $\mathfrak{a u t}(\mathcal{C}, x)$ the Lie algebra of infinitesimal automorphisms of $\mathcal{C}$, which consists of germs of vector fields whose local flow preserves $\mathcal{C}$ near $x$. Note that the action of $\operatorname{Aut}^{0}(X)$ on $X$ sends minimal rational curves to minimal rational curves, hence it preserves the VMRT structure, which gives a natural inclusion $\mathfrak{a u t}(X) \subset$ $\mathfrak{a u t}(\mathcal{C}, x)$ for $x \in X$ general.

The following result is a combination of Propositions 5.10, 5.12, 5.14 and 6.13 in [FH12].

ON FANO MANIFOLDS OF PICARD NUMBER ONE WITH BIG AUTOMORPHISM GROUPS 9
Proposition 3.6. Let $X$ be an n-dimensional smooth Fano variety of Picard number one. Assume that the VMRT $\mathcal{C}_{x}$ at a general point $x \in X$ is smooth irreducible and non-degenerate. Then

$$
\operatorname{dim} \mathfrak{a u t}(X) \leq n+\operatorname{dim} \mathfrak{a u t}\left(\hat{\mathcal{C}_{x}}\right)+\operatorname{dim} \mathfrak{a u t}\left(\hat{\mathcal{C}_{x}}\right)^{(1)}
$$

The equality holds if and only if the VMRT structure $\mathcal{C}$ is locally flat, or equivalently if and only if $X$ is an equivariant compactification of $\mathbb{C}^{n}$.

We recall the following result from Theorem 7.5 [FH18].
Theorem 3.7. Let $S \subsetneq \mathbb{P} V$ be an irreducible smooth non-degenerate variety such that $\mathfrak{a u t}(\widehat{S})^{(1)} \neq 0$. Then $S \subset \mathbb{P} V$ is projectively equivalent to one in the following list.
(1) The VMRT of an irreducible Hermitian symmetric space of compact type of rank $\geq 2$.
(2) The VMRT of a symplectic Grassmannian.
(3) A smooth linear section of $\operatorname{Gr}(2,5) \subset \mathbb{P}^{9}$ of codimension $\leq 2$.
(4) A $\mathbb{P}^{4}$-general linear section of $\mathbb{S}_{5} \subset \mathbb{P}^{15}$ of codimension $\leq 3$.
(5) Biregular projections of (1) and (2) with nonzero prolongation, which are completely described in Section 4 of [FH12].

Proposition 3.8. Let $S \subsetneq \mathbb{P} V$ be an irreducible smooth non-degenerate variety such that $\mathfrak{a u t}(\widehat{S})^{(1)} \neq 0$. Let $n=\operatorname{dim} V$. Then
(a) we have $\operatorname{dim} \mathfrak{a u t}(\hat{S})+\operatorname{dim} \mathfrak{a u t}(\hat{S})^{(1)} \leq \frac{n(n-1)}{2}$ unless $S \subset \mathbb{P} V$ is projectively equivalent to the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{2}$ or the natural embedding of $\mathbb{Q}^{n-2} \subset \mathbb{P}^{n-1}(n \geq 3)$.
(b) The equality holds if and only if $S \subset \mathbb{P} V$ is projectively equivalent to the second Veronese embedding of $\mathbb{P}^{2}$ or a general hyperplane section of the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{2}$.

Proof. Consider case (1) in Theorem 3.7, Let $M$ be an IHSS and $S \subset \mathbb{P V}$ its VMRT at a general point. As the VMRT structure is locally flat, we have $\operatorname{dim} \mathfrak{a u t}(\hat{S})+$ $\operatorname{dim} \mathfrak{a u t}(\hat{S})^{(1)}=\operatorname{dim} \mathfrak{a u t}(M)-\operatorname{dim} M=\operatorname{dim} \mathfrak{a u t}(M)-n$ by Proposition 3.6. Now the claim follows from Lemma 3.3.

