Iwasawa Main Conjecture for Rankin-Selberg $p$-adic $L$-functions

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Abstract

In this paper we prove that the $p$-adic $L$-function that interpolates the Rankin-Selberg product of a general modular form and a CM form of higher weight divides the characteristic ideal of the corresponding Selmer group. This is one divisibility of the Iwasawa main conjecture for this $p$-adic $L$-function. We prove this conjecture using congruences between Klingen Eisenstein series and cusp forms on the group $GU(3,1)$, following the strategy of a recent work of C. Skinner and E. Urban. The actual argument is, however, more complicated due to the need to work with general Fourier-Jacobi expansions. This theorem is used to deduce a converse of Gross-Zagier-Kolyvagin theorem and the $p$-adic part of the precise BSD formula in the rank one case.

1 Introduction

Let $p$ be an odd prime. Let $\mathcal{K} \subset \mathbb{C}$ be an imaginary quadratic field such that $p$ splits in $\mathcal{K}$ as $(p) = v_0\mathfrak{o}_0$. We fix an isomorphism $\iota: \mathbb{C}_p \simeq \mathbb{C}$ and suppose $v_0$ is determined by $\iota$. There is a unique $\mathbb{Z}_p^2$-extension $\mathcal{K}_\infty/\mathcal{K}$ unramified outside $p$. Let $\Gamma_\mathcal{K} := \text{Gal}(\mathcal{K}_\infty/\mathcal{K})$. Suppose $\mathbf{f}$ is a Hida family of ordinary cuspidal eigenforms new outside $p$ with coefficient ring $\mathcal{O}_L$. Suppose $\xi$ is an $\mathcal{O}_L[[\Gamma_\mathcal{K}]]$-valued Hecke character of $\mathcal{A}_\mathcal{K}/\mathcal{K}$ whose infinite type is $(\xi_{0,\infty}, -\frac{\kappa}{2})$ for some even integer $\kappa \geq 6$ and such that $\text{ord}_{v_0}(\text{cond}(\xi_{0,v_0})) \leq 1$ and $\text{ord}_{v_0}(\text{cond}(\xi_{0,v_0})) \leq 1$. Denote by $\xi$ the $\mathcal{O}_L[[\Gamma_\mathcal{K}]]$-adic family of Hecke characters containing $\xi$ as some specialization (we make this precise later). In this paper we associate with $\mathbf{f}, \mathcal{K}$ and $\xi$ a dual Selmer group $X_{\mathbf{f}, \mathcal{K}, \xi}$, which is a finite module over the ring $\mathbb{Z}[[\Gamma_\mathcal{K}]]$. On the other hand, there is a $p$-adic $L$-function $L_{\mathbf{f}, \mathcal{K}, \xi} \in \mathbb{F}[[\Gamma_\mathcal{K}]]$ interpolating the algebraic parts of the special $L$-values $L_\mathcal{K}(f_\phi, \xi_\phi, \frac{\kappa}{2})$, where $f_\phi$ and $\xi_\phi$ are elements in the families $\mathbf{f}$ and $\xi$ and $F_A$ is the fraction field of $A$ ($f_\phi$ has weight 2 and $\xi_\phi$ has infinite type $(\kappa/2, -\kappa/2)$). We let $L_{f_0, \mathcal{K}, \xi}$ be the specialization of $L_{\mathbf{f}, \mathcal{K}, \xi}$ to a single form $f_0$ of weight 2 and trivial character in the family $\mathbf{f}$, which we assume is defined over $L$. We write $\hat{\mathcal{O}}^\text{ur}_{L}$ for the completion of the maximal unramified extension of $\mathcal{O}^\text{ur}_{L}$ and $\hat{\mathcal{O}}^\text{ur}_{\mathcal{K}}$ for the normalization of the ring corresponding to an irreducible component of $\hat{\mathbb{Z}} \otimes_{\mathcal{O}_L} \hat{\mathcal{O}}^\text{ur}_{L}$. For a finite set of primes $\Sigma$ containing all bad primes, we construct in this paper the “$\Sigma$-primitive” $p$-adic $L$-functions $L_{\mathbf{f}, \xi, \mathcal{K}}^\Sigma \in \hat{\mathcal{O}}^\text{ur}_{L}[[\Gamma_\mathcal{K}]]$, $L_{\mathbf{f}, \xi, \mathcal{K}}^\Sigma \in \hat{\mathcal{O}}^\text{ur}_{\mathcal{K}}[[\Gamma_\mathcal{K}]]$ using the doubling method. The general case is obtained by putting back the local Euler factors at primes in $\Sigma$. In Section 7 we also recall closely related $p$-adic $L$-function constructed by Hida: $L_{\mathbf{f}, \xi, \mathcal{K}}^{\Sigma, \text{Hida}}, L_{\mathbf{f}, \xi, \mathcal{K}}^{\text{Hida}}$. The Iwasawa-Greenberg main conjecture for $L_{\mathbf{f}, \mathcal{K}, \xi}$ essentially states that the characteristic ideal (to be defined later) of $X_{\mathbf{f}, \mathcal{K}, \xi}$ is generated by $L_{\mathbf{f}, \mathcal{K}, \xi}$. We also associate with $f, \mathcal{K}, \xi$ a dual Selmer group $X_{f, \mathcal{K}, \xi}$ over $\hat{\mathcal{O}}^\text{ur}_{L}[[\Gamma_\mathcal{K}]]$. The Iwasawa-Greenberg main conjecture says that its
characteristic ideal of $X_{f,K,ξ}$ is generated by $L_{f,K,ξ}$.

Let $\bar{Q} \subset \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$ and let $G_\mathbb{Q} = \text{Gal}(\bar{Q}/\mathbb{Q})$ be the Galois group. Let $G_p \subset G_\mathbb{Q}$ be the decomposition group determined by the inclusion $\bar{Q} \subset \bar{Q}_p$ coming from $\iota$. We write $\epsilon$ for the cyclotomic character and $\omega$ for the Teichmüller character of $G_\mathbb{Q}$.

Let $g$ be a cuspidal eigenform on $GL_2/\mathbb{Q}$ with the associated $p$-adic Galois representation $\rho_g : G_\mathbb{Q} \to GL_2(\mathcal{O}_L)$. We say $g$ satisfies (irred) if:

- The residual representation $\bar{\rho}_g$ is absolutely irreducible.

If $g$ is nearly ordinary at $p$, then $\rho_g|_{G_p}$ is equivalent to an upper triangular representation and we say it satisfies (dist) if:

- The characters of $\rho_g|_{G_p}$ on the diagonal are distinct modulo the maximal ideal of $\mathcal{O}_L$.

We will see later (in Section 7) that if the CM form $g_ξ$ associated to $ξ$ satisfies (irred) and (dist) then $L_{f,K,ξ} \in \hat{\mathcal{I}}[[\Gamma_K]]$.

In this paper, under certain conditions on $f,ξ,K$, we prove one inclusion (or divisibility) of the Iwasawa-Greenberg main conjecture for $L_{f,K,ξ}$. Our first theorem is a three-variable result for Hida families.

Theorem 1.1. Let $f$ be a Hida family of ordinary eigenforms that are new outside $p$ of square-free tame level $N$, and suppose $f$ has a weight two specialization $f$ that has trivial nebentypus and is the ordinary stabilization of a newform of level $N$. Let $\bar{\rho}$ be the mod $p$ residual $G_\mathbb{Q}$-representation associated with the Hida family $f$. Let $ξ$ be a Hecke character of $K^× \backslash \mathbb{A}^× K$ with infinite type $(\frac{κ}{2}, -\frac{κ}{2})$ for some $κ \geq 6$. If

(a) $p \geq 5$;

(b) $ξ|_{\mathbb{A}^× K} = \omega \circ \text{Nm}$ and $κ \equiv 0(\text{mod } 2(p-1))$;

(c) $\bar{\rho}_f|_{G_K}$ is irreducible;

(d) there exists $q|N$ that does not split in $K$ and such that $\bar{\rho}_f$ is ramified at $q$;

(e) the CM eigenform $g_ξ$ associated to the character $ξ$ satisfies (dist) and (irred);

(f) For each non-split prime $v$ of $\mathbb{Q}$ we have the conductor of $ξ_v$ is not $(\varpi_v)$ where $\varpi_v$ is a uniformizer of the integer ring of $K_v$, and that

$$e(ξ_v, \frac{1}{2}) = 1.$$  

(As in [18] the $e(ξ_v, \frac{1}{2})$ is the local root number for the base change of $ξ_v$ to $K_v$ twisted by $ξ_v$. It differs from the local root number for the Rankin-Selberg product of $ξ_v$ and $g_ξ,v$ by a factor $χ_K/\mathbb{Q}(-1).$)

Then $L_{f,K,ξ}^{\text{Hida}} \in \hat{\mathcal{I}}[[\Gamma_K]]$ and $(L_{f,K,ξ}^{\text{Hida}}) \supset \text{char}_{\hat{\mathcal{I}}[[\Gamma_K]]}(X_{f,K,ξ})$ as ideals of $\hat{\mathcal{I}}[[\Gamma_K]]$. Here char means the characteristic ideal.
We also have a two variable theorem for a single form.

