RIGIDITY OF WONDERFUL GROUP COMPACTIFICATIONS UNDER FANO DEFORMATIONS

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Abstract. For a complex connected semisimple linear algebraic group G of adjoint type and of rank n, De Concini and Procesi constructed its wonderful compactification $\bar{G}$, which is a smooth Fano $G \times G$-variety of Picard number n enjoying many interesting properties. In this paper, it is shown that the wonderful compactification $\bar{G}$ is rigid under Fano deformations. Namely, for any family of smooth Fano varieties over a connected base, if one fiber is isomorphic to $\bar{G}$, then so are all other fibers.

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1. Introduction

Throughout this paper, we work over the complex number field. A smooth projective variety X is said *rigid* if for any smooth projective family over a connected base $\mathcal{X} \to B$ with one fiber $\mathcal{X}_{b_0}$ isomorphic to X, then all fibers $\mathcal{X}_b$ are isomorphic to X. For example, the projective space $\mathbb{P}^n$ is rigid, as any Kähler manifold homeomorphic to $\mathbb{P}^n$ is isomorphic to $\mathbb{P}^n$ by a result of Hirzebruch-Kodaira and Yau. But in general, the rigidity is a strong property which is difficult to check. We refer to surveys [S] and [H06] for an account of the history and the development of this problem.

To show the rigidity of a smooth projective variety X, it is natural to prove first that X is *locally rigid*, namely $\mathcal{X}_b \simeq X$ for all b in an open neighborhood of b_0. By Kodaira-Spencer deformation theory, a smooth Fano variety X is locally rigid if and only if $H^1(X, T_X) = 0$. The latter can be effectively computed in many concrete cases. For example, rational homogeneous spaces G/P or more generally regular Fano G-varieties are locally rigid ([BB, Proposition 4.2]).

For a locally rigid variety X, it is rigid if and only if it is *rigid under specializations*, namely if for any smooth projective family over the unit disk $\mathcal{X} \to \Delta$ with $\mathcal{X}_t \simeq X$ for all $t \neq 0$, we have $\mathcal{X}_0 \simeq X$. It is a difficult and subtle problem to prove the rigidity under specializations and there is no cohomological approach to it. Even for rational homogeneous varieties G/P of Picard number one, the rigidity does not always hold. To wit, let B_3/P_2 be the variety of lines on a 5-dimensional smooth hyperquadric $\mathbb{Q}^5$. Pasquier and Perrin constructed in [PP] an explicit family specializing B_3/P_2 to a smooth projective G_2-variety. In [HL2], it is shown that this is the only smooth non-isomorphic specialization of B_3/P_2. It turns out that B_3/P_2 is the only exception among all G/P with Picard number one, as shown by the following

Theorem 1.1 ([H97], [HM98], [HM02], [HM05])**.** *A rational homogeneous variety of Picard number 1 is rigid except for B_3/P_2.*

The main ingredient for the proof is the VMRT theory developed by Hwang and Mok. For a uniruled projective manifold X and a fixed family of minimal rational curves $\mathcal{K}$, the variety of minimal rational tangents (VMRT for short) of X (with respect to $\mathcal{K}$) at a general point x is the closed subvariety $\mathcal{C}_x \subset \mathbb{P}T_xX$ consisting of all tangent directions at x of curves in $\mathcal{K}$ passing through x. The projective geometry of $\mathcal{C}_x \subset \mathbb{P}T_xX$ encodes a lot of global geometry of X and in some cases, we can reconstruct X from its VMRT at general points, which is the case for all G/P of Picard number one by series of works ([M], [HH], [HL1], [HLT]).

Let us recall the main strategy of the proof of Theorem 1.1. As G/P is locally rigid, we need to check the rigidity under specializations $\mathcal{X} \to \Delta$. The family of minimal rational curves on $G/P \simeq \mathcal{X}_t$ deforms to a family of minimal rational curves on $\mathcal{X}_0$. The key step is to show that the VMRT of $\mathcal{X}_0$ at a general point is projectively equivalent to that of G/P, then we can apply the aforementioned theorem to reconstruct G/P from its VMRT. The rational homogeneous variety B_3/P_2 is an exception in Theorem 1.1 precisely because its VMRT is isomorphic to $\mathbb{P}^1 \times \mathbb{Q}^1$, which can be deformed to the Hirzebruch surface $S_{1,3} = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(1) \oplus \mathcal{O}(3))$ as embedded projective subvarieties.

For higher Picard number case, the rigidity problem becomes even more difficult. As a simple example, the surface $\mathbb{P}^1 \times \mathbb{P}^1$ can be specialized to any even-degree Hirzebruch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(-a) \oplus \mathcal{O}(a))$. This is the reason that we will only consider *rigidity under Fano deformations*, namely the fibers of $\mathcal{X} \to B$ are assumed to be Fano. We feel this is the right framework for higher Picard number cases. Note that for Picard number one case, the two notions of rigidity are the same.

One of the nice features of Fano specializations is that they preserve product structures, as shown by the second-named author in [L1]. In particular, a product of Fano manifolds is rigid under Fano deformations if and only if so is each factor. As an immediate corollary, a product of rational homogeneous varieties of Picard number one is rigid under Fano deformations if and only if none of its factors is isomorphic to B_3/P_2.

It is proven in [WW] that complete flag varieties G/B are rigid under Fano deformations, by using the characterization of complete flag varieties as the only projective Fano manifolds whose elementary Mori contractions are all $\mathbb{P}^1$-fibrations. Recently, the second-named author has proven in [L2] the rigidity under Fano deformations of many rational homogeneous varieties of higher Picard numbers by using Cartan connections. However, one should bear in mind that the rigidity under Fano deformations does not always hold, and a typical example of higher Picard number is $\mathbb{P}T_{\mathbb{P}^{2n}}$, which can be specialized to $\mathbb{P}_{\mathbb{P}^{2n}}(\mathcal{N}(1) \oplus \mathcal{O}(2))$, where $\mathcal{N}$ denotes the null-correlation bundle on $\mathbb{P}^{2n}$. It is still an open problem to classify rational homogeneous varieties which are rigid under Fano deformations.

The main purpose of this paper is to study the rigidity under Fano deformations of wonderful compactifications of semisimple linear algebraic groups. For any semisimple adjoint linear algebraic group G of rank n, there exists ([dCP]) a unique smooth Fano $G \times G$-equivariant compactification of G, say $\bar{G}$ with the following properties: (1) the boundary divisor $\partial \bar{G} = \cup_{i=1}^n D_i$ is a union of n smooth divisors with normal crossing; (2) the $G \times G$-orbit closures in $\partial \bar{G}$ are exactly intersections of these boundary divisors. The main result in this paper is the following:

Theorem 1.2. *The wonderful compactification $\bar{G}$ of a semisimple linear algebraic group G of adjoint type is rigid under Fano deformations.*

As shown in [BB, Proposition 4.2], $\bar{G}$ is locally rigid, hence we only need to prove the rigidity under Fano specializations. Write $G = G_1 \times \cdots \times G_k$ into product of simple adjoint algebraic groups, then we have a decomposition of wonderful compactifications $\bar{G} = \bar{G}_1 \times \cdots \times \bar{G}_k$. By the aforementioned result in [L1], $\bar{G}$ is rigid under Fano deformations if and only if so is each factor $\bar{G}_i$. This reduces the proof of Theorem 1.2 to the case where G is simple, a situation which we will focus on from now on. Note that if G is of rank 1, then $\bar{G}$ is isomorphic to $\mathbb{P}^3$, which is rigid. Hence in the following, we always assume that the rank of G is at least 2.

The key ingredient to the proof is again the VMRT theory. For the wonderful compactification $\bar{G}$ of a simple adjoint group, it is shown in [BF] that it has a unique family of minimal rational curves. When G is not of type A, the associated VMRT at a general point of $\bar{G}$ is the unique closed orbit $\mathbb{P}\mathscr{O} \subset \mathbb{P}\mathfrak{g}$ in the projective adjoint representation of G. For type A_n, the VMRT at a general point of $\bar{A}_n$ is projectively equivalent to the projection from a general point of the Segre variety $\mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n^2+2n}$.

Now consider a family of Fano manifolds $\mathcal{X} \to \Delta$ such that $\mathcal{X}_t \simeq \bar{G}$ for $t \neq 0$ and $\mathcal{X}_0$ is Fano. We first show that the family of minimal rational curves on $\bar{G}$ deforms to a family of minimal rational curves on $\mathcal{X}_0$ (Proposition 3.3), by using the uniqueness of family of minimal rational curves on $\bar{G}$. Then the first key step is to show the invariance of VMRT, namely the VMRT of $\mathcal{X}_0$ at a general point is projectively equivalent to that of $\bar{G}$. Note that for G not of type A, the VMRT $\mathbb{P}\mathscr{O}$ of $\bar{G}$ at a general point is a rational homogeneous space of Picard number one. Take a section $\sigma : \Delta \to \mathcal{X}$ through general points, then the normalized VMRTs (which are smooth) along σ give a family of smooth projective varieties, with general fibers isomorphic to $\mathbb{P}\mathscr{O}$. Whence we can apply Theorem 1.1 to obtain the invariance of the normalized VMRT and then the invariance of VMRT (Proposition 3.7). For type A_n, its wonderful compactification $\bar{A_n}$ can be explicitly constructed by successive blowups from $\mathbb{P}\mathrm{End}(\mathbb{C}^{n+1})$. We extend this construction to $\mathcal{X}$ and then show the invariance of VMRT through a careful study (Proposition 3.13).

The above method to prove the invariance of VMRT does not apply to B_3, as in this case $\mathbb{P}\mathscr{O} = B_3/P_2$, which is not rigid. To remedy this, we consider the variety of lines $\mathbf{L}_{z_0} \subset \mathbb{P}^5$ on $\mathcal{C}_{x_0}$ passing through a general point z_0 on it, where $\mathcal{C}_{x_0}$ is the VMRT of $\mathcal{X}_0$ through a general point x_0. It turns out that $\mathbf{L}_{z_0} \subset \mathbb{P}^5$ is a nondegenerate surface of degree 4, which has only two possibilities: either $\mathbb{P}^1 \times \mathbb{Q}^1$ or $S_{1,3}$. For the former case, we conclude that the VMRT of $\mathcal{X}_0$ is isomorphic to B_3/P_2 by using the recognization of B_3/P_2 by its VMRT ([M, Main Theorem]). It remains to exclude the possibility of $\mathbf{L}_{z_0} \simeq S_{1,3}$. To that end, we first give a geometric construction (Proposition 6.4) of the wonderful compactification $\bar{B}_n$ by successive blowups from the spinor variety $\mathrm{OG(2n+1, W_1 \oplus W_2)}$, where (W_i, o_i) are vector spaces of dimension $2n+1$ endowed with a nondegenerate symmetric quadric form o_i. This gives in particular a birational map $\phi : \bar{B}_3 \to \mathbb{S} := \mathrm{OG}(7, \mathbb{C}^{14})$, which extends to a birational morphism $\Phi : \mathcal{X} \to \mathcal{S}$. The key point is to show that $\mathcal{S}$ is an isotrivial family with fibers isomorphic to $\mathbb{S}$. Passing to the variety of lines on VMRT, this gives an injective holomorphic map $S_{1,3} \simeq \mathbf{L}_{z_0} \to \mathbb{P}^1 \times \mathbb{P}^4$, which is a specialization of the natural embedding $\mathbb{P}^1 \times \mathbb{Q}^1 \to \mathbb{P}^1 \times \mathbb{P}^4$. This gives an embedding of $S_{1,3}$ into $\mathbb{P}^4$, which is clearly not possible, concluding the proof of the invariance of VMRT (Proposition 6.18).

Now we borrow an idea from [Par], where the rigidity of odd Lagrangian Grassmannians is proven. The observation is that if $H^1(\mathcal{X}_0, T_{\mathcal{X}_0}) = 0$, then $\mathcal{X}_0$ is locally rigid, hence $\mathcal{X}_0 \simeq \mathcal{X}_t$ for small t and we are done. To show the vanishing of $H^1(\mathcal{X}_0, T_{\mathcal{X}_0})$, we note that $\chi(\mathcal{X}_t, T_{\mathcal{X}_t})$ is invariant and $H^i(\mathcal{X}_t, T_{\mathcal{X}_t}) = 0$ for all $i \geq 2$ by the Nakano vanishing, which gives

$$h^1(\mathcal{X}_0, T_{\mathcal{X}_0}) = h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) - \chi(\mathcal{X}_t, T_{\mathcal{X}_t}) = h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) - 2\dim G.$$

We can now use VMRT theory to bound $h^0(\mathcal{X}_0, T_{\mathcal{X}_0})$, which gives that if G is not of type C, then $h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) \leq 2\dim G + 1$, with equality if and only if the VMRT structure is locally flat. As a consequence, when $\mathfrak{g}$ is not of type C, we have either $\mathcal{X}_0$ is isomorphic to $\bar{G}$ or $\mathcal{X}_0$ is an equivariant compactification of the vector group $\mathbb{G}_a^g$ with $g = \dim G$.

To exclude the latter case, we observe that if $\mathcal{X}_0$ is an equivariant compactification, then its Picard group is freely generated by its boundary divisors. By the invariance of pseudo-effective cones of divisors under Fano deformations (cf. Theorem 3.2), the Picard group of $\bar{G}$ is generated by the boundary divisors of $\bar{G}$. As the boundary divisors of $\bar{G}$ correspond to simple roots of G and $\mathrm{Pic}(\bar{G})$ coincides with weight lattice, this implies that the root lattice and the weight lattice of G are the same, which is not possible except for G_2, F_4 or E_8, whence this proves the rigidity if G is not C_n, G_2, F_4 or E_8 (Theorem 4.9). For G_2, F_4 and E_8, we take another approach to exclude the equivariant compactification case. Let T be a maximal torus of G, and T' the subtorus of dimension two that is annihilated by $\alpha_2, \ldots, \alpha_{n-1}$, where $\alpha_1, \ldots, \alpha_n$ is a set of simple roots in Bourbaki's numbering order. Applying a generalized version of Białynicki-Birula decomposition theorem in [JS], we obtain a family of smooth projective surfaces $\mathcal{Y}'/\Delta$ such that the central fiber is an equivariant compactification of $\mathbb{G}_a^2$ while the general fiber is isomorphic to $\bar{T}'$, the closure of T' in the wonderful compactification $\bar{G}$. The stabilizer $W_{T'}$ of T', under the action of the Weyl group $W(G)$ on T, acts on the family $\mathcal{Y}'/\Delta$ and stabilizes the open orbit of each fiber $\mathcal{Y}'_t$. There is a subgroup W' of $W_{T'}$ such that the set of prime boundary divisors of $\mathcal{Y}'_0$ consists of two W'-orbits, and each orbit contains at least two elements (Proposition 4.18). This is impossible, because there does not exist any equivariant compactification of $\mathbb{G}_a^2$ admitting such a finite group action (Proposition 4.3), whence we conclude the proof of the rigidity for wonderful compactifications of G_2, F_4 and E_8 (Theorem 4.19).

It remains to prove the rigidity under Fano deformations for the wonderful compactification of C_n, for which we use the theory of spherical varieties. Let (W_i, Ω_i) be two symplectic vector spaces of dimension $2n$ and $\mathrm{LG}(2n, W_1 \oplus W_2)$ the Lagrangian Grassmannian. One notices that both $\bar{C}_n$ and $\mathrm{LG}(2n, W_1 \oplus W_2)$ have the same locally flat VMRT-structure. By using the theory of spherical varieties, we show that there exists a birational morphism $\phi : \bar{C}_n \to Z := \mathrm{LG}(2n, W_1 \oplus W_2)/\tau$ which is the composition of successive blowups along explicit strata, where τ is the involution induced from $(1_{W_1}, -1_{W_2}) \in \mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2)$. The morphism ϕ extends to a morphism $\Phi : \mathcal{X} \to \mathcal{Z}$ by the invariance of Mori cones under Fano deformations (cf. Theorem 3.2). We then show that $\mathcal{Z}_0$ is in fact isomorphic to Z, making $\mathcal{Z} \to \Delta$ an isotrivial family, with a $G \times G$-action. As Φ_0 is birational, the vector fields of $G \times G$-action on $\mathcal{Z}_0$ lift to an open subset of $\mathcal{X}_0$, which coincide with vector fields on $\mathcal{X}|_{\Delta^*}$ coming from the $G \times G$-action. By Hartogs extension theorem, these vector fields extend to the whole of $\mathcal{X}$, making $\mathcal{X}_0$ a $G \times G$-variety. It follows that $\mathcal{X}_0$ is a spherical variety under the $G \times G$-action. Then we show its colored cone is the same as that of $\bar{C}_n$, which concludes our proof (Theorem 5.18).

It is worthwhile to point out that the construction of wonderful compactifications of $\bar{B}_n$ by successive blowups from the spinor variety $\mathrm{OG(2n+1, W_1 \oplus W_2)}$ also works for type D. Together with similar results for A_n and C_n, we obtain explicit constructions of wonderful compactifications of all classical types by successive blowups, which could be of independent interest. As pointed out by Michel Brion, there are constructions of smooth log homogeneous compactifications of classical groups by successive blow-ups in [Hu]. It turns out that our construction of $\bar{B}_n$ is the same as that in [Hu], while the other cases can be deduced from constructions in [Hu]. We have kept our construction in this paper as it is more transparent and we hope this makes our paper more self-contained.

The paper is organized as follows: after recalling the study of minimal rational curves on wonderful compactifications of simple adjoint groups from [BF] in Section 2, we prove the invariance of VMRT (except B_3) in Section 3, where the difficulty is to deal with A_n case, as its VMRT is a projected Segre variety. In Section 4, we prove Theorem 1.2 for all G except C_n, while the C_n case is proven in Section 5. In the last section, we prove the invariance of VMRT for B_3.

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2. Minimal rational curves on wonderful compactifications of simple groups

2.1. **Minimal rational curves and geometric structures.** For a uniruled projective manifold X, let $\mathrm{RatCurves}^{\mathrm{n}}(X)$ denote the normalization of the space of rational curves on X (see [Ko, II.2.11]). Every irreducible component $\mathcal{K}$ of $\mathrm{RatCurves}^{\mathrm{n}}(X)$ is a (normal) quasi-projective variety equipped with a quasi-finite morphism to the Chow variety of X; the image consists of the Chow points of irreducible, generically reduced rational curves. There is a universal family $\mathcal{U}$ with projections $\mu' : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$, and μ' is a $\mathbb{P}^1$-bundle (for these results, see [Ko, II.2.11, II.2.15]).

For any $x \in X$, let $\mathcal{U}_x := \mu^{-1}(x)$ and $\mathcal{K}_x := \mu'(\mathcal{U}_x)$. We call $\mathcal{K}$ a *family of minimal rational curves* if $\mathcal{K}_x$ is non-empty and projective for a general point x. There is a rational map $\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}T_xX$ (the projective space of lines in the tangent space at x) that sends any curve which is smooth at x to its tangent direction. The closure of the image of τ_x is denoted by $\mathcal{C}_x$ and called the *variety of minimal rational tangents* (VMRT) at the point x. By [HM04, Thm. 1] and [Ke, Thm. 3.4], composing τ_x with the normalization map $\mathcal{K}_x^{\mathrm{n}} \to \mathcal{K}_x$ yields the normalization of $\mathcal{C}_x$. Also, $\mathcal{K}_x^{\mathrm{n}}$ is a union of components of the variety $\mathrm{RatCurves}^{\mathrm{n}}(x, X)$ defined in [Ko, II.2.11.2], and hence is smooth for $x \in X$ general by [Ko, II.3.11.5]. In this case, we have $\mathcal{U}_x \simeq \mathcal{K}_x^n$. By abuse of notation, we denote by $\mathcal{K}_x$ its normalization, which is then smooth with a birational morphism $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$.

The closure $\mathcal{C} \subset \mathbb{P}TX$ of the union of $\mathcal{C}_x \subset \mathbb{P}T_xX$ is the *VMRT-structure* on X. The natural projection $\mathcal{C} \to X$ is a proper surjective morphism. The VMRT-structure $\mathcal{C}$ is said *locally flat* if for a general $x \in X$, there exists an analytical open subset U of X containing x with an open immersion $\phi : U \to \mathbb{C}^n, n = \dim X$, and a projective subvariety $S \subset \mathbb{P}^{n-1}$ with $\dim S = \dim \mathcal{C}_x$ such that $\phi_* : \mathbb{P}TU \to \mathbb{P}T\mathbb{C}^n$ maps $\mathcal{C}|_U$ into the trivial fiber subbundle $\mathbb{C}^n \times S$ of the trivial projective bundle $\mathbb{P}T\mathbb{C}^n = \mathbb{C}^n \times \mathbb{P}^{n-1}$.

Example 2.1. Let $\mathfrak{g}$ be a simple Lie algebra on which its adjoint group G acts by adjoint action. In the projective space $\mathbb{P}\mathfrak{g}$, there is a unique closed G-orbit $\mathbb{P}\mathscr{O}$, which is a homogeneous Fano contact manifold. When $\mathfrak{g}$ is not of type A, the variety $\mathbb{P}\mathscr{O}$ is of Picard number one, thence of the form G/P for some maximal parabolic subgroup P of G. The contact structure on $\mathbb{P}\mathscr{O}$ gives a corank one subbundle $\mathcal{W} \subset TG/P$ and the isotropy action of P on T_oG/P at the base point o induces an irreducible representation on $\mathcal{W}_o$. It turns out that the VMRT of $\mathbb{P}\mathscr{O}$ is the highest weight variety of this representation. In particular, the VMRT of $\mathbb{P}\mathscr{O}$ is linearly degenerated in this case. For example, when $\mathfrak{g}$ is of type B_n (with $n \geq 3$), the orbit $\mathbb{P}\mathscr{O}$ is isomorphic to B_n/P_2, which is the variety of lines on the smooth hyperquadric $\mathbb{Q}^{2n-1}$. Its VMRT is given by the Segre embedding of $\mathbb{P}^1 \times \mathbb{Q}^{2n-5}$ to a hyperplane in $\mathbb{P}T_oG/P$.

Definition 2.2. Let V be a complex vector space and $\mathfrak{g} \subset \mathrm{End}(V)$ a Lie subalgebra. The *k-th prolongation* (denoted by $\mathfrak{g}^{(k)}$) of $\mathfrak{g}$ is the space of symmetric multi-linear homomorphisms $A : \mathrm{Sym}^{k+1} V \to V$ such that for any fixed $v_1, \cdots, v_k \in V$, the endomorphism $A_{v_1,\ldots,v_k} : V \to V$ defined by

$$v \in V \mapsto A_{v_1,\ldots,v_k,v} := A(v, v_1, \cdots, v_k) \in V$$

is in $\mathfrak{g}$. In other words, $\mathfrak{g}^{(k)} = \mathrm{Hom}(\mathrm{Sym}^{k+1} V, V) \cap \mathrm{Hom}(\mathrm{Sym}^k V, \mathfrak{g})$.

We are interested in the prolongation of infinitesimal automorphisms of a projective subvariety. Let $S \subsetneq \mathbb{P}V$ be a smooth non-degenerate projective subvariety and let $\hat{S} \subset V$ be the corresponding affine cone. Let $\mathfrak{aut}(\hat{S}) \subset \mathfrak{gl}(V)$ be the Lie algebra of infinitesimal linear automorphisms of $\hat{S}$, which is the subalgebra of $\mathfrak{gl}(V)$ preserving $\hat{S}$. It is shown in [HM05] that the second prolongation of $\mathfrak{aut}(\hat{S})$ always vanishes, namely $\mathfrak{aut}(\hat{S})^{(2)} = 0$.

For a uniruled projective manifold X with a VMRT structure $\mathcal{C} \subset \mathbb{P}TX$, we denote by $\mathfrak{aut}(\mathcal{C}, x)$ the Lie algebra of infinitesimal automorphisms of $\mathcal{C}$, which consists of germs of vector fields on X whose local flow preserves $\mathcal{C}$ near x. Note that the action of $\mathrm{Aut}^0(X)$ on X sends minimal rational curves to minimal rational curves, hence it preserves the VMRT structure, which gives a natural inclusion $\mathfrak{aut}(X) \subset \mathfrak{aut}(\mathcal{C}, x)$ for $x \in X$ general.

The following result is a combination of Propositions 5.10, 5.12 and 5.14 in [FH].

Proposition 2.3. *Let X be a smooth Fano variety. Assume that the VMRT $\mathcal{C}_x \subset \mathbb{P}T_xX$ at a general point $x \in X$ is smooth irreducible and non-degenerate. Then*

$$\dim \mathfrak{aut}(\mathcal{C}, x) \leq \dim X + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x) + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)}.$$

The equality holds if and only if the VMRT structure $\mathcal{C}$ is locally flat.

Example 2.4. An irreducible Hermitian symmetric space (IHSS for short) is a rational homogeneous variety G/P of Picard number one such that the isotropic representation of P on $T_o(G/P)$ is irreducible. Assume that G/P is different from projective spaces, namely G/P is of rank at least two. The highest weight variety of this representation is the VMRT $\mathcal{C}_o$ of G/P at the base point o. The VMRT-structure on G/P is locally flat and the equality in Proposition 2.3 holds. Furthermore, in this case we have $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} \simeq T_o^*(G/P)$. The rank of G/P is the least number r such that a general point of $\mathbb{P}T_oG/P$ is contained in the linear span of r points on $\mathcal{C}_o$. The following is the list of IHSS with their VMRT and ranks.

IHSS G/P	Gr(a, a+b)	D_n/P_n	C_n/P_n	$\mathbb{Q}^r$	E_6/P_1	E_7/P_7
VMRT $\mathcal{C}_o$	$\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$	$\mathrm{Gr}(2, n)$	$\mathbb{P}^{n-1}$	$\mathbb{Q}^{r-2}$	D_5/P_5	E_6/P_1
$\mathcal{C}_o \subset \mathbb{P}T_o(G/P)$	Segre	Plücker	second Veronese	Hyperquadric	Spinor	Severi
rank of G/P	min $\{a, b\}$	$[\frac{n}{2}]$	n	2	2	3

Definition 2.5. Let M be an m-dimensional complex manifold and $G \subset \mathrm{GL}(\mathbb{C}^m)$ a complex Lie subgroup.

(i) The *frame bundle* of M is the principal $\mathrm{GL}(\mathbb{C}^m)$-bundle $\mathcal{F}(M)$, whose fiber at a point $x \in M$ is $\mathcal{F}(M)_x = \mathrm{Isom}(\mathbb{C}^m, T_xM)$.

(ii) A *G-structure* on M is a G-principal subbundle $\mathcal{G}$ of $\mathcal{F}(M)$.

(iii) A G-structure $\mathcal{G}$ on M is said *locally flat* if for $x \in M$, there exists an analytic open subset $U_x \subset M$ such that the restricted G-structure $\mathcal{G}|_{U_x}$ is equivalent to the trivial G-structure on some open subset of $\mathbb{C}^m$.

Consider a uniruled projective manifold X of dimension m with a minimal rational component $\mathcal{K}$ of minimal rational curves. Assume there exists a subvariety $S \subset \mathbb{P}^{m-1}$ such that the VMRT $\mathcal{C}_x$ is projectively equivalent to $S \subset \mathbb{P}^{m-1}$ for all x in an open subset X^o of X. In this case, the VMRT structure $\mathcal{C} \subset \mathbb{P}TX$ induces a G-structure $\mathcal{G}$ on X^o, given by $\mathcal{G}_x = \mathrm{Isom}(\hat{S}, \hat{\mathcal{C}}_x)$, where G is the linear automorphism group of $\hat{S} \subset \mathbb{C}^m$. One notices that the local flatness of VMRT-structure and that of its associated G-structure are equivalent.

Example 2.6. Let G/P be an IHSS of rank ≥ 2 with $G = \mathrm{Aut}^0(G/P)$. Let $G_0 \subset \mathrm{GL}(T_oG/P)$ be the image of the representation of P. Then G/P carries naturally a G_0-structure $\mathcal{G}_o$ coming from the VMRT-structure, which is locally flat by Example 2.4. Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the filtration associated to P. It is known that $\mathfrak{g}_0$ is the Lie algebra of G_0, $\mathfrak{p} = \mathfrak{g}_1 \oplus \mathfrak{g}_0$, $\mathfrak{g}_1 = \mathfrak{aut}(\hat{\mathcal{C}}_s)^{(1)}$ and $\mathfrak{g}_{-1} \simeq T_oG/P$. It follows from [FH, Proposition 5.14] that $\mathfrak{aut}(\mathcal{G}_o, o) \simeq \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and the set of vector fields vanishing at o is isomorphic to $\mathfrak{p}$.