Consider case (2) in Theorem 3.7, By Example 3.5, we have

$$
\operatorname{dim} \mathfrak{a u t}(\hat{S})+\operatorname{dim} \mathfrak{a u t}(\hat{S})^{(1)}=m^{2}+k^{2}+k m+k(k+1) / 2
$$

with $k \geq 2$ and $n=k m+k(k+1) / 2$. Note that $n \geq k m+3$. Assume first that $m \geq 2$, then we have $m^{2}+k^{2}<(k m+3) k m / 2 \leq n(n-3) / 2$, which gives the claim. Now assume $m=1$, then it is easy to check that $1+k^{2} \leq n(n-3) / 2$ with equality if and only if $(k, m)=(2,1)$. By Lemma 3.6 [FH12], this implies that $S \subset \mathbb{P} V$ is projectively equivalent to a general hyperplane section of the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{2}$.

Consider case (3) in Theorem 3.7. If $S$ is the hyperplane section of $\operatorname{Gr}(2,5)$, then we have $\operatorname{dim} \mathfrak{a u t}(\hat{S})=16$ and $\operatorname{dim} \mathfrak{a u t}(\hat{S})^{(1)}=5$ by Propositions 3.11 and 3.12 in [FH12]. Now assume that $S$ is a codimension 2 linear section of $\operatorname{Gr}(2,5)$, then $\operatorname{dim} \mathfrak{a u t}(\hat{S})=9$ and $\operatorname{dim} \mathfrak{a u t}(\hat{S})^{(1)}=1$ by Lemma 4.6 BFM18]. The claim follows immediately.

Consider case (4) in Theorem 3.7. If $S$ is the hyperplane section of $\mathbb{S}_{5}$, then $\operatorname{dim} \mathfrak{a u t}(\hat{S})=31$ and $\operatorname{dim} \mathfrak{a u t}(\hat{S})^{(1)}=7$ by Propositions 3.9 and 3.10 in [FH12]. Now assume that $S_{k}$ is a $\mathbb{P}^{4}$-general linear section of $\mathbb{S}_{5}$ of codimension $k=2,3$. By

Propositions 4.7 and 4.11 BFM18, we have $\operatorname{dim} \mathfrak{a u t}\left(\hat{S}_{2}\right)=19$ and $\operatorname{dim} \mathfrak{a u t}\left(\hat{S}_{3}\right)=12$. By Theorem 1.1.3 [HM05], we have $\operatorname{dim} \mathfrak{a u t}(\hat{S})^{(1)} \leq \operatorname{dim} V^{*}$, hence $\operatorname{dim} \mathfrak{a u t}\left(\hat{S}_{2}\right)^{(1)} \leq$ 14. By a similar argument as Lemma 4.6 [BFM18], we have $\operatorname{dim} \mathfrak{a u t}\left(\hat{S}_{3}\right)^{(1)}=1$. Now the claim follows immediately.

The case (5) follows from Proposition 3.9.
Proposition 3.9. Let $S \subsetneq \mathbb{P} V$ be an irreducible linearly-normal non-degenerate smooth variety such that $\mathfrak{a u t}(\widehat{S})^{(1)} \neq 0$. Let $L \subset \mathbb{P V}$ be a linear subspace such that the linear projection $p_{L}: \mathbb{P} V \rightarrow \mathbb{P}(V / L)$ maps $S$ isomorphically to $p_{L}(S)$. Assume that $\mathfrak{a u t}\left(p_{L}(S)\right)^{(1)} \neq 0$. Then $\operatorname{dim} \mathfrak{a u t}\left(\widehat{p_{L}(S)}\right)+\operatorname{dim} \mathfrak{a u t}\left(\widehat{p_{L}(S)}\right)^{(1)}<\frac{\ell(\ell-1)}{2}$, where $\ell=\operatorname{dim}(V / L)$.

Proof. By [FH12] (Section 4), $S \subset \mathbb{P} V$ is one of the followings: VMRT of IHSS of type (I), (II), (III) or VMRT of the symplectic Grassmanninans. We will do a case-by-case check based on the computations in FH12] (Section 4).