**Theorem 1.2.** Let $N$, $f$, $\kappa$ and $\xi$ be as before. If

(a) $p \geq 5$;

(b) the $p$-adic avatar of $|\xi| \cdot |\nu/2(\omega^{-1} \circ \text{Nm})$ factors through $\Gamma_K$ and $\kappa \equiv 0 \pmod{2(p-1)}$;

(c) $\bar{\rho}_f|_{G_K}$ is irreducible;

(d) there exists $q | N$ that does not split in $K$ and such that $\bar{\rho}_f$ is ramified at $q$.

Then

$$(L_{f,K,\xi}) \supset \text{char}_{\hat{O}_L[[\Gamma_K]]} \otimes_{\hat{O}_L} L(X_{f,K,\xi})$$

is true as fractional ideals of $\hat{O}_L[[\Gamma_K]] \otimes_{\hat{O}_L} L$.

Hida’s $p$-adic $L$-functions $L_{f,K,\xi}^{Hida}$ are more canonical than the $L_{f,K,\xi}$ in that there is a constant in $\hat{Q}_p$ showing up in our interpolation formula (see Proposition 7.5) that depends on some choices. Under the assumptions of Theorem 1.1 we show that Hida’s $p$-adic $L$-function is integral: it belongs to $\Gamma[[\Gamma_K]]$. Note that in the setting of Theorem 1.2 we do not know if $L_{f_0,K,\xi}$ is actually in $\hat{O}_L[[\Gamma_K]]$.

Hypothesis (b) of Theorem 1.2 means that $L_{f,K,\xi}$ can be evaluated at the trivial character of $\Gamma_K$, though it is not a point at which it interpolates classical $L$-values. As a result, Theorem 1.2 has interesting applications for the usual Bloch-Kato Selmer group of $f$.

Skinner [44] has recently been able to use Theorem 1.2 to prove a converse of the Gross-Zagier-Kolyvagin theorem: if the Mordell-Weil rank of an elliptic curve over $\mathbb{Q}$ is exactly one and the Shafarevich-Tate group is finite, then its $L$-function vanishes to exactly order one at the central critical point. The author has been able to prove an anti-cyclotomic main conjecture of Perrin-Riou when the root number is $-1$ [19] (by comparing the Selmer group in the theorem with the one studied by Perrin-Riou, using the Poitou-Tate long exact sequence and applying F. Castella’s generalization [2] of a formula of Bertolini-Darmon-Prasanna relating the different $p$-adic $L$-functions). There is also an ongoing joint work of the author with Skinner and Jetchev that uses Theorem 1.2 to deduce the $p$-adic part of the precise BSD formula in the rank one case [50]. Finally the methods of this paper can be adapted (with some additional arguments) to the case when $f$ is non-ordinary as well. This forms the foundation of the author’s recent proof of the Iwasawa main conjecture for supersingular elliptic curves formulated by Kobayashi (see [51]). We remark that in all of the above mentioned applications one can not appeal to the main conjecture proved in [45] since the global sign of the $L$-functions has to be $+1$ in loc.cit.

Our proofs of Theorem 1.1 and Theorem 1.2 use Eisenstein congruences on the unitary group $U(3,1)$, which first appeared in Hsieh’s paper [19]. Recent works with a similar flavor include Skinner-Urban’s [45] using the group $U(2,2)$, and the work of M. Hsieh [16] for CM characters using the group $U(2,1)$. The difference between our results and Skinner-Urban’s is that they studied
the $p$-adic $L$-function of Rankin-Selberg product of a general modular form and a CM form such that the weight of the CM form is lower, while in our case the weight of the CM form is higher. This is the very reason we work with unitary groups of different signature.

We also mention there are works establishing the other divisibility of the main conjecture using Euler systems ([49], [31]) under some more restrictions. Together with Theorem 1.1 and Theorem 1.2 these give the full equality of the main conjecture in the case when all hypotheses are satisfied.

For the reader’s convenience we briefly discuss our proof of the theorems. The proof follows the main outline of Skinner-Urban’s proof in [45] (which in turn followed the main outline of Wiles’ proof of the Iwasawa main conjecture for totally real fields). But carrying this out requires new arguments. The main steps are: (1) constructing a $p$-adic family of Eisenstein series whose constant terms are essentially the $p$-adic Rankin-Selberg $L$-function $L_{f,K,\xi}^\Sigma$; (2) proving that the Eisenstein series is coprime to $p$-adic $L$-function (that is, modulo any divisor of the $p$-adic $L$-function it is still non-zero), which shows that its congruences with cuspforms is ‘measured’ by the $p$-adic $L$-function; (3) the Galois argument. The main differences between our proof and that of Skinner-Urban are in steps (1) and (2). First of all we need to work with the unitary group $U(3,1)$ instead of $U(2,2)$ which is used in [45]. The reason is that by our assumption that the CM form has higher weight than $f$, the $L$-values interpolated by the $p$-adic $L$-function $L_{f,K,\xi}$ show up in the constant terms of holomorphic Eisenstein series on the group $U(3,1)$ that are induced from the Klingen parabolic subgroup with Levi $U(2) \times K$. The cuspidal representations on $U(2)$ is determined by the automorphic representation $\pi_f$ and a Hecke character of $\mathbb{A}_K^\times$ whose restriction to $\mathbb{A}_\mathbb{Q}^\times$ is the central character of $\pi_f$. As a result, the construction of the $p$-adic families of Eisenstein series via the pullback formula requires finding the right Siegel section at $p$ (which turns out to be different from the one used in [45]). To have the right pullback and to make the Fourier-Jacobi coefficient computation not too hard, such choice of section is quite subtle. The idea for our choice is similar to that in [3] and is inspired by the formula for differential operators on Fourier expansions.

Step (2) is the core of the whole argument. In [45], the Klingen Eisenstein series on $U(2,2)$ has a Fourier expansion $E_{\text{Kling}} = \sum_T a_T q^T$, with $T$ running over $2 \times 2$ Hermitian matrices. By the pullback formula we have $a_T = \langle F_{JT} E_{\text{sieg}}, \varphi_{\pi} \rangle_{U(1,1)}$, where $E_{\text{sieg}}$ is a Siegel Eisenstein series on $U(3,3)$, $F_{JT} E_{\text{sieg}}$ is its $T$-Fourier-Jacobi coefficient (regarded as a form on $U(1,1)$), and $\varphi_{\pi}$ is a vector in the $U(1,1)$ automorphic representation $\pi$ considered in [45] (again determined by $\pi_f$ and a Hecke character of $\mathbb{A}_K^\times$). Computation tells us $F_{JT} E_{\text{sieg}}$ is essentially a product of an Eisenstein series and a theta function, and thus this pairing, and hence $a_T$ is essentially a Rankin-Selberg product. In our case, forms on $U(3,1)$ only have Fourier-Jacobi expansions (instead of Fourier expansions):

$$F \mapsto \sum_{n \in \mathbb{Q}} a_n(F) q^n := FJ(F)$$

with $a_n(F) \in H^0(\mathcal{B}, L(n))$, where $\mathcal{B}$ is a 2-dimensional abelian variety which is the abelian part of the universal semiabelian scheme over a point in the boundary of a toroidal compactification Shimura variety for GU(3,1). The $L(n)$ is a line bundle on $\mathcal{B}$. We can view each $a_n(F)$ as an automorphic form on the group $U(2) \cdot N$, where $U(2)$ is the definite unitary group appearing as a factor of the Levi of the Klingen parabolic subgroup of $U(3,1)$, and $N$ is the unipotent radical of
the parabolic, which is a Heisenberg group. It consists of matrices of the form \[
\begin{pmatrix}
1 & x & x & x \\
1 & x \\
x & 1 \\
x & 1 \\
1 & 1
\end{pmatrix}.
\]

To study \(a_n(F)\) we use a functional \(l_{\theta_1}\) on \(H^0(B, L(n))\). This is just the pairing over \(N\) (modulo its center) with an explicit theta function \(\theta_1\) on \(U(2) \cdot N\). We first compute the \(n\)-th Fourier-Jacobi coefficient of a Siegel Eisenstein series, considered as a form on the Jacobi group \(N' \cdot U(2, 2) \subseteq U(3, 3)\) with \(N'\) a unipotent subgroup of \(U(3, 3)\). It consists of matrix of the form \[
\begin{pmatrix}
1 & x & x & x \\
1 & x \\
x & 1 \\
x & 1 \\
1 & 1
\end{pmatrix}.
\]

This turns out to be of the form \(E \cdot \Theta\), with \(E\) a Siegel Eisenstein series on \(U(2, 2)\) and \(\Theta\) a theta function on the Jacobi group. Next we restrict this \(n\)-th Fourier-Jacobi coefficient to the group \((N \cdot U(2)) \times U(2) \subseteq U(3, 1) \times U(2) \cap N' \cdot U(2, 2)\).