Remark 2.7. There is a cohomological way to detect the local flatness of a G-structure. To wit, for a G-structure $\mathcal{G}$, we can define certain vector-valued functions $c^k, k = 0, 1, 2, \cdots$ on $\mathcal{G}$, the vanishing of which implies the local flatness of $\mathcal{G}$ (cf. [FH, Theorem 5.11]). The G_0-structure on an IHSS G/P of rank ≥ 2 is locally flat, and the corresponding functions c^k vanish (cf. proof of Proposition 5.12 in [FH]).

2.2. **Wonderful compactifications of simple groups.** Let us first recall some basic constructions and properties of wonderful compactifications of simple adjoint groups from [BK] and [dCP].

Let G be a simple linear algebraic group of adjoint type and of rank n, and let $\mathfrak{g}$ be the corresponding Lie algebra. Fix a Borel subgroup $B \subset G$ as well as a maximal torus $T \subset B$. We denote by R the root system of (G, T) and by $R^+ \subset R$ the subset of positive roots consisting of roots of B. The corresponding set of simple roots is denoted by $\{\alpha_1, \cdots, \alpha_n\}$, where we use Bourbaki's numbering order for simple roots. The half-sum of positive roots is denoted by $\boldsymbol{\rho}$. Let $\boldsymbol{\theta}$ be the highest root of R. We denote by $P_i \subset G$ the standard maximal parabolic subgroup corresponding to α_i. Then G/P_i is a rational homogeneous variety of Picard number one.

We denote by Λ the weight lattice, with the submonoid Λ^+ of dominant weights, and the fundamental weights $\omega_1, \ldots, \omega_n$. For any $\lambda \in \Lambda^+$, we denote by V_λ the irreducible representation of the simply-connected cover of G with highest weight λ. This gives a projective representation

$$\varphi_\lambda : G \to \mathrm{PGL}(V_\lambda).$$

Moreover, the G-orbit of the highest weight line in V_λ yields the unique closed orbit in the projectivization $\mathbb{P}V_\lambda$; it is isomorphic to G/P_λ, where the parabolic subgroup P_λ only depends on the type of λ, i..e., the set of simple roots that are orthogonal to that weight.

Note that we have a natural open embedding $\mathrm{PGL}(V_\lambda) \subset \mathbb{P}\,\mathrm{End}\, V_\lambda$. The closure of the image of φ_λ in $\mathbb{P}\,\mathrm{End}\, V_\lambda$ will be denoted by X_λ. This is a projective variety on which $G \times G$ acts via its action on $\mathbb{P}\,\mathrm{End}\, V_\lambda$ by left and right multiplication.

When the dominant weight λ is regular, X_λ turns out to be smooth and independent of the choice of λ; this defines the wonderful compactification X of G, which is sometimes denoted by $\bar{G}$. The identity component of $\mathrm{Aut}(X)$ is $G\times G$. The boundary $\partial X := X\backslash G$ is a union of n smooth irreducible divisors $D_1, \cdots, D_n$ with simple normal crossings. The $G\times G$-orbits in X are indexed by the subsets of $\{1,\dots,n\}$, by assigning to each such subset I the unique open orbit in the partial intersection $D_I := \cap_{i\in I} D_i$. The orbit closure D_I is equipped with a $G\times G$-equivariant fibration $f_I : D_I \to G/P_\lambda^- \times G/P_\lambda$, where $\lambda := \sum_{i\in I}\omega_i \in \Lambda^+$ and P_λ^- is the opposite parabolic subgroup of P_λ. The fiber of f_I at the base point is isomorphic to the wonderful compactification of the adjoint group of $L_I := P_\lambda \cap P_\lambda^-$ (a Levi subgroup of both). In particular, there exists a unique closed $G\times G$-orbit $D_{1,2,\cdots,n} := \cap_{i=1}^n D_i$, which is isomorphic to $G/B^- \times G/B$.

For an arbitrary λ, the variety X_λ is usually singular. The homomorphism φ_λ extends to a $G\times G$-equivariant morphism $X \to X_\lambda$ that we shall still denote by φ_λ. The pull-backs $\mathcal{L}_X(\lambda) := \varphi_\lambda^*\mathcal{O}_{\mathbb{P}\,\mathrm{End}\,V_\lambda}(1)$, $\lambda \in \Lambda^+$, are exactly the globally generated line bundles on X; moreover, $\mathcal{L}_X(\lambda)$ is ample if and only if λ is regular. In particular, X admits a unique minimal ample line bundle, namely, $\mathcal{L}_X(\boldsymbol{\rho})$. The assignment $\lambda \mapsto \mathcal{L}_X(\lambda)$ extends to an isomorphism $\Lambda \overset{\cong}{\to} \mathrm{Pic}(X)$.

We shall index the boundary divisors so that $\mathcal{O}_X(D_i) = \mathcal{L}_X(\alpha_i)$ for $i = 1,\dots,n$. The anti-canonical bundle of X is given by $-K_X = \mathcal{L}_X(2\boldsymbol{\rho} + \sum_i \alpha_i)$, which is in particular ample, hence X is Fano. By [BB], X satisfies $H^1(X, T_X) = 0$, hence it is locally rigid.

Example 2.8. If G is of type A_1, i.e. $G = \mathrm{PGL}(\mathbb{C}^2)$, its wonderful compactification X is the projective space $\mathbb{P}(\mathrm{End}(\mathbb{C}^2)) = \mathbb{P}^3$. In this case, $\mathcal{L}_X(\omega_1) = \mathcal{O}_{\mathbb{P}^3}(1)$, $\mathcal{O}_X(D_1) = \mathcal{O}_{\mathbb{P}^3}(2)$, and $-K_X = \mathcal{O}_{\mathbb{P}^3}(4)$. Furthermore, the projective space is known to be rigid by the theorem of Hirzebruch-Kodaira and Yau.

2.3. **Minimal rational curves on wonderful group compactifications.** For any $\alpha \in R$, we denote by U_α the corresponding root subgroup of G (with Lie algebra the root subspace $\mathfrak{g}_\alpha \subset \mathfrak{g}$) and by C_α the closure of U_α in X. This gives a rational curve on X, which is in general not minimal. The main result in [BF] shows that for the highest root $\boldsymbol{\theta}$, the deformations of $C_{\boldsymbol{\theta}}$ form the unique minimal rational component of X. More precisely, we have

Theorem 2.9 ([BF]). *Let X be the wonderful compactification of a simple algebraic group G of adjoint type. Let $e \in G$ be the identity element and $\boldsymbol{\theta}$ the highest root. Then*
(i) *$C_{\boldsymbol{\theta}}$ is the unique B-stable irreducible curve on X through e.*
(ii) *There exists a unique family of minimal rational curves $\mathcal{K}$ on X, and it consists of deformations of $C_{\boldsymbol{\theta}}$. Moreover, $\mathcal{K}_e$ is smooth and the normalization map $\tau : \mathcal{K}_e \to \mathcal{C}_e$ is an isomorphism.*
(iii) *$\mathcal{C}_e$ is the unique closed G-orbit in $\mathbb{P}\mathfrak{g}$, if G is not of type A.*
(iv) *When G is of type A_n, so that $G = \mathrm{PGL}(V)$ for a vector space V of dimension $n+1$, the VMRT $\mathcal{C}_e$ is the image of $\mathbb{P}V \times \mathbb{P}V^*$ under the Segre embedding $\mathbb{P}V \times \mathbb{P}V^* \to \mathbb{P}\,\mathrm{End}(V)$, followed by the projection $\mathbb{P}\,\mathrm{End}(V) \dashrightarrow \mathbb{P}(\mathrm{End}(V)/\mathbb{C}\mathrm{id}) \simeq \mathbb{P}\mathfrak{g}$.*

The following is a reformulation of Remark 3.6 in [BF].

Corollary 2.10. *For any irreducible curve $C \subseteq X$ such that $C \not\subseteq \partial X$ and for any nef line bundle L on X, we have $L \cdot C_{\boldsymbol{\theta}} \leq L \cdot C$. If L is moreover ample, then $L \cdot C_{\boldsymbol{\theta}} = L \cdot C$ if and only if $C \in \mathcal{K}$.*

Proof. We may assume C passes through the point $e \in G \subset X$. By Remark 3.6 of [BF], C is rationally equivalent to an effective B-stable 1-cycle C_1 through e, which is equal to $mC_{\boldsymbol{\theta}} + C'$ for some positive integer m and some effective B-stable cycle C' not containing e by Theorem 2.9(i). Since L is nef, we have $L \cdot C = mL \cdot C_{\boldsymbol{\theta}} + L \cdot C' \geq mL \cdot C_{\boldsymbol{\theta}}$. When L is moreover ample, $L \cdot C' = 0$ if and only if $C' = 0$, completing the proof. □

Let $\mathscr{O} \subset \mathfrak{g}$ be the minimal nilpotent orbit, then $\mathbb{P}\mathscr{O} \subset \mathbb{P}\mathfrak{g}$ is the unique closed G-orbit, which is the VMRT of X at a general point if $\mathfrak{g}$ is not of type A by Theorem 2.9 (iii). For $x \in \mathscr{O}$, the tangent space $T_x\mathscr{O}$ is naturally identified with $[\mathfrak{g}, x]$. The variety of tangential lines of $\mathscr{O} \subset \mathfrak{g}$ is given by

$$\mathcal{T}_{\mathscr{O}} = \{x \wedge [z, x] | x \in \mathscr{O}, z \in \mathfrak{g}\} \subset \wedge^2 \mathfrak{g}.$$

Lemma 2.11. *The variety of tangential lines $\mathcal{T}_{\mathscr{O}} \subset \wedge^2\mathfrak{g}$ is non-degenerate.*

Proof. Note that for $\mathfrak{g} = \mathfrak{sl}_2$, $\mathbb{P}\mathscr{O} \subset \mathbb{P}\mathfrak{g}$ is the conic curve in $\mathbb{P}^2$, whose variety of tangential lines is non-degenerate. Now consider the general case. For any $x \in \mathscr{O}$, there exists an $\mathfrak{sl}_2$ triplet (x, y, h) by the Jacobson-Morozov theorem, namely the Lie sub algebra $\mathfrak{l} := \mathbb{C}\langle x, y, h\rangle$ is isomorphic to $\mathfrak{sl}_2$. Let $\mathfrak{n} \subset \mathfrak{l}$ be the set of all non-zero nilpotent elements, then the variety of tangential lines of $\mathfrak{n}$ is non-degenerate in $\wedge^2\mathfrak{l}$. As a consequence, the vector $x \wedge y$ is in the linear span of $\mathcal{T}_{\mathfrak{n}}$, hence also in the linear span of $\mathcal{T}_{\mathscr{O}} \subset \wedge^2\mathfrak{g}$. As $\mathcal{T}_{\mathscr{O}}$ is G-invariant, we get that $G \cdot (x \wedge y)$ is contained in the linear span of $\mathcal{T}_{\mathscr{O}}$.

On the other hand, $G \cdot (x, y) \subset \mathscr{O} \times \mathscr{O}$ is dense by [KY] (p. 69). In particular, $G \cdot (x \wedge y)$ is dense in $\{x \wedge x' | x, x' \in \mathscr{O}\}$. As $\mathscr{O} \subset \mathfrak{g}$ is non-degenerate, the set $\{x \wedge x' | x, x' \in \mathscr{O}\}$ is non-degenerate in $\wedge^2\mathfrak{g}$. This implies that $G \cdot (x \wedge y)$ is non-degenerate in $\wedge^2\mathfrak{g}$, concluding the proof. □

To conclude this section, we prove a geometrical property of the VMRT of wonderful compactifications. For a smooth projective subvariety $Z \subset \mathbb{P}V$, the variety of tangential lines of Z is the subvariety $\mathcal{T}_Z \subset \mathrm{Gr}(2, V) \subset \mathbb{P}\wedge^2 V$ consisting of tangential lines of Z.

Proposition 2.12. *Let $Z \subset \mathbb{P}\mathfrak{g}$ be the VMRT at a general point of the wonderful compactification X. Then the variety of tangential lines $\mathcal{T}_Z \subset \mathbb{P}\wedge^2 \mathfrak{g}$ is linearly non-degenerate.*

Proof. By Theorem 2.9, we have $\mathbb{P}\mathscr{O} \subset Z \subset \mathbb{P}\mathfrak{g}$ in all cases. Then the variety of tangential lines $\mathcal{T}_Z \subset \mathbb{P}\wedge^2 \mathfrak{g}$ is linearly non-degenerate by Lemma 2.11. □

The relevance of this property to us is the following integrability result from [HM98, Proposition 9].

Proposition 2.13. *Let M be a uniruled quasi-projective manifold and $\mathcal{K}$ a family of minimal rational curves on M. Let $\mathcal{C}_x \subset \mathbb{P}T_xM$ be the VMRT at a general point and $W_x \subset \mathbb{P}T_xM$ its linear span. Assume that the variety of tangential lines of $\mathcal{C}_x \subset W_x$ is linearly non-degenerate in $\mathbb{P}\wedge^2 W_x$. Then the distribution $\mathcal{W}$ defined by W_x is integrable on an open dense subset of M.*

3. Invariance of varieties of minimal rational tangents

3.1. **Rigidity properties of Fano deformations.** Let X be a normal projective variety. Consider the $\mathbb{R}$-space $N^1(X) := (\mathrm{Pic}(X)/\equiv) \otimes \mathbb{R}$, where $\equiv$ is the numerical equivalence. The *nef cone* $\mathrm{Nef}(X) \subset N^1(X)$ is the closure of the cone spanned by ample classes. The closure of the cone spanned by effective classes in $N^1(X)$ is the *pseudo-effective cone* $\mathrm{PEff}(X) \subset N^1(X)$. The *movable cone* $\mathrm{Mov}(X) \subset N^1(X)$ is the closure of the cone spanned by classes of divisors moving in a linear system with no fixed components. The inclusion relation among these cones is given by

$$\mathrm{Nef}(X) \subset \mathrm{Mov}(X) \subset \mathrm{PEff}(X) \subset N^1(X).$$

The *Mori cone* $\overline{\mathrm{NE}(X)} \subset N_1(X)$ is the dual of the nef cone $\mathrm{Nef}(X) \subset N^1(X)$, which by Kleiman's criterion is the closure of the cone spanned by classes of effective curves.

When X is a smooth Fano variety, all these cones are rational polyhedral cones.

Example 3.1. Let X be the wonderful compactification of a simple algebraic group G. The Picard group $\mathrm{Pic}(X)$ is identified with the weight lattice Λ of G. By [B07, Section 2], the extremal rays of the pseudo-effective cone $\mathrm{PEff}(X)$ are generated by simple roots, while those of the nef cone $\mathrm{Nef}(X)$ are generated by fundamental weights. In particular, the nef cone $\mathrm{Nef}(X)$ is then identified with the positive Weyl chamber.

It turns out that these cones behave well under Fano deformations, which follows from a series of works of Siu, Wiśniewski and de Fernex-Hacon. We refer to the survey [dFH] and the references therein for more details.

Theorem 3.2 ([dFH])**.** *Let $\pi : \mathcal{X} \to \Delta$ be a holomorphic family of Fano manifolds. Then*

(i) $N^1(\mathcal{X}/\Delta) \simeq N^1(\mathcal{X}_t) \simeq H^2(\mathcal{X}_t, \mathbb{R})$ *and* $\mathrm{Pic}(\mathcal{X}/\Delta) \simeq \mathrm{Pic}(\mathcal{X}_t)$ *for any* $t \in \Delta$.

(ii) *The nef cones, movable cones, pseudo-effective cones are constant in the family under the isomorphism in (i).*

(iii) *For any* $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}/\Delta)$ *and any* $t \in \Delta$, *the homomorphism* $H^0(\mathcal{X}, \mathcal{L}) \to H^0(\mathcal{X}_t, \mathcal{L}|_{\mathcal{X}_t})$ *is surjective, and thus* $h^0(\mathcal{X}_t, \mathcal{L}|_{\mathcal{X}_t}) = h^0(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$.

3.2. **Invariance of VMRT for general cases.** Let $\pi : \mathcal{X} \to \Delta \ni 0$ be a holomorphic family of smooth Fano varieties such that $\mathcal{X}_t \simeq X$ for all $t \in \Delta^* := \Delta \setminus 0$, where X is the wonderful compactification of a simple algebraic group G of adjoint type.

Proposition 3.3. *Let $\mathcal{K}$ be the irreducible component of the relative Chow scheme of $\mathcal{X}/\Delta$ containing the unique minimal rational component of $\mathcal{X}_t$ for $t \neq 0$. Then $\mathcal{K}^0 := \{[C] \in \mathcal{K} | C \subset \mathcal{X}_0\}$ is a minimal rational component of $\mathcal{X}_0$.*

Proof. As a general minimal rational curve C_t on $\mathcal{X}_t$ is free on $\mathcal{X}_t$, it is also free on $\mathcal{X}$, hence its deformation covers $\mathcal{X}_0$. Take a family $\{C_t\}_{t \in \Delta^*}$ of such curves deforming to $C_0 \subset \mathcal{X}_0$ such that C_0 passes through a very general point x of $\mathcal{X}_0$. We claim that C_0 is a minimal rational curve on $\mathcal{X}_0$.

Take a relative ample line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}/\Delta)$. For all t, the line bundle $\mathcal{L}_t := \mathcal{L}|_{\mathcal{X}_t}$ is ample on $\mathcal{X}_t$ and $\mathcal{L}_t \cdot C_t = \mathcal{L}_0 \cdot C_0$. Let $C_0' \subset C_0$ be an irreducible reduced component of

C_0 through x, then C_0' is a free rational curve on $\mathcal{X}_0$ as $x \in \mathcal{X}_0$ is very general. It follows that C_0' is also free on $\mathcal{X}$, hence it deforms to C_t' on $\mathcal{X}_t$ such that

$$\mathcal{L}_t \cdot C_t' = \mathcal{L}_0 \cdot C_0' \leq \mathcal{L}_0 \cdot C_0 = \mathcal{L}_t \cdot C_t.$$

By Corollary 2.10, $\mathcal{L}_t \cdot C_t' \geq \mathcal{L}_t \cdot C_t$, implying that $C_0 = C_0'$ is irreducible and reduced.

In a similar way, one shows that C_0 cannot break into a cycle passing through x with several components having lower intersection numbers with $\mathcal{L}_0$, hence C_0 is a minimal rational curve. □

The first key step to the proof of Theorem 1.2 is the following:

Theorem 3.4. *Let G be a simple linear algebraic group of adjoint type and X its wonderful compactification. Let $\pi : \mathcal{X} \to \Delta$ be a family of Fano manifolds such that $\mathcal{X}_t \simeq X$ for all $t \neq 0$. Let $x \in \mathcal{X}_0$ be a general point and $\mathcal{K}^0$ the family of minimal rational curves on $\mathcal{X}_0$ constructed by Proposition 3.3. Then the VMRT of $\mathcal{X}_0$ at x is projectively equivalent to the VMRT of X at a general point.*

The case of type A will be proved in the next subsection (Proposition 3.13), by using the explicit construction of the wonderful compactification $\bar{A}_n$ by successive blowups, while we defer the case of type B_3 to the last section (Proposition 6.18), as its proof is more involved. For all other types, we will use the following result of deformation rigidity of rational homogeneous contact manifolds of Picard number one, which is a special case of Theorem 1.1.

Theorem 3.5 ([H97])**.** *Let $\mathfrak{g}$ be a simple Lie algebra, which is neither of type A nor of type B_3. Let $\mathbb{P}\mathscr{O} \subset \mathbb{P}\mathfrak{g}$ be the projectivized minimal nilpotent orbit, which is a homogeneous contact manifold of Picard number one. Let $\mathcal{Y} \to \Delta$ be a smooth projective family such that $\mathcal{Y}_t \cong \mathbb{P}\mathscr{O}$ for all $t \neq 0$. Then $\mathcal{Y}_0 \cong \mathbb{P}\mathscr{O}$.*

Corollary 3.6. *Assume $\mathfrak{g}$ is neither of type A nor of type B_3. Let $x \in \mathcal{X}_0$ be a general point and $\mathcal{K}^0$ the family of minimal rational curves on $\mathcal{X}_0$ constructed by Proposition 3.3. Then the normalized Chow space $\mathcal{K}_x$ of $\mathcal{X}_0$ at x is isomorphic to $\mathbb{P}\mathscr{O}$.*

Proof. Take a general section $\sigma : \Delta \to \mathcal{X}$ of π passing through the general point x in $\mathcal{X}_0$. Shrinking Δ if necessary, we can assume that $\sigma(t) \notin \partial\mathcal{X}_t$ for each $t \neq 0$. The normalized Chow spaces $\mathcal{K}_{\sigma(t)}$ along this section gives a family of smooth projective varieties such that $\mathcal{K}_{\sigma(t)} \simeq \mathbb{P}\mathscr{O}$ for $t \neq 0$ by Theorem 2.9. Now the claim follows from Theorem 3.5. □

Proposition 3.7. *Assume $\mathfrak{g}$ is neither of type A nor of type B_3. Let $x \in \mathcal{X}_0$ be a general point and $\mathcal{K}^0$ the family of minimal rational curves on $\mathcal{X}_0$ constructed by Proposition 3.3. Then the VMRT of $\mathcal{X}_0$ at x is projectively equivalent to $\mathbb{P}\mathscr{O} \subset \mathbb{P}\mathfrak{g}$.*

Proof. Take a general section $\sigma : \Delta \to \mathcal{X}$ of π passing through the general point x in $\mathcal{X}_0$ such that $\sigma(t) \notin \partial\mathcal{X}_t$ for each $t \neq 0$. Then for $t \neq 0$, the following map

$$\mathcal{K}_{\sigma(t)} \simeq \mathcal{C}_{\sigma(t)} \hookrightarrow \mathbb{P}T_{\sigma(t)}\mathcal{X}_t$$

is induced by the linear system $\mathcal{O}_{\mathbb{P}\mathscr{O}}(1)$ (or $\mathcal{O}_{\mathbb{P}\mathscr{O}}(2)$ in the case of type C) under the identification $\mathcal{C}_{\sigma(t)} \simeq \mathbb{P}\mathscr{O}$. This implies that the map $\mathcal{K}_x \to \mathcal{C}_x \hookrightarrow \mathbb{P}T_x\mathcal{X}_0$ is given by a sublinear system of $\mathcal{O}_{\mathbb{P}\mathscr{O}}(1)$ (or $\mathcal{O}_{\mathbb{P}\mathscr{O}}(2)$ in the case of type C). Let $W_x \subset \mathbb{P}T_x\mathcal{X}_0$ be the linear span of

$\mathcal{C}_x$. Then $\mathcal{C}_x \subset W_x$ is a linear projection of $\mathbb{P}\mathscr{O} \subset \mathbb{P}\mathfrak{g}$. If $W_x = \mathbb{P}T_x\mathcal{X}_0$, then the sublinear system is complete and the claim follows. It remains to show that $W_x = \mathbb{P}T_x\mathcal{X}_0$.

As the variety of tangential lines of $\mathbb{P}\mathscr{O} \subset \mathbb{P}\mathfrak{g}$ is linearly non-degenerate by Proposition 2.12, so is the variety of tangential lines of $\mathcal{C}_x \subset W_x$. By Proposition 2.13, the distribution $\mathcal{W}$ defined by W_x is integrable on an open dense subset of $\mathcal{X}_0$, which induces a dominant rational fibration $f : \mathcal{X}_0 \dashrightarrow B$. For $C \in \mathcal{K}^0$ general on $\mathcal{X}_0$, the projective curve C is disjoint from the indeterminant locus of f, and $f(C)$ is a single point.

If $W_x \neq \mathbb{P}T_x\mathcal{X}_0$, then $\dim B > 0$. Take an ample divisor H_B on B, then $H := f^*H_B$ is a movable divisor on $\mathcal{X}_0$ such that $H \cdot C = 0$ for $C \in \mathcal{K}$. By Theorem 3.2, this gives a rational fibration $f' : X \dashrightarrow B'$ with $\dim B' > 0$ such that the corresponding movable divisor H' satisfies $H' \cdot C_{\boldsymbol{\theta}} = 0$, i.e. f' contracts general minimal rational curves on X. As the VMRT on X is non-degenerate, any two general points are chain connected by minimal rational curves, hence they are contracted to the same point by f'. This implies $\dim B' = 0$, a contradiction. □

3.3. **Invariance of VMRT for A_n.** Fix an integer $n \geq 2$. Let V be a vector space of dimension $n+1$ and $G = \mathrm{PGL}(V)$. Let $Z = \mathbb{P}\mathrm{End}(V)$ and $Z_i \subset Z$ the locus of elements in $\mathrm{End}(V)$ of rank $\leq i$, where $i = 1, \cdots, n$. Note that $Z_1 \simeq \mathbb{P}V \times \mathbb{P}V^*$ and the embedding $Z_1 \subset \mathbb{P}\mathrm{End}(V)$ is the Segre embedding. The subvariety Z_n is the determinant hypersurface, which is of degree $n+1$.

The wonderful compactification X of G is given by the composition of successive blowups $\phi : X \to Z$ along the strict transforms of Z_i, from the smallest Z_1 to the biggest Z_{n-1}. We denote by $D_1, \cdots, D_{n-1}$ the exceptional divisors and by D_n the strict transform of Z_n. Let $\mathcal{K}$ be the family of minimal rational curves on X, whose members are strict transforms of lines in Z intersecting Z_1.

The following is straight-forward.

Proposition 3.8. *(1) The secant variety of Z_1 is Z_2, which is different from Z if $n \geq 2$.*
(2) We have $H^2(X, \mathbb{Q}) = \oplus_{i=1}^n \mathbb{Q}[D_i]$.
(3) For any $C \in \mathcal{K}$, we have $D_1 \cdot C = D_n \cdot C = 1$ and $D_j \cdot C = 0$ for $j = 2, \cdots, n-1$.
(4) The pseudo-effective cone $\mathrm{PEff(X)} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathbb{R}^+\mathrm{D_i}$ is simplicial.
(5) The pull-back ϕ^*H of any hyperplane $H \subset Z$ is equal to $\sum_{i=1}^n (1 - \frac{i}{n+1}) D_i$ in $H^2(X, \mathbb{Q})$.

Let $\pi : \mathcal{X} \to \Delta$ be a Fano deformation of X such that $\mathcal{X}_t \simeq X$ for all $t \neq 0$. Let $\mathcal{K}$ be the minimal rational component on $\mathcal{X}$, then for $x \in \mathcal{X}_t$ general, we have $\mathcal{K}_x \xrightarrow{\simeq} \mathcal{C}_x \simeq \mathbb{P}V \times \mathbb{P}V^*$.

Lemma 3.9. *For $x \in \mathcal{X}_0$ general, we have $\mathcal{K}_x \simeq \mathbb{P}V \times \mathbb{P}V^*$.*

Proof. Let $\mathcal{U} \to \mathcal{K}$ be the universal family and $\mathcal{U} \to \mathcal{X}$ the evaluation map. Let $\mathcal{X}^\circ \subset \mathcal{X}$ be the open subset such that for $t \neq 0$, $\mathcal{X}_t^\circ$ is the open $G \times G$-orbit in $\mathcal{X}_t$, and for $x \in \mathcal{X}_0^\circ$, $\mathcal{K}_x$ is smooth and the map $\tau_x : \mathcal{K}_x \simeq \mathcal{U}_x \to \mathcal{C}_x$ is the normalization map. Let $\mathcal{U}^\circ$ be the pre-image of $\mathcal{X}^\circ$ and $\tau : \mathcal{U}^\circ \to \mathcal{C}^\circ \subset \mathbb{P}(T_{\mathcal{X}^\circ/\Delta})$ the universal tangent map. Let $\mathcal{N}$ be the pull-back to $\mathcal{U}^\circ$ of the relative line bundle $\mathcal{O}(1)$ on $\mathbb{P}(T_{\mathcal{X}^\circ/\Delta})$ via the map τ. As τ is a finite map, $\mathcal{N}$ is relative ample for the map $\mathcal{U}^\circ \to \mathcal{X}^\circ$, i.e. for any $x \in \mathcal{X}^\circ$, the line bundle $\mathcal{N}|_{\mathcal{U}_x^\circ}$ is ample.