Consider the case of VMRT of IHSS of type (I). Let $A, B$ be two vector spaces of dimension $a \geq 2$ (resp. $b \geq 2$ ). Then $S \simeq \mathbb{P} A^{*} \times \mathbb{P} B \subset \mathbb{P} \operatorname{Hom}(A, B)$. For a linear subspace $L \subset \operatorname{Hom}(A, B)$, we define $\operatorname{Ker}(L)=\cap_{\phi \in L} \operatorname{Ker}(\phi)$ and $\operatorname{Im}(L) \subset B$ the linear span of $\cup_{\phi \in L} \operatorname{Im}(\phi)$. By Proposition 4.10 [FH12], we have $\left.\mathfrak{a u t} \widehat{p_{L}(S)}\right)^{(1)} \simeq$ $\operatorname{Hom}(B / \operatorname{Im}(L), \operatorname{Ker}(L))$. As $p_{L}$ is an isomorphism from $S$ to $p_{L}(S), \mathbb{P} L$ is disjoint from $\operatorname{Sec}(S)$, hence elements in $L$ have rank at least 3. This implies that $\operatorname{dim} \mathfrak{a u t}\left(\widehat{p_{L}(S)}\right)^{(1)} \leq(b-3)(a-3)$ and $a, b \geq 4$. As $\mathfrak{a u t}\left(\widehat{p_{L}(S)}\right) \subset \mathfrak{a u t}(\hat{S})=$ $\mathfrak{g l}\left(A^{*}\right) \oplus \mathfrak{g l}(B)$, we obtain

$$
\operatorname{dim} \mathfrak{a u t}\left(\widehat{p_{L}(S)}\right)+\operatorname{dim} \mathfrak{a u t} \widehat{\left(p_{L}(S)\right)^{(1)}}<a^{2}+b^{2}+(b-3)(a-3)
$$

Note that $\ell=\operatorname{dim} \operatorname{Hom}(A, B)-\operatorname{dim} L \geq 3(a+b)-9$ by Proposition 4.10 [FH12]. Now it is straightforward to check that $a^{2}+b^{2}+(b-3)(a-3)<\ell(\ell-1) / 2$ since $a+b \geq 8$.

Consider the case of VMRT of IHSS of type (II), then $S=\operatorname{Gr}(2, W)$, where $W$ is a vector space of dimension $r \geq 6$. Let $L \subset \wedge^{2} W$ be a linear subspace. We denote by $\operatorname{Im}(L)$ the linear span of all $\cup_{\phi \in L} \operatorname{Im}(\phi)$, where $\phi \in L$ is regarded as an element in $\operatorname{Hom}\left(W^{*}, W\right)$. By Proposition 4.11 [FH12], we have $\mathfrak{a u t}\left(\widehat{p_{L}(S)}\right)^{(1)} \simeq$ $\wedge^{2}(W / \operatorname{Im}(L))^{*}$. As $p_{L}$ is biregular on $S$, the rank of an element in $L$ is at least 5 , hence $\operatorname{dim} \mathfrak{a u t}\left(\widehat{p_{L}(S)}\right)^{(1)} \leq(r-5)(r-6) / 2$. On the other hand, we have $\ell=$ $\operatorname{dim} \wedge^{2} W-\operatorname{dim} L \geq 6 r-11$ by Proposition 4.11 [FH12]. As $\operatorname{dim} \mathfrak{a u t}\left(\widehat{p_{L}(S)}\right)<r^{2}$, we check easily that $r^{2}+(r-5)(r-6) / 2<\ell(\ell-1) / 2$.

Consider the case of VMRT of IHSS of type (III), then $S=\mathbb{P} W$, where $W$ is a vector space of dimension $r \geq 4$. By Proposition 4.12 [FH12], we have $\mathfrak{a u t}\left(\widehat{p_{L}(S)}\right)^{(1)} \simeq$ $\operatorname{Sym}^{2}(W / \operatorname{Im}(L))^{*}$, which has dimension at most $(r-2)(r-3) / 2$. On the other hand, we have $\ell \geq 3 r-3$ by loc. cit., hence we have $r^{2}+(r-2)(r-3) / 2<\ell(\ell-1) / 2$.