Another computation shows that \(\Theta\) essentially restricts to a form \(\theta_2 \times \theta_3\) on \((N \cdot U(2)) \times U(2)\) (the actual situation is slightly more complicated, see Lemma 6.32). Applying \(l_{\theta_1}\), we show that \(\langle \theta_2, \theta_1 \rangle_N\) is a constant function on \(U(2)\) (which we manage to make nonzero). So using the pullback formula, we get

\[l_{\theta_1}(a_n(F)) = \langle E_{U(2) \times U(2)}, \varphi \pi \cdot \theta_3 \rangle_{1 \times U(2)}\]

regarded as a form on the first \(U(2)\), which is the \(U(2)\) in the Levi of the Klingen parabolic. To study its \(p\)-adic property, we pair it with an auxiliary (Hida family of) form \(h\) on \(U(2)\):

\[\langle \langle E_{U(2) \times U(2)}, \varphi \pi \cdot \theta_3 \rangle_{1 \times U(2)}, h \rangle_{U(2)} = (\ast) \cdot \langle h, \varphi \pi \cdot \theta_3 \rangle.
\]

To obtain this formula we use the doubling method formula for \((U(2) \times U(2)) \hookrightarrow U(2, 2)\) applied to \(h\). The \(\langle \ast \rangle\) is some \(p\)-adic \(L\)-function factor for \(h\) coming from this. The pairing on the right hand side is just a triple product integral \(\int_{U(2)} h(g) \theta_3(g)f(g)dg\). The fact that the \(\theta_3\) can be taken to be an eigenform follows from considering the central character (see the proof of Proposition 8.22). We use Ichino’s formula to evaluate this:

\[
\langle \langle E_{U(2) \times U(2)}, \varphi \pi \cdot \theta_3 \rangle_{1 \times U(2)}, h \rangle_{U(2)} = \langle h, \tilde{\theta}_3 \rangle \cdot \int_{[U(2)]} h(g) \theta_3(g)f(g)dg = \langle h, \tilde{\theta}_3 \rangle \cdot \int_{[U(2)]} \tilde{h}(g) \tilde{\theta}_3(g)\tilde{f}(g)dg.
\]

Here \(\tilde{\cdot}\) means some forms or vectors in the contragredient representation of the automorphic representation for \(h, \theta_3\) and \(f\). The factor \(I_v\) is a local integral defined by Ichino, and \(\chi_1\) and \(\chi_2\) are two CM Hecke characters showing up in the computation. We interpolate everything in \(p\)-adic families and compare it to the product of several \(p\)-adic \(L\)-functions of modular forms or Hecke characters. Furthermore:
• We can choose \( h \) and \( \theta_3 \) so that these \( p \)-adic \( L \)-functions are essentially units in \( \hat{I}^{ur}[[\Gamma_K]]^\times \).

• The ratio of the triple product and the product of these \( p \)-adic \( L \)-functions is a product of local factors (we show that the triple product is a \( p \)-adic analytic function, so the product of these local factors is a \( p \)-adic meromorphic function). We make the local choices such that for inert or ramified primes these local factors involve only the Hida-family variable of \( f \) (which has nothing to do with \( \Gamma_+^K \) or \( \Gamma_-^K \)). For split primes we compute these local factors explicitly.

The constructions above finally provide a nonzero element of \( \mathbb{I} \), which is good enough for our use. After this we can use the same argument as in \[45\] to deduce our main theorem: by a geometric argument we construct a cuspidal family on \( U(3,1) \) congruent modulo the \( p \)-adic \( L \)-function \( L_f, K, \xi \) to the Eisenstein family constructed as above. Passing to the Galois side, we get a family of Galois representations coming from cuspidal forms that is congruent to the Galois representations coming from our Klingen Eisenstein family, but which is “more irreducible” than the Eisenstein Galois representations. Then an argument (the “lattice construction”) of Urban gives the required elements in the dual Selmer group.

Remark 1.3. We emphasize here that the \( \theta_1 \) is fixed throughout the whole \( p \)-adic family (instead of varying). Note also that the space of theta functions with given Archimedean kernel function and level group at finite places is finite dimensional. The space \( H^0(B, L(n)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) is generated by a finite number of such theta functions. We will show in the text that by pairing the \( \hat{I}^{ur}[[\Gamma_K]] \)-adic Fourier-Jacobi coefficient with one rational theta function (not necessarily \( p \)-integral!), we get an element in \( \hat{I}^{ur}[[\Gamma_K]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \). We then show that by choosing the datum properly, this element is the product of a unit in \( \hat{I}^{ur}[[\Gamma_K]] \) and a non-zero element in \( \mathbb{Q}_p \), and proving it is prime to the \( p \)-adic \( L \)-function we study. Such strategy is notably different from the one adopted in \[18\], where Hsieh argued \( p \)-integrally and proved stronger result that the Fourier-Jacobi coefficient is already a unit. This is the very reason why we do not need to study the theory of \( p \)-integral theta functions.

Remark 1.4. In \[54\] the special \( L \)-value showing up in the Fourier-Jacobi expansion is the near central point of the Rankin-Selberg \( L \)-function, while in our case it is the central value of the triple product \( L \)-function. Moreover the Fourier-Jacobi coefficient considered in \[54\] is non-zero only when \( f \) is a CM form (see Theorem 4.12 in loc.cit). This is due to the fact that we are paring the Fourier-Jacobi coefficient with the product of a theta function and an auxiliary form \( h \) on \( U(2) \), while Zhang paired it with the theta function only (i.e. taking the \( h \) in our case to be the constant function). Our strategy has the advantage that these central \( L \)-values are accessible to various results of non-vanishing modulo \( p \) by Hsieh.

This rest of this paper is organized as follows. In section 2 we recall some backgrounds and formulate the main conjecture. In section 3 we discuss automorphic forms and \( p \)-adic automorphic forms on various unitary groups. In section 4 we recall the notion of theta functions which plays an important role in studying Fourier-Jacobi expansions as outlined above. In sections 5 and 6 we make the local and global calculations for Siegel and Klingen Eisenstein series using the pullback formula of Shimura. In section 7 we interpolate our previous calculations \( p \)-adically and construct the families. In section 8 we prove the co-primeness of (the Fourier-Jacobi coefficients of) the Klingen-Eisenstein series and the \( p \)-adic \( L \)-function. Finally, we deduce the main theorem in section 9.

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2 Backgrounds

We first introduce some notations. We will usually take a finite extension \( L/\mathbb{Q}_p \) and write \( \mathcal{O}_L \) for its integer ring and \( \varpi_L \) for a uniformizer. Let \( G_Q \) and \( G_K \) be the absolute Galois groups of \( Q \) and \( K \). Let \( \Gamma_K^\pm \) be the subgroups of \( \Gamma_K \) such that the complex conjugation acts by \( \pm 1 \). We take topological generators \( \gamma^\pm \) so that \( \text{rec}^{-1}(\gamma^+) = ((1 + p)^{\frac{1}{2}}, (1 + p)^{\frac{1}{2}}) \) and \( \text{rec}^{-1}(\gamma^-) = ((1 + p)^{\frac{1}{2}}, (1 + p)^{-\frac{1}{2}}) \) where \( \text{rec} : A_K^\times \to G_K^{ab} \) is the reciprocity map normalized by the geometric Frobenius. Let \( \Psi_K \) be the composition

\[
G_K \to \Gamma_K \hookrightarrow \mathbb{Z}_p[[\Gamma_K]]^\times.
\]

Define \( \Lambda_K := \mathcal{O}_L[[\Gamma_K]] \). Recall we defined a character \( \xi \) in the introduction. We will write \( \sigma_\xi \) for the Galois character corresponding to \( \xi \) via class field theory. We also let \( \mathbb{Q}_\infty \) be the cyclotomic \( \mathbb{Z}_p \) extension of \( Q \) and let \( \Gamma_Q = \text{Gal}(\mathbb{Q}_\infty/Q) \). Define \( \Psi_Q \) to be the composition \( G_Q \to \Gamma_Q \hookrightarrow \mathbb{Z}_p[[\Gamma_Q]]^\times \).

We also define \( \varepsilon_K \) and \( \varepsilon_Q \) to be the compositions \( \mathcal{K}^\times \backslash A_K^\times \overset{\text{rec}}{\to} G_K^{ab} \to \mathbb{Z}_p[[\Gamma_K]]^\times \) and \( \mathbb{Q}^\times \backslash A_Q^\times \overset{\text{rec}}{\to} G_Q^{ab} \to \mathbb{Z}_p[[\Gamma_Q]]^\times \) where the second arrows are the \( \Psi_K \) and \( \Psi_Q \) defined above. Let \( \omega \) and \( \varepsilon \) be the Teichmüller character and the cyclotomic character.

2.1 \( p \)-adic Families for \( \text{GL}_2/\mathbb{Q} \)

Let \( M \) be a positive integer prime to \( p \) and \( \chi \) a character of \( (\mathbb{Z}/p\mathbb{Z})^\times \). Let \( \Lambda_Q := \mathbb{Z}_p[[W]] \) (we call \( \Lambda_Q \) the weight space). Let \( \mathbb{I} \) be a domain finite over \( \Lambda_Q \). A point \( \phi \in \text{Spec}(\mathbb{I}) \) is called arithmetic if the image of \( \phi \) in \( \text{Spec}(\mathbb{Q}_p) \) is the continuous \( \mathbb{Z}_p \)-homomorphism sending \( (1 + W) \mapsto \zeta(1 + p)^{\kappa - 2} \) for some \( \kappa \geq 2 \) and \( \zeta \) a \( p \)-power root of unity. We usually write \( \kappa_\phi \) for this \( \kappa \), called the weight of \( \phi \). We also define \( \chi_\phi \) to be the character of \( \mathbb{Z}_p^\times \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times (1 + p\mathbb{Z}_p) \) that is trivial on the first factor and given by \( (1 + p) \mapsto \zeta \) on the second factor.