Consider the line bundle $\mathcal{A} = \mathcal{N}^{\otimes(n+1)} \otimes K_{\mathcal{U}^\circ/\mathcal{X}^\circ}$. For $y \in \mathcal{X}_t^\circ$ with $t \neq 0$, $\tau_y : \mathcal{U}_y^\circ \to \mathcal{C}_y$ is an isomorphism and $\mathcal{N}|_{\mathcal{U}_y^\circ} \simeq \mathcal{O}(1,1)$ on $\mathbb{P}V \times \mathbb{P}V^*$, hence $\mathcal{A}_y = 0$ in $H^2(\mathcal{U}_y^\circ, \mathbb{Q})$. This implies that $\mathcal{A} = 0$ in $H^2(\mathcal{U}^\circ/\mathcal{X}^\circ, \mathbb{Q})$. As a consequence, we have $K^{-1}_{\mathcal{U}^\circ_x} = \mathcal{N}^{\otimes(n+1)}|_{\mathcal{U}^\circ_x}$ is ample, which shows that $\mathcal{U}_x$ is Fano.

Take a section $\sigma : \Delta \to \mathcal{X}^\circ$ through x, then $\mathcal{U}_{\sigma(t)}$ is a smooth Fano deformation of $\mathbb{P}V \times \mathbb{P}V^*$, hence $\mathcal{K}_x \simeq \mathcal{U}_x$ is itself isomorphic to $\mathbb{P}V \times \mathbb{P}V^*$ by [L1]. □

By Theorem 3.2(i), there exists a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}_t$ is the pull-back line bundle $\phi^*(\mathcal{O}_Z(H))$ for $t \neq 0$, where H is a hyperplane in Z. Let $\mathcal{Z} = \mathbb{P}(\pi_*\mathcal{L}^*) \simeq Z \times \Delta$ and $\Phi : \mathcal{X} \dashrightarrow \mathcal{Z}$ the rational map induced by the linear system $|\mathcal{L}|$. Then for $t \neq 0$, the map Φ_t coincides with ϕ and the indeterminate locus F of Φ is properly contained in $\mathcal{X}_0$. Note that Φ_0 is induced from the linear system $|\mathcal{L}_0| := |\mathcal{L}|_{\mathcal{X}_0}|$, hence by Theorem 3.2(iii) the indeterminate locus of Φ_0 is F.

Lemma 3.10. *(i) For any divisor $D \in |\mathcal{L}_0|$ through a general point $x \in \mathcal{X}_0$, D is smooth at x.*

(ii) The rational map $\Phi_0 : \mathcal{X}_0 \dashrightarrow \mathcal{Z}_0$ is birational.

Proof. (i) Fix a point $y \in \mathcal{X}_0 \setminus D$. Take two sections $\sigma_i : \Delta \to \mathcal{X}$ such that $\sigma_1(0) = x$ and $\sigma_2(0) = y$. Let $\mathbf{l}_t$ be the line in $\mathcal{Z}_t$ joining the two points $\Phi(\sigma_1(t))$ and $\Phi(\sigma_2(t))$ and let C_t be the strict transform of $\mathbf{l}_t$ for $t \neq 0$. Denote by C_0 the limit cycle of C_t on $\mathcal{X}_0$, which joins x to y. As $\mathcal{L} \cdot C_t = 1$ for $t \neq 0$, we have $\mathcal{L} \cdot C_0 = 1$. Let $C_0 = C_0^1 \cup \cdots \cup C_0^k$ be the irreducible components, with $x \in C_0^1$. There exists a component, say C_0^i, such that $C_0^i \nsubseteq D$ and $C_0^i \cap D \neq \emptyset$, which implies that $D \cdot C_0^i \geq 1$. As D is nef by Theorem 3.2(ii) and as $D \cdot C_0 = 1$, we have $D \cdot C_0^i = 1$ and $D \cdot C_0^j = 0$ for all $j \neq i$. As C_0^1 passes through a general point x of $\mathcal{X}_0$, it is free on $\mathcal{X}_0$ and deforms to a free curve C_t^1 on $\mathcal{X}_t$ with $t \neq 0$. By Corollary 2.10, we have $D \cdot C_0^1 = \mathcal{L}_t \cdot C_t^1 \geq \mathcal{L}_t \cdot C_{\boldsymbol{\theta}} = 1$. Hence $C_0^1 = C_0^i$ and $D \cdot C_0^1 = 1$, implying D is smooth at x.

(ii) Let us first show that Φ_0 is dominant. If not, then the closure Z_0 of $\Phi(\mathcal{X}_0)$ is properly contained in $\mathcal{Z}_0$. The embedded tangent space $T_z Z_0$ at $z = \Phi_0(x) \in Z_0$ is contained in some hyperplane H of $\mathcal{Z}_0$. By Theorem 3.2(iii), Z_0 is linearly non-degenerate in $\mathcal{Z}_0$. Then $D_H := \Phi_0^{-1}(H)$ is a divisor of $\mathcal{X}_0$ that is singular at x. It contradicts the claim (i), hence Φ_0 is dominant.

Let $\mathcal{Y}$ be the graph closure of Φ and $\mathcal{X} \xleftarrow{\Psi_1} \mathcal{Y} \xrightarrow{\Psi_2} \mathcal{Z}$ the two projections. Let $Y_0 \subset \mathcal{Y}_0$ be the strict transform of $\mathcal{X}_0$. As Ψ_2 is projective and dominant, it is surjective. In particular, $\Psi_{2,0} : \mathcal{Y}_0 \to \mathcal{Z}_0$ is surjective. As Ψ_2 is birational and $\mathcal{Z}$ is smooth, Ψ_2 has connected fibers. This implies that $\Psi_{2,0} : \mathcal{Y}_0 \to \mathcal{Z}_0$ is birational. As a consequence, $\Phi_0 : \mathcal{X}_0 \dashrightarrow \mathcal{Z}_0$ is birational. □

Let $\mathcal{B} \subset \mathcal{Z}$ be the irreducible subvariety such that $\mathcal{B}_t = Z_1$ under the isomorphism $\mathcal{Z}_t \simeq Z$ for all $t \neq 0$. Similarly, we denote by $\mathcal{D}_i \subset \mathcal{X}$ the irreducible divisor corresponding to D_i on X. Then similar results as those in Proposition 3.8 hold for $\mathcal{D}_i$ and also for the specialization $\mathcal{D}_{i,0} := \mathcal{D}_i|_{\mathcal{X}_0}$ to $t = 0$.

Lemma 3.11. *We have $\mathrm{PEff}(\mathcal{X}_0) = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathbb{R}^+\mathcal{D}_{\mathrm{i,0}}$. The divisor $\mathcal{D}_{1,0}$ is irreducible, and $C \cdot \mathcal{D}_{1,0} = 1$ for $[C] \in \mathcal{K}^0$ on $\mathcal{X}_0$.*

Proof. By Proposition 3.8(3), we have $C \cdot \mathcal{D}_{1,0} = 1$ for $C \in \mathcal{K}^0$ on $\mathcal{X}_0$. Since pseudo-effective cones are invariant under Fano deformations by Theorem 3.2, we know from Proposition

3.8(4) that the pseudo-effective cone $\mathrm{PEff}(\mathcal{X}_0) = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathbb{R}^+\mathrm{D}_{\mathrm{i,0}}$ with $\mathbb{R}^+\mathcal{D}_{1,0}$ being an extremal ray, and each irreducible component E of $\mathcal{D}_{1,0}$ satisfies that $E \in \mathbb{R}^+\mathcal{D}_{1,0}$, implying that $C \cdot E \geq 1$. Hence $\mathcal{D}_{1,0} = E$ and it is irreducible. □

Lemma 3.12. *The subvariety $\mathcal{B}_0 \subset \mathcal{Z}_0$ is linearly non-degenerate.*

Proof. Assume that $\mathcal{B}_0$ is contained in a hyperplane H of $\mathcal{Z}_0$. Note that Φ_0 is the limit of Φ_t, which is defined outside F. The map Φ_0 sends the irreducible divisor $\mathcal{D}_{1,0}$ to $\mathcal{B}_0$, as so is Φ_t. Then the pull-back Φ_0^*H can be written as $k\mathcal{D}_{1,0} + E$, where $k \geq 1$ and E is an effective divisor. Take a general member $[C] \in \mathcal{K}^0$ on $\mathcal{X}_0$. As $C \cdot \Phi_0^*H = 1$ and $C \cdot E \geq 0$, we have $k = 1$ and $C \cdot E = 0$. By Theorem 3.2(ii) and Proposition 3.8(4), we have $E \in \oplus_{i=2}^{n-1}\mathbb{Q}[\mathcal{D}_{i,0}]$, implying that $\Phi_0^*H \in \oplus_{i=1}^{n-1}\mathbb{Q}[\mathcal{D}_{i,0}]$. It contradicts the conclusion (5) of Proposition 3.8. □

Proposition 3.13. *Assume $\mathfrak{g} = \mathfrak{sl}(V)$ for a vector space V of dimension $n + 1 \geq 3$. Let $x \in \mathcal{X}_0$ be a general point and $\mathcal{K}^0$ the family of minimal rational curves on $\mathcal{X}_0$ constructed by Proposition 3.3. Then the VMRT of $\mathcal{X}_0$ at x is projectively equivalent to the projection from a general point of the Segre embedding $\mathbb{P}V \times \mathbb{P}V^* \subset \mathbb{P}(V \otimes V^*)$.*

Proof. Let $z = \Phi_0(x) \in \mathcal{Z}_0$ and $p_z : \mathcal{Z}_0 \dashrightarrow \mathbb{P}T_z\mathcal{Z}_0$ the projection from z. Note that the secant variety of $\mathcal{B}_0$ is contained in the limit of secant varieties of $\mathcal{B}_t$, which is not the full space by Proposition 3.8(1). We may assume that z is not in the secant variety of $\mathcal{B}_0$.

Take a curve $[C] \in \mathcal{K}_x$ and deform it to $C_t \subset \mathcal{X}_t$, then the image $\Phi_0(C)$ is the limit of $\Phi_t(C_t)$, the latter being lines in $\mathcal{Z}_t$ meeting $\mathcal{B}_t$. This implies that $\Phi_0(C)$ is again a line through z in $\mathcal{Z}_0$ meeting $\mathcal{B}_0$. As z is not in the secant variety of $\mathcal{B}_0$, this line intersects $\mathcal{B}_0$ at a unique point, which gives a map $\mathcal{K}_x \to \mathcal{B}_0$. Note that the line $\Phi_0(C)$ is uniquely determined by its tangent direction at z, thence this induces a map $f_x : \mathcal{C}_x \to p_z(\mathcal{B}_0)$ which fits into the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_x & \xrightarrow{f_x} & p_z(\mathcal{B}_0) \\ \downarrow & & \downarrow \\ \mathbb{P}T_x\mathcal{X}_0 & \underset{\simeq}{\xrightarrow{d_x\Phi_0}} & \mathbb{P}T_z\mathcal{Z}_0 \end{array}$$

Note that f_x is injective and $\mathcal{C}_x$ has the same dimension as $\mathcal{B}_0$, hence the image $f_x(\mathcal{C}_x)$ is an irreducible component of $p_z(\mathcal{B}_0)$. On the other hand, the degree of $\mathcal{C}_x$ is the same as that of $Z_1 \subset Z$, which is also the same degree as $\mathcal{B}_t$ for all t. Hence $f_x(\mathcal{C}_x)$ has the same degree as that of $\mathcal{B}_0$, which implies that f_x is also surjective, hence f_x is bijective and $\mathcal{B}_0$ is irreducible.

Take a general section $\sigma : \Delta \to \mathcal{X}$ through x and consider the composition map $g_t : \mathcal{K}_{\sigma(t)} \to \mathcal{C}_{\sigma(t)} \to \mathcal{B}_t \to \mathcal{Z}_t$. For $t \neq 0$, this is induced from the line bundle $\mathcal{O}(1,1)$ on $\mathbb{P}V \times \mathbb{P}V^*$. This implies that the central map g_0 is induced from a linear subsystem (say L) of $\mathcal{O}(1,1)$. As $\mathcal{B}_0$ is non-degenerate in $\mathcal{Z}_0$ by Lemma 3.12, the linear system L must be the complete linear system of $\mathcal{O}(1,1)$. This implies that g_0 is the Segre embedding of $\mathcal{K}_x$, hence the maps τ_x and f_x are both isomorphisms, concluding the proof. □

4. Rigidity under Fano deformations

4.1. **Equivariant compactifications of vector groups.** A vector group of dimension g is the additive group $\mathbb{G}_a^g$. An equivariant compactification of $\mathbb{G}_a^g$ is a smooth projective $\mathbb{G}_a^g$-variety Y which admits an open $\mathbb{G}_a^g$-orbit O isomorphic to $\mathbb{G}_a^g$. The boundary $\partial Y = Y \setminus O$ is a union of irreducible reduced divisors $\cup_j E_j$.

Typical examples of equivariant compactifications of $\mathbb{G}_a^g$ are $\mathbb{P}^g$ and its blowups along a smooth subvariety contained in a hyperplane. It turns out that $\mathbb{P}^g$ can even have infinitely many different $\mathbb{G}_a^g$-equivariant compactification structures as soon as $g \geq 6$ ([HT]). As easily seen, there exists a unique equivariant compactification of $\mathbb{G}_a$, which is given by $\mathbb{P}^1$.

Example 4.1. Consider the case $g = 2$. Let Y be an equivariant compactification of $\mathbb{G}_a^2$ and $f : Y \to Y_{min}$ the natural birational morphism to the minimal model Y_{min} of Y.

(i) By [HT, Proposition 5.1], Y_{min} is an equivariant compactification of $\mathbb{G}_a^2$ and the map f is $\mathbb{G}_a^2$-equivariant, which is the composition of successive blowups along $\mathbb{G}_a^2$-fixed points.

(ii) By [HT, Proposition 5.2], Y_{min} is isomorphic to $\mathbb{P}^2$ or a Hirzebruch surface $F_k := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(k))$ with $k \geq 0$.

(iii) The plane $\mathbb{P}^2$ admits two different equivariant compactification structures and the boundary for both is a line.

(iv) The product $\mathbb{P}^1 \times \mathbb{P}^1$ admits a unique equivariant compactification structure, induced from the one on each $\mathbb{P}^1$ by [HT, Proposition 5.5]. The boundary divisor consists of two components isomorphic to $\mathbb{P}^1$.

(v) The Hirzebruch surface F_k with $k > 0$ admits two different equivariant compactification structures by [HT, Proposition 5.5] and the boundary divisor consists of two components.

The following result is from [HT, Theorem 2.5, Theorem 2.7].

Proposition 4.2. *Let Y be an equivariant compactification of $\mathbb{G}_a^g$ with boundary divisors $\cup_{i=1}^l E_i$. Then*

(i) the Picard group of Y is freely generated by $E_1, \cdots, E_l$ and the pseudo-effective cone $\mathrm{PEff}(Y)$ *is given by $\oplus_{i=1}^l \mathbb{R}_{\geq 0} E_i$;*

(ii) the anti-canonical divisor of Y is given by $-K_Y = \sum_{i=1}^l a_i E_i$ with $a_i \geq 2$ for all i.

Proposition 4.3. *Let Y be a smooth projective surface which is an equivariant compactification of the vector group $\mathbb{G}_a^2$ and $O \subset Y$ the open $\mathbb{G}_a^2$-orbit. Let $A \subset \mathrm{Aut}(Y)$ be a finite subgroup such that $A \cdot O = O$. Let S be the set of irreducible components of the boundary $\partial Y := Y \setminus O$. If S consists of two A-orbits, then there exists an irreducible component of ∂Y which is stabilized by A.*

Proof. By Example 4.1, Y is a successive blowup of Y_{min} along $\mathbb{G}_a^2$-fixed points, namely $Y = Y_\ell \xrightarrow{f_\ell} Y_{\ell-1} \to \cdots \to Y_1 \xrightarrow{f_1} Y_{min} = Y_0$, where $f_i : Y_i \to Y_{i-1}$ is the blowup along a $\mathbb{G}_a^2$-fixed point $y_{i-1} \in Y_{i-1}$. Let us denote by $E_i \subset Y_i$ the exceptional fiber of f_i. By abuse of notation, we also denote by E_i its strict transformation in Y_j for $j \geq i$. There are three possibilities for Y_{min}, namely $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or a Hirzebruch surface F_k (with $k > 0$).

Write H (resp. H_1, H_2) for irreducible components of the boundary divisor of $\mathbb{P}^2$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$ and F_k). Then the boundary set of Y consists of H (resp. H_1, H_2) and the exceptional divisors E_j. As $-K_Y$ is A-invariant and the boundary divisor set S has only two A-orbits, there are at most two coefficients in the expression of $-K_Y$ by Proposition 4.2.

Assume $Y_{min} = \mathbb{P}^2$. The boundary of Y_{min} is a line H, which is also the fixed point set of the $\mathbb{G}_a^2$-action. By abuse of notation, we denote by H its strict transform to any Y_j. As the blowup point y_0 lies on H, we have $-K_{Y_1} = 3H + 2E_1$. The expression of $-K_Y$ implies that each blowup center y_i is on H but not in any other irreducible component of E_i, since otherwise there would be at least three different coefficients in the expression of $-K_Y$. It follows that $-K_Y = 3H + 2\sum_i E_i$, and thus H is the A-stable irreducible component.

Suppose now $Y_{min} = \mathbb{P}^1 \times \mathbb{P}^1$. The boundary of Y_{min} has two irreducible components H_1, H_2, which are fibers of two $\mathbb{P}^1$-fibrations $Y_{min} \to \mathbb{P}^1$ respectively. Note that $-K_{Y_{min}} = 2H_1 + 2H_2$ and $y_0 = H_1 \cap H_2$ is the unique $\mathbb{G}_a^2$-fixed point on Y_{min}. Then f_1 must be the blow-up at y_0, thence $-K_{Y_1} = 3E_1 + 2H_1 + 2H_2$. The expression of $-K_Y$ implies that each blowup center y_i is in E_1 but not in any other irreducible component of ∂Y_i. Hence $-K_Y = 3E_1 + 2(H_1 + H_2 + \sum_{j\geq 2} E_j)$, and E_1 is the A-stable irreducible component.

Now assume $Y_{min} = F_k$ with $k \geq 1$. The boundary of Y_{min} has two irreducible components H_1, H_2, which are respectively a fiber and the minimal section of the $\mathbb{P}^1$-fibrations $F_k \to \mathbb{P}^1$. Note that $-K_{Y_{min}} = (k+2)H_1 + 2H_2$ and all $\mathbb{G}_a^2$-fixed points lie in H_1. The expression of $-K_Y$ implies that $k = 1$ and each blowup center y_i belongs to H_1 but not in any other irreducible component of ∂Y_i. This gives that $-K_Y = 3H_1 + 2(\sum_j E_j + H_2)$, thence H_1 is the A-stable irreducible component. $\square$

Remark 4.4. It can be seen from the proof of Proposition 4.3 that if the action of A on S is transitive, then Y is isomorphic to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 4.5. *Let Y be a smooth uniruled projective variety of dimension d.*

(i) If Y is an equivariant compactification of $\mathbb{G}_a^d$, then the VMRT structure is locally flat.

(ii) Assume that (a) Y is of Picard number one; (b) the VMRT structure is locally flat; and (c) the VMRT at a general point is smooth irreducible and non-degenerate, then Y is an equivariant compactification of $\mathbb{G}_a^d$.

(iii) Take a general point $y \in Y$. Assume that (a) the VMRT $\mathcal{C}_y$ is smooth irreducible and non-degenerate; (b) the linear space $\mathfrak{aut}(\hat{\mathcal{C}}_y)^{(1)} = 0$; and (c) $\dim \mathfrak{aut}(Y) = \dim Y + \dim \mathfrak{aut}(\hat{\mathcal{C}}_y)$. Then Y is an equivariant compactification of $\mathbb{G}_a^d$.

Proof. (i) Let $O \subset Y$ be the open $\mathbb{G}_a^d$-orbit, which is isomorphic to $\mathbb{C}^d$. As the $\mathbb{G}_a^d$-action preserves the VMRT structure $\mathcal{C} \subset \mathbb{P}TY$, its action on O trivializes $\mathcal{C}|_O$ as a trivial subbundle of $\mathbb{P}TY|_O \simeq O \times \mathbb{P}^{d-1}$, hence the VMRT structure is locally flat.

(ii) This follows from [FH, Proposition 6.13].

(iii) As $\mathrm{Aut}^o(Y)$ preserves the VMRT structure on Y, we have $\mathfrak{aut}(Y) \subset \mathfrak{aut}(\mathcal{C}, y)$. The assumptions (b) and (c) imply that the equality in Proposition 2.3 holds, hence Y has locally flat VMRT structure and $\mathfrak{aut}(Y) = \mathfrak{aut}(\mathcal{C}, y)$. By [FH, Proposition 5.14], we have $\mathfrak{aut}(Y) = \mathfrak{aut}(\mathcal{C}, y) = \mathbb{C}^d \oplus \mathfrak{aut}(\hat{\mathcal{C}}_y)$. By considering the natural representation $\mathrm{Aut}(Y) \to \mathrm{GL}(H^0(Y, T_Y)) = \mathrm{GL}(\mathfrak{aut}(Y))$, we know that the adjoint group of $\mathrm{Aut}^0(Y)$ is $\mathbb{G}_a^d \rtimes \mathrm{Aut}^0(\hat{\mathcal{C}}_y)$. It gives rise to a subgroup $\mathbb{G}_a^d$ of $\mathrm{Aut}^0(Y)$, and the orbit $\mathbb{G}_a^d \cdot y$ is isomorphic to $\mathbb{C}^d$, which is open and dense in Y by dimension reason. $\square$

4.2. **Rigidity for general cases.** We start with the following result, whose proof is similar to [Par].

Lemma 4.6. *Assume $\mathfrak{g}$ is not of type C. Let $\pi : \mathcal{X} \to \Delta \ni 0$ be a smooth family of Fano varieties such that $\mathcal{X}_t \cong X$ for $t \neq 0$, where X is the wonderful compactification of the simple algebraic group G of adjoint type. Then either $\mathcal{X}_0 \cong X$ or the VMRT-structure on $\mathcal{X}_0$ is locally flat.*

Proof. By the Nakano vanishing, we have $H^q(\mathcal{X}_t, T_{\mathcal{X}_t}) = 0$ for all $q \geq 2$, hence $\chi(\mathcal{X}_t, T_{\mathcal{X}_t}) = h^0(\mathcal{X}_t, T_{\mathcal{X}_t}) - h^1(\mathcal{X}_t, T_{\mathcal{X}_t})$. Since $\chi(\mathcal{X}_t, T_{\mathcal{X}_t})$ is invariant under deformations, and $H^1(X, T_X) = 0$ by [BB, Proposition 4.2], we have

$$h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) - h^1(\mathcal{X}_0, T_{\mathcal{X}_0}) = h^0(X, T_X) - h^1(X, T_X) = h^0(X, T_X) = 2 \dim \mathfrak{g}.$$

This gives that $h^1(\mathcal{X}_0, T_{\mathcal{X}_0}) = h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) - 2 \dim \mathfrak{g} = \dim \mathfrak{aut}(\mathcal{X}_0) - 2 \dim \mathfrak{g}$. Note that by Theorem 3.4, the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x\mathcal{X}_0$ is projectively equivalent to the VMRT of X at a general point. If $\mathfrak{g}$ is not of type C, then $\mathfrak{aut}(\hat{\mathcal{C}}_x) \simeq \mathfrak{g} \oplus \mathbb{C}$ and $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} = 0$ by [BF, Proof of Proposition 6.1]. Then we know from Proposition 2.3 that

$$\dim \mathfrak{aut}(\mathcal{X}_0) \leq \dim \mathfrak{g} + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x) + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} = 2 \dim \mathfrak{g} + 1.$$

We deduce that $h^1(\mathcal{X}_0, T_{\mathcal{X}_0}) \leq 1$. If $h^1(\mathcal{X}_0, T_{\mathcal{X}_0}) = 0$, then $\mathcal{X}_0$ is locally rigid, hence it is isomorphic to $\mathcal{X}_t$ for t small enough, namely $\mathcal{X}_0 \simeq X$. If $h^1(\mathcal{X}_0, T_{\mathcal{X}_0}) = 1$, then the equality in Proposition 2.3 holds, which implies that the VMRT-structure on $\mathcal{X}_0$ is locally flat. □

Remark 4.7. When $\mathfrak{g}$ is C_n, the VMRT of $\bar{C}_n$ is isomorphic to the second Veronese embedding of $\mathbb{P}^{2n-1}$, hence $\mathfrak{aut}(\hat{\mathcal{C}}_x) \simeq \mathfrak{sl}_{2n} \oplus \mathbb{C}$ which is much bigger than $\mathfrak{sp}_{2n} \oplus \mathbb{C}$. This is why the type C case must be treated separately.

By applying Proposition 4.5, we have the following

Corollary 4.8. *In the setting of Lemma 4.6, if $\mathcal{X}_0$ is not isomorphic to X, then $\mathfrak{aut}(\mathcal{X}_0) \simeq \mathbb{C}^g \rtimes (\mathfrak{g} \oplus \mathbb{C})$ and $\mathcal{X}_0$ is an equivariant compactification of $\mathbb{G}_a^g$, where $g = \dim \mathfrak{g}$.*

Theorem 4.9. *Let $\pi : \mathcal{X} \to \Delta \ni 0$ be a smooth family of Fano varieties such that $\mathcal{X}_t \cong X$ for $t \neq 0$, where X is the wonderful compactification of a simple algebraic group G of adjoint type. Assume that $\mathfrak{g}$ is different from the following: C_n, G_2, F_4, E_8. Then $\mathcal{X}_0 \cong X$.*

Proof. Let n be the rank of G, which is assumed to be at least 2. By Corollary 4.8, we may assume that $\mathcal{X}_0$ is an equivariant compactification of the vector group $\mathbb{G}_a^g$. Its boundary is given by $\partial\mathcal{X}_0 = \cup_{i=1}^n E_i$. By Proposition 4.2, the Picard group of $\mathcal{X}_0$ is freely generated by $E_1, \cdots, E_n$ and the pseudo-effective cone $\mathrm{PEff}(\mathcal{X}_0)$ is given by $\oplus_i \mathbb{R}_{\geq 0} E_i$. By Example 3.1, $\mathrm{PEff}(X)$ is generated by the boundary divisors D_i. By Theorem 3.2, the pseudo-effective cones are invariant, hence $\mathrm{PEff}(\mathcal{X}_0) = \mathrm{PEff}(X)$.

Let $\mathcal{D}_i \in \mathrm{Pic}(\mathcal{X}/\Delta) \simeq \mathrm{Pic}(\mathcal{X}_t)$ be the divisor such that $\mathcal{D}_i|_{\mathcal{X}_t} = D_i$ for $t \neq 0$. As D_i and E_i are in the extremal rays of the pseudo-effective cones, we have $\mathcal{D}_i|_{\mathcal{X}_0} = a_i E_i$ for some $a_i \in \mathbb{N}$, up to re-ordering E_i. On the other hand, when $\mathfrak{g}$ is not of type C_n, α_i is primitive in the weight lattice. The latter is identified with $\mathrm{Pic}(X)$ by Example 3.1, hence the divisor $D_i = \mathcal{O}_X(\alpha_i) \in \mathrm{Pic}(X)$ is primitive. This forces $a_i = 1$ for all i.

As $\mathrm{Pic}(\mathcal{X}_0) = \oplus_{i=1}^n \mathbb{Z}E_i$ and the Picard group is invariant, we have $\mathrm{Pic}(X) = \oplus_{i=1}^n \mathbb{Z}D_i$, hence $\mathrm{Pic}(X)$ is the same as the root lattice of G. This shows that the weight lattice equals to the root lattice for G, which is true if and only if G is of type G_2, F_4 or E_8. Hence if G is not G_2, F_4 or E_8, then this leads to a contradiction, which concludes the proof. □

4.3. **Families of quasi-homogeneous submanifolds.** Fix a simple adjoint linear algebraic group G of dimension g and of rank n, which is assumed to be different from C_n. Let $\pi : \mathcal{X} \to \Delta$ be a family of connected Fano manifolds such that $\mathcal{X}_t \simeq \bar{G}$ for all $t \neq 0$. We assume that $\mathcal{X}_0$ is not isomorphic to $\bar{G}$. By Corollary 4.8, $\mathcal{X}_0$ is an equivariant compactification of the vector group $\mathbb{G}_a^g$ and $\mathfrak{aut}(\mathcal{X}_0) \simeq \mathbb{C}^g \rtimes (\mathfrak{g} \oplus \mathbb{C})$.