Now consider the VMRT of symplectic Grassmannians. We use the notations in Example 3.5 with $V=(W \otimes Q)^{*} \oplus \operatorname{Sym}^{2} W^{*}$, where $\operatorname{dim} W=k \geq 2$ and $\operatorname{dim} Q=m$. By Lemma 4.19 [FH12], we have $k \geq 3$, hence $k+m \geq 4$. Let $L \subset V$ be a linear subspace, then by Proposition 4.18 [FH12], we have $\ell \geq 3(k+m)-3$ and $\mathfrak{a u t}\left(\widehat{p_{L}(S)}\right)^{(1)} \simeq \operatorname{Sym}^{2}\left(W / \operatorname{Im}_{W}(L)\right)^{*}$ which has dimension at most $k(k+1) / 2$. As $\operatorname{dim} \mathfrak{a u t}(\hat{S})=k^{2}+m^{2}+k m$, we have

$$
\operatorname{dim} \mathfrak{a u t}\left(\widehat{p_{L}(S)}\right)+\operatorname{dim} \mathfrak{a u t}\left(\widehat{p_{L}(S)}\right)^{(1)}<k^{2}+m^{2}+k m+k(k+1) / 2 .
$$

Put $s=k+m$, then $k^{2}+m^{2}+k m+k(k+1) / 2=s^{2}-k m+k(k+1) / 2<3 s^{2} / 2-k m<$ $\ell(\ell-1) / 2$ since $k+m \geq 4$.

Now we can complete the proof of Theorem 1.2.
Proof of Theorem 1.2. By Proposition 3.6, we have

$$
\operatorname{dim} \mathfrak{a u t}(X) \leq n+\operatorname{dim} \mathfrak{a u t}\left(\hat{\mathcal{C}_{x}}\right)+\operatorname{dim} \mathfrak{a u t}\left(\hat{\mathcal{C}_{x}}\right)^{(1)}
$$

If $\mathfrak{a u t}\left(\hat{\mathcal{C}_{x}}\right)^{(1)}=0$, then $\mathcal{C}_{x} \subset \mathbb{P}^{n-1}$ is not a hyperquadric, which implies that $\operatorname{dim} \mathfrak{a u t}\left(\hat{\mathcal{C}_{x}}\right) \leq n(n-1) / 2-2$ by Theorem 1.4. This gives that

$$
\operatorname{dim} \mathfrak{a u t}(X) \leq n+\operatorname{dim} \mathfrak{a u t}\left(\hat{\mathcal{C}_{x}}\right) \leq n+n(n-1) / 2-2<n(n+1) / 2
$$

Now assume $\mathfrak{a u t}\left(\hat{\mathcal{C}_{x}}\right)^{(1)} \neq 0$. If $\mathcal{C}_{x}=\mathbb{P} T_{x} X$ for general $x \in X$, then $X$ is isomorphic to $\mathbb{P}^{n}$ by CMSB02. If $\mathcal{C}_{x} \subset \mathbb{P}_{x} X$ is a hyperquadric $\mathbb{Q}^{n-2}$, then $X \simeq \mathbb{Q}^{n}$ by Mok08. If $\mathcal{C}_{x} \subset \mathbb{P} T_{x} X$ is isomorphic to the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{2}$, then $X \simeq \operatorname{Gr}(2,5)$ by Mok08]. Now assume $\mathcal{C}_{x} \subset \mathbb{P} T_{x} X$ is not one of the previous varieties, then by Proposition 3.8 we have $\operatorname{dim} \mathfrak{a u t}(X) \leq n+n(n-1) / 2=n(n+1) / 2$, which proves claim (a) in Theorem 1.2,

Now assume the equality $\operatorname{dim} \mathfrak{a u t}(X)=n(n+1) / 2$ holds, then by Proposition 3.6, the VMRT structure is locally flat. By Proposition 3.8, $\mathcal{C}_{x} \subset \mathbb{P} T_{x} X$ is either the second Veronese embedding of $\mathbb{P}^{2}$ or a general hyperplane section of $\mathbb{P}^{1} \times \mathbb{P}^{2}$. Note that these are also the VMRT of $\operatorname{Lag}(6)$ and a general hyperplane section of $\operatorname{Gr}(2,5)$, which have locally flat VMRT structure. This implies that $X$ is isomorphic to $\operatorname{Lag}(6)$ or a general hyperplane section of $\operatorname{Gr}(2,5)$ respectively by the Cartan-Fubini extension theorem [HM01].

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