Definition 2.1. An \( \mathbb{I} \)-adic family of modular forms of tame level \( M \) and character \( \chi \) is a formal \( q \)-expansion \( f = \sum_{n=0}^\infty a_n q^n, a_n \in \mathbb{I} \), such that for a Zariski dense set of arithmetic points \( \phi \in \text{Spec}(\mathbb{I}) \) the specialization \( f_\phi = \sum_{n=0}^\infty \phi(a_n)q^n \) of \( f \) at \( \phi \) is the \( q \)-expansion of a modular form of weight \( \kappa_\phi \), character \( \chi\chi_\phi \omega^{2-\kappa_\phi} \) (where \( \omega \) is the Teichmüller character), and level \( Mp^{t_\phi} \) for some \( t_\phi \geq 0 \).

The \( U_p \) operator is defined on both the spaces of modular forms and families. It is given by:

\[
U_p(\sum_{n=0}^\infty a_n q^n) = \sum_{n=0}^\infty a_{pn} q^n.
\]

Note that \( (U_p \cdot f_\phi) = U_p \cdot f_\phi \). Hida’s ordinary idempotent \( e_p \) is defined by \( e_p := \lim_{n \to \infty} U_p^n \). A form \( f \) or family \( f \) is called ordinary if \( e_p f = f \) or \( e_p f = f \). (See for instance \cite{H19} Page 550.) A well known fact is that every ordinary eigenform fits into an ordinary family of eigenforms \( f \) (\cite{H19} Theorem II for example). By results of Deligne, Langlands, Shimura et al there is a Galois representation \( \rho_f : G_Q \to \text{GL}_2(\mathbb{Q}_p) \) for \( f \). If the residual representation \( \bar{\rho}_f \) is irreducible then one can construct a Galois representation \( \rho_f : G_Q \to \text{GL}_2(\mathbb{I}) \) such that it specializes to the Galois representation \( \rho_{f_\phi} \) of \( f_\phi \) at each arithmetic specialization \( \phi \in \text{Spec}(\mathbb{I}) \).
2.2 The Main Conjecture

Before formulating the main conjecture, we first define the characteristic ideals and the Fitting ideals. We let \( A \) be a Noetherian ring. We write \( \text{Fitt}_A(X) \) for the Fitting ideal in \( A \) of a finitely generated \( A \)-module \( X \). This is the ideal generated by the determinant of the \( r \times r \) minors of the matrix giving the first arrow in a given presentation of \( X \):

\[
A^s \to A^r \to X \to 0.
\]

If \( X \) is not a torsion \( A \)-module, then \( \text{Fitt}_A(X) = 0 \).

Fitting ideals behave well with respect to base change. For \( I \subset A \) an ideal

\[
\text{Fitt}_{A/I}(X/IX) = \text{Fitt}_A(X) \mod I.
\]

Now suppose \( A \) is a Krull domain (a domain which is Noetherian and normal). Then the characteristic ideal is defined by:

\[
\text{char}_A(X) := \{ x \in A : \text{ord}_Q(x) \geq \text{length}_Q(X) \text{ for any height one prime } Q \text{ of } A \}.
\]

If \( X \) is not a torsion \( A \)-module, then we define \( \text{char}_A(X) = 0 \).

We consider the Galois representation:

\[
T_{f, K, \xi} := T_{f, K, \xi} \hat{\otimes} \mathbb{Z}_p \Lambda_{K}
\]

with the \( G_K \) action given by \( \rho_{f, \sigma, \xi}(\Psi_K) = \sigma(\Psi_K) \hat{\otimes} \Lambda_{K} \). We define the Selmer group (recall \( \kappa \) is assumed to be even)

\[
\text{Sel}_{f, K, \xi} := \ker \{ H^1(K, T_{f, K, \xi} \otimes \mathbb{I}[[\Gamma_K]]^*) \to H^1(I_{0, \xi}, T_{f, K, \xi} \otimes \mathbb{I}[[\Gamma_K]]^*) \times \prod_{v \neq p} H^1(I_{v, \xi}, T_{f, K, \xi} \otimes \mathbb{I}[[\Gamma_K]]^*) \}
\]

where \( * \) means the Pontryagin dual \( \text{Hom}_{\mathbb{Z}_p}(-, \mathbb{Q}_p/\mathbb{Z}_p) \). We also define the \( \Sigma \)-primitive Selmer groups:

\[
\text{Sel}_{f, K, \xi}^\Sigma := \ker \{ H^1(K, T_{f, K, \xi} \otimes \mathbb{I}[[\Gamma_K]]^*) \to H^1(I_{0, \xi}, T_{f, K, \xi} \otimes \mathbb{I}[[\Gamma_K]]^*) \times \prod_{v \in \Sigma} H^1(I_{v, \xi}, T_{f, K, \xi} \otimes \mathbb{I}[[\Gamma_K]]^*) \}
\]

We let

\[
X_{f, K, \xi} := (\text{Sel}_{f, K, \xi})^*,
\]

\[
X_{f, K, \xi}^\Sigma := (\text{Sel}_{f, K, \xi}^\Sigma)^*.
\]

These are finitely generated \( \mathbb{I}[[\Gamma_K]] \)-modules (see e.g. [15, Lemma 3.3]). We take the extension of scalars of them to \( \hat{\mathbb{I}}[[\Gamma_K]] \) and still denote them using the same notations. In section 7 we are going to construct \( p \)-adic \( L \)-functions \( L_{f, K, \xi}^{Hida} \) and \( L_{f, K, \xi}^{\Sigma, Hida} \) which are elements in \( \hat{\mathbb{I}}[[\Gamma_K]] \) or its fraction field. Their interpolation formulas are given in equation (7) (see also Remark 7.6). The 3-variable Iwasawa main conjecture is
Conjecture 2.2. $X_{f,K,\xi}$ and $X^\Sigma_{f,K,\xi}$ are torsion $\hat{I}_{ur}[[\Gamma_K]]$-modules and

$$\text{char}_{I_{ur}[[\Gamma_K]]} X_{f,K,\xi} = (L_{f,K,\xi}^{Hida}),$$

$$\text{char}_{\hat{I}_{ur}[[\Gamma_K]]} X^\Sigma_{f,K,\xi} = (L^\Sigma_{f,K,\xi}^{Hida}).$$

We can also replace $f$ with a single form $f_0$ and have the two-variable main conjecture

Conjecture 2.3. $X_{f_0,K,\xi}$ and $X^\Sigma_{f_0,K,\xi}$ are torsion $\hat{O}_L^{ur}[[\Gamma_K]]$-modules and

$$\text{char}_{\hat{O}_L^{ur}[[\Gamma_K]]} X_{f_0,K,\xi} = (L_{f_0,K,\xi}^{Hida}),$$

$$\text{char}_{\hat{O}_L^{ur}[[\Gamma_K]]} X^\Sigma_{f_0,K,\xi} = (L^\Sigma_{f_0,K,\xi}^{Hida}).$$

2.3 Control of Selmer Groups

In this subsection we prove a control theorem of Selmer groups which will be used to prove Theorem 1.2. Let $\phi_0 \in \text{Spec}[[\Gamma_K]](\bar{\mathbb{Q}}_p)$ to be the point mapping $\gamma^\pm$ to 1 and such that $\phi_0|\Sigma$ corresponds to a form $f_0$. Let $\phi = \ker\phi_0|\Sigma$ of weight two and trivial character. Then we prove the following proposition.

Proposition 2.4. There is an exact sequence of $O_L[[\Gamma_K]]$-modules

$$M \to X^\Sigma_{f,K,\xi}/\phi X^\Sigma_{f,K,\xi} \to X^\Sigma_{f_0,K,\xi} \to 0$$

where $M \otimes_{O_L} L$ has support of codimension at least 2 in $\text{Spec}O_L[[\Gamma_K]] \otimes L$.

Proof. We write $I_K$ for $I[[\Gamma_K]]$ for simplicity. Write $T = T_{f,K,\xi}$ as an $I_K$-module. Let $T$ be the $\Lambda_K$-module $T_{f_0,K,\xi}$. Recall that $p = v_0\bar{v}_0$. We have an exact sequence

$$0 \to T \otimes_{\Lambda_K} \Lambda_K^* \to T \otimes_{I_K} I_K^* \to T \otimes_{I_K} (\phi I_K)^* \to 0.$$

Write $G_{K,\xi}$ for the Galois group over $K$ of the maximal algebraic extension of $K$ unramified outside $\Sigma$. From this we deduce

$$H^1(G_{K,\xi},T \otimes_{\Lambda_K} \Lambda_K^*) \xrightarrow{\sim} H^1(G_{K,\xi},T \otimes_{I_K} I_K^*)[\phi]$$

as in Proposition 3.7. We also have an exact sequence:

$$H^0(I_{\bar{v}_0},T \otimes_{I[[\Gamma_K]]} (I[[\Gamma_K]]^*)) \xrightarrow{s} H^0(I_{\bar{v}_0},T \otimes_{I[[\Gamma_K]]} (\phi[[\Gamma_K]]^*))$$

$$\to H^1(I_{\bar{v}_0},T \otimes_{O_L[[\Gamma_K]]} (O_L[[\Gamma_K]]^*)) \to H^1(I_{\bar{v}_0},T \otimes_{I[[\Gamma_K]]} (I[[\Gamma_K]]^*)).$$

From these we deduce an exact sequence of $\Lambda_K$-modules

$$M := ((\text{cokers})^{G_{v_0}}/\phi(\text{cokers})^{G_{v_0}}) \to X^\Sigma_{f,K,\xi}/\phi X^\Sigma_{f,K,\xi} \to X^\Sigma_{f_0,K,\xi} \to 0.$$