Fix a section $\sigma : \Delta \to \mathcal{X}$ such that $x_t := \sigma(t)$ is a general point in $\mathcal{X}_t$ for all t. For $t \in \Delta$, we define $j_t^0 : \mathfrak{aut}(\mathcal{X}_t) = H^0(\mathcal{X}_t, T_{\mathcal{X}_t}) \to T_{x_t}\mathcal{X}_t$ by sending a global vector field on $\mathcal{X}_t$ to its value at the point x_t. Define the jet map: $j_t^1 : \mathrm{Ker}(j_t^0) \to \mathfrak{gl}(T_{x_t}\mathcal{X}_t)$ to be the isotropic representation. The following is immediate from the construction and from our previous discussions.

Lemma 4.10. *(i) For all t, the map $\varphi_t : \mathfrak{aut}(\mathcal{X}_t) \to \mathfrak{aut}(\mathcal{C}_t, x_t)$ is an isomorphism of Lie algebras, where $\mathcal{C}_t$ is the VMRT structure on $\mathcal{X}_t$.*

(ii) For all t, the map j_t^0 is surjective and the map j_t^1 is injective.

(iii) For $t \neq 0$, we have $\mathrm{Aut}^0(\mathcal{X}_t) \simeq G \times G$, $\mathfrak{aut}(\mathcal{X}_t) \simeq \mathfrak{g} \oplus \mathfrak{g}$ and $\mathrm{Ker}(j_t^0) \simeq \mathrm{diag}(\mathfrak{g})$. The map j_t^1 is the adjoint representation of $\mathfrak{g}$.

(iv) For $t = 0$, we have $\mathrm{Aut}^0(\mathcal{X}_0) \simeq \mathbb{G}_a^g \rtimes (G \times \mathbb{C}^*)$, $\mathfrak{aut}(\mathcal{X}_0) \simeq \mathbb{C}^g \oplus (\mathfrak{g} \oplus \mathbb{C})$ and $\mathrm{Ker}(j_t^0) \simeq \mathfrak{g} \oplus \mathbb{C}$. The map j_t^1 is the adjoint representation on $\mathfrak{g}$ (by identifying $\mathbb{C}^g$ with $\mathfrak{g}$ as modules) and the dilation representation on $\mathbb{C}$.

To avoid confusions, we denote by $\mathfrak{g}_l$ (resp. $\mathfrak{g}_r$, $\mathfrak{g}_{ad}$) the representation of $\mathfrak{g}$ which corresponds to the left (resp. right, adjoint) G-action. Let $\mathcal{V} = \pi_* T_{\mathcal{X}/\Delta}$, which is a vector bundle over Δ such that $\mathcal{V}_t \simeq \mathfrak{aut}(\mathcal{X}_t)$ for $t \neq 0$. Let $\mathcal{W} \subset \mathcal{V}$ be the subbundle such that $\mathcal{W}_t \simeq \mathrm{Ker}(\mathrm{j}_\mathrm{t}^0) \simeq \mathfrak{g}_{\mathrm{ad}}$ for $t \neq 0$.

Lemma 4.11. *(i) The vector bundle $\mathcal{W}$ is isomorphic to the trivial bundle with fiber $\mathfrak{g}_{ad}$.*

(ii) For all $t \in \Delta$, the fiber $\mathcal{V}_t$ is a $\mathfrak{g}_{ad}$-module, which is isomorphic to $\mathfrak{g}_l \oplus \mathfrak{g}_r$.

Proof. (i) Fix a holomorphic family of linear isomorphisms $\xi_t : T_{x_t}\mathcal{X}_t \to \mathfrak{g}$ sending $\hat{\mathcal{C}}_{x_t}$ to the variety $\hat{\mathcal{C}}_e$ constructed in Theorem 2.9. By Lemma 4.10, the composition $\mathrm{Ker}(j_t^0) \xrightarrow{j_t^1} \mathfrak{gl}(T_{x_t}\mathcal{X}_t) \xrightarrow{\zeta_t = (\xi_t)_*} \mathfrak{gl}(\mathfrak{g})$ is injective for all $t \in \Delta$. For $t \neq 0$, we have $\zeta_t \circ j_t^1(\mathcal{W}_t) = \mathfrak{g}_{ad}$, hence by continuity, we have $\zeta_0 \circ j_0^1(\mathcal{W}_0) \subseteq \mathfrak{g}_{ad}$. As both sides have the same dimension, we have the equality. Hence the map $\zeta_t \circ j_t^1$ gives the trivialization of the bundle $\mathcal{W}$.

(ii) For all $t \in \Delta$, $\mathcal{W}_t \simeq \mathfrak{g}_{ad}$ is a Lie subalgebra of $\mathcal{V}_t$. For $t \neq 0$, $\mathcal{V}_t \simeq \mathfrak{g} \oplus \mathfrak{g}$ as $\mathfrak{g}_{ad}$-modules. Note that $\mathfrak{aut}(\mathcal{X}_0)$ is completely reducible as a $\mathfrak{g}_{ad}$-module, with irreducible factors $\mathbb{C}^g, \mathfrak{g}_{ad}$ and $\mathbb{C}$. By dimension reason, we have $\mathcal{V}_0 \simeq \mathbb{C}^g \rtimes \mathfrak{g}_{ad}$, which is isomorphic to $\mathfrak{g}_l \oplus \mathfrak{g}_r$ as $\mathfrak{g}_{ad}$-modules. □

Let $\tilde{G}$ be the simply-connected cover of G. By Lemma 4.10, $\mathrm{Ker}(j_t^0)$ contains $\mathfrak{g}$ for all t, which shows that $\tilde{G}$ acts on $\mathcal{X}$ fixing the section $\sigma(\Delta)$. As the center of $\tilde{G}$ acts trivially on $\mathcal{X}_t$ for $t \neq 0$, so is on $\mathcal{X}_0$ by continuity. Then we have an action of G on $\mathcal{X}$ fixing the section

$\sigma(\Delta)$. Let $T \subset G$ be a maximal torus and $\mathfrak{h} \subset \mathfrak{g}$ its Lie algebra. Then T acts on $\mathcal{X}/\Delta$. Let $\mathcal{Y}$ be the connected component of the fixed locus $\mathcal{X}^T$ which contains $\sigma(\Delta)$. Then the map $\mathcal{Y} \to \Delta$ is a family of smooth projective varieties by Białynicki-Birula ([B]).

For a Lie algebra $\mathfrak{g}$, we denote by $\mathbb{G}_a(\mathfrak{g})$ the vector group associated to $\mathfrak{g}$ (viewed as a vector space). In particular, for a Lie sub-algebra $\mathfrak{h} \subset \mathfrak{g}$, we have $\mathbb{G}_a(\mathfrak{h}) \subset \mathbb{G}_a(\mathfrak{g})$. The adjoint action of T on $\mathcal{Y}$ is trivial by construction.

Proposition 4.12. *(1) For $t \neq 0$, the subgroup $T \times T$ of $\mathrm{Aut}^0(\mathcal{X}_t) = G \times G$ stabilizes $\mathcal{Y}_t$. The left T-action gives rise to an equivariant open embedding $T \simeq T \cdot x_t \subset \mathcal{Y}_t$. Furthermore, $\mathcal{Y}_t = \bar{T}$ is the toric variety with fan consisting of Weyl chambers of G and their faces.*

(2) For $t = 0$, the subgroup $\mathbb{G}_a(\mathfrak{h}) \rtimes T$ of $\mathrm{Aut}^0(\mathcal{X}_0) = \mathbb{G}_a^g \rtimes (G \times \mathbb{C}^*)$ stabilizes $\mathcal{Y}_0$. The $\mathbb{G}_a(\mathfrak{h})$-action gives rise to an equivariant open embedding $\mathfrak{h} \simeq \mathbb{G}_a(\mathfrak{h}) \cdot x_0 \subset \mathcal{Y}_0$. In particular, $\mathcal{Y}_0$ is an equivariant compactification of the vector group $\mathbb{G}_a(\mathfrak{h})$.

(3) There is an action of the Weyl group $W(G)$ on the family $\mathcal{Y}/\Delta$, which is induced by the G-action on the family $\mathcal{X}/\Delta$ fixing the section $\sigma(\Delta)$. Furthermore, the action of $W(G)$ on $T \subset \mathcal{Y}_t$ with $t \neq 0$ and that on $\mathfrak{h} \subset \mathcal{Y}_0$ are the natural ones.

Proof. (1) For $t \neq 0$, the fiber $\mathcal{X}_t$ is an equivariant compactification of G. The intersection $G \cap \mathcal{Y}_t$ is equal to the subgroup $T \subset G$, and it is stable under the $T \times T$-action. Then the left T-action gives rise to an equivariant embedding $T \simeq T \cdot x_t \subset \mathcal{Y}_t$. The description of the fan of $\mathcal{Y}_t = \bar{T}$ follows from [BK, Lemma 6.1.6].

(2) Denote by $\mathfrak{h}_l \subset \mathfrak{g}_l$, $\mathfrak{h}_r \subset \mathfrak{g}_r$ and $\mathfrak{h}_{ad} \subset \mathfrak{g}_{ad}$ the Lie subalgebras identified with $\mathfrak{h} \subset \mathfrak{g}$. Let $\mathcal{U}$ be the vector subbundle of $\mathcal{V} := \pi_* T_{\mathcal{X}/\Delta}$ such that $\mathcal{U}_t = \mathfrak{h}_l \oplus \mathfrak{h}_r$ for $t \neq 0$. Since $\mathcal{U}_t$ is a trivial T-module of dimension $2 \dim \mathfrak{h}$ for each $t \neq 0$, so is $\mathcal{U}_0$. Then by Lemma 4.10(iv) and Lemma 4.11, we have $\mathcal{U}_0 = \mathfrak{h} \rtimes \mathfrak{h}_{ad}$. When $t \neq 0$, the manifold $\mathcal{Y}_t$ is stable under the holomorphic vector fields given by elements in $\mathcal{U}_t$. By continuity, the same holds for $t = 0$. In particular, $\mathbb{G}_a(\mathfrak{h}) \subset \mathrm{Aut}^0(\mathcal{Y}_0)$ and $\mathfrak{h} \cong \mathbb{G}_a(\mathfrak{h}) \cdot x_0 \subset \mathcal{Y}_0$.

(3) Let $N_G(T)$ be the normalizer of T in G, which stabilizes $\mathcal{Y}_t = \bar{T}$ for each $t \neq 0$. By continuity, $N_G(T)$ stabilizes $\mathcal{Y}$. The torus T acts trivially on a dense open subset of $\mathcal{Y}$, hence it acts trivially on $\mathcal{Y}$. This gives rise to the action of $W(G)$ on $\mathcal{Y}/\Delta$. When $t \neq 0$, the action of $N_G(T)$ on the open orbit $G \subset \mathcal{X}_t$ is from the inner automorphisms, and thus the induced action of $W(G)$ on $T = G \cap \mathcal{Y}_t$ is the natural one. By Lemma 4.10(iv), the action of $N_G(T)$ on the open orbit $\mathfrak{g} \simeq \mathbb{G}_a^g \subset \mathcal{X}_t$ is from the adjoint representation. Then the action of $W(G)$ on $\mathfrak{h} = \mathfrak{g} \cap \mathcal{Y}_0$ is the natural one. □

Example 4.13. Given a Weyl chamber $\mathfrak{C}$ of G, denote by $\alpha_1, \ldots, \alpha_n$ the associated simple roots. By [BK, Lemma 6.1.6], there is a unique T-stable affine space $\mathbb{A}^n$ in $\bar{T}$ whose fan consists of $\mathfrak{C}$ and all its faces. The inclusion $T \subset \mathbb{A}^n$ is given by $t \mapsto (\alpha_1(t), \ldots, \alpha_n(t))$

In [JS], a generalized version of Białynicki-Birula's theorem is proven. In the following, we only state a special case of this generalization, which is sufficient for our purpose.

Theorem 4.14. [JS, Theorem 1.5] *Let Y be a smooth H-variety, where H is a connected reductive group that is the product of a torus and a finite number of general linear groups. Then the fixed locus Y^H is smooth.*

Now we have the following analogue of Proposition 4.12.

Proposition 4.15. *Suppose there is a homomorphism of algebraic groups $\iota : G' \to G$ such that G' is the product of a torus and a finite number of general linear groups, and the image $\iota(G')$ contains the maximal torus T of G. Then the G-action on the family $\mathcal{X}/\Delta$ fixing the section $\sigma(\Delta)$ induces an action of G' on this family. Let $\mathcal{Y}'$ be the connected component of the fixed locus $\mathcal{X}^{G'}$ containing the section $\sigma(\Delta)$, which is a subfamily of the family $\mathcal{Y}/\Delta$ given in Proposition 4.12. Then the followings hold.*

(i) The morphism $\pi' : \mathcal{Y}' \to \Delta$ is smooth and projective.

(ii) The identity component T' of the centralizer of $\iota(G')$ in G is a subtorus of T.

(iii) For $t \neq 0$, the left T-action on $\mathcal{Y}_t$ induces an equivariant compactification $T' \simeq T' \cdot x_t \subset \mathcal{Y}'_t$.

(iv) For $t = 0$, the $\mathbb{G}_a(\mathfrak{h})$-action on $\mathcal{Y}_0$ induces an equivariant compactification $\mathbb{G}_a(\mathfrak{h}') \simeq \mathbb{G}_a(\mathfrak{h}') \cdot x_0 \subset \mathcal{Y}'_0$, where $\mathfrak{h}'$ is the Lie algebra of T'.

(v) Let $W_{T'}$ be the stabilizer of T' under the $W(G)$-action on T. Then the $W(G)$-action on the family $\mathcal{Y}/\Delta$ induces an action of $W_{T'}$ on the family $\mathcal{Y}'/\Delta$. Furthermore, the $W_{T'}$-action on $T' \subset \mathcal{Y}'_t$ with $t \neq 0$ and that on $\mathfrak{h}' \simeq \mathbb{G}_a(\mathfrak{h}') \subset \mathcal{Y}_0$ are the natural ones.

Proof. Since $T \subset \iota(G')$ and the identity component of the centralizer $C_G(T)$ is T itself, we know T' is a subtorus of T. By definition, $\mathcal{Y}'$ is a closed subvariety of $\mathcal{Y}$ and thus π' is projective. The smoothness of π' follows from Theorem 4.14. Note that $T \subset \iota(G')$, $\mathcal{Y}' \subset \mathcal{Y}$ and the relative dimension of $\mathcal{Y}'/\Delta$ is the same as $\dim T'$. Then the rest follows from Proposition 4.12. □

4.4. **Rigidity of $\bar{G}_2$, $\bar{F}_4$ and $\bar{E}_8$.** Given a root α of G, we denote by $\mathfrak{g}_\alpha$ and $\mathfrak{h}_{\alpha^\vee}$ the 1-dimensional root subspace and coroot subspace of $\mathfrak{g}$. Given a coweight $\omega^\vee$ of G, say $\omega^\vee : \mathbb{C} \to \mathfrak{h}$, denote by $\mathfrak{h}_{\omega^\vee}$ the corresponding 1-dimensional subspace of $\mathfrak{h}$. In particular, $\mathfrak{h}_{\alpha^\vee}$ and $\mathfrak{h}_{\omega^\vee}$ are Lie algebras of 1-dimensional torus subgroups of the fixed maximal torus T, and $\mathfrak{g}_\alpha$ is the Lie algebra of a connected subgroup of G which is isomorphic to the 1-dimensional additive group $\mathbb{G}_a$.

We need the following technical result, which will be proven in the following subsections.

Proposition 4.16. *Let n be the rank of G, and suppose G is G_2, F_4 or E_8. Then there exists an algebraic group G' which is a torus or a product of general linear groups satisfying the following properties:*

(i) There exists a homomorphism of algebraic groups $\iota : G' \to G$ such that (a) $\iota(G')$ contains the maximal torus T; (b) the identity component T' of the centralizer $C_G(\iota(G'))$ is the 2-dimensional torus with Lie algebra $\mathfrak{h}' := \mathfrak{h}_{\omega_1^\vee} \oplus \mathfrak{h}_{\omega_n^\vee}$.

(ii) Let $\bar{T}'$ be the closure of T' in the wonderful compactification $\bar{G}$. Then the Picard number ρ' of the toric surface $\bar{T}'$ is equal to 10 (in G_2 case) or 6 (in F_4 and E_8 cases).

(iii) There is a subgroup W' of $W_{T'}$ of order $\rho' + 2$ such that its action on the set of irreducible components of the boundary divisor $\partial\bar{T}' := \bar{T}' \setminus T'$ has two orbits.

We need the following elementary result on finite group actions.

Lemma 4.17. *Let Y be a smooth projective variety, and let $A \subset \mathrm{Aut}(Y)$ be a finite subgroup. Suppose the Picard group $\mathrm{Pic}(Y)$ is generated by prime divisors $D_{i,j} (1 \leq i \leq m, 1 \leq j \leq k_i)$, where for each i, the set $\{D_{i,1}, \ldots, D_{i,k_i}\}$ is an A-orbit. Then $\mathrm{Pic}(Y)^A \otimes \mathbb{Q}$ is generated by $D_i := \sum_{j=1}^{k_i} D_{i,j}, i = 1, \cdots, m$.*

Proof. The inclusion $\sum_{i=1}^m \mathbb{Z}D_i \subset \mathrm{Pic}(Y)^A$ is straight-forward. Any element in $\mathrm{Pic}(Y)$ can be written as $\mathcal{O}(D)$ with $D = \sum_{i=1}^m \sum_{j=1}^{k_i} c_{i,j} D_{i,j}$ for some $c_{i,j} \in \mathbb{Z}$. Since $\sum_{a\in A} a \cdot D_{i,j} = \frac{|A|}{k_i} D_i \in \mathbb{Z}D_i$, we have $\sum_{a\in A} a \cdot D = \sum_{i=1}^m \sum_{j=1}^{k_i} \frac{c_{i,j}|A|}{k_i} D_i \in \sum_{i=1}^m \mathbb{Z}D_i$. If $\mathcal{O}(D) \in \mathrm{Pic}(Y)^A$, then $\mathcal{O}(|A|D) = \mathcal{O}(\sum_{a\in A} a \cdot D)$, which implies that $\mathcal{O}(D) \in \sum_i \mathbb{Z}\frac{1}{|A|} D_i$. □

Take the reductive group G', the torus T' and the finite group W' as in Proposition 4.16. Define the smooth family $\mathcal{Y}'/\Delta$ as in Proposition 4.15, which is the connected component of $\mathcal{X}^{G'}$ containing the section $\sigma(\Delta)$.

Proposition 4.18. *Suppose G is G_2, F_4 or E_8. Then the set of irreducible components of the boundary $\partial\mathcal{Y}'_0 := \mathcal{Y}'_0 \setminus \mathfrak{h}'$ consists of two W'-orbits, and each orbit contains at least two elements.*

Proof. Since the W'-action on the set of irreducible components of $\partial\bar{T}'$ has two orbits, the rank of $\mathrm{Pic}(\bar{T}')^{W'}$ is at most two by Lemma 4.17. The W'-action on the family $\mathcal{Y}'/\Delta$ induces identifications $\mathrm{Pic}(\bar{T}')^{W'} = \mathrm{Pic}(\mathcal{Y}'/\Delta)^{W'} = \mathrm{Pic}(\mathcal{Y}'_0)^{W'}$.

Being the equivariant compactification of a vector group, the Picard group of $\mathcal{Y}'_0$ is freely generated by elements in S, where S is the set of irreducible components of the boundary divisor $\partial\mathcal{Y}'_0 := \mathcal{Y}'_0 \setminus \mathfrak{h}'$. By Lemma 4.17 again, S consists of one or two W'-orbits. Now the Picard number of $\mathcal{Y}'_0$ is $\rho' \in \{6, 10\}$ and the order of W' is $\rho' + 2$. Since ρ' does not divide $\rho' + 2$, the set S is not W'-transitive. In the case $\rho' = 6$ (resp. $\rho' = 10$), the set S consists of two orbits of cardinalities 2 and 4 (resp. 4 and 6) respectively. □

Now we can prove the rigidity under Fano deformations of G_2, F_4 and E_8.

Theorem 4.19. *Assume G is G_2, F_4 or E_8 and X its wonderful compactification. Let $\pi : \mathcal{X} \to \Delta \ni 0$ be a smooth family of Fano varieties such that $\mathcal{X}_t \cong X$ for $t \neq 0$. Then $\mathcal{X}_0 \cong X$.*

Proof. Assume $\mathcal{X}_0$ is not isomorphic to X, then by Corollary 4.8, $\mathcal{X}_0$ is an equivariant compactification of $\mathbb{G}_a^g$. Propositions 4.15 and 4.16 give a family of smooth projective surfaces $\mathcal{Y}'/\Delta$ whose general fibers are toric. Now Proposition 4.18 implies that the central fiber is an equivariant compactification of $\mathbb{G}_a^2$ with a property which contradicts Proposition 4.3. This concludes the proof. □

Now we prove Proposition 4.16 through a case-by-case argument.

4.4.1. *The case of G_2.* Let e_1, e_2 and e_3 be an orthonormal basis of the inner product space $\mathbb{R}^3$. The short roots of G_2 are $e_i - e_j$, $i \neq j$, and the long roots are $\pm(e_1 + e_2 + e_3 - 3e_k)$, $k = 1, 2, 3$. The simple roots (in Bourbaki's numbering order) are $\alpha_1 = e_1 - e_2$ and $\alpha_2 = -e_1 + 2e_2 - e_3$. The fundamental coweights are $\omega_1^\vee = e_1 - e_3$ and $\omega_2^\vee = \frac{1}{3}(e_1 + e_2 - 2e_3)$. The Weyl group $W(G_2) = \langle -id \rangle \times \mathfrak{S}_3$, where $\mathfrak{S}_3$ is the group of permutations of the set $\{e_1, e_2, e_3\}$. The action of the Weyl group $W(G_2)$ on fundamental coweights is given by: $W(G_2) \cdot \omega_1^\vee = \{e_i - e_j \mid i \neq j\}$ and $W(G_2) \cdot \omega_2^\vee = \{\pm\frac{1}{3}(e_1 + e_2 + e_3 - 3e_k) \mid k = 1, 2, 3)\}$.

In this case, take $G' = T'$ to be the maximal torus T and let W' be the whole Weyl group $W(G_2)$ (of order 12). By Proposition 4.12, the fan of $\bar{T}$ consists of the Weyl chambers and their faces. Hence $\bar{T}$ is a toric surface of Picard number $12 - 2 = 10$. Now consider the action of W' on the boundary divisors. An irreducible component of the boundary $\partial\bar{T}$ corresponds to a unique ray of the fan, and this ray is generated by a unique element in $W(G_2)\cdot\{\omega_1^\vee,\omega_2^\vee\}$. It follows that W' acts on the irreducible components of $\partial\bar{T}'$ with two orbits.

4.4.2. *The case of F_4.* Let us first recall some facts on F_4 from [A] (page 93-94). Let e_1, e_2, e_3, e_4 be an orthonormal basis of the inner product space $\mathbb{R}^4$, then the 24 long roots of F_4 are $\pm e_i \pm e_j (i<j)$ while the 24 short roots are $\pm e_i, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$. The simple roots (in Bourbaki's numbering order) are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3$ and $\alpha_4 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4)$. The fundamental coweights are $\omega_1^\vee = e_1 + e_4$, $\omega_2^\vee = e_1 + e_2 + 2e_4$, $\omega_3^\vee = e_1 + e_2 + e_3 + 3e_4$ and $\omega_4^\vee = 2e_4$.

The root system of F_4 contains a root subsystem of type D_4 such that simple roots of D_4 are given by (in Bourbaki's numbering order): $e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4$. In particular, the long roots of F_4 are exactly the roots of D_4. The Weyl group $W(F_4) \simeq W(D_4) \rtimes \mathfrak{S}_3$, where $\mathfrak{S}_3$ is the permutation group of 3 letters, being the symmetries of Dynkin diagram of D_4. Furthermore, the Weyl group $W(D_4) = H_3 \rtimes \mathfrak{S}_4$, where $\mathfrak{S}_4$ is the group of permutations of the set $\{e_1, e_2, e_3, e_4\}$, and H_3 is the kernel of the map $\{\pm 1\}^4 \to \pm 1$ given by $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \mapsto \Pi_i \varepsilon_i$. Recall that $\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i} \oplus \mathfrak{h}_{\alpha_i^\vee}$ is an $\mathfrak{sl}_2$ Lie subalgebra of $\mathfrak{g}$.

Lemma 4.20. *(i) The Lie subalgebra $\mathfrak{gl}_2(\alpha_i) := \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i} \oplus \mathfrak{h}_{\alpha_i^\vee} \oplus \mathfrak{h}_{\omega_{i+1}^\vee}$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}(\mathbb{C}^2)$ for $i \in \{0,3\}$, where $\alpha_0 := \alpha_2 + \alpha_3 = e_2$ is a short root of F_4.*

(ii) There is a homomorphism of algebraic groups $\iota : G' := \mathrm{GL}(\mathbb{C}^2) \times \mathrm{GL}(\mathbb{C}^2) \to \mathrm{G}$ such that the Lie algebra of $\iota(G')$ is $\mathfrak{g}' := \mathfrak{gl}_2(\alpha_0) \oplus \mathfrak{gl}_2(\alpha_3)$.

(iii) We have $\mathfrak{h} \subset \mathfrak{g}'$, hence $\iota(G')$ contains a maximal torus of G. Furthermore we have $\mathfrak{g}^{\mathfrak{g}'} = \mathfrak{h}_{\omega_1^\vee} \oplus \mathfrak{h}_{\omega_4^\vee}$, where $\mathfrak{g}^{\mathfrak{g}'} := \{v \in \mathfrak{g} \mid [v, \mathfrak{g}'] = 0\}$.

Proof. (i) By considering the Cartan pairings, there is a Lie subalgebra of type B_2 such that $\{\alpha_1, \alpha_0\}$ is an ordered set of simple roots, and the restriction of the adjoint representation induces $\mathfrak{gl}_2(\alpha_0) \subset \mathfrak{gl}(\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_0} \oplus \mathfrak{g}_{\alpha_1+2\alpha_0})$, that is the irreducible representation $\mathrm{Sym}^2(\mathbb{C}^2)$ of $\mathfrak{gl}(\mathbb{C}^2)$. Similarly, there is a Lie subalgebra of type A_2 such that $\{\alpha_3, \alpha_4\}$ is a set of simple roots, whose adjoint representation induces an identity $\mathfrak{gl}_2(\alpha_3) = \mathfrak{gl}(\mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_3+\alpha_4})$.

(ii) For $i \in \{0,3\}$, the adjoint representation of $\mathfrak{g}$ gives rise to the homomorphism $\iota_i : \mathrm{GL}(\mathbb{C}^2) \to \mathrm{G}$ of finite kernel such that the Lie algebra of $\iota_i(\mathrm{GL}(\mathbb{C}^2))$ is $\mathfrak{gl}_2(\alpha_i)$. Since the Cartan pairing $\langle \alpha_0, \alpha_3 \rangle = 0$, we have $[\mathfrak{gl}_2(\alpha_0), \mathfrak{gl}_2(\alpha_3)] = 0$ and thus $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$. Now $\iota : G' \ni (a,b) \mapsto \iota_0(a) \cdot \iota_3(b) \in G$ is the required homomorphism.

(iii) Since the coweights $\omega_1^\vee$, $\omega_4^\vee$, $\alpha_0^\vee$ and $\alpha_3^\vee$ are linearly independent, we have $\mathfrak{h} = \mathfrak{h}_{\omega_1^\vee} \oplus \mathfrak{h}_{\omega_4^\vee} \oplus \mathfrak{h}_{\alpha_0^\vee} \oplus \mathfrak{h}_{\alpha_3^\vee}$, implying that $\mathfrak{h} \subset \mathfrak{g}'$ and $\mathfrak{g}^{\mathfrak{g}'} \subset \mathfrak{g}^{\mathfrak{h}} = \mathfrak{h}$. The roots $\alpha_i : \mathfrak{h} \to \mathbb{C}$ satisfy that $\mathfrak{h}_{\omega_1^\vee} \oplus \mathfrak{h}_{\omega_4^\vee} = \ker \alpha_2 \cap \ker \alpha_3 = \ker \alpha_0 \cap \ker \alpha_3$. It follows that $\mathfrak{g}^{\mathfrak{g}'} = \mathfrak{h}_{\omega_1^\vee} \oplus \mathfrak{h}_{\omega_4^\vee}$. □

Lemma 4.21. *Let ϵ_k be the linear transformation on $\mathbb{R}^4$ such that $\epsilon_k(e_i) = (-1)^{\delta_{ik}} e_i$, and let (e_i, e_j) be the linear transformation on $\mathbb{R}^4$ induced by permuting e_i and e_j.*

(i) Let W' be the subgroup of $W(F_4)$ generated by $\tau_1 := (e_1, e_4)$ and $\tau_2 := \epsilon_1\epsilon_2$. Then W' is of order 8.