Let $K_{\infty,v_0}$ (resp. $K_{\infty,\bar{v}_0}$) be the $\mathbb{Z}_p$-extension of $K$ unramified outside $v_0$ (resp. $\bar{v}_0$) and let $\Gamma_{\bar{v}_0} = \text{Gal}(K_{\infty,\bar{v}_0}/K)$ (resp. $\Gamma_{v_0} = \text{Gal}(K_{\infty,v_0}/K)$). Let $\gamma_{\bar{v}_0} \in \Gamma_{\bar{v}_0}$ and $\gamma_{v_0} \in \Gamma_{v_0}$ be topological generators. Note that we have

$$0 \to T^+ \to T \to T/T^+ \to 0.$$
as $G_{\mathbb{Q}_p}$-modules. By the description of the Galois action, there is a $\gamma \in I_{\ell_0}$ such that $\gamma - 1$ acts invertibly on $T^+ \otimes \mathbb{Q}[\Gamma_K]$ ($\mathbb{Q}[[\Gamma_K]]^*$). We take a basis $(v_1, v_2)$ such that $v_1$ generates $T^+$ and the action of $\gamma$ on $T$ is diagonal under this basis. Then it is not hard to see (by looking at the $I_{\ell_0}$-action) that if

$$v \in H^0(I_{\ell_0}, T \otimes \mathbb{Q}[\Gamma_K])$$

we have $v \in (\mathbb{Q}[[\Gamma_{\ell_0}]]^*)^* \cdot v_2$ and if $v \in H^0(I_{\ell_0}, T \otimes \mathbb{Q}[\Gamma_K] (\varphi \mathbb{Q}_K)^*)$ then $v \in (\varphi \mathbb{Q}_K)^* v_2$. From the above discussion we know that $((\text{cokers}_1)^{G_{\ell_0}}/\ker \varphi_0(\text{cokers}_1)^{G_{\ell_0}}$ is supported in

$$\text{Spec} \mathcal{O}_L[[\Gamma_{\ell}]] \otimes L.$$

Moreover by looking at the action of $\text{Frob}_{\ell_0}$ we see it is killed by the function $a_p^{-1}R - 1$ where $a_p$ is the invertible function in $\mathbb{Q}$ which gives the $U_p$-eigenvalue of $f$ and $R$ is the image in $\Gamma_{\ell}$ of $\text{Frob}_{\ell_0}$ under class field theory. But $a_p(\phi_0) \neq 1$ and $R(\phi_0) = 1$ so $a_p^{-1}R - 1$ is non zero at $\phi_0$. So the support of $M \otimes \mathcal{O}_L L$ has support of dimension at most zero and this proves the proposition.

3 Unitary Groups

In this section we introduce our notation for unitary groups and develop Hida theory on them. We define $S_p(R)$ to be the set of $n \times n$ Hermitian matrices with entries in $\mathcal{O}_K \otimes \mathbb{Z} R$. We define a map $e_{\mathbb{A}} = \prod_v e_v : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}^\times$ where for each place $v$ of $\mathbb{Q}$ $e_v$ is the usual exponential map at $v$. We refer to [16 Section 2.8] for the discussion of the CM period $\Omega_\infty$ and the $p$-adic period $\Omega_p$.

For two automorphic forms $f_1, f_2$ on $U(2)$ we write $(f_1, f_2) = \int_{[U(2)]} f_1(g) f_2(g) dg$. Here the Harr measure is normalized so that at finite places $U(2)(\mathbb{Z}_\ell)$ has measure 1, and at $\infty$ the compact set $U(1)(\mathbb{R}) \setminus U(2)(\mathbb{R})$ has measure 1.

3.1 Groups

Let $\delta \in K$ be a totally imaginary element such that $-i\delta$ is positive. Let $d = \text{Nm}(\delta)$ which we assume to be a $p$-adic unit. Let $U(2) = U(2, 0)$ (resp. $GU(2) = GU(2, 0)$) be the unitary group (resp. unitary similitude group) associated to the skew-Hermitian matrix $\zeta = \begin{pmatrix} s\delta & \delta \\ -\delta & \delta \end{pmatrix}$ for some $s \in \mathbb{Z}_+$ prime to $p$. More precisely $GU(2)$ is the group scheme over $\mathbb{Z}$ defined by: for any $\mathbb{Z}$ algebra $A$,

$$GU(2)(A) = \{ g \in \text{GL}_2(A \otimes \mathbb{Q}) | \bar{g} \zeta g = \lambda(g) \zeta, \lambda(g) \in A^\times \}. $$

The map $\mu : GU(2) \to \mathbb{Z}_m$, $g \mapsto \lambda(g)$ is called the similitude character and $U(2) \subseteq GU(2)$ is the kernel of $\mu$. Let $G = GU(3, 1)$ (resp. $U(3, 1)$) be the similarly defined unitary similitude group (resp. unitary group) over $\mathbb{Z}$ associated to the skew-Hermitian matrix

$$\begin{pmatrix} 1 & \zeta \\ -1 & \zeta \end{pmatrix}.$$ 

Let $P \subseteq G$ be the parabolic subgroup of $GU(3, 1)$ consisting of those matrices in $G$ of the form

$$\begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix}. $$

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Let $N_P$ be the unipotent radical of $P$. Then if $X_K$ is the 1-dimensional space over $K$,

$$M_P := \text{GL}(X_K) \times \text{GU}(2) \hookrightarrow \text{GU}(V), \ (a, g_1) \mapsto \text{diag}(a, g_1, \mu(g_1)a^{-1})$$

is the Levi subgroup. Let $G_P := \text{GU}(2)(\subset M_P) \hookrightarrow \text{GU}(V)$, $(a, g_1) \mapsto \text{diag}(a, g_1, \mu(g_1))$. Let $\delta_P$ be the modulus character for $P$. We usually use a more convenient character $\delta$ such that $\delta^3 = \delta_P$.

Since $p$ splits as $v_0\bar{v}_0$ in $K$, $\text{GL}_4(\mathcal{O}_K \otimes \mathbb{Z}_p) \sim \to \text{GL}_4(\mathcal{O}_{K_{v_0}} \times \mathcal{O}_{K_{\bar{v}_0}})$. Here

$$U(3, 1)(\mathbb{Z}_p) \sim \to \text{GL}_4(\mathbb{Z}_p)$$

with the projection onto the first factor. Let $B$ and $N$ be the upper triangular Borel subgroup of $G(\mathbb{Q}_p)$ and its unipotent radical, respectively. Let

$$K_p = \text{GU}(3, 1)(\mathbb{Z}_p) \simeq \text{GL}_4(\mathbb{Z}_p),$$

and for any $n \geq 1$ let $K^n_1 \subset K^n_0$ be the subgroup of matrices whose diagonal elements are 1 modulo $p^n$.

The group GU(2) is closely related to a division algebra. Put

$$D = \{g \in M_2(\mathcal{K})|g^t \bar{g} = \det(g)\zeta\}.$$ 

Then $D$ is a definite quaternion algebra over $\mathbb{Q}$ with local invariants $\text{inv}_v(D) = (-s, -D_{\mathcal{K}/\mathbb{Q}})_v$ (the Hilbert symbol). The relation between GU(2) and $D$ is explained by

$$\text{GU}(2) = D^X \times_{\mathbb{G}_m} \text{Res}_{\mathcal{K}/\mathbb{Q}} \mathbb{G}_m.$$ 

For each finite place $v$ we write $D^1_v$ for the set of elements $g_v \in D^X_v$ such that $|\text{Nm}(g_v)|_v = 1$, where $\text{Nm}$ is the reduced norm.

Let $\Sigma$ be a finite set of primes containing all the primes at which $\mathcal{K}/\mathbb{Q}$ or $\xi$ is ramified, the primes dividing the level of $f_0$ (as in the introduction), the primes dividing $s$, the primes such that $U(2)(\mathbb{Q}_v)$ is compact and the prime 2. Let $\Sigma^1$ and $\Sigma^2$, respectively be the set of non-split primes in $\Sigma$ such that $U(2)(\mathbb{Q}_v)$ is non-compact, and compact. We will sometimes write $[D^X]$ for $D^X(\mathbb{Q}) \setminus D^X(A_{\mathbb{Q}})$. We similarly write $[U(2)], [\text{GU}(2, 0)],$ etc.

We define $G_n = \text{GU}(n, n)$ for the unitary similitude group for the skew-Hermitian matrix $\begin{pmatrix} 1_n \\ -1_n \end{pmatrix}$

and $U(n, n)$ for the corresponding unitary groups.

### 3.2 Hermitian Spaces and Automorphic Forms

Let $(r, s) = (3, 3)$ or $(3, 1)$ or $(2, 0)$. Then the unbounded Hermitian symmetric domain for GU$(r, s)$ is

$$X^+ = X_{r, s} = \{\tau = \begin{pmatrix} x \\ y \end{pmatrix} | x \in M_s(\mathbb{C}), y \in M_{(r-s) \times s}(\mathbb{C}), i(x^* - x) > iy^* \zeta^{-1} y\}.$$
We use $x_0$ to denote the Hermitian symmetric domain for GU(2), which is just a point. We have

$$\iota: X_{3,1} \times X_{2,0} \hookrightarrow X_{3,3}$$

where $Z_\tau = \begin{pmatrix} x & 0 \\ y & \frac{\zeta}{2} \end{pmatrix}$ for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $G_{r,s} = GU(r,s)$ and $H_{r,s} = GL_r \times GL_s$. Let $G_{r,s}(\mathbb{R})^+$ be the subgroup of elements of $G_{r,s}(\mathbb{R})$ whose similitude factors are positive. If $s \neq 0$ we define a cocycle:

$$J: G_{r,s}(\mathbb{R})^+ \times X^+ \rightarrow H_{r,s}(\mathbb{C})$$

by $J(\alpha, \tau) = (\kappa(\alpha, \tau), \mu(\alpha, \tau))$, where for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\alpha = \begin{pmatrix} a & b & c \\ g & e & f \\ h & l & d \end{pmatrix}$ (blocks matrix with respect to the partition $(s + (r - s) + s)$),

$$\kappa(\alpha, \tau) = \begin{pmatrix} \bar{h}x + \bar{d} \\ -\bar{\zeta}^{-1}g'x + \bar{f} \\ -\bar{\zeta}^{-1}l'y + \bar{\zeta}^{-1}e\zeta \end{pmatrix}, \ \mu(\alpha, \tau) = hx + ly + d$$

in the GU(3,1) case and

$$\kappa(\alpha, \tau) = \bar{h}x + \bar{d}, \ \mu(\alpha, \tau) = hx + d$$

in the GU(3,3) case. Let $i \in X^+$ be the point $\begin{pmatrix} 1s \\ 0 \end{pmatrix}$. Let $K^+_{\infty}$ be the compact subgroup of $U(r, s)(\mathbb{R})$ stabilizing $i$ and let $K^+_{\infty}$ be the groups generated by $K^+_{\infty}$ and $\text{diag}(1_{r+s}, -1_s)$. Then

$$K^+_{\infty} \rightarrow H(\mathbb{C}), \ k_\infty \mapsto J(k_\infty, i)$$

defines an algebraic representation of $K^+_{\infty}$.

**Definition 3.1.** A weight $\underline{k}$ is defined to be an $(r + s)$-tuple

$$\underline{k} = (c_{r+s}, \ldots, c_{s+1}; c_1, \ldots, c_s) \in \mathbb{Z}^{r+s}$$

with $c_1 > \ldots > c_{r+s}$.

We refer to [16 Section 3.1] for the definition of the algebraic representation $L_\underline{k}(\mathbb{C})$ of $H$ with the action denoted by $\rho_\underline{k}$ (note the different index for weight) and define a model $L_\underline{k}(\mathbb{C})$ of the representation $H(\mathbb{C})$ with the highest weight $\underline{k}$ as follows. The underlying space of $L_\underline{k}(\mathbb{C})$ is $L_\underline{k}(\mathbb{C})$ and the group action is defined by

$$\rho_\underline{k}(h) = \rho_\underline{k}(h^{-1}), h \in H(\mathbb{C}).$$

Our convention for identifying a weight with a tuple of integers is different from others in literature. For example our $c_{s+i}$ (1 $\leq i \leq r$) and $c_j$ (1 $\leq j \leq s$) corresponds to $-a_{r+1-i}$ and $b_{s+1,j}$ in [16 Section 3.1]. We also note that if each $\underline{k} = (0, \ldots, 0; \kappa, \ldots, \kappa)$ then $L_\underline{k}(\mathbb{C})$ is one dimensional.
For a weight \( k \), define \( \|k\| \) by:
\[
\|k\| := -c_{s+1} - \ldots - c_{s+r} + c_1 + \ldots + c_s
\]
and \( |k| \) by:
\[
|k| = (c_1 + \ldots + c_s)\sigma - (c_{s+1} + \ldots + c_{s+r})\sigma c \in \mathbb{Z}^I.
\]
Here \( I \) is the set of embeddings \( K \hookrightarrow \mathbb{C} \) and \( \sigma \) is the Archimedean place of \( K \) determined by our fixed embedding \( K \hookrightarrow \mathbb{C} \). Let \( \chi \) be a Hecke character of \( K \) with infinite type \( |k| \), i.e. the Archimedean part of \( \chi \) is given by:
\[
\chi_{\infty}(z) = (z^{c_1 + \ldots + c_s} \cdot z^{-c_{s+1} + \ldots + c_{s+r}}).
\]

**Definition 3.2.** Let \( U \) be an open compact subgroup in \( G(\mathbb{A}_f) \). We denote by \( M_k(U, \mathbb{C}) \) the space of holomorphic \( L_k^{\mathbb{C}}(\mathbb{C}) \)-valued functions \( f \) on \( X^+ \times G(\mathbb{A}_f) \) such that for \( \tau \in X^+ \), \( \alpha \in G(\mathbb{Q})^+ \) and \( u \in U \) we have
\[
f(\alpha \tau, \alpha gu) = \mu(\alpha)^{-\|k\|} \rho_k^\mathbb{C}(J(\alpha, \tau)) f(\tau, g).
\]

Now we consider automorphic forms on unitary groups in the adelic language. The space of automorphic forms of weight \( k \) and level \( U \) with central character \( \chi \) consists of smooth and slowly increasing functions \( F : G(\mathbb{A}) \to L_k^\mathbb{C}(\mathbb{C}) \) such that for every \( (\alpha, k_\infty, u, z) \in G(\mathbb{Q}) \times K_\infty^+ \times U \times Z(\mathbb{A}), \)
\[
F(zagk_\infty u) = \rho_k^\mathbb{C}(J(k_\infty, \mathfrak{i}^{-1}) F(g) \chi^{-1}(z).
\]
We can associate a \( L_k^\mathbb{C} \)-valued function on \( X^+ \times G(\mathbb{A}_f)/U \) given by
\[
f(\tau, g) := \chi_f(\mu(g)) \rho_k^\mathbb{C}(J(g_\infty, \mathfrak{i})) F((f_\infty, g))
\]
where \( g_\infty \in G(\mathbb{R}) \) such that \( g_\infty(\mathfrak{i}) = \tau \). If this function is holomorphic then we say that the automorphic form \( F \) is holomorphic.

### 3.3 Galois representations Associated to Cuspidal Representations

In this section we follow [45, Theorem 7.1, Lemma 7.2] to discuss the Galois representations associated to cuspidal automorphic representations on \( GU(r, s)(\mathbb{A}_\mathbb{Q}) \). Let \( \pi \) be an irreducible automorphic representation of \( GU(r, s)(\mathbb{A}_\mathbb{Q}) \) generated by a holomorphic cuspidal eigenform with weight \( k = (c_{r+s}, \ldots, c_{s+1}; c_1, \ldots, c_s) \) and central character \( \chi_\pi \). Let \( \Sigma(\pi) \) be a finite set of primes of \( \mathbb{Q} \) containing all the primes at which \( \pi \) is unramified and all the primes dividing \( p \). Then for some \( L \) finite over \( \mathbb{Q}_p \) there is a Galois representation (see [42], [33] and [43]):
\[
R_p(\pi) : G_K \to GL_n(L)
\]
such that:

(a) \( R_p(\pi)^c \simeq R_p(\pi)^\vee \otimes \rho_{p, \chi_\pi^1} \epsilon^{1-n} \) where \( \rho_{p, \chi_\pi^1} \) denotes the \( p \)-adic Galois character associated to \( \chi_\pi^1 \) by class field theory and \( \epsilon \) is the cyclotomic character.

(b) \( R_p(\pi) \) is unramified at all finite places not above primes in \( \Sigma(\pi) \), and for such a place \( w \nmid p \):
\[
\det(1 - R_p(\pi)(\text{Frob}_w)q_w^{-s}) = L(BC(\pi)_w \otimes \chi_\pi^c, s + \frac{1-n}{2})^{-1}
\]
Here the Frobenius is the geometric Frobenius and BC means the base change from $U(r, s)$ to $GL_{r+s}$. Suppose $\pi_v$ is nearly ordinary (see Subsection 3.8) and unramified at all primes $v$ dividing $p$ with respect to $k$. Recall $v_0$ corresponds to $i : \mathbb{C} \cong \mathbb{C}_p$. If we write $\kappa_i = s - i + c_i$ for $1 \leq i \leq s$ and $\kappa_i = c_i + s + r + s - i$ for $s + 1 \leq i \leq r + s$, then

\[
R_p(\pi)|_{G_{K, v_0}} \simeq \begin{pmatrix}
\xi_{r+s, v} e^{-\kappa_{r+s}} & * & * & * \\
* & \xi_{r+s-1, v} e^{\kappa_{r+s-1}} & * & * \\
* & * & \ldots & * \\
0 & * & * & \xi_{1, v} e^{-\kappa_1}
\end{pmatrix}
\]

where $\xi_{i, v}$ are unramified characters. Using the fact (a) above we also know that $R_p(\pi)_{v_0}$ is equivalent to an upper triangular representation as well (with the Hodge-Tate weight being $-(\kappa_1 + 1 - r - s - |k|), \ldots, -(\kappa_{r+s} + 1 - r - s - |k|)$ (in our geometric convention $\epsilon^{-1}$ has Hodge-Tate weight one).