(ii) The sublattice $\mathbb{Z}\omega_1^\vee \oplus \mathbb{Z}\omega_4^\vee$ of the coweight lattice of F_4 is stable under the action of W'. Moreover, $W' \cdot \omega_1^\vee = \{\pm e_1 \pm e_4\}$ and $W' \cdot \omega_4^\vee = \{\pm 2e_1, \pm 2e_4\}$.

(iii) Let $\mathbb{F}'$ be the fan in $\mathbb{R}\omega_1^\vee \oplus \mathbb{R}\omega_4^\vee$ whose rays are those generated by $\pm(2e_1)$, $\pm(2e_4)$, $\pm e_1 \pm e_4$ respectively. Then $\{a \cdot (\mathbb{R}^+\omega_1^\vee + \mathbb{R}^+\omega_4^\vee) \mid a \in W'\}$ is the set of maximal cones.

Proof. A direct calculation shows that $W' = \{id, \tau_1, \tau_2, \tau_{2,1}, \tau_{1,2}, \tau_{1,2,1}, \tau_{2,1,2}, \tau_{2,1,2,1}\}$, where $\tau_{i_1,i_2,\ldots,i_k} = \tau_{i_1} \circ \tau_{i_2} \circ \cdots \circ \tau_{i_k}$. The other claims are straight-forward. □

Remark 4.22. In the setting of Lemma 4.21, we can identify $\mathbb{Z}\omega_1^\vee \oplus \mathbb{Z}\omega_4^\vee$ with the coweight lattice of B_2 such that $\omega_1^\vee$ and $\omega_4^\vee$ are the first and second coweights of B_2, and W' coincides with the Weyl group of B_2. The fan $\mathbb{F}'$ is in fact the fan consisting of Weyl chambers of B_2 and their faces. The associated toric surface has 8 boundary divisors, hence it is of Picard number 6.

Proposition 4.23. *Let T' be the subtorus of T with Lie algebra $\mathfrak{h}' := \mathfrak{h}_{\omega_1^\vee} \oplus \mathfrak{h}_{\omega_4^\vee} \subset \mathfrak{h}$, and $\bar{T}'$ its closure in the wonderful compactifcation $\bar{F}_4$. Then the group W' stabilizes the subtorus $T' \subset T$, and the fan of the toric surface $\bar{T}'$ is exactly $\mathbb{F}'$.*

Proof. The subtorus $T' \subset T$ is stabilized by W', since so is the lattice $\mathbb{Z}\omega_1^\vee \oplus \mathbb{Z}\omega_4^\vee$. Now $\bar{T}'$ is a closed subvariety of the toric variety $\bar{T}$, and the fan of $\bar{T}$ consists of Weyl chambers of F_4 and their faces by Proposition 4.12(1). The open subset $\bar{T}^{\text{aff}}$ of $\bar{T}$ associated with the positive Weyl chamber $\sum_{i=1}^4 \mathbb{R}^+\omega_i^\vee$ is isomorphic to the affine space $\mathbb{A}^4$. More precisely, we define $T \to \mathbb{A}^4$ by sending $t \in T$ to $(\alpha_1(t), \ldots, \alpha_4(t)) \in \mathbb{A}^4$, which makes $\mathbb{A}^4$ a toric variety associated with the fan consisting of the positive Weyl chamber $\sum_{i=1}^4 \mathbb{R}^+\omega_i^\vee$ and its faces, see Example 4.13. Hence, $\bar{T'}^{\text{aff}} := \bar{T}' \cap \bar{T}^{\text{aff}}$ is an affine plane $\mathbb{A}^2 = \{(v_1, 0, 0, v_4) \in \mathbb{A}^4 \mid v_1, v_4 \in \mathbb{C}\} \subset \mathbb{A}^4 = \bar{T}^{\text{aff}}$. Furthermore, the fan of $\bar{T'}^{\text{aff}}$ consists of the positive cone $\mathbb{R}^+\omega_1^\vee + \mathbb{R}^+\omega_4^\vee$ and its faces. Take any $w \in W'$. Then $w \cdot \bar{T'}^{\text{aff}}$ is a T'-stable open subset of $\bar{T}'$, whose fan consists of the cone $w \cdot (\mathbb{R}^+\omega_1^\vee + \mathbb{R}^+\omega_4^\vee)$. By Lemma 4.21, the cones $W' \cdot (\mathbb{R}^+\omega_1^\vee + \mathbb{R}^+\omega_4^\vee)$ cover the space $\mathbb{R}\omega_1^\vee + \mathbb{R}\omega_4^\vee$, and thus $W' \cdot \bar{T'}^{\text{aff}}$ gives rise to an open covering of $\bar{T}'$, completing the proof. □

Now Proposition 4.16 for type F_4 follows from Lemma 4.20, Lemma 4.21 and Proposition 4.23.

4.4.3. *The case of E_8.* Let us first recall some facts on E_8 from [A, Page 56-59]. Let $e_1, \ldots, e_8$ be an orthonormal basis of the inner product space $\mathbb{R}^8$. A set of simple roots of E_8 in Bourbaki's numbering order can be given as follows: $\alpha_1 = \frac{1}{2}\sum_{i=1}^2 e_i - \frac{1}{2}\sum_{j=3}^8 e_i$, $\alpha_2 = e_2 + e_3$, $\alpha_k = -e_{k-1} + e_k$ for $3 \le k \le 8$. The roots of E_8 are of form $\pm e_i \pm e_j$ with $i \neq j$ or of the form $\frac{1}{2}\sum_{k=1}^8 \pm e_k$ with an even number of $-$ signs. There are 112 roots of first form, and 128 roots of second form. The fundamental dominant coweights of E_8 are $\omega_1^\vee = 2e_1$, $\omega_2^\vee = \frac{5}{2}e_1 + \frac{1}{2}\sum_{j=2}^8 e_j$, $\omega_3^\vee = \frac{7}{2}e_1 - \frac{1}{2}e_2 + \frac{1}{2}\sum_{j=3}^8 e_j$, $\omega_k^\vee = (9-k)e_1 + \sum_{j=k}^8 e_j$ for $4 \le k \le 8$. The Weyl group of E_8 is $W(E_8) = \{a \in SO(\mathbb{R}^8) \mid a(R) = R\}$ and it is of order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$, where R is the set of roots of E_8. The action of $W(E_8)$ on R is transitive.

Lemma 4.24. *(i) There is a Lie subalgebra $\mathfrak{f}_1 \subset \mathfrak{g}$ of type A_3 such that $\{\alpha_3, \alpha_4, \alpha_2\}$ is the ordered set of simple roots, and $\mathfrak{g}_1' := \mathfrak{f}_1 \oplus \mathfrak{h}_{\omega_1^\vee}$ is a Lie algebra isomorphic to $\mathfrak{gl}(\mathbb{C}^4)$.*

(ii) There is a Lie subalgebra $\mathfrak{f}_2 \subset \mathfrak{g}$ of type A_3 such that $\{\alpha_0, \alpha_6, \alpha_7\}$ is the ordered set of simple roots, and $\mathfrak{g}_2' := \mathfrak{f}_2 \oplus \mathfrak{h}_{\omega_5^\vee}$ is a Lie algebra isomorphic to $\mathfrak{gl}(\mathbb{C}^4)$, where $\alpha_0 := -\sum_{i=2}^7 \alpha_i - \sum_{j=4}^6 \alpha_j = -e_6 - e_7$ is a root of E_8.

(iii) There is a homomorphism of algebraic groups $\iota : G' := \mathrm{GL}(\mathbb{C}^4) \times \mathrm{GL}(\mathbb{C}^4) \to \mathrm{G}$ *such that the Lie algebra of* $\iota(G')$ *is* $\mathfrak{g}' := \mathfrak{g}_1' \oplus \mathfrak{g}_2'$.

(iv) We have $\mathfrak{h} \subset \mathfrak{g}'$*, hence* $\iota(G')$ *contains a maximal torus of* G*. Furthermore we have* $\mathfrak{g}^{\mathfrak{g}'} = \mathfrak{h}_{\omega_1^\vee} \oplus \mathfrak{h}_{\omega_8^\vee}$*, where* $\mathfrak{g}^{\mathfrak{g}'} := \{v \in \mathfrak{g} \mid [v, \mathfrak{g}'] = 0\}$.

Proof. (i) By considering Cartan pairings, there is a Lie subalgebra $\mathfrak{l}_1 \subset \mathfrak{g}$ of type A_4 such that $\{\alpha_1, \alpha_3, \alpha_4, \alpha_2\}$ is the ordered set of simple roots. Let V be the 4-dimensional subspace of $\mathfrak{l}_1$ generated by root spaces $\mathfrak{g}_\beta$ such that $\beta = \alpha_1 + \sum_{j=2}^4 c_j\alpha_j$ for some c_j. The restriction of adjoint representation induces $\mathfrak{g}_1' = \mathfrak{gl}(V)$, and $\mathfrak{f}_1 = \mathfrak{sl}(V)$.

(ii) By considering Cartan pairings, there is a Lie subalgebra $\mathfrak{l}_2 \subset \mathfrak{g}$ of type D_4 such that $\{\alpha_0, \alpha_6, \alpha_5, \alpha_7\}$ is the ordered set of simple roots. Let V be the 6-dimensional subspace of $\mathfrak{l}_2$ generated by root spaces $\mathfrak{g}_\beta$ such that the root β can be written as $\alpha_5 + c_6\alpha_6 + c_7\alpha_7 + c_0\alpha_0$ for some c_j. The restriction of adjoint representation induces $\mathfrak{g}_2' \subset \mathfrak{gl}(V)$, identified with the representation $\wedge^2\mathbb{C}^4$ of $\mathfrak{gl}(\mathbb{C}^4)$. Moreover, $\mathfrak{f}_2 = [\mathfrak{g}_2', \mathfrak{g}_2'] \simeq \mathfrak{sl}(\mathbb{C}^4)$.

(iii) For $i \in \{1, 2\}$, the adjoint representation of $\mathfrak{g}$ gives rise to the homomorphism $\iota_i : \mathrm{GL}(\mathbb{C}^4) \to G$ of finite kernel such that the Lie algebra of $\iota_i(\mathrm{GL}(\mathbb{C}^4))$ is $\mathfrak{g}_i'$. Since $[\mathfrak{g}_1', \mathfrak{g}_2'] = 0$ and $\mathfrak{g}_1' \cap \mathfrak{g}_2' = 0$, the space $\mathfrak{g}' := \mathfrak{g}_1' \oplus \mathfrak{g}_2'$ is a Lie subalgebra of $\mathfrak{g}$. Now $\iota : (a, b) \in G' \mapsto \iota_1(a) \cdot \iota_2(b) \in G$ is the required homomorphism.

(iv) Since the coweights $\omega_1^\vee$, $\omega_5^\vee$, and $\alpha_j^\vee$, $j \in J := \{0, 2, 3, 4, 6, 7\}$, are linearly independent, we have $\mathfrak{h} = \mathfrak{h}_{\omega_1^\vee} \oplus \mathfrak{h}_{\omega_5^\vee} \oplus (\oplus_{j\in J}\mathfrak{h}_{\alpha_j^\vee})$, implying that $\mathfrak{h} \subset \mathfrak{g}'$ and $\mathfrak{g}^{\mathfrak{g}'} \subset \mathfrak{g}^{\mathfrak{h}} = \mathfrak{h}$. The roots $\alpha_i : \mathfrak{h} \to \mathbb{C}$ satisfy that $\mathfrak{h}_{\omega_1^\vee} \oplus \mathfrak{h}_{\omega_8^\vee} = \cap_{j=2}^7 \ker \alpha_j = \cap_{j\in J} \ker \alpha_j$, implying the conclusion. □

As in the F_4 case, define $\tau_1 := (e_1, e_8)$ and $\tau_2 := \epsilon_1\epsilon_2$ and W' the finite subgroup of $W(E_8)$ generated by τ_1 and τ_2. The analogue statements of Lemma 4.21 and Proposition 4.23 hold for E_8-case, which proves Proposition 4.16 for the E_8 case. Note that in this case, the toric surface $\bar{T}'$ is the same as that for F_4, which is in fact the toric surface with fan consisting of Weyl chambers of B_2 and their faces.

5. Rigidity of the wonderful compactification of C_n

5.1. **A recap on spherical varieties.** Wonderful compactifications of simple algebraic groups of adjoint type are spherical varieties. In the following we recall some basic notions and results of spherical varieties. One can consult surveys of Knop [Kn] or Perrin [Pe] for more details.

Let L be a connected reductive group, and $B \subset L$ a Borel subgroup. For a closed subgroup $H \subset L$, the homogeneous variety L/H is said spherical if it admits an open B-orbit. The B-stable prime divisors on L/H are called colors of L/H, the set of whom is denoted by $\mathfrak{D}(L/H)$. Let $\mathbb{C}(L/H)^{(B)}$ be the set of nonzero rational functions f on L/H such that there exists a character χ_f on B satisfying $b \cdot f = \chi_f(b)f$ for all $b \in B$. The constant functions form a subgroup $\mathbb{C}^*$ of $\mathbb{C}(L/H)^{(B)}$, and the quotient group $\Lambda(L/H) := \mathbb{C}(L/H)^{(B)}/\mathbb{C}^*$ is called the weight lattice of L/H. Given a valuation ν on $\mathbb{C}(L/H)$, we can define $\varrho_\nu : \Lambda(L/H) \to \mathbb{Q}$ by sending f to $\nu(f)$. This induces a map ϱ from the set of valuations on $\mathbb{C}(L/H)$ to $\Lambda^\vee_{\mathbb{Q}}(L/H) := \mathrm{Hom}(\Lambda(L/H), \mathbb{Z}) \otimes \mathbb{Q}$. The set of images of all L-invariant valuations is called the valuation cone of L/H and denoted by $\mathcal{V}(L/H) \subset \Lambda^\vee_{\mathbb{Q}}(L/H)$.

A colored cone is a pair $(\mathfrak{C}, \mathfrak{D})$ such that $\mathfrak{D} \subset \mathfrak{D}(L/H)$, $\mathfrak{C}$ is a cone in $\Lambda^\vee_{\mathbb{Q}}(L/H)$ generated by $\varrho(\mathfrak{D})$ and finitely many elements in $\mathcal{V}(L/H)$ satisfying $\mathfrak{C}^o \cap \mathcal{V}(L/H) \neq \emptyset$, where $\mathfrak{C}^o$ denotes the interior domain of the cone $\mathfrak{C}$. The pair $(\mathfrak{C}_0, \mathfrak{D}_0)$ is called a colored face of $(\mathfrak{C}, \mathfrak{D})$ if $\mathfrak{C}_0$ is a face of $\mathfrak{C}$, $\mathfrak{C}_0^o \cap \mathcal{V}(L/H) \neq \emptyset$ and $\mathfrak{D}_0 = \mathfrak{D} \cap \varrho^{-1}(\mathfrak{C}_0)$. A colored fan $\mathbb{F}$ is a nonempty finite set of colored cones such that colored faces of colored cones in $\mathbb{F}$ also belong to $\mathbb{F}$, and $\mathfrak{C}_1^o \cap \mathfrak{C}_2^o \cap \mathcal{V}(L/H) = \emptyset$ for pairwise different colored cones $(\mathfrak{C}_1, \mathfrak{D}_1)$ and $(\mathfrak{C}_2, \mathfrak{D}_2)$ in $\mathbb{F}$. The colored fan $\mathbb{F}$ is called strictly convex if $(0, \emptyset) \in \mathbb{F}$.

An L-spherical variety is a normal variety X which admits an L-equivariant open embedding $L/H \subset X$ for a homogeneous spherical variety L/H. Sometimes we call such X a spherical L/H-embedding. In this case, B-stable prime divisors on X consist of closures of B-stable prime divisors on L/H and those contained in the boundary $\partial X := X \setminus L/H$. There are finitely many L-orbits on X. Given an L-orbit Y in X, we can define the associated colored cone $(\mathfrak{C}_Y, \mathfrak{D}_Y)$ such that $\mathfrak{C}_Y$ is the cone in $\Lambda^\vee_{\mathbb{Q}}(L/H)$ generated by those $\varrho(D)$, where D runs over the set of B-stable prime divisors on X containing Y, and $\mathfrak{D}_Y$ is the set of $D \in \mathfrak{D}(L/H)$ such that its closure satisfies $\bar{D} \supset Y$. The set of colored cones associated to all L-orbits on X is a strictly convex colored fan in $\Lambda^\vee_{\mathbb{Q}}(L/H)$, which is called the colored fan of X, and denoted by $\mathbb{F}_X$. Given two L-orbits Y and Z on X, then $\bar{Y} \subset \bar{Z}$ if and only if $(\mathfrak{C}_Z, \mathfrak{D}_Z)$ is a colored face of $(\mathfrak{C}_Y, \mathfrak{D}_Y)$. In particular, $Y = Z$ if and only if $(\mathfrak{C}_Y, \mathfrak{D}_Y) = (\mathfrak{C}_Z, \mathfrak{D}_Z)$. For the convenience of discussions, we also write $\mathfrak{C}_{\bar{Y}} := \mathfrak{C}_Y$ and $\mathfrak{D}_{\bar{Y}} := \mathfrak{D}_Y$ for the closure $\bar{Y}$ of an orbit Y.

Here we summarize some of the basic results of the theory of spherical varieties.

Proposition 5.1. *Let L/H be a homogeneous spherical variety.*

(i) Given a strictly convex colored fan $\mathbb{F}$ in $\Lambda^\vee_{\mathbb{Q}}(L/H)$, there is a unique (up to L-equivariant isomorphisms) spherical L/H-embedding X such that $\mathbb{F}_X = \mathbb{F}$.

(ii) Let X be a smooth L/H-embedding, and S_Y the set of B-stable prime divisors containing the orbit Y on X. Then $\{\varrho(\nu_D) \mid D \in S_Y\}$ is a linearly independent set in $\Lambda^\vee_{\mathbb{Q}}(L/H)$.

(iii) A spherical L/H-embedding X is complete if and only if $\mathcal{V}(L/H)$ is contained in the union of colored cones of X.

(iv) The Chow group $A^1(X)$ of a spherical variety X is generated by B-stable prime divisors. The rational equivalences are generated by relations $\mathrm{Div}(f) = 0$, for $f \in \mathbb{C}(L/H)^{(B)}$.

(v) The cycle map $A^1(X) \to H^2(X, \mathbb{Z})$ of a smooth projective spherical variety X is an isomorphism.

(vi) The anti-canonical divisor of a spherical L/H-embedding X is given by $-K_X = \sum_{D \in \mathfrak{D}(L/H)} m_D D + \sum_{j=1}^n D_j$, where the coefficients m_D only depend on the open orbit L/H, and $D_1, \ldots, D_n$ are the prime boundary divisors.

Take a spherical L/H-embedding X and a spherical L/H'-embedding X' with $H \subset H'$. The natural morphism $\psi : L/H \to L/H'$ induces a homomorphism $\Lambda(L/H') \to \Lambda(L/H)$, and thus a homomorphism $\psi_* : \Lambda^\vee_{\mathbb{Q}}(L/H) \to \Lambda^\vee_{\mathbb{Q}}(L/H')$. We say ψ_* sends $\mathbb{F}_X$ to $\mathbb{F}_{X'}$ if given any $(\mathfrak{C}, \mathfrak{D}) \in \mathbb{F}_X$ there exists $(\mathfrak{C}', \mathfrak{D}') \in \mathbb{F}_{X'}$ such that $\psi_*(\mathfrak{C}) \subset \mathfrak{C}'$, and each element $D \in \mathfrak{D}$ either dominates L/H' or satisfies $\psi_*(D) \in \mathfrak{D}'$.

Proposition 5.2. *The morphism $\psi : L/H \to L/H'$ can be extended to an L-equivariant morphism $\Psi : X \to X'$ if and only if ψ_* sends $\mathbb{F}_X$ to $\mathbb{F}_{X'}$.*

Example 5.3. [B07, Example 2.1.3] Let X be the wonderful compactification of a simple linear algebraic group G of adjoint type and of rank n. It is a $(G\times G)$-spherical variety with open orbit being $O=(G\times G)/\mathrm{diag}(G)\simeq G$. The weight lattice of O as a homogeneous spherical variety coincides with the root lattice of G, i.e. $\Lambda(O)=\oplus_{i=1}^n\mathbb{Z}\alpha_i$, where $\alpha_1,\dots,\alpha_n$ is the set of simple roots. The dual lattice $\Lambda^\vee(O)=\oplus_{i=1}^n\mathbb{Z}\omega_i^\vee$ with $(\omega_i^\vee,\alpha_j)=\delta_{ij}$ is identified with the coweight lattice of G. Recall that $\mathrm{Pic}(X)$ is identified with the weight lattice of G. The boundary divisor D_i associated to the root α_i (i.e. $D_i=\alpha_i\in\mathrm{Pic}(X)$) satisfies that $\varrho(D_i)=-\omega_i^\vee$ and the color $D(\omega_j)$ associated to ω_j (i.e. $D(\omega_j)=\omega_j\in\mathrm{Pic}(X)$) satisfies that $\varrho(D(\omega_j))=\alpha_j^\vee$. Let R be the root system of G. Then $\alpha_1^\vee,\dots,\alpha_n^\vee$ is the set of simple roots of the dual root system $R^\vee$ of R, and $\omega_1^\vee,\dots,\omega_n^\vee$ is the set of fundamental dominant weights of $R^\vee$. The valuation cone of O is the negative Weyl chamber, i.e. $\mathcal{V}(O)=\mathrm{cone}\{-\omega_1^\vee,\dots,-\omega_n^\vee\}$. The colored fan $\mathbb{F}_X$ of X consists of all colored faces of $(\mathcal{V}(O),\emptyset)$. The colored cone of the open orbit D_I^o of $D_I:=\cap_{i\in I}D_I$ is $(\mathrm{cone}\{-\omega_i\mid i\in I\},\emptyset)$. In this case, the anti-canonical divisor $-K_X$ is given by $-K_X=2\sum_{i=1}^n D(\omega_i)+\sum_{j=1}^n D_j$.

Lemma 5.4. [BK, Propositon 6.1.11] *Let X be the wonderful compactification of a simple linear algebraic group G of adjoint type. The restriction of the color $D(\omega_k)$ to the closed orbit $D_{1,\dots,n}:=\cap_{i=1}^n D_i=G/B^-\times G/B$ is the Cartier divisor $\mathcal{O}(-w_0\omega_k,\omega_k)$, where w_0 is the longest element in the Weyl group of G.*

5.2. **Lagrangian Grassmannians.** Let (W,Ω) be a symplectic vector space of dimension $2n$. Take two copies $(W_i,\Omega_i), i=1,2$, of (W,Ω) and endow the vector space $W_1\oplus W_2$ the symplectic form given by $p_1^*\Omega_1-p_2^*\Omega_2$, where $p_i:W_1\oplus W_2\to W_i$ is the natural i-th projection. Let $\mathrm{LG(2n,W_1\oplus W_2)}$ be the Lagrangian Grassmannian associated to the symplectic vector space $W_1\oplus W_2$, which parameterizes Lagrangian subspaces in $W_1\oplus W_2$. Note that $\dim\mathrm{LG(2n,W_1\oplus W_2)}=n(2n+1)$, which is the same as $\dim\mathrm{Sp}(2n)$. We will identify W_i with the i-th summand in $W_1\oplus W_2$.

Lemma 5.5. *Let $[V]\in\mathrm{LG(2n,W_1\oplus W_2)}$, then $\dim V\cap W_1=\dim V\cap W_2$.*

Proof. Let $V_i=V\cap W_i$. Note that V_1 is the kernel of $p_2|_V:V\to W_2$. Let $I_2=p_2(V)$ be the image, then $\dim I_2=\dim V/V_1$, which gives that $\dim W_2\cap I_2^\perp=\dim V_1$. As $V\subset W_1\oplus I_2$, we get $V=V^\perp\supset W_1^\perp\cap I_2^\perp=W_2\cap I_2^\perp$, which gives $V_2=V\cap W_2\supset W_2\cap I_2^\perp$. This implies that $\dim V_2\geq\dim V_1$. By symmetry, we have $\dim V_1=\dim V_2$. □

For any $0\leq k\leq n$, we denote by $\mathrm{IG}(k,W_i)$ the k-th isotropic Grassmannian in W_i, which parameterizes k-dimensional isotropic linear subspaces in W_i. Define

$$\mathrm{LG}(k):=\{[V]\in\mathrm{LG(2n,W_1\oplus W_2)}\,|\,\dim V\cap W_1=\dim V\cap W_2=k\}.$$

Consider the map $\pi_k:\mathrm{LG}(k)\to\mathrm{IG}(k,W_1)\times\mathrm{IG}(k,W_2)$ defined by $[V]\mapsto([V\cap W_1],[V\cap W_2])$, which is $\mathrm{Sp}(W_1)\times\mathrm{Sp}(W_2)$-equivariant.

Lemma 5.6. (i) *The fiber of π_k is isomorphic to* $\mathrm{Sp}(\mathbb{C}^{2n-2k})$.
(ii) *The variety* $\mathrm{LG}(k)$ *is an* $\mathrm{Sp}(W_1)\times\mathrm{Sp}(W_2)$*-orbit and* $\mathrm{LG(2n,W_1\oplus W_2)}=\bigsqcup_{k=0}^n\mathrm{LG}(k)$ *is the decomposition of* $\mathrm{LG(2n,W_1\oplus W_2)}$ *into* $\mathrm{Sp}(W_1)\times\mathrm{Sp}(W_2)$*-orbits.*

Proof. Consider first the case $k=0$. Any element $[V]\in\mathrm{LG}(0)$ is the graph of a unique linear isomorphism $\ell_V:W_1\to W_2$. Note that V is Lagrangian if and only if ℓ_V is a symplectomorphism, which shows that $\mathrm{LG}(0)\simeq\mathrm{Sp}(W)$.

In general, fix $[V_i] \in \mathrm{IG}(k, W_i)$. Let $\bar{W}_i = V_i^{\perp, W_i}/V_i$, which is a symplectic vector space of dimension $2n-2k$. Any element $[V] \in \pi_k^{-1}([V_1],[V_2])$ is uniquely determined by $\bar{V} := V/(V_1 \oplus V_2)$. Note that V is Lagrangian if and only if so is $\bar{V} \subset \bar{W}_1 \oplus \bar{W}_2$, while the latter satisfies $\bar{V} \cap \bar{W}_i = \{0\}$. This shows that $\pi_k^{-1}([V_1],[V_2]) \simeq \mathrm{Sp}(\bar{W}_i)$, proving (i). To prove (ii), we note that the stabilizer in $\mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2)$ of $([V_1],[V_2])$ surjects to $\mathrm{Sp}(\bar{W}_1)$, hence $\mathrm{LG}(k)$ is $\mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2)$-homogeneous. □

Recall that $\dim \mathrm{IG}(k, W) = \frac{k(4n+1-3k)}{2}$ and $\dim \mathrm{Sp}(2n) = 2n^2+n$, which implies immediately the following

Corollary 5.7. (a) *We have* $\dim \mathrm{LG}(k) = (2n^2+n) - k^2$, *i.e.* $\mathrm{codim}(\mathrm{LG}(k)) = k^2$.

(b) *The subvariety* $\mathrm{LG}(0) \subset \mathrm{LG}(2\mathrm{n}, \mathrm{W}_1 \oplus \mathrm{W}_2)$ *is open and the closure of* $\mathrm{LG}(1)$ *is the unique* $\mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2)$*-stable prime divisor. The unique closed* $\mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2)$*-orbit of* $\mathrm{LG}(2\mathrm{n}, \mathrm{W}_1 \oplus \mathrm{W}_2)$ *is* $\mathrm{LG}(n)$*, which is isomorphic to* $\mathrm{LG}(n, W_1) \times \mathrm{LG}(n, W_2)$.

Lemma 5.8. *We have* $\overline{\mathrm{LG}(k)} = \cup_{j \geq k} \mathrm{LG}(j)$.