### 3.4 Shimura varieties

Now we consider the group $GU(3, 1)$. For any open compact subgroup $K = K_p K^p$ of $GU(3, 1) (\mathbb{A}_f)$ whose $p$-component is $K_p = GU(3, 1)(\mathbb{Z}_p)$, we refer to [16 Section 2.1] for the definition and arithmetic models of the associated Shimura variety, which we denote as $S_G(K)/\mathcal{O}_{K, v_0}$. The scheme $S_G(K)$ represents the following functor: for any $\mathcal{O}_{K, v_0}$-algebra $R$, $A(R) = \{(A, \lambda, \iota, \bar{\eta}^p)\}$ where $A$ is an abelian scheme over $R$ of relative dimension four with CM by $\mathcal{O}_K$ given by $\iota$, $\lambda$ is an orbit of prime-to-$p$ polarizations and $\bar{\eta}^p$ is an orbit of prime-to-$p$ level structures. There is also a theory of compactifications of $S_G(K)$ developed in [29]. We denote by $\bar{S}_G(K)$ a toroidal compactification and $S^*_G(K)$ the minimal compactification. We refer to [16 Section 2.7] for details. The boundary components of $S^*_G(K)$ is in one-to-one correspondence with the set of cusp labels defined below. For $K = K_p K^p$ as above we define the set of cusp labels to be:

\[
C(K) := (\text{GL}(X_K) \times G_P(\mathbb{A}_f)) N_P(\mathbb{A}_f) \backslash G(\mathbb{A}_f)/K.
\]

This is a finite set. We denote by $[g]$ the class represented by $g \in G(\mathbb{A}_f)$. For each such $g$ whose $p$-component is 1 we define $K^g_p = G_P(\mathbb{A}_f) \cap gKg^{-1}$ and denote $S_{[g]} := \bar{S}_{GP}(K^g_p)$ the corresponding Shimura variety for the group $G_P$ with level group $K^g_p$. By strong approximation we can choose a set $C(K)$ of representatives of $C(K)$ consisting of elements $g = pk^0$ for $p \in P(\mathbb{A}_f^\Sigma)$ and $k^0 \in K^0$ for $K^0$ the maximal compact subgroup of $G(\mathbb{A}_f)$ defined in [16 Section 1.10].

### 3.5 Igusa varieties and $p$-adic automorphic forms

Now we recall briefly the notion of Igusa varieties in [16 Section 2.3]. Let $M$ be the standard lattice of $V$ and $M_p = M \otimes \mathbb{Z}_p$. Let $\text{Pol}_p = \{N^{-1}, N^0\}$ be a polarization of $M_p$. Recall that this means that $N^{-1}$ and $N^0$ are maximal isotropic $\mathcal{O}_K \otimes \mathbb{Z}_p$-submodules in $M_p$ such that they are dual to each other with respect to the Hermitian metric on $V$, and

\[
\text{rank}_{\mathbb{Z}_p} N_{v_0}^{-1} = \text{rank}_{\mathbb{Z}_p} N_{v_0}^0 = 3, \text{rank}_{\mathbb{Z}_p} N_{v_0} = 1.
\]

We mainly follow [16 Section 2.3] in this Subsection. The Igusa variety of level $p^n$ is the scheme over $\mathcal{O}_{K, v_0}$ representing the quadruple $A(R) = ((A, \lambda, \iota, \bar{\eta}^p))$ for Shimura variety of $GU(3, 1)$ as
above, together with an injection of group schemes

\[ j : \mu_{p^n} \otimes_{\mathbb{Z}} \mathbb{N}^0 \hookrightarrow A[p^n] \]

over \( R \) which is compatible with the \( \mathcal{O}_K \)-action on both hand sides. Note that the existence of \( j \) implies that \( A \) must be ordinary along the special fiber. There is also a theory of Igusa varieties over \( S_{G}(K) \). Let \( \omega \) be the automorphic vector bundle on \( S_{G}(K) \) as defined in [?, 2.7.3]. As in loc.cit let \( \tilde{H}_{p-1} \in H^0(S_G(K)/\mathbb{F}, \det(\omega)^{p-1}) \) be the Hasse invariant. Over the minimal compactification, some power (say the \( t \)-th) of the Hasse invariant can be lifted to \( \mathcal{O}_m \), by the ampleness of \( \det \omega \). We denote such a lift by \( E \). By the Koecher principle we can regard \( E \) as in \( H^0(S_G(K), \det(\omega^{(p-1)}) \). Let \( \mathcal{O}_m := \mathcal{O}_{\mathcal{K},v_0}/p^m \mathcal{O}_{\mathcal{K},v_0} \). Set \( T_{O,n,m} := S_G(K)[1/E]/\mathcal{O}_m \). For any positive integer \( n \) define \( T_{n,m} := T_{G(K^n)/\mathcal{O}_m} \). Let \( V_{\infty,m} = \lim_{\to\leftarrow} T_{n,m,m} \). Then \( V_{\infty,m} \) is a Galois cover over \( T_{n,m} \) with Galois group \( \mathcal{H} \simeq GL_3(\mathbb{Z}_p) \times GL_1(\mathbb{Z}_p) \). Let \( \mathcal{N} \subset \mathcal{H} \) be the upper triangular unipotent radical. Define:

\[ V_{n,m} = H^0(T_{n,m}, \mathcal{O}_{T_{n,m}}). \]

Let \( V_{n,m} = \lim_{\to\leftarrow} V_{n,m} \) and \( V_{\infty,\infty} = \lim_{\to\leftarrow} V_{n,m} \) be the space of \( p \)-adic automorphic forms on \( GU(3, 1) \) with level \( K \). We also define \( W_{n,m} = V_{n,m}^N, \quad W_{\infty,m} = V_{\infty,m}^N \) and \( W = \lim_{\to\leftarrow} W_{n,m} \). We define \( V_{n,m}^0, \) etc., to be the cuspidal part of the corresponding spaces.

We can make similar definitions for the definite unitary similitude groups \( G_P \) as well and define \( V_{n,m,P}, V_{\infty,\infty,P}, V_{n,m,P}^N, W_P, \) etc.

Let \( K_0^n \) and \( K_1^n \) be the subgroup of \( \mathcal{H} \) consisting of matrices which are in \( B_3 \times \ll B_1 \) or \( N_3 \times N_1 \) modulo \( p^n \). (These notations are already used for level groups of automorphic forms. The reason for using the same notation here is that automorphic forms with level group \( K^n_0 \) are \( p \)-adic automorphic forms of level group \( K^n_1 \).) We sometimes denote \( I_G(K_0^n) = I_G(K^n_0)K^n_1 \) and \( I_G(K_1^n) = I_G(K^n_1)K^n_0 \).

We can define the Igusa varieties for \( G_P \) as well. For \( \bullet = 0, 1 \) we let \( K^n_{P,\bullet} := gK^n_{\bullet}g^{-1} \cap G_P(\mathbb{A}_f) \) and let \( I_g(K^n_{\bullet}) := I_G(K^n_{P,\bullet}) \) be the corresponding Igusa variety over \( S_{[g]} \). We denote \( A^n_{[g]} \) the coordinate ring of \( I_g(K^n_{\bullet}) \). Let \( A^n_{[g]} = \lim_{\to\leftarrow} A^n_{[g]} \) and let \( \hat{A^n}_{[g]} \) be the \( p \)-adic completion of \( A^n_{[g]} \). This is the space of \( p \)-adic automorphic forms for the group \( GU(2, 0) \) of level group \( gK^{-1}_0 \cap G_P(\mathbb{A}_f) \).

For Unitary Groups

Assume the tame level group \( K \) is neat. For any \( c \) an element in \( \mathbb{Q}_+ \setminus A^\infty_{\mathbb{Q},f}/\mu(K) \). We refer to [?? 2.5] for the notion of \( c \)-Igusa schemes \( P^0_{U(2)}(K, c) \) for the unitary groups \( U(2, 0) \) (not the similitude group). It parameterizes quintuples \( (\lambda, \iota, \eta^{(p)}, j) \) similar to the Igusa schemes for unitary similitude groups but requiring \( \lambda \) to be a prime to \( p \) \( c \)-polarization of \( A \) such that \( (\lambda, \hat{\lambda}, \iota, \eta^{(p)}, j) \) is a quintuple as in the definition of Shimura varieties for \( GU(2) \). For \( g_e \) with \( \mu(g) \in A^\infty_{\mathbb{Q},f} \) in the class of \( c \). Let \( K = g_e K g_e^{-1} \cap U(2)(\mathbb{A}_Q, f) \). Then the space \( P^0_{U(2)}(K, c) \) is isomorphic to the space of forms on \( P^0_{U(2)}(\mathcal{O}_K, 1) \) (see loc.cit).

Embedding of Igusa Schemes

In order to use the pullback formula geometrically we need a map from the Igusa scheme of \( U(3, 1) \times U(0, 2) \) to that of \( U(3, 3) \) (or from the Igusa scheme of \( U(2, 0) \times U(0, 2) \) to that of \( U(2, 2) \)) given by:

\[ i([A_1, \lambda_1, \iota_1, \eta_{11}^p K_1, j_1], [A_2, \lambda_2, \iota_2, \eta_{12}^p K_2, j_2]) = [(A_1 \times A_2, \lambda_1 \times \lambda_2, \iota_1, \iota_2, (\eta_{11}^p \times \eta_{12}^p) K_3, j_1 \times j_2)]. \]
We define an element \( \Upsilon \in U(3,3)(\mathbb{Q}_p) \) such that \( \Upsilon v_0 = S^{-1}_v \) and \( \Upsilon' v_0 = S^{-1'}_v \), where \( S \) is defined at the end of Section 6.2. Similar to [16], we know that under the complex uniformization, taking the change of polarization into consideration the above map is given by

\[
i([\tau, g], [x_0, h]) = [Z_{\tau}, (g, h)\Upsilon]
\]

(see [16] Section 2.6.)