Proof. By Corollary 5.7, we have $\overline{\mathrm{LG}(0)} = \mathrm{LG}(2\mathrm{n}, \mathrm{W}_1 \oplus \mathrm{W}_2) = \cup_{j \geq 0} \mathrm{LG}(j)$. For $[V_i] \in \mathrm{IG}(k, W_i)$, let $\bar{W}_i = V_i^{\perp, W_i}/V_i$. Then

$$\overline{\pi_k^{-1}([V_1],[V_2])} = \mathrm{LG}(2n-2k, \bar{W}_1 \oplus \bar{W}_2),$$

which implies that $\overline{\mathrm{LG}(k)} \cap \mathrm{LG}(j) \neq \emptyset$ if $j \geq k$. As $\overline{\mathrm{LG}(k)}$ is $\mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2)$-invariant and $\mathrm{LG}(j)$ is an orbit, we have $\mathrm{LG}(j) \subset \overline{\mathrm{LG}(k)}$. This gives a chain

$$\mathrm{LG}(n) \subset \overline{\mathrm{LG}(n-1)} \subset \cdots \subset \overline{\mathrm{LG}(0)} = \mathrm{LG}(2\mathrm{n}, \mathrm{W}_1 \oplus \mathrm{W}_2),$$

which proves the claim. □

Consider the symplectic involution $\tau := (1_{W_1}, -1_{W_2}) \in \mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2)$, which acts naturally on $\mathrm{LG}(2\mathrm{n}, \mathrm{W}_1 \oplus \mathrm{W}_2)$. Note that an element $[V] \in \mathrm{LG}(2\mathrm{n}, \mathrm{W}_1 \oplus \mathrm{W}_2)$ is fixed by τ if and only if $V = V_1 \oplus V_2$, where $V_i = V \cap W_i$. This gives that $\mathrm{LG}(2\mathrm{n}, \mathrm{W}_1 \oplus \mathrm{W}_2)^\tau = \mathrm{LG}(n) \simeq \mathrm{LG}(n, W_1) \times \mathrm{LG}(n, W_2)$. Let $Z = \mathrm{LG}(2\mathrm{n}, \mathrm{W}_1 \oplus \mathrm{W}_2)/\tau$ and $\psi : \mathrm{LG}(2\mathrm{n}, \mathrm{W}_1 \oplus \mathrm{W}_2) \to Z$ the natural projection. Denote $Z^\circ = \mathrm{LG}(0)/\tau$, $Z_j^o := \mathrm{LG}(j)/\tau$ and $Z_j := \overline{\mathrm{LG}(j)}/\tau$ for $j \geq 1$, then $Z_n \subset Z_{n-1} \subset \cdots \subset Z_1 \subset \overline{Z^\circ} = Z$ by Lemma 5.8. Let $K = \mathrm{PSp}(W_1) \times \mathrm{PSp}(W_2)$, which acts on Z, preserving each Z_j.

Proposition 5.9. *(i) The Lagrangian Grassmannian* $\mathrm{LG}(2\mathrm{n}, \mathrm{W}_1 \oplus \mathrm{W}_2)$ *is spherical under the* $\mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2)$*-action with the unique open orbit* $\mathrm{LG}(0) \simeq \mathrm{Sp}(W)$ *and the unique closed orbit* $\mathrm{LG}(n) \simeq \mathrm{LG}(n, W_1) \times \mathrm{LG}(n, W_2)$.

(ii) The variety Z *is spherical under the* K*-action, with unique open orbit* $Z^\circ \simeq \mathrm{PSp}(W)$ *and unique closed orbit* $Z_n \simeq \mathrm{LG}(n, W_1) \times \mathrm{LG}(n, W_2)$*. The boundary* $Z_1 = Z \setminus Z^\circ$ *is an irreducible divisor and each* Z_i *is a* K*-orbit closure in* Z.

Proof. Let $W' = \{(w,w) \in W_1 \oplus W_2 | w \in W\}$, which is Lagrangian in $W_1 \oplus W_2$. The isotropy group of $[W']$ is the diagonal subgroup of $\mathrm{Sp}(W) \times \mathrm{Sp}(W)$, hence the orbit of $[W']$ is isomorphic to $\mathrm{Sp}(W)$. This implies that $\mathrm{LG}(2\mathrm{n}, \mathrm{W}_1 \oplus \mathrm{W}_2)$ is spherical, which proves (i) by Corollary 5.7.

We denote by $[W'_\tau] \subset Z$ the image of $[W']$ in Z. Then the stabilizer $K_{[W'_\tau]}$ is isomorphic to the diagonal subgroup of $\mathrm{PSp}(W_1) \times \mathrm{PSp}(W_2)$, hence the open orbit Z° is isomorphic to $\mathrm{PSp}(W)$ and Z is spherical. Then the claim follows from (i). □

5.3. **A construction of wonderful compactification of C_n.** Let X be the wonderful compactification of $\mathrm{PSp}(W)$ and X° the open orbit. By Example 5.3, X is a spherical variety under the K-action. Both Z and X are spherical varieties under the K-action, with an isomorphic open orbit $Z^\circ \simeq X^\circ \simeq \mathrm{PSp}(W)$. The boundary divisor $\partial X := X \setminus X^\circ$ has a decomposition into irreducible components $\cup_{i=1}^n D_i$, where D_i is the divisor corresponding to the simple root α_i.

Proposition 5.10. *The natural isomorphism $\phi^\circ : X^\circ \to Z^\circ$ extends to a K-equivariant birational morphism $\phi : X \to Z$.*

Proof. The isomorphism ϕ° induces an identity $\phi^\circ_* : \Lambda^\vee_{\mathbb{Q}}(X^\circ) = \Lambda^\vee_{\mathbb{Q}}(Z^\circ)$. By Example 5.3, $\mathbb{F}_X$ consists of colored faces of $(\mathcal{V}(\mathrm{PSp}(W)), \emptyset)$, where $\mathcal{V}(\mathrm{PSp}(W)) = \mathrm{cone}\{-\omega_1^\vee, \ldots, -\omega_n^\vee\}$. As Z is projective, the valuation cone $\mathcal{V}(\mathrm{PSp}(W))$ is contained in the union of the colored cones of Z by Proposition 5.1(iii). As Z has a unique closed orbit Z_n, we have $\mathfrak{C}_{Z_n} \supset \mathcal{V}(\mathrm{PSp}(W))$. By definition ϕ°_* sends any element $(\mathfrak{C}, \mathfrak{D}) \in \mathbb{F}_X$ to $(\mathfrak{C}_{Z_n}, \mathfrak{D}_{Z_n})$, thence ϕ°_* sends $\mathbb{F}_X$ to $\mathbb{F}_Z$. It follows that ϕ° can be extended to a K-equivariant birational morphism $\phi : X \to Z$ by Proposition 5.2. □

Lemma 5.11. *The morphism ϕ sends the open orbit D_1^o of D_1 isomorphically to the open orbit of Z_1, and the exceptional locus of ϕ is $\cup_{i=2}^n D_i$.*

Proof. By Proposition 5.9(ii), the boundary Z_1 of Z is irreducible. Since ϕ is a K-equivariant birational morphism with connected fibers, there exists a unique boundary divisor D_k such that ϕ sends its open orbit D_k^o isomorphically to the open orbit of Z_1, and the exceptional locus is $\mathrm{Exc}(\phi) = \cup_{i \neq k} D_i$. By the description of orbits closures on wonderful compactifications in Section 2.2, the orbit D_k^o admits a $\mathrm{PSp}(\mathbb{C}^{2n-2k})$-fibration over $\mathrm{LG}(k, W) \times \mathrm{LG}(k, W)$. It follows that $k = 1$ by Lemma 5.6. □

Lemma 5.12. *The dual weight lattice of Z^o is $\Lambda^\vee(Z^o) = \oplus_{i=1}^n \mathbb{Z}\omega_i^\vee$, and the valuation cone $\mathcal{V}(Z^o) = \mathrm{cone}\{-\omega_1^\vee, \ldots, -\omega_n^\vee\}$. The colored cone of the orbit Z_k^o with $1 \leq k \leq n$ is $(\mathrm{cone}\{-\omega_1^\vee, \alpha_1^\vee, \ldots, \alpha_{k-1}^\vee\}, \{D(\omega_1), \ldots, D(\omega_{k-1})\})$.*

Proof. The descriptions of $\Lambda^\vee(Z^o)$ and $\mathcal{V}(Z^o)$ are clear from Example 5.3, since they depend only on the open orbit $Z^o = \mathrm{PSp}(W)$. Set $\mathfrak{C}_k := \mathrm{cone}\{-\omega_1^\vee, \alpha_1^\vee, \ldots, \alpha_{k-1}^\vee\}$ and $\mathfrak{D}_k := \{D(\omega_1), \ldots, D(\omega_{k-1})\}$ for $1 \leq k \leq n$. We claim that the colored cone of Z_n is $(\mathfrak{C}_n, \mathfrak{D}_n)$. By the claim, there is a chain of elements in $\mathbb{F}_Z$ as follows: $(0, \emptyset) \subset (\mathfrak{C}_1, \mathfrak{D}_1) \subset \cdots (\mathfrak{C}_{n-1}, \mathfrak{D}_{n-1}) \subset (\mathfrak{C}_n, \mathfrak{D}_n)$. By the bijective correspondence with orbit closures, these are all colored cones of Z and the colored cone of Z_k is $(\mathfrak{C}_k, \mathfrak{D}_k)$.

To verify the claim, we consider the closed orbit $D_{1,\ldots,n} := G/B^- \times G/B$ on X, where $G = \mathrm{PSp}(W)$ and B is the Borel subgroup. To simplify the notations, let $X_n = D_{1,\ldots,n}$. The map ϕ sends X_n to the unique closed orbit $Z_n = G/P_n^- \times G/P_n$ on Z, which is induced by the inclusions $B \subset P_n$ and $B^- \subset P_n^-$. Since $\mathcal{O}_X(D(\omega_k))|_{X_n} = \mathcal{O}_{X_n}(\omega_k, \omega_k)$ by Lemma 5.4, we know that $\phi(D(\omega_k) \cap X_n) = Z_n$ if and only if $k \neq n$. By the definition of colored cones, $\mathfrak{D}_{Z_n} = \mathfrak{D}_n$ and $\mathfrak{C}_{Z_n}$ is the cone in $\Lambda^\vee(Z^o) = \oplus_{i=1}^n \mathbb{Z}\omega_i^\vee$ generated by $\varrho(\mathfrak{D}_n)$ and $\varrho(Z_1)$.

By Example 5.3 and Lemma 5.11, $\varrho(Z_1) = -\omega_1^\vee$ and $\varrho(\mathfrak{D}_n) = \{\alpha_1^\vee, \dots, \alpha_{n-1}^\vee\}$, completing the proof. □

Lemma 5.13. *The dual weight lattice of* $\mathrm{LG}(0)$ *is* $\Lambda^\vee(\mathrm{LG}(0)) = \oplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$*, and the valuation cone is* $\mathcal{V}(Z^o) = \mathrm{cone}\{-\omega_1^\vee, \dots, -\omega_n^\vee\}$*. For each* k*, the colored cone of the orbit* $\mathrm{LG}(k)$ *is given by* $\mathfrak{C}_{\mathrm{LG}(k)} = \mathrm{cone}\{-\omega_1^\vee, \alpha_1^\vee, \dots, \alpha_{k-1}^\vee\}$ *and* $\mathfrak{D}_{\mathrm{LG}(k)} = \{D(\omega_1), \dots, D(\omega_{k-1})\}$*.*

Proof. The surjective quotient map $\mathrm{LG}(0) \to Z^o$ induces $\Lambda^\vee(Z^o) = \oplus_{i=1}^n \mathbb{Z}\omega_i^\vee \supset \Lambda^\vee(\mathrm{LG}(0)) = \oplus_{i=1}^n \mathbb{Z}\alpha_j^\vee$. Under the identity $\Lambda^\vee_{\mathbb{Q}}(\mathrm{LG}(0)) = \Lambda^\vee_{\mathbb{Q}}(Z^o)$, we have $\mathcal{V}(\mathrm{LG}(0)) = \mathcal{V}(Z^o)$, $\mathfrak{C}_{\mathrm{LG}(k)} = \mathfrak{C}_{Z_k^o}$, $\mathfrak{D}_{\mathrm{LG}(k)} = \mathfrak{D}_{Z_k^o}$ for each k. Then the claim follows from Lemma 5.12. □

Proposition 5.14. *(1) The morphism* ϕ *satisfies* $\phi(D_k) = Z_k$ *for each* k*.*

(2) Let Y *be the fiber product* $X \times_Z \mathrm{LG(2n, W_1 \oplus W_2)}$ *and* $X \xleftarrow{\psi'} Y \xrightarrow{\phi'} \mathrm{LG(2n, W_1 \oplus W_2)}$ *the two projections. Then* $\psi' : Y \to X$ *is a double cover with branch loci* D_n*.*

(3) The birational map ϕ *is the composition of successive blow-ups of* Z *along the strict transforms of* $Z_n, Z_{n-1}, \cdots, Z_2$*, from the smallest to the biggest.*

(4) The degree 2 rational map $\psi \circ (\phi')^{-1} : \mathrm{LG(2n, W_1 \oplus W_2)} \dashrightarrow X$ *preserves the VMRT-structures.*

Proof. (1) By Example 5.3, $\varrho(D_k) = -\omega_k^\vee$ and $\varrho(D(\omega_k)) = \alpha_k^\vee$ for all k. It follows that $\varrho(D_k) = k\varrho(D_1) + \sum_{j=1}^{k-1}(k-j)\varrho(D(\omega_j))$ in $\Lambda^\vee_{\mathbb{Q}}(\mathrm{PSp}(W))$. Then $\mathfrak{C}^o_{D_k} \subset \mathfrak{C}^o_{Z_k}$ by Lemma 5.12, which implies that ϕ sends the orbit D_k^o onto the orbit Z_k^o, and thus $\phi(D_k) = Z_k$.

(2) Since $\psi : \mathrm{LG(2n, W_1 \oplus W_2)} \to Z$ is a double cover with branch loci Z_n, the morphism $\psi' : Y \to X$ is a double cover with branch loci $\phi^{-1}(Z_n) = D_n$.

(3) For $0 \le i \le n-1$, set $\mathfrak{C}_i' := \mathrm{cone}(\alpha_1^\vee, \dots, \alpha_i^\vee, -\omega_1^\vee, -\omega_{i+2}^\vee, \dots, -\omega_n^\vee)$ and $\mathfrak{D}_i' := \{D(\omega_1), \dots, D(\omega_i)\}$. The set $\mathbb{F}_i$ of all colored faces of $(\mathfrak{C}_i', \mathfrak{D}_i')$ is a colored fan in $\Lambda^\vee_{\mathbb{Q}}(\mathrm{PSp}(W))$. Let Y_i be the unique complete spherical $K/\mathrm{diag}(\mathrm{PSp}(W))$-embedding such that $\mathbb{F}_{Y_i} = \mathbb{F}_i$. There is a chain of K-equivariant birational morphism with connected fibers as follows:

$$X \cong Y_0 \xrightarrow{\pi_0} Y_1 \xrightarrow{\pi_1} Y_2 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{n-2}} Y_{n-1} \cong Z.$$

For each $1 \le k \le n-1$, $-\omega_{k+1}^\vee$ lies in the relative interior of $\mathfrak{C}_k'' := \mathrm{cone}(-\omega_1^\vee, \alpha_1^\vee, \dots, \alpha_k^\vee)$, and the two colored fans $\{(\mathfrak{C}, \mathfrak{D}) \in \mathbb{F}_{k-1} \mid -\omega_{k+1}^\vee \notin \mathfrak{C}\}$ and $\{(\mathfrak{C}, \mathfrak{D}) \in \mathbb{F}_k \mid \mathfrak{C}_k'' \nsubseteq \mathfrak{C}\}$ are the same one. Thus π_{k-1} is a birational morphism with connected fibers such that the exceptional loci is the strict transform $\widetilde{Z}_{k+1} \subset Y_k$ of Z_{k+1} and the exceptional locus on Y_{k-1} is a prime divisor which is the birational image of the boundary divisor D_{k+1}.

The blow-up of Y_k along $\widetilde{Z}_{k+1}$ is again a spherical $K/\mathrm{diag}(\mathrm{PSp}(W))$-embedding Y_{k-1}'. The morphisms $Y_{k-1} \to Y_{k-1}' \to Y_k$ induce maps of colored fans $\mathbb{F}_{k-1} \to \mathbb{F}_{Y_{k-1}'} \to \mathbb{F}_k$. There is no proper intermediate colored cone between $(\mathfrak{C}_{D_{k+1}}, \emptyset)$ and $(\mathfrak{C}_{\widetilde{Z}_{k+1}}, \mathfrak{D}_{\widetilde{Z}_{k+1}})$, where $\mathfrak{C}_{D_{k+1}} = \mathrm{cone}\{-\omega_{k+1}^\vee\}$, $\mathfrak{C}_{\widetilde{Z}_{k+1}} = \mathfrak{C}_k''$, and $\mathfrak{D}_{\widetilde{Z}_{k+1}} = \mathfrak{D}_k'$. Hence $\mathbb{F}_{k-1} = \mathbb{F}_{Y_{k-1}'}$ and $Y_{k-1} = Y_{k-1}'$.

(4) The restriction of $\psi \circ (\phi')^{-1}$ on the open orbit is the surjective group morphism $\mathrm{Sp}(W) \to \mathrm{PSp}(W)$. The VMRT on open orbits of $\mathrm{LG(2n, W_1 \oplus W_2)}$ and X coincides with the minimal nilpotent orbit $\mathbb{P}\mathscr{O} \subset \mathbb{P}\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of both $\mathrm{Sp}(W)$ and $\mathrm{PSp}(W)$. This completes the proof. □

5.4. **Rigidity of $\bar{C}_n$.** Let G/P be an IHSS (cf. Example 2.4). By Example 2.6, there exists a decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let $G_0 \subset G$ be the Lie subgroup corresponding to $\mathfrak{g}_0$. Then G/P carries a locally flat G_0-structure $\mathcal{G}_o$ which is given by the VMRT structure. The following result is collected from the proof of Case (A) of [M, Proposition 2.3].

Proposition 5.15. *Let G/P be an IHSS of rank ≥ 2, and let $\mathcal{V}$ be a holomorphic vector bundle over Δ. Let $\mathcal{S} \subset \mathbb{P}\mathcal{V}$ be an irreducible closed subvariety such that $\mathcal{S}_t \simeq G/P$ for $t \neq 0$. Suppose there exists a proper closed subvariety $N \subsetneq \mathcal{S}_0$ and a G_0-structure $\mathcal{G}$ on $\mathcal{S} \setminus N$ such that $\mathcal{S} \setminus N$ is a smooth family over Δ, and for $t \neq 0$, the G_0-structure $\mathcal{G}|_{\mathcal{S}_t}$ is the natural G_0-structure $\mathcal{G}_o$ on G/P.*

(i) Take any point $x \in \mathcal{S}_0 \setminus N$. Then after shrinking Δ to a smaller disk containing 0 *(denoted again by Δ), there is a meromorphic map $h : G/P \times \Delta \dashrightarrow \mathcal{S}$ preserving the G_0-structure such that for $t \neq 0$, h_t is a biholomorphic map, and the image of h_0 contains an open neighborhood of $x \in \mathcal{S}_0$.*

(ii) Suppose moreover that for $t \neq 0$, the embedding $\mathcal{S}_t \subset \mathbb{P}\mathcal{V}_t$ is given by the complete linear system $|\mathcal{O}_{G/P}(k)|$ for some $k \geq 1$, and that the germ of $\mathcal{S}_0$ at x is linearly non-degenerate in $\mathbb{P}\mathcal{V}_0$. Then the map h in (i) is biholomorphic. In particular, $\mathcal{S}_0 \simeq G/P$ and the inclusion $\mathcal{S}_0 \subset \mathbb{P}\mathcal{V}_0$ is induced by the complete linear system $|\mathcal{O}_{G/P}(k)|$.

Proof. For the convenience of readers, we repeat the proof given in [M]. Recall from Remark 2.7, the local flatness of a G_0-structure can be detected by the vanishing of certain vector-valued function $c^k, k = 0, 1, 2, \cdots$. Furthermore, these functions c^k vanish for the G_0-structure on G/P. As the function c^k is continuous, it vanishes on the whole $\mathcal{S} \setminus N$, which implies that the G_0-structure $\mathcal{G}$ on $\mathcal{S} \setminus N$ is locally flat. Then there is an analytic neighbohood U of x in $\mathcal{S}$ and a holomorphic map $f : U \to G/P$ whose restriction $f_t : U \cap \mathcal{S}_t \to G/P$ is a developing map (i.e. f_t preserves G_0-structures) whenever $U \cap \mathcal{S}_t \neq \emptyset$. If necessary, shrinking Δ to a smaller disk containing 0 (denoted as Δ again). Then there is a holomorphic map $h : G/P \times \Delta^* \to \mathbb{P}\mathcal{V}$ respecting canonical projections such that h induces a biholomorphic map from G/P onto $\mathcal{S}_t \subset \mathbb{P}\mathcal{V}_t$ for $t \in \Delta^* := \Delta \setminus \{0\}$, and such that h extends holomorphically across some nonempty subset U_0 of $G/P \times \{0\}$ with $h(U_0)$ contains an open neighborhood U_x of x in $\mathcal{S}_0$. By Hartogs extension of meromorphic functions, h extends to a meromorphic map from $G/P \times \Delta$ into $\mathbb{P}\mathcal{V}$ respecting canonical projections, to be denoted by the same symbol h. Since $h(G/P \times \Delta^*)$ is contained in $\mathcal{S}$, so is $h(G/P \times \{0\})$, verifying (i). The assertion (ii) follows from [M, Lemma 2.3] immediately. □

As a direct consequence of Proposition 5.15, we have the following result.

Proposition 5.16. *Let G/P be an IHSS of rank ≥ 2. Let $\pi : \mathcal{S} \to \Delta$ be a projective flat map such that $\mathcal{S}_t \simeq G/P$ for $t \neq 0$ and the central fiber $\mathcal{S}_0$ is irreducible and reduced. Assume that there exists a proper closed subset $N \subsetneq \mathcal{S}_0$ and a G_0-structure $\mathcal{G}$ on $\mathcal{S} \setminus N$ such that for $t \neq 0$, the G_0-structure $\mathcal{G}|_{\mathcal{S}_t}$ is the natural G_0-structure $\mathcal{G}_o$ on G/P. Then $\mathcal{S}_0$ is isomorphic to G/P.*

Proof. Take a holomorphic very ample line bundle $\mathcal{L}$ on $\mathcal{S}/\Delta$. This induces a closed immersion $\mathcal{S} \subset \mathbb{P}^r_\Delta$ over Δ such that $\mathcal{O}_{\mathbb{P}^r_\Delta}(1)|_{\mathcal{S}} = \mathcal{L}$. Then the Hilbert polynomial of $\mathcal{S}_t$ satisfies that $P_{\mathcal{S}_t}(m) = h^0(\mathcal{S}_t, \mathcal{O}_{\mathcal{S}_t}(m))$ for all sufficiently large integers m. Since π is a flat family, Hilbert polynomials are independent of $t \in \Delta$. Fix a sufficiently large integer m. The line bundle $\mathcal{O}(m) \in \mathrm{Pic}(\mathcal{S}/\Delta)$ gives rise to a closed immersion $f : \mathcal{S} \to \mathbb{P}^r_\Delta$ over Δ, where $r := P_{\mathcal{S}_t}(m)$ is independent of $t \in \Delta$. For each t, the morphism $f_t : \mathcal{S}_t \to \mathbb{P}^r = \mathbb{P}(H^0(\mathcal{S}_t, \mathcal{O}_{\mathcal{S}_t}(m))^\vee)$ is the morphism induced from the r-dimensional complete linear system $|\mathcal{O}_{\mathcal{S}_t}(m)|$ on the irreducible and reduced variety $\mathcal{S}_t$. In particular, $f_t(\mathcal{S}_t)$ is a linearly non-degenerate closed subvariety of $\mathbb{P}^r$ for each t. When we consider the closed immersion $\mathcal{S} \subset \mathbb{P}^r_\Delta$ induced by $\mathcal{O}(m)$, all the assumptions in Proposition 5.15(ii) are fulfilled, implying that $\mathcal{S}_0 \simeq G/P$. □

Proposition 5.17. *Let X be the wonderful compactification of an adjoint simple group G of rank n. Let $\pi : \mathcal{X} \to \Delta$ be a holomorphic family of connected projective manifolds such that $\mathcal{X}_t \simeq X$ for all $t \neq 0$. Assume (i) the $G \times G$-action on $\mathcal{X}_t$ extends to a $G \times G$-action on $\mathcal{X}$; (ii) there exists a $G \times G$-stable open subset $\mathcal{U} \subset \mathcal{X}$ such that $\pi|_\mathcal{U} : \mathcal{U} \to \Delta$ is a $(G \times G)/diag(G)$-fibration. Then $\mathcal{X}_0 \simeq X$.*

Proof. By assumption, $\mathcal{X}_0$ is also a $G \times G$-spherical variety. Set $\partial\mathcal{X}_t := \mathcal{X}_t \setminus \mathcal{U}_t$ for each $t \in \Delta$. By Proposition 5.1 (v), we have $\mathrm{Pic}(\mathcal{X}_t) = H^2(\mathcal{X}_t, \mathbb{Z})$ for each $t \in \Delta$. Then the invariance of cohomology under deformations implies that $\mathrm{Pic}(\mathcal{X}_0) \simeq \mathrm{Pic}(X)$. Since the open orbit $(G \times G)/\mathrm{diag}(G)$ is affine, the boundary $\partial\mathcal{X}_0$ is the union of n prime divisors $E_1, \ldots, E_n$.

Let $\mathcal{D}_i$ (resp. $\mathcal{D}(\omega_j)$) be the irreducible divisor on $\mathcal{X}$ which restricts to D_i (resp. $D(\omega_j)$) on $\mathcal{X}_t$ for $t \neq 0$. Since $\mathcal{U}/\Delta$ is a trivial fibration, so is $(\mathcal{D}(\omega_j) \cap \mathcal{U})/\Delta$. Then by Example 5.3, $\varrho(\mathcal{D}(\omega_j)|_{\mathcal{X}_0}) = \varrho(D(\omega_j)) = \alpha_j^\vee \in \Lambda^\vee_\mathbb{Q}(G)$. We have $\partial\mathcal{X}_t = -K_{\mathcal{X}_t} - 2\sum_{j=1}^n \mathcal{D}(\omega_j)|_{\mathcal{X}_t}$ for each t by Proposition 5.1(vi). The right hand side of this formula is invariant in $\mathrm{Pic}(\mathcal{X}/\Delta) \simeq \mathrm{Pic}(\mathcal{X}_t)$, then so is the left hand side. Combining with the fact $\sum_{i=1}^n \mathcal{D}_i|_{\mathcal{X}_t} = \partial\mathcal{X}_t \in \mathrm{Pic}(\mathcal{X}_t)$ for $t \neq 0$, we have $\sum_{i=1}^n \mathcal{D}_i|_{\mathcal{X}_0} = \partial\mathcal{X}_0 = \sum_{k=1}^n E_k \in \mathrm{Pic}(\mathcal{X}_0)$. Since $\mathcal{D}_i|_{\mathcal{X}_0} = \sum_{k=1}^n c_{ik} E_k$ for some nonnegative integers c_{ij} with $\sum_{k=1}^n c_{ik} \geq 1$, the formula above implies that $\mathcal{D}_i|_{\mathcal{X}_0} = E_i$, up to reordering $E_1, \ldots, E_n$. It follows that $\varrho(E_i) = \varrho(D_i) = -\omega_i^\vee \in \Lambda^\vee_\mathbb{Q}(G)$ for each i.

Let $\mathcal{M}$ be the irreducible closed subvariety on $\mathcal{X}$ which restricts to the closed orbit $D_{1,\ldots,n} := \cap_{i=1}^n D_i$ on $\mathcal{X}_t$ for $t \neq 0$. Take a closed orbit F that is contained in $\mathcal{M}_0$. Since $\mathcal{M}_t \subset \mathcal{D}_i|_{\mathcal{X}_t}$ for $t \neq 0$, we have $F \subset \mathcal{D}_i|_{\mathcal{X}_0} = E_i$ by continuity. Set $\mathcal{X}'_0 := \mathcal{X}_0 \setminus (\cup Y)$, where Y runs over the set of orbits on $\mathcal{X}_0$ such that $F \not\subseteq \overline{Y}$. Then the colored fan $\mathbb{F}_{\mathcal{X}'_0}$ is exactly the set of colored faces of the colored cone $(\mathfrak{C}_F, \mathfrak{D}_F)$ of F, and the cone $\mathfrak{C}_F$ is generated by $-\omega_1^\vee, \ldots, -\omega_n^\vee$, and $\varrho(\mathfrak{D}_F)$. Since $-\omega_1^\vee, \ldots, -\omega_n^\vee$ form a basis of $\Lambda^\vee_\mathbb{Q}(G)$, we have $\mathfrak{D}_F = \emptyset$ by Proposition 5.1(ii). It follows that $(\mathfrak{C}_F, \mathfrak{D}_F) = (\mathcal{V}(G), \emptyset) = (\mathfrak{C}_{D_{1,\ldots,n}}, \mathfrak{D}_{D_{1,\ldots,n}})$, and thus the colored fans satisfy $\mathbb{F}_{\mathcal{X}'_0} = \mathbb{F}_X$ by Example 5.3. This gives $\mathcal{X}'_0 \cong X$, implying $\mathcal{X}_0 \cong X$. □

Theorem 5.18. *Let (W, ω) be a symplectic vector space. Let $\pi : \mathcal{X} \to \Delta \ni 0$ be a smooth family of Fano varieties such that $\mathcal{X}_t \cong X$ for $t \neq 0$, where X is the wonderful compactification of* $\mathrm{PSp(W)}$*. Then $\mathcal{X}_0 \cong X$.*

Proof. By Theorem 3.2, the contraction $\phi : X \to Z$ constructed by Proposition 5.10 extends to a contraction (over Δ) $\Phi : \mathcal{X} \to \mathcal{Z}$ such that for $t \neq 0$, Φ_t coincides with ϕ. Let $\mathcal{D}$ be the irreducible divisor on $\mathcal{X}$ which restricts to D_n on $\mathcal{X}_t$ for $t \neq 0$. Let $\Psi' : \mathcal{Y} \to \mathcal{X}$ be the double cover of $\mathcal{X}$ ramified along $\mathcal{D}$, then by Proposition 5.14, $\mathcal{Y}_t \simeq Y$ for $t \neq 0$. Take the Stein factorization of $\mathcal{Y} \to \mathcal{Z}$ as $\mathcal{Y} \xrightarrow{\Phi'} \mathcal{Z}' \xrightarrow{\Psi} \mathcal{Z}$. Then for $t \neq 0$, $\mathcal{Z}'_t \simeq \mathrm{LG(2n, W_1 \oplus W_2)}$ and Φ'_t (resp. Ψ'_t) coincides with ϕ' (resp. ψ'). By construction, there is a nef and big Cartier divisor $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}/\Delta)$ such that $\mathcal{Z} = \mathrm{Proj}_\Delta(R(\mathcal{L}))$, where $R(\mathcal{L}) = \oplus_{m\geq 0} H^0(\mathcal{X}, \mathcal{L}^{\otimes m})$. Then Φ' is induced by the nef and big Cartier divisor $\tilde{\mathcal{L}} := (\Psi')^*\mathcal{L} \in \mathrm{Pic}(\mathcal{Y}/\Delta)$ such that $\mathcal{Z}' = \mathrm{Proj}_\Delta(R(\tilde{\mathcal{L}}))$, where $R(\tilde{\mathcal{L}}) = \oplus_{m\geq 0} H^0(\mathcal{Y}, \tilde{\mathcal{L}}^{\otimes m})$. In particular, the fiber $\mathcal{Z}'_t$ at each $t \in \Delta$ is a normal projective variety. As Φ' has connected fibers and $\mathcal{Z}'_0$ has the same dimension as $\mathcal{Y}_0$, the map $\Phi'_0 : \mathcal{Y}_0 \to \mathcal{Z}'_0$ is also birational.

We first show that $\mathcal{Z}'_0$ is isomorphic to $\mathrm{LG(2n, W_1 \oplus W_2)}$. The VMRT of $\mathrm{LG(2n, W_1 \oplus W_2)}$ is the second Veronese embedding $\nu_2 : \mathbb{P}W \to \mathbb{P}\mathrm{Sym}^2 W$, which induces a G_0-structure on $\mathrm{LG(2n, W_1 \oplus W_2)}$ with G_0 being the automorphism group of the affine cone of $\nu_2(\mathbb{P}W)$. Let $\mathcal{C} \subset \mathbb{P}(T_{\mathcal{X}/\Delta})$ be the VMRT-structure on $\mathcal{X}$, and denote by $\mathcal{C}' := h^*\mathcal{C} \subset \mathbb{P}(T_{\mathcal{Z}'/\Delta})$ the pulling back by $h := \Psi \circ (\Phi')^{-1} : \mathcal{Z}' \dashrightarrow \mathcal{X}$. For each t, the rational map $h_t : \mathcal{Z}'_t \dashrightarrow \mathcal{X}_t$ is generically finite, and by Proposition 3.7 the VMRT at a general point of $\mathcal{X}_t$ is projectively equivalent to that of $\mathrm{LG(2n, W_1 \oplus W_2)}$, which implies that $\mathcal{C}'_z \subset \mathbb{P}(T_z\mathcal{Z}'_t)$ at a general point $z \in \mathcal{Z}'_t$ is projectively equivalent to the VMRT of $\mathrm{LG(2n, W_1 \oplus W_2)}$. It induces a G_0-structure $\mathcal{G}^\circ \to \mathcal{Z}'$ whose defining domain contains a Zariski open dense subset of $\mathcal{Z}'_t$ for each t. By Proposition 5.14, $\mathcal{C}'|_{\mathcal{Z}'_t}$ is itself the VMRT structure on $\mathcal{Z}'_t$ for $t \neq 0$. Hence the G_0-structure $\mathcal{G}^\circ$ is well-defined on $\mathcal{Z}'|_{\Delta^*}$, and the indeterminate locus of $\mathcal{G}^\circ$ is a proper closed subset of $\mathcal{Z}'_0$. Recall that the central fiber $\mathcal{Z}'_0$ is normal, and thus irreducible and reduced. Then $\mathcal{Z}'_0$ is isomorphic to $\mathrm{LG(2n, W_1 \oplus W_2)}$ by Proposition 5.16.

Up to shrinking Δ, we may assume that $\mathcal{Z}'$ is the trivial family $\mathrm{LG}(2n, V) \times \Delta$, where V is a symplectic vector space of dimension $4n$ (and thus isomorphic to $W_1 \oplus W_2$). As $\Psi : \mathcal{Z}' \to \mathcal{Z}$ is a degree 2 map, it induces an involution of $\mathcal{Z}'$ and $\mathcal{Z}$ is the quotient. The involution is given by $\iota : \Delta \to \mathrm{Aut}(\mathrm{LG}(2n, V)) \simeq \mathrm{PSp}(V)$. As $\mathrm{Sp}(V) \to \mathrm{PSp}(V)$ is étale, we can choose a lift $\tilde{\iota} : \Delta \to \mathrm{Sp}(V)$. For $t \neq 0$, the morphism Ψ_t is the quotient map of $\mathrm{LG(2n, W_1 \oplus W_2)}$ given by $\tau = (1_{W_1}, -1_{W_2}) \in \mathrm{Sp}(W_1 \oplus W_2)$. In other words, there exists a symplectomorphism $\theta(t) : V \to W_1 \oplus W_2$ such that $\tilde{\iota}(t) = \theta(t)^{-1} \circ \tau \circ \theta(t)$. Then for each $t \neq 0$, V is a direct sum of $2n$-dimensional eigenspaces of $\tilde{\iota}(t)$ with eigenvalues 1 and -1 respectively. By continuity, the same holds for $t = 0$. The latter gives a symplectomorphism $\theta(0) : V \to W_1 \oplus W_2$ such that $\tilde{\iota}(0) = \theta(0)^{-1} \circ \tau \circ \theta(0)$. This implies that $\mathcal{Z}_0$ is also isomorphic to Z, the quotient of $\mathrm{LG(2n, W_1 \oplus W_2)}$ by τ. In particular, $\mathcal{Z}$ is normal. As $\Phi : \mathcal{X} \to \mathcal{Z}$ is birational, Φ has connected fibers, hence $\Phi_0 : \mathcal{X}_0 \to \mathcal{Z}_0$ is also birational.

Recall that $K = \mathrm{PSp}(W_1) \times \mathrm{PSp}(W_2)$ acts on X and on Z such that $\phi : X \to Z$ is K-equivariant. This induces K-actions on $\mathcal{Z}$ and on $\mathcal{X}^\circ := \mathcal{X}|_{\Delta^*}$. Let $\mathfrak{k}$ be the Lie algebra of K. For $v \in \mathfrak{k}$, it induces a vector field on $\mathcal{X}^\circ$, which descends to the vector field on $\mathcal{Z} = Z \times \Delta$. As $\Phi_0 : \mathcal{X}_0 \to \mathcal{Z}_0$ is birational, this vector field extends to an open subset of $\mathcal{X}_0$. By Hartogs extension theorem, we get a vector field on $\mathcal{X}$, hence on $\mathcal{X}_0$. This shows that the universal covering $\tilde{K}$ of K acts on $\mathcal{X}_0$. As the center of $\tilde{K}$ acts trivially on $\mathcal{X}_t$, we get a K-action on $\mathcal{X}_0$ whose stabilizer at a general point contains the diagonal subgroup $\mathrm{diag}(G)$ of K as a subgroup of finite index. By descending the K-action onto $\mathcal{Z}_0 = Z$, the index is one. In particular, the open K-orbit in $\mathcal{X}_0$ is isomorphic to $(G \times G)/\mathrm{diag}(G)$. By Proposition 5.17, we have $\mathcal{X}_0 \simeq X$, concluding the proof. □

6. Invariance of varieties of minimal rational tangents for B_3

6.1. **A construction of wonderful compactification of B_n.** Let (W, o) be a vector space of dimension $2n+1$ endowed with a nondegenerate symmetric quadric form o. Take two copies $(W_i, o_i), i = 1, 2$, of (W, o) and endow the vector space $W_1 \oplus W_2$ the quadratic form given by $p_1^* o_1 - p_2^* o_2$, where $p_i : W_1 \oplus W_2 \to W_i$ is the natural i-th projection. Let $W' = \{(w, w) | w \in W\} \subset W_1 \oplus W_2$, which is an isotropic vector space of dimension $2n+1$. There are two families of isotropic subspaces of dimension $2n+1$ in $W_1 \oplus W_2$, each of which is parameterized by a Spinor variety. Let $\mathrm{OG(2n+1, W_1 \oplus W_2)}$ be the irreducible component containing $[W']$.

In this subsection, we will give a construction of the wonderful compactification of B_n by successive blowups from a spinor variety. As the construction is similar to that in Section 5.3, we will only give the corresponding statements while omitting the proofs. The following is analogous to Lemma 5.5.

Lemma 6.1. *Let* $[V] \in \mathrm{OG(2n+1, W_1 \oplus W_2)}$*, then* $\dim V \cap W_1 = \dim V \cap W_2$*.*

For any $0 \le k \le n$, denote by $\mathrm{OG}(k, W_i)$ the k-th isotropic Grassmannian in W_i, which parameterizes k-dimensional orthogonal linear subspaces in W_i. Define

$$\mathrm{OG}(k) := \{[V] \in \mathrm{OG(2n+1, W_1 \oplus W_2)} \,|\, \dim V \cap W_1 = \dim V \cap W_2 = k\}.$$

Consider the map $\pi_k : \mathrm{OG}(k) \to \mathrm{OG}(k, W_1) \times \mathrm{OG}(k, W_2)$ defined by $[V] \mapsto ([V \cap W_1], [V \cap W_2])$, which is $\mathrm{SO}(W_1) \times \mathrm{SO}(W_2)$-equivariant.

Similar to Lemma 5.6 and Lemma 5.8 , we have the following:

Lemma 6.2. (i) *Each* $\mathrm{OG}(k)$ *is an* $\mathrm{SO}(W_1) \times \mathrm{SO}(W_2)$*-orbit and* $\mathrm{OG(2n+1, W_1 \oplus W_2)} = \bigsqcup_{k=0}^{n} \mathrm{OG}(k)$ *is the decomposition of* $\mathrm{OG(2n+1, W_1 \oplus W_2)}$ *into* $\mathrm{SO}(W_1) \times \mathrm{SO}(W_2)$*-orbits.*

(ii) *We have* $\overline{\mathrm{OG}(k)} = \cup_{j \ge k} \mathrm{OG}(j)$*.*

By a similar way as Proposition 5.9, we have

Proposition 6.3. *The spinor variety* $\mathrm{OG(2n+1, W_1 \oplus W_2)}$ *is spherical under the* $\mathrm{SO}(W_1) \times \mathrm{SO}(W_2)$*-action with the unique open orbit* $\mathrm{OG}(0) \simeq \mathrm{SO}(W)$ *and the unique closed orbit* $\mathrm{OG}(n) \simeq \mathrm{OG}(n, W_1) \times \mathrm{OG}(n, W_2)$*.*

The following result is analogous to Proposition 5.10 and Proposition 5.14.

Proposition 6.4. *Let* (W, o) *be an orthogonal vector space of dimension* $2n+1$ *and let* $\bar{B}_n$ *be the wonderful compactification of* $\mathrm{SO}(W)$*. Then*

(i) the natural isomorphism $\phi^\circ : \mathrm{SO}(W) \to \mathrm{OG}(0)$ *extends to an* $\mathrm{SO}(W) \times \mathrm{SO}(W)$*-equivariant birational morphism* $\phi : \bar{B}_n \to \mathrm{OG(2n+1, W_1 \oplus W_2)}$*.*

(ii) The morphism ϕ *is the composition of successive blowups along the strict transforms of* $\overline{\mathrm{OG}(k)}$ *from the smallest to the biggest.*

As in Example 5.3, we denote by $D(\omega_1), \ldots, D(\omega_n)$ the colors on $\bar{B}_n$, and by $D_1, \ldots, D_n$ the prime boundary divisors. Now ϕ gives an isomorphism between the open obit on $\bar{B}_n$ and that on $\mathrm{OG}(2n+1, W_1 \oplus W_2)$. To distinguish, let $D(\omega_i)^{\mathrm{OG}}$ be the image of $D(\omega_i)$, which is the corresponding color on $\mathrm{OG(2n+1, W_1 \oplus W_2)}$. As an analogue of Lemma 5.11 and Lemma 5.12, we have the following result.

Lemma 6.5. *(i) The morphism ϕ sends the open orbit D_1^o of D_1 isomorphically to* $\mathrm{OG}(1)$*, and the exceptional locus of ϕ is $\cup_{i=2}^n D_i$.*

(ii) The dual weight lattice of $\mathrm{OG}(0)$ *is $\Lambda^\vee(\mathrm{OG}(0)) = \oplus_{i=1}^n \mathbb{Z}\omega_i^\vee$, and the valuation cone is $\mathcal{V}(\mathrm{OG}(0)) = \mathrm{cone}\{-\omega_1^\vee, \ldots, -\omega_n^\vee\}$. The colored cone $(\mathfrak{C}_{\mathrm{OG}(k)}, \mathfrak{D}_{\mathrm{OG}(k)})$ of $\mathrm{OG}(k)$ is given by $\mathfrak{C}_{\mathrm{OG}(k)} = \mathrm{cone}\{-\omega_1^\vee, \alpha_1^\vee, \ldots, \alpha_{k-1}^\vee\}$ and $\mathfrak{D}_{\mathrm{OG}(k)} = \{D(\omega_1)^{\mathrm{OG}}, \ldots, D(\omega_{k-1})^{\mathrm{OG}}\}$.*

Corollary 6.6. *We have $\phi^*(D(\omega_n)^{\mathrm{OG}}) = D(\omega_n)$.*

Proof. The description of colored cones in Lemma 6.5 implies that there is no $\mathrm{SO}(W_1) \times \mathrm{SO}(W_2)$-orbit $\mathrm{OG}(k)$ contained in the divisor $D(\omega_n)^{\mathrm{OG}}$. Since the morphism ϕ is the composition of successive blowups along the strict transforms of $\overline{\mathrm{OG}(k)}$, the pulling back of $D(\omega_n)^{\mathrm{OG}}$ is equal to its strict transform, verifying that $\phi^*(D(\omega_n)^{\mathrm{OG}}) = D(\omega_n)$. ☐

Lemma 6.7. *Under the minimal embedding $\mathrm{OG}(2\mathrm{n}+1, \mathrm{W}_1 \oplus \mathrm{W}_2) \subset \mathbb{P}^{2^{2n}-1}$, we have $\overline{\mathrm{OG}(1)} = \mathcal{O}(2)$, $D(\omega_n)^{\mathrm{OG}} = \mathcal{O}(1)$, and $D(\omega_j)^{\mathrm{OG}} = \mathcal{O}(2)$ for $j \neq n$.*

Proof. The unique prime boundary divisor of $\mathrm{OG}(2\mathrm{n}+1, \mathrm{W}_1 \oplus \mathrm{W}_2)$ is $\overline{\mathrm{OG}(1)}$, and the colors are $D(\omega_1)^{\mathrm{OG}}, \ldots, D(\omega_n)^{\mathrm{OG}}$. By Proposition 5.1(iv), these divisors generate the Picard group of $\mathrm{OG}(2\mathrm{n}+1, \mathrm{W}_1 \oplus \mathrm{W}_2)$, and the rational equivalence relations among them are defined by elements in the weight lattice $\Lambda(\mathrm{OG}(0))$. In the dual lattice $\Lambda^\vee(\mathrm{OG}(0)) = \oplus_{k=1}^n \mathbb{Z}\omega_k^\vee$, we have $\varrho(\overline{\mathrm{OG}(1)}) = -\omega_1^\vee$, and $\varrho(D(\omega_j)^{\mathrm{OG}}) = \alpha_j^\vee$ for $j = 1, \ldots, n$. The rational equivalence relation in the Picard group of $\mathrm{OG}(2\mathrm{n}+1, \mathrm{W}_1 \oplus \mathrm{W}_2)$ defined by the root $\alpha_k \in \Lambda(\mathrm{OG}(0))$ is $-\delta_{1,k}\overline{\mathrm{OG}(1)} + \sum_{j=1}^n \langle \alpha_k, \alpha_j \rangle D(\omega_j)^{\mathrm{OG}} = 0$, where $\langle , \rangle$ is the Cartan pairing. It follows that $\overline{\mathrm{OG}(1)} = D(\omega_j)^{\mathrm{OG}} = 2D(\omega_n)^{\mathrm{OG}}$ for $j = 1, \ldots, n-1$. Since the Picard group is generated by these divisors, we have $D(\omega_n)^{\mathrm{OG}} = \mathcal{O}(1)$, implying the conclusion. ☐

Proposition 6.8. *The morphism ϕ sends minimal rational curves on $\bar{B}_n$ to lines on* $\mathrm{OG}(2\mathrm{n}+1, \mathrm{W}_1 \oplus \mathrm{W}_2)$.

Proof. Under the identification of the Picard group of $\bar{B}_n$ with the weight lattice of B_n, we have $D_i = \alpha_i$ and $D(\omega_j) = \omega_j$ for each i and j. It follows that $D(\omega_n) = \frac{1}{2}\sum_{k=1}^n kD_k$ in the Picard group of $\bar{B}_n$. Let C be a minimal rational curve on $\bar{B}_n$. By Lemma 3.4 of [BF], $D_2 \cdot C = 1$ and $D_j \cdot C = 0$ for $j \neq 2$, hence $D(\omega_n) \cdot C = 1$. This gives $D(\omega_n)^{\mathrm{OG}} \cdot \phi_*(C) = \phi^* D(\omega_n)^{\mathrm{OG}} \cdot C = D(\omega_n) \cdot C = 1$, implying that $\phi_*(C)$ is a line on $\mathrm{OG}(2\mathrm{n}+1, \mathrm{W}_1 \oplus \mathrm{W}_2)$. ☐

6.2. **Invariance of VMRT for** B_3**.** Let $\mathbb{S}$ be the Spinor variety $\mathrm{OG}(7, \mathbb{C}^{14})$, which is a 21-dimensional Fano manifold of Picard number one. The VMRT of $\mathbb{S}$ is the Plücker embedding $\mathrm{Gr}(2,7) \subset \mathbb{P}^{20}$.

By Proposition 6.4, there exists a birational contraction $\phi : \bar{B}_3 \to \mathbb{S}$. Let $\pi : \mathcal{X} \to \Delta \ni 0$ be a smooth family of Fano varieties such that $\mathcal{X}_t \cong \bar{B}_3$ for $t \neq 0$. By Theorem 3.2, the map ϕ extends to a morphism $\Phi : \mathcal{X} \to \mathcal{S}$, such that for $t \neq 0$, we have $\mathcal{S}_t \simeq \mathbb{S}$ and $\Phi_t = \phi$. We denote by $\mathcal{K}^{\mathcal{X}}$ (resp. $\mathcal{K}^{\mathcal{S}}$) the family of minimal rational curves on $\mathcal{X}$ (resp. $\mathcal{S}$) and denote by $\mathcal{U}^{\mathcal{X}}$ (resp. $\mathcal{U}^{\mathcal{S}}$) the corresponding universal family.

Let $\sigma : \Delta \to \mathcal{X}$ be a section such that $x_t := \sigma(t) \in \mathcal{X}_t$ is general for all t. We denote by $s_t = \Phi(x_t) \in \mathcal{S}_t$. Let $\mathcal{C}_\sigma^{\mathcal{X}} = \cup_t \mathcal{C}_{x_t}^{\mathcal{X}}$ be the family of VMRT of $\mathcal{X}$ along the section σ. Note that for $t \neq 0$, the VMRT $\mathcal{C}_{x_t}^{\mathcal{X}}$ is isomorphic to the orthogonal Grassmannian $B_3/P_2 = \mathrm{OG}(2,7)$

by Theorem 2.9, which is covered by lines. As a consequence, the central fiber $\mathcal{C}^{\mathcal{X}}_{x_0} \subset \mathbb{P}T_{x_0}\mathcal{X}_0$ is also covered by lines. Take $\epsilon : \Delta \to \mathcal{C}^{\mathcal{X}}_{\sigma}$ a general section and put $z_t = \epsilon(t) \in \mathcal{C}^{\mathcal{X}}_{x_t}$. We denote by $\mathbf{L}^{\mathcal{X}}_{z_t} \subset \mathbb{P}T_{z_t}\mathcal{C}^{\mathcal{X}}_{x_t}$ the lines on $\mathcal{C}^{\mathcal{X}}_{x_t}$ passing through z_t. Then for $t \neq 0$, we have $\mathbf{L}^{\mathcal{X}}_{z_t} \simeq \mathbb{P}^1 \times \mathbb{Q}^1 \subset \mathbb{P}^5$ by Example 2.1, which is a surface of degree 4 in $\mathbb{P}^5$. As a consequence, the central fiber $\mathbf{L}^{\mathcal{X}}_{z_0} \subset \mathbb{P}T_{z_0}\mathcal{C}^{\mathcal{X}}_{x_0}$ is a surface of degree 4. In [HL2], there is a description of $\mathbf{L}^{\mathcal{X}}_{z_0}$ as follows.

Proposition 6.9. *The subvariety $\mathbf{L}^{\mathcal{X}}_{z_0} \subset \mathbb{P}T_{z_0}\mathcal{C}^{\mathcal{X}}_{x_0} = \mathbb{P}^6$ is degenerated and in its linear span, it is projectively equivalent to one of the rational normal scrolls in $\mathbb{P}^5$: $S_{2,2} := \mathbb{P}^1 \times \mathbb{Q}^1$ or $S_{1,3} := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(1) \oplus \mathcal{O}(3))$. Furthermore, in the former case, we have $\mathcal{C}^{\mathcal{X}}_{x_0} \simeq \mathrm{OG}(2,7)$.*

Proof. This is from [HL2]. For the convenience of readers, we give a sketch of the proof. Let $F^{\mathcal{X}} \subset \mathrm{Chow}(\mathcal{U}^{\mathcal{X}}_{\sigma}/\Delta)$ be the family of lines on the family $\mathcal{U}^{\mathcal{X}}_{\sigma} := \cup_t \mathcal{U}^{\mathcal{X}}_{x_t}$, and let $V^{\mathcal{X}} \to F^{\mathcal{X}}$ be the universal map. The given general section $\epsilon : \Delta \to \mathcal{C}^{\mathcal{X}}_{\sigma}$ lifts uniquely to a section $\varepsilon : \Delta \to \mathcal{U}^{\mathcal{X}}_{\sigma}$ sending $t \in \Delta$ to $z'_t \in \mathcal{U}^{\mathcal{X}}_{x_t}$. Now $V^{\mathcal{X}}_{\varepsilon} := \cup_t V^{\mathcal{X}}_{z'_t} \to \Delta$ is a family of smooth projective surfaces. The family of tangent maps $\tau_{z_t} : V^{\mathcal{X}}_{z'_t} \to \mathbb{P}T_{z'_t}\mathcal{U}^{\mathcal{X}}_{x_t} \to \mathbb{P}T_{z_t}\mathcal{C}^{\mathcal{X}}_{x_t}$, $t \in \Delta$, satisfies that τ_{z_t} sends $V^{\mathcal{X}}_{z'_t}$ biholomorphically onto $\mathbf{L}^{\mathcal{X}}_{z_t}$ for each $t \neq 0$, and τ_{z_0} is the normalization map of $\mathbf{L}^{\mathcal{X}}_{z_0}$. As a smooth deformation of $V^{\mathcal{X}}_{z'_t} \simeq \mathbf{L}^{\mathcal{X}}_{z_t} \simeq \mathbb{P}^1 \times \mathbb{Q}^1$, the variety $V^{\mathcal{X}}_{z'_0}$ is a Hirzebruch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(-k) \oplus \mathcal{O}(k))$ for some $k \geq 0$. We can determine $\tau^*_{z_0}\mathcal{O}(1) \in \mathrm{Pic}(V^{\mathcal{X}}_{z'_0})$ from the fact $\tau^*_{z_t}\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)$ for $t \neq 0$. By counting degrees of the surfaces and curves on them, one has $k = 0$ or 1, and thus $V^{\mathcal{X}}_{z'_0}$ is biholomorphic to $S_{2,2}$ or $S_{1,3}$ respectively. By an analogue with the proof of [HM98, Proposition 9], we can show that the linear span of $\mathbf{L}^{\mathcal{X}}_{z_0}$ in $\mathbb{P}T_{z_0}\mathcal{C}^{\mathcal{X}}_{x_0}$ is of dimension 5. Hence in its linear span, the variety $\mathbf{L}^{\mathcal{X}}_{z_0}$ is projectively equivalent to one of the rational normal scrolls: $S_{2,2}$ or $S_{1,3}$. By the main theorem of [M], the normalized VMRT $\mathcal{U}^{\mathcal{X}}_{x_0} \simeq \mathrm{OG}(2,7)$ in the former case. By an argument similar to that in the proof of Proposition 3.7, we have $\mathcal{C}^{\mathcal{X}}_{x_0} \simeq \mathrm{OG}(2,7)$. ☐

From now on, we will assume that $\mathbf{L}^{\mathcal{X}}_{z_0} \simeq S_{1,3}$ and then deduce a contradiction, which will then conclude our proof of the invariance of VMRT for type B_3.

Let $\mathcal{G} \subset \mathbb{P}T\mathcal{X}$ be the pull-back of the VMRT structure on $\mathcal{S}$, which is given by $\mathcal{G}_x = d\Phi^{-1}(\mathcal{C}^{\mathcal{S}}_{\Phi(x)})$ for $x \in \mathcal{X}$ general. As Φ preserves minimal rational curves by Proposition 6.8, we have $\mathcal{C}^{\mathcal{X}} \subset \mathcal{G}$ at general (hence all) points of $\mathcal{X}$. For $t \neq 0$, let $\mathbf{L}^{\mathcal{G}}_{z_t} \subset \mathbb{P}T_{z_t}\mathcal{G}_{x_t}$ be the lines on $\mathcal{G}_{x_t}$ through z_t, which is projectively equivalent to the Segre variety $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$. For $t \neq 0$, let Y_t be the unique subvariety isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$ which satisfies $\mathbb{P}^1 \times \mathbb{Q}^1 \simeq \mathbf{L}^{\mathcal{X}}_{z_t} \subset Y_t \subset \mathbf{L}^{\mathcal{G}}_{z_t} \simeq \mathbb{P}^1 \times \mathbb{P}^4$. Set $Y = \overline{\cup_{t\neq 0} Y_t}$ and Y_0 the central fiber of Y. We will first describe Y_0 in more detail.

Recall $\mathbf{L}^{\mathcal{X}}_{z_0} \simeq S_{1,3}$, hence $S_{1,3} \subset Y_0$. The linear span of $S_{1,3}$ is a 5-dimensional projective space, denoted by $\mathbb{P}^5_0$. Inside $\mathbb{P}^5_0$, there exists a line C and the rational normal cubic $\nu_3(C)$ contained in a $\mathbb{P}^3$ of $\mathbb{P}^5_0$ disjoint from C. Then the rational normal scroll $S_{1,3}$ is the union of lines $\cup_{\lambda \in C}\mathbf{l}_\lambda$, where $\mathbf{l}_\lambda$ is the line $\overline{\lambda\nu_3(\lambda)}$. Now we can describe Y_0 as follows:

Lemma 6.10. *Let $\mathbb{P}^2_\lambda = \langle C, \mathbf{l}_\lambda \rangle$ be the plane generated by C and $\mathbf{l}_\lambda$. Then $Y_0 = \cup_{\lambda \in C}\mathbb{P}^2_\lambda$ and it is linearly non-degenerate in $\mathbb{P}^5_0$.*

Proof. Since $Y_t = \mathbb{P}^1 \times \mathbb{P}^2$ for $t \neq 0$, the variety Y_0 is the union of the limits of $\{pt\} \times \mathbb{P}^2 \subset Y_t$. The latter is a linear span of $\{pt\} \times \mathbb{Q}^1 \subset \mathbf{L}^{\mathcal{X}}_{z_t}$. The limit of $\{pt\} \times \mathbb{Q}^1 \subset Y_t$ is the cycle $C + \mathbf{l}_\lambda$ for some $\lambda \in C$, whose linear span is $\mathbb{P}^2_\lambda$. It follows that $Y_0 = \cup_{\lambda \in C} \mathbb{P}^2_\lambda$. Since the subvariety $S_{1,3}$ is linearly non-degenerate in $\mathbb{P}^5_0$, so is Y_0. $\square$

The two factors $\mathbb{P}^1$ and $\mathbb{P}^2$ of $Y_t \simeq \mathbb{P}^1 \times \mathbb{P}^2$ with $t \neq 0$ induce two integrable distributions $\mathcal{T}'$ and $\mathcal{T}''$ of ranks 1 and 2 respectively on an open subset of Y, which extend to meromorphic distributions on Y with singular loci properly contained in Y_0. Being degenerations of $\mathbb{P}^1$ and $\mathbb{P}^2$, the leaf closures of $\mathcal{T}'$ and $\mathcal{T}''$ at a general point $y \in Y_0$ is a projective line and a projective plane respectively, denoted by $\mathbb{P}^1_y$ and $\mathbb{P}^2_y$ respectively. Given a general point $\beta_0 \in Y_0$, we can take a section $\beta : \Delta \to Y$ passing through β_0. Then $\mathbb{P}^1_{\beta_0} = \lim_{t \to 0} \mathbb{P}^1_{\beta_t}$ and $\mathbb{P}^2_{\beta_0} = \lim_{t \to 0} \mathbb{P}^2_{\beta_t}$.

Lemma 6.11. *For general $\beta_0 \in Y_0$, we have $\mathbb{P}^1_{\beta_0} \not\subseteq \mathbb{P}^2_{\beta_0}$.*

Proof. Assume $\mathbb{P}^1_{\beta_0} \subset \mathbb{P}^2_{\beta_0}$, we will derive a contradiction. For $t \neq 0$, we can write $\beta_t = (\beta^1_t, \beta^2_t) \in \mathbb{P}^1 \times \mathbb{P}^2 = Y_t$. The plane $\mathbb{P}^2_{\beta_t}$ is the linear span of $\{\beta^1_t\} \times \mathbb{Q}^1$ and the limit of $\{\beta^1_t\} \times \mathbb{Q}^1$ is the cycle $C + \mathbf{l}_b$ for some $b \in C$. It follows that $\mathbb{P}^2_{\beta_0} = \mathbb{P}^2_b$. As β_0 is general in Y_0, we have $\beta_0 \notin C$, and thus $\mathbb{P}^2_{\beta_0} = \mathbb{P}^2_b = \langle C, \mathbb{P}^1_{\beta_0} \rangle$.

Take a family of lines $l_t \subset \mathbb{P}^2_{\beta_t}$ through β_t such that $l_0 \neq \mathbb{P}^1_{\beta_0}$ in $\mathbb{P}^2_{\beta_0}$. Consider the family of quadric surfaces $M_t = \mathbb{P}^1 \times l_t \subset Y_t = \mathbb{P}^1_{\beta_t} \times \mathbb{P}^2_{\beta_t}$ and set $M_0 = \lim_{t \to 0} M_t$. As the linear span of M_t is of dimension three for $t \neq 0$, M_0 is linearly degenerate in $\langle S_{1,3} \rangle = \langle Y_0 \rangle = \mathbb{P}^5_0$.

Now we will show that $M_0 \subset \mathbb{P}^5_0$ is linearly non-degenerate, which will conclude the proof. Since $\mathbb{P}^2_b \cap M_0$ contains two distinct lines l_0 and $\mathbb{P}^1_{\beta_0}$, the plane $\mathbb{P}^2_b$ is contained in the linear span of M_0. Take λ in a small neighborhood of $b \in C$. The plane $\mathbb{P}^2_\lambda \subset Y_0$ is the limit of $\{a_t\} \times \mathbb{P}^2 \subset Y_t$ for a suitable choice of $a_t \in \mathbb{P}^1$. Then the limit of $\{a_t\} \times l_t \subset M_t$ is a line l'_λ in $\mathbb{P}^2_\lambda$. In particular, $l'_\lambda \neq C$, because $l'_\lambda \subset \mathbb{P}^2_\lambda$ is a small deformation of $l_0 \subset \mathbb{P}^2_b$ and $l_0 \neq C$.

Take a section $\gamma : \Delta \to Y$ such that $y := \gamma_0 \in l'_\lambda \setminus (l'_\lambda \cap C)$ is general. By assumption, $\mathbb{P}^1_y = \lim_{t \to 0} \mathbb{P}^1_{\gamma_t}$ is a line in $\mathbb{P}^2_\lambda = \mathbb{P}^2_y = \lim_{t \to 0} \mathbb{P}^2_{\gamma_t}$ which is different from l'_λ, because they are small deformations of two distinct lines $l_0 \subset \mathbb{P}^2_{\beta_0}$ and $\mathbb{P}^1_{\beta_0} \subset \mathbb{P}^2_{\beta_0}$ respectively. For $t \neq 0$, the point $\gamma_t \in M_t$ and thus the line $\mathbb{P}^1_{\gamma_t} = \mathbb{P}^1 \times \{pt\} \subset M_t$. It follows that we have $\mathbb{P}^1_y = \lim_{t \to 0} \mathbb{P}^1_{\gamma_t} \subset M_0$, so $\mathbb{P}^2_\lambda \cap M_0$ contains two distinct lines l'_λ and $\mathbb{P}^1_y$. This implies that the plane $\mathbb{P}^2_\lambda$ is contained in the linear span of M_0. By Lemma 6.10, the variety Y_0 is contained in the linear span of M_0, and thus M_0 is linearly non-degenerate in $\mathbb{P}^5_0$. $\square$

Let $\zeta : \Delta \to \mathcal{G}$ be a general section over the section $\sigma : \Delta \to \mathcal{X}$ and set $p_t = \zeta(t) \in \mathcal{G}_{x_t}$. Let $\mathbf{L}^{\mathcal{G}}_\zeta := \cup_t \mathbf{L}^{\mathcal{G}}_{p_t}$, which is a flat family of projective subvarieties, with general fiber being $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$. The two factors $\mathbb{P}^1$ and $\mathbb{P}^4$ induce two integrable distributions $\mathcal{D}^{\mathcal{G}}$ and $\mathcal{D}'^{\mathcal{G}}$ of rank respectively 1 and 4 on an open subset of $\mathbf{L}^{\mathcal{G}}_\zeta$, which extend to meromorphic distributions on $\mathbf{L}^{\mathcal{G}}_\zeta$ with singular loci properly contained in $\mathbf{L}^{\mathcal{G}}_{p_0}$. Let $\mathcal{D}^{\mathcal{G}}_0$ (resp. $\mathcal{D}'^{\mathcal{G}}_0$) be the restriction of $\mathcal{D}^{\mathcal{G}}$ (resp. $\mathcal{D}'^{\mathcal{G}}$) to $\mathbf{L}^{\mathcal{G}}_{p_0}$, which are of ranks 1 and 4 respectively. Denote by $\mathbb{P}^1_u$ (resp. $\mathbb{P}^4_u$) the leaf closure of $\mathcal{D}^{\mathcal{G}}$ (resp. $\mathcal{D}'^{\mathcal{G}}$) at a point u in the domain, which are projective spaces of dimension 1 and 4 respectively.

Lemma 6.12. *We have $\mathcal{D}_0^{\mathcal{G}} \not\subseteq \mathcal{D'}_0^{\mathcal{G}}$ on $\mathbf{L}_{p_0}^{\mathcal{G}}$. In other words, the two distributions $\mathcal{D}_0^{\mathcal{G}}$ and $\mathcal{D'}_0^{\mathcal{G}}$ generate the tangent sheaf of $\mathbf{L}_{p_0}^{\mathcal{G}}$ at general points.*

Proof. Suppose not. Given a general section $\gamma : \Delta \to \mathbf{L}_\zeta^{\mathcal{G}}$, the projective line $\mathbb{P}^1_{\gamma_0} = \lim_{t\to 0} \mathbb{P}^1_{\gamma_t}$ is contained in the 4-dimensional projective space $\mathbb{P}^4_{\gamma_0} = \lim_{t\to 0} \mathbb{P}^4_{\gamma_t}$. The given general sections $\epsilon : \Delta \to \mathcal{C}_\sigma^{\mathcal{X}}$ and $\beta : \Delta \to Y \subset \mathbf{L}_\epsilon^{\mathcal{G}} = \cup_t \mathbf{L}_{\epsilon(t)}^{\mathcal{G}}$ are specializations of ζ and γ respectively. Since the inclusion condition is preserved by specializations, the projective line $\mathbb{P}^1_{\beta_0} = \lim_{t\to 0} \mathbb{P}^1_{\beta_t}$ is contained in the projective space $\mathbb{P}^4_{\beta_0} = \lim_{t\to 0} \mathbb{P}^4_{\beta_t}$.

For $t \neq 0$, $Y_t \simeq \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbf{L}_{z_t}^{\mathcal{G}} \simeq \mathbb{P}^1 \times \mathbb{P}^4$. In particular, $\mathbb{P}^2_{\beta_0} = \lim_{t\to 0} \mathbb{P}^2_{\beta_t}$ is contained in $\mathbb{P}^4_{\beta_0} = \lim_{t\to 0} \mathbb{P}^4_{\beta_t}$. By Lemma 6.10, the 3-dimensional variety Y_0 is not contained in $\mathbb{P}^4_{\beta_0}$. Since $\beta_0 \in Y_0$ is general, the plane $\mathbb{P}^2_{\beta_0}$ is the unique irreducible component of $\mathbb{P}^4_{\beta_0} \cap Y_0$ passing through β_0. On the other hand, $\mathbb{P}^1_{\beta_0} = \lim_{t\to 0} \mathbb{P}^1_{\beta_t}$ is contained in $Y_0 = \lim_{t\to 0} Y_t$, and thus $\beta_0 \in \mathbb{P}^1_{\beta_0} \subset Y_0 \cap \mathbb{P}^4_{\beta_0}$. It follows that $\mathbb{P}^1_{\beta_0} \subset \mathbb{P}^2_{\beta_0}$, contradicting Lemma 6.11. □

Now we consider the analogous statement for $\mathcal{S}$. Let $\xi : \Delta \to \mathcal{U}^{\mathcal{S}}$ be a general section over the section $\varsigma : \Delta \ni t \mapsto s_t \in \mathcal{S}_t$ and set $q_t = \xi(t) \in \mathcal{U}^{\mathcal{S}}_{s_t}$. Let $\mathbf{L}^{\mathcal{S}}_{q_t} \subset \mathbb{P}T_{q_t}\mathcal{U}^{\mathcal{S}}_{s_t}$ be the variety of lines on $\mathcal{U}^{\mathcal{S}}_{s_t}$ through q_t. Let $\mathbf{L}^{\mathcal{S}}_\xi := \cup_t \mathbf{L}^{\mathcal{S}}_{q_t}$, which is a flat family of projective subvarieties, with general fiber being $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$. The two factors $\mathbb{P}^1$ and $\mathbb{P}^4$ induce two integrable distributions $\mathcal{D}$ and $\mathcal{D}'$ of rank respectively 1 and 4 on an open subset of $\mathbf{L}^{\mathcal{S}}_\xi$, which extend to meromorphic distributions on $\mathbf{L}^{\mathcal{S}}_\xi$. As an immediately corollary of Lemma 6.12, we have

Corollary 6.13. *The two distributions $\mathcal{D}$ and $\mathcal{D}'$ generate the tangent sheaf of $\mathbf{L}^{\mathcal{S}}_{q_0}$ at general points.*

Lemma 6.14. *For $s \in \mathcal{S}_0$ and $q \in \mathcal{U}^{\mathcal{S}}_s$ general points, there exists a birational map $f_0 : \mathbb{P}^1 \times \mathbb{P}^4 \dashrightarrow \mathbf{L}^{\mathcal{S}}_q \subset \mathbb{P}T_q\mathcal{U}^{\mathcal{S}}_s$ such that $f_0^*(\mathcal{O}(1)) = \mathcal{O}(1,1)$.*

Proof. Consider a general section $\varsigma : \Delta \to \mathcal{S}$ through s, i.e. $s = \varsigma(0) \in \mathcal{S}_0$. Set $s_t = \varsigma(t) \in \mathcal{S}_t$. Take a general section $\xi : \Delta \to \mathcal{U}^{\mathcal{S}}$ over the section $\varsigma : \Delta \to \mathcal{S}$ such that $q_t := \xi(t) \in \mathcal{U}^{\mathcal{S}}_{s_t}$ and $\xi(0) = q$. Fix a general section $\eta : \Delta \to \mathbf{L}^{\mathcal{S}}_\xi := \cup_t \mathbf{L}^{\mathcal{S}}_{q_t}$, and set $\mathbb{P}^{1,4}_\eta := \cup_t \mathbb{P}^1_{\eta_t} \times \mathbb{P}^4_{\eta_t}$, which is a $\mathbb{P}^1 \times \mathbb{P}^4$-bundle over Δ. Let $\Gamma \subset \mathbf{L}^{\mathcal{S}}_\xi \times_\Delta \mathbb{P}^{1,4}_\eta$ be the closure of those $(\alpha_t, a, b) \in \Gamma_t$ such that $t \in \Delta$ is arbitrary, $\alpha_t \in \mathbf{L}^{\mathcal{S}}_{q_t}$ is general, $a \in \mathbb{P}^4_{\alpha_t} \cap \mathbb{P}^1_{\eta_t}$ and $b \in \mathbb{P}^1_{\alpha_t} \cap \mathbb{P}^4_{\eta_t}$.

Both projections of Γ_t are isomorphisms for $t \neq 0$, implying that both projections of Γ_0 have connected fibers. The fiber of $pr'_0 : \Gamma_0 \to \mathbf{L}^{\mathcal{S}}_{q_0}$ at $\eta_0 \in \mathbf{L}^{\mathcal{S}}_{q_0}$ consists of a single point, namely (η_0, η_0, η_0). By semi-continuity of dimensions of fibers, there exists a Zariski open subset O_{η_0} of $\eta_0 \in \mathbf{L}^{\mathcal{S}}_{q_0}$ whose inverse image in Γ_0 is biholomorphically sent onto it.

By Corollary 6.13, there is an analytic open neighborhood U_{η_0} of $\eta_0 \in O_{\eta_0}$ such that the parameter spaces of leaves of $\mathcal{D}'|_{U_{\eta_0}}$ (resp. $\mathcal{D}|_{U_{\eta_0}}$) admit an open embedding into $\mathbb{P}^1_{\eta_0}$ (resp. $\mathbb{P}^4_{\eta_0}$), defined by sending the leaf at a point $\alpha \in U_{\eta_0}$ to the point $\mathbb{P}^4_\alpha \cap \mathbb{P}^1_{\eta_0}$ (resp. $\mathbb{P}^1_\alpha \cap \mathbb{P}^4_{\eta_0}$).

Now $(\eta_0, \eta_0, \eta_0) \in \Gamma_0$ is the unique point in the fiber of $pr'_0 : \Gamma_0 \to \mathbf{L}^{\mathcal{S}}_{q_0}$ at $\eta_0 \in \mathbf{L}^{\mathcal{S}}_{q_0}$. Suppose the fiber of $pr''_0 : \Gamma_0 \to \mathbb{P}^1_{\eta_0} \times \mathbb{P}^4_{\eta_0}$ at (η_0, η_0) consists of at least two points. Since pr''_0 has connected fibers, there is an irreducible curve $A \subset (pr''_0)^{-1}(\eta_0, \eta_0)$ passing through the point $(\eta_0, \eta_0, \eta_0) \in \Gamma_0$, which is sent onto an irreducible curve $A' \subset \mathbf{L}^{\mathcal{S}}_{p_0}$ passing through $\eta_0 \in U_{\eta_0}$. By the descriptions of parameter spaces in last paragraph, $\mathbb{P}^1_\beta = \mathbb{P}^1_{\eta_0}$ and $\mathbb{P}^4_\beta = \mathbb{P}^4_{\eta_0}$ for all $\beta \in A' \cap U_{\eta_0}$. It follows that $A' \cap U_{\eta_0} \subset \mathbb{P}^1_{\eta_0} \cap \mathbb{P}^4_{\eta_0}$, which contradicts the fact $\mathbb{P}^1_{\eta_0} \cap \mathbb{P}^4_{\eta_0} = \{\eta_0\}$. Then the fiber of pr''_0 at $(\eta_0, \eta_0) \in \mathbb{P}^1_{\eta_0} \times \mathbb{P}^4_{\eta_0}$ consists of a single point in Γ_0, namely (η_0, η_0, η_0).

By semi-continuity of dimensions of fibers, there exists a Zariksi open neighborhood of $(\eta_0, \eta_0) \in \mathbb{P}^1_{\eta_0} \times \mathbb{P}^4_{\eta_0}$ whose inverse image in Γ_0 is biholomorphically sent onto it. Through the incidence variety $\Gamma \subset \mathbf{L}^{\mathcal{S}}_\xi \times_\Delta \mathbb{P}^{1,4}_\eta$, we obtain a family of birational maps $f_t : \mathbb{P}^1_{\eta_t} \times \mathbb{P}^4_{\eta_t} \dashrightarrow \mathbf{L}^{\mathcal{S}}_{q_t} \subset \mathbb{P}T_{q_t}\mathcal{U}^{\mathcal{S}}_{s_t}$, $t \in \Delta$, such that for each $t \neq 0$, f_t is an isomorphism sending $(\eta_t, \eta_t) \in \mathbb{P}^1_{\eta_t} \times \mathbb{P}^4_{\eta_t}$ to $\eta_t \in \mathbf{L}^{\mathcal{S}}_{q_t}$, and $f_0(\eta_0, \eta_0) = \eta_0 \in \mathbf{L}^{\mathcal{S}}_{q_0}$. Since $f_t^*\mathcal{O}(1) = \mathcal{O}(1,1)$ for $t \neq 0$, one has $f_0^*\mathcal{O}(1) = \mathcal{O}(1,1)$. □

Proposition 6.15. *The birational map f_0 is an isomorphism, which induces a projective equivalence between $\mathbf{L}^{\mathcal{S}}_q \subset \mathbb{P}T_q\mathcal{U}^{\mathcal{S}}_s$ and $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$.*

Proof. By Lemma 6.14, $\mathbf{L}^{\mathcal{S}}_q \subset \mathbb{P}T_q\mathcal{U}^{\mathcal{S}}_s \simeq \mathbb{P}^9$ is a linear projection of $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$, hence it suffices to show that $\mathbf{L}^{\mathcal{S}}_q$ is linearly non-degenerate in $\mathbb{P}T_q\mathcal{U}^{\mathcal{S}}_s$.

Consider the meromorphic distribution $\mathcal{W} \subset T\mathcal{U}^{\mathcal{S}}_s$ given by the linear span of $\mathbf{L}^{\mathcal{S}}_q$ for general points $q \in \mathcal{U}^{\mathcal{S}}_s$. By a similar argument as [HM98, Proposition 10], the variety of tangential lines of $\widehat{\mathbf{L}^{\mathcal{S}}_q} \subset T_q\mathcal{U}^{\mathcal{S}}_s$ is contained in the kernel of the Frobenius bracket $[\cdot,\cdot] : \wedge^2\mathcal{W}_q \to T_q\mathcal{U}^{\mathcal{S}}_s/\mathcal{W}_q$.

As the variety of tangential lines of $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$ is linearly non-degenerate, so is the variety of tangential lines of its linear projection $\mathbf{L}^{\mathcal{S}}_q \subset \mathbb{P}\mathcal{W}_q$, which implies that $[\mathcal{W}_q, \mathcal{W}_q] \subset \mathcal{W}_q$, namely $\mathcal{W}$ is an integrable distribution. As for $t \neq 0$, the variety $\mathcal{U}^{\mathcal{S}}_{s_t} \simeq \mathrm{Gr}(2,7)$ is chain connected by lines, so is the central fiber $\mathcal{U}^{\mathcal{S}}_s$. Note that lines through general points of $\mathcal{U}^{\mathcal{S}}_s$ are tangent to $\mathcal{W}$ and the latter is integrable, hence we have $\mathcal{W} = T\mathcal{U}^{\mathcal{S}}_s$. This shows that $\mathbf{L}^{\mathcal{S}}_q$ is linearly non-degenerate in $\mathbb{P}T_q\mathcal{U}^{\mathcal{S}}_s$, concluding the proof. □

Remark 6.16. There is a statement very similar to Proposition 6.15 in [HM98, Proposition 8], but in our situation, the central fiber $\mathbf{L}^{\mathcal{S}}_q$ (as well as $\mathcal{U}^{\mathcal{S}}_s$) is not known a priori smooth. Our approach here is an adaption of the proof from [M].

Proposition 6.17. *For $s \in \mathcal{S}_0$ general, we have*

(i) The normalized VMRT $\mathcal{U}^{\mathcal{S}}_s$ is isomorphic to $\mathrm{Gr}(2,7)$.

(ii) The VMRT $\mathcal{C}^{\mathcal{S}}_s \subset \mathbb{P}T_s\mathcal{S}_0$ is projectively equivalent to $\mathrm{Gr}(2,7) \subset \mathbb{P}^{20}$.

(iii) The central fiber $\mathcal{S}_0$ is isomorphic to $\mathbb{S}$.

Proof. (i) Take a section $\varsigma : \Delta \to \mathcal{S}$ through s such that $s_t := \varsigma(t)$ is a general point in $\mathcal{S}_t$. Let $\mathcal{U}^{\mathcal{S}}_\varsigma = \cup_t \mathcal{U}^{\mathcal{S}}_{s_t}$, which is a family of projective varieties with general fiber isomorphic to $\mathrm{Gr}(2,7)$. Let $G = (\mathrm{GL}(2) \times \mathrm{GL}(5))/\{(\lambda id, \lambda^{-1} id) \mid \lambda \in \mathbb{C}^*\}$, then the VMRT structure on $\mathrm{Gr}(2,7)$ induces a G-structure on $\mathcal{U}^{\mathcal{S}}_\varsigma$ outside the central fiber. By Proposition 6.15, it extends to a G-structure on $\mathcal{U}^{\mathcal{S}}_\varsigma$ outside a codimension at least 2 subset. By Proposition 5.16, we have that $\mathcal{U}^{\mathcal{S}}_s$ is isomorphic to $\mathrm{Gr}(2,7)$.

(ii) Consider the family of tangent maps $\tau_t : \mathcal{U}^{\mathcal{S}}_{s_t} \to \mathcal{C}^{\mathcal{S}}_{s_t} \subset \mathbb{P}T_{s_t}\mathcal{S}_t$, which is an isomorphism for $t \neq 0$ and $\tau_t^*\mathcal{O}(1) = \mathcal{O}(1)$. Hence $\mathcal{C}^{\mathcal{S}}_s \subset \mathbb{P}T_s\mathcal{S}_0$ is a linear projection of $\mathrm{Gr}(2,7) \subset \mathbb{P}^{20}$. As $\mathrm{Gr}(2,7) \subset \mathbb{P}^{20}$ has non-degenerate variety of tangential lines, so is $\mathcal{C}^{\mathcal{S}}_s \subset \mathbb{P}T_s\mathcal{S}_0$. By an analogue of the proof of Proposition 6.15, one can see that $\mathcal{C}_s^{\mathcal{S}} \subset \mathbb{P}T_s\mathcal{S}_0$ is linearly non-degenerate, hence τ_0 is an isomorphism.

(iii) The VMRT structure on $\mathbb{S}$ induces a G-structure on $\mathcal{S}|_{\Delta^*}$. By (ii), this G-structure extends to an open subset of $\mathcal{S}_0$. By Proposition 5.16, $\mathcal{S}_0$ is isomorphic to $\mathbb{S}$. □

Proposition 6.18. *For $x \in \mathcal{X}_0$ general, the VMRT $\mathcal{C}_x^{\mathcal{X}} \subset \mathbb{P}T_x\mathcal{X}_0$ is projectively equivalent to* $\mathrm{OG}(2,7) \subset \mathbb{P}^{20}$.

Proof. By Proposition 6.9, we may assume $\mathbf{L}_{z_0}^{\mathcal{X}} \simeq S_{1,3}$ and then deduce a contradiction. The birational map $\Phi : \mathcal{X} \to \mathcal{S}$ induces an injective morphism $\Psi_t : \mathcal{C}_{x_t}^{\mathcal{X}} \to \mathcal{C}_{s_t}^{\mathcal{S}} \simeq \mathrm{Gr}(2,7)$ for all $t \in \Delta$ by Proposition 6.17.

Let $\epsilon : \Delta \to \mathcal{C}^{\mathcal{X}}$ be a section over $\sigma : \Delta \to \mathcal{X}$ such that $z_t := \epsilon(t) \in \mathcal{C}_{x_t}^{\mathcal{X}}$ is a general point. Let $\varepsilon : \Delta \to \mathcal{C}^{\mathcal{S}}$ be the composition of ϵ with Ψ, which is a section over $\varsigma : \Delta \to \mathcal{S}$. Put $z_t' = \varepsilon(t) \in \mathcal{C}_{s_t}^{\mathcal{S}}$. The family of lines $\mathbf{L}_\epsilon^{\mathcal{X}} = \cup_t \mathbf{L}_{z_t}^{\mathcal{X}}$ has $\mathbb{P}^1 \times \mathbb{Q}^1$ as general fibers and $S_{1,3}$ as central fiber. The family of lines $\mathbf{L}_\varepsilon^{\mathcal{S}} = \cup_t \mathbf{L}_{z_t'}^{\mathcal{S}}$ is the trivial family of $(\mathbb{P}^1 \times \mathbb{P}^4) \times \Delta$. By previous discussions, we have an injective holomorphic map: $g : \mathbf{L}_\epsilon^{\mathcal{X}} \to \mathbf{L}_\varepsilon^{\mathcal{S}}$. When $t \neq 0$, the map $g_t : \mathbb{P}^1 \times \mathbb{Q}^1 \to \mathbb{P}^1 \times \mathbb{P}^4$ is induced from $\mathbb{Q}^1 \subset \mathbb{P}^2 \subset \mathbb{P}^4$, in particular, the composition $h_t : \mathbb{P}^1 \times \mathbb{Q}^1 \xrightarrow{g_t} \mathbb{P}^1 \times \mathbb{P}^4 \to \mathbb{P}^1$ contracts the sections $\mathbb{P}(\mathcal{O}(2))$. By deforming to $t = 0$, we see that the morphism $h_0 : S_{1,3} \xrightarrow{g_0} \mathbb{P}^1 \times \mathbb{P}^4 \to \mathbb{P}^1$ contracts both the minimal section and the ruling of $S_{1,3}$, hence it contracts $S_{1,3}$ to one point. As g_0 is injective, this implies that $S_{1,3}$ is contained in $\mathbb{P}^4$, which is not possible, concluding the proof. □

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