### 3.6 Fourier-Jacobi expansions

**Analytic Fourier-Jacobi Coefficients:**
Let us go back to the Fourier-Jacobi coefficients of automorphic forms on \( G := \text{GU}(3,1) \). Let \( \beta \in \mathbb{Q}_+^{\times} \). Over \( \mathbb{C} \) we have the \( \beta \)-analytic Fourier-Jacobi coefficient for a holomorphic automorphic form \( f \) given by:

\[
FJ_{\beta}(f, g) = \int_{\mathbb{Q}\backslash \mathbb{A}} f\left(\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} g \right) e_{\mathbb{A}}(-\beta n) \, dn.
\]

The Harr measure is normalized so that the set \( (\mathbb{Q}\backslash \mathbb{A}) \) has measure 1.

**\( p \)-adic Cusps**
As in [16] each pair \((g, j) \in C(K) \times H\) can be regarded as a \( p \)-adic cusp, i.e. cusps of the Igusa tower. In the following we are going to give the algebraic Fourier-Jacobi expansion at \( p \)-adic cusps.

**Algebraic Theory for Fourier-Jacobi expansions**
We follow [16] Pages 16-17 to give some backgrounds about the algebraic theory for the group \( G = \text{GU}(r,1) \). Recall \([g] \) is a cusp label corresponding to class \( g \in G(\mathbb{A}_f) \). One defines \( \mathcal{Z}_{[g]} \) a group scheme over \( S_{[g]} \) using the universal abelian variety as in loc.cit and denote \( \mathcal{Z}_{[g]}^0 \) the connected component over \( S_{[g]} \). It is well known that there is a Fourier-Jacobi expansion for modular forms by evaluating the form at the Mumford family \((M, \omega_M)\) over the ring

\[
\prod_{\beta} (\hat{A}_{[g]}^\infty \otimes_{\mathcal{O}} R) \otimes_{A_{[g]}} H^0(\mathcal{Z}_{[g]}^0, \mathcal{L}(\beta))
\]

(we again refer to loc.cit).

Now let \( f \in H^0(I_G(K_f^1)_{/R}, \omega_\kappa) \) be a scalar weight \( \kappa \geq 6 \) (i.e. of weight \( (0,0,0; \kappa) \)) modular form over an \( \mathcal{O} \) algebra \( R \), then by ([16] 3.6.2) there is a Fourier-Jacobi expansion of \( f \) at the \( p \)-adic cusp \((g, h)\) for \( h \in H\):

\[
FJ_{[g]}^h(f) = \sum_{\beta \in \mathcal{Z}_{[g]}} a_{[g]}^h(\beta, f) q^\beta
\]

where

\[
a_{[g]}^h(\beta, f) \in (\hat{A}_{[g]}^\infty \otimes_{\mathcal{O}} R) \otimes_{A_{[g]}} H^0(\mathcal{Z}_{[g]}^0, \mathcal{L}(\beta))
\]

and \( \mathcal{Z}_{[g]} \) is a sub-lattice of \( \mathbb{Q} \) determined by the level subgroup. This is given by evaluating \( f \) at the Mumford family \((\mathfrak{M}, h^{-1}j_M, \omega_M)\) where \( j_M \) is a fixed level structure (see [16] 2.7.4). (note that we do not have the subscript \( N_H^1 \) since it is a scalar weight \( \kappa \).)
Siegel Operators

We have a Siegel $\Phi$ operator at the $p$-adic cusp $(g, h)$ defined by:
\[
\Phi^h_{[g]} : H^0(I_G(K_n^I)/R, \omega_K) \rightarrow A^p_{[g]} \otimes R
\]
\[
f \mapsto \Phi^h_{[g]}(f) := a^h_{[g]}(f) := a_{[g]}(0, f).
\]
The Siegel operator at $[g]$ can be defined analytically as follows:
For any $g \in G(\mathbb{A}_f)$ we define:
\[
\Phi_{P,g}(f) = \int_{N_P(\mathbb{Q}) \backslash N_P(\mathbb{A}_Q)} f(ng)dn.
\]
We fix the Haar measure on $N_P(\mathbb{Q}) \backslash N_P(\mathbb{A}_Q)$ as in [15, Section 8.2]. The relation between the algebraic and analytic Siegel operator is given in [16 (3.12)].

3.7 Weight Space for $GU(3,1)$

Let $H = GL_3 \times GL_1$ and $T \subseteq H$ be the diagonal torus. Then $H \simeq H(\mathbb{Z}_p)$. We let $\Lambda_2 = \Lambda$ be the completed group algebra $\mathbb{Z}_p[[T(1 + \mathbb{Z}_p)]]$. This is (non-canonically) isomorphic to a formal power series ring with four variables. There is an action of $T(\mathbb{Z}_p)$ on the Igusa scheme given by its action on the embedding $j : \mu_{p^n} \otimes \mathbb{Z} N^0 \rightarrow A[p^n]$. (See [16, 3.4]) This gives the spaces of $p$-adic modular forms for $GU(3,1)$ a structure of $\Lambda$-algebra. A $\mathbb{Q}_p$-point $\phi$ of Spec$\Lambda$ is call arithmetic if it is determined by a character $[k] : [\zeta]$ of $T(1 + p\mathbb{Z}_p) \simeq (1 + p\mathbb{Z}_p)^4$ where $k$ is a weight and $\zeta = (\zeta_1, \zeta_2, \zeta_3; \zeta_4)$ for $\zeta_i \in \mu_{p^n}$. Here $[k]$ is the character by regarding $k$ as a character of $T(1 + \mathbb{Z}_p)$ by $[k](t_1, t_2, t_3, t_4) = (t_1^{-c_4}t_2^{-c_3}t_3^{-c_2}t_4^{-c_1})$ and $[\zeta]$ is the finite order character given by mapping $(1 + p)$ to $\zeta_i$ at the corresponding entry of $T(\mathbb{Z}_p)$. We often write this point $k_{\zeta}$. We also define $\omega_{[k]}$ a character of the torsion part of $T(\mathbb{Z}_p)$ (canonically isomorphic to $(\mathbb{F}_p^\times)^4$) given by $\omega_{[k]}(t_1, t_2, t_3, t_4) = \omega(t_1^{-c_4}t_2^{-c_3}t_3^{-c_2}t_4^{-c_1})$.

We can define the weight ring $\Lambda_P$ for the definite unitary group $G_P$ as well.

3.8 Nearly Ordinary Forms

We refer to [16, 3.8.3, 4.3] for the notion and definition of Hida’s idempotent $e$ acting on the space $V_{\infty, \infty}$ of $p$-adic automorphic forms on $GU(3,1)$ and the nearly ordinary subspace of the space of the $p$-adic modular forms. The spaces of nearly ordinary automorphic forms are denoted as $W_{\text{ord}}$, $W^q_{\text{ord}}$, etc. For $q = 0$ or $\emptyset$ we let $V^q_{\text{ord}}$ be the Pontryagin dual of $W^q_{\text{ord}}$. Then we have the following theorem ([16, Theorem 4.21])

**Theorem 3.3.** Let $q = 0$ or $\emptyset$. Then:
(1) $V^q_{\text{ord}}$ is a free $\Lambda$ module of finite rank;
(2) For any $k$ very regular we have nature isomorphisms:
\[
\mathcal{M}^q_{\text{ord}}(K, \Lambda) \otimes \Lambda/P_k \xrightarrow{\sim} eM^q_k(K, \mathcal{O}_p)
\]
where $\mathcal{M}^q_{\text{ord}}(K, \Lambda)$ is defined in Definition 3.3. Here we identify $eM^q_k(K, \mathcal{O}_v)$ with its image in the space of $p$-adic automorphic forms of weight $k$ under $\beta_k$.
Remark 3.4. If $K$ is a general CM field, then the statement of the corresponding result is more complicated; see [16, Section 4.5].

3.9 $\Lambda$-adic forms

**Definition 3.5.** For any finite $\Lambda$ algebra $A$, and $q = 0$ or $\emptyset$ we define the space of $A$-adic ordinary forms to be:

$$\mathcal{M}_q^{\text{ord}}(K, A) := \text{Hom}_A(V^q_{\text{ord}}, A).$$

Similarly, if $A$ is a $\Lambda_P$-algebra, then we define:

$$\mathcal{M}_{\text{ord}, [g], P}(K, P, [g], A) := \text{Hom}_A(V^\text{ord, P, [g]}, A).$$

Here the subscript means the prime to $P$ level is $K_{P}$ defined before.

For any $f \in \mathcal{M}_{\text{ord}}(K, A)$ we have an $A$-adic Fourier-Jacobi expansion:

$$F_{Jh}^g(f) = \sum_{\beta \in \pi^g} a_h^g(\beta, f) q^\beta$$

obtained from the Fourier-Jacobi expansion on $W^q_{\text{ord}}$, where $a_h^g(\beta, f) \in A \hat{\otimes} \hat{\otimes} A_{\text{ord}, [g]} H^0(L^g_0, \mathcal{L}(\beta))$ (see [16, 4.6.1]). We also have a $\Lambda$-adic Siegel operator which we denote as $\hat{\Phi}^h_g$. Let $w'_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ \in GL_4(Z_p) \approx U(3, 1)(Z_p). (Notice that we used the place $v_0$ to identify GL_4(Z_p) with U(3, 1)(Z_p) here. The matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ itself is not in U(3, 1)(Z_p). We use $w'_3$ instead of $w_3$ as in [16, Page 35] to distinguish it from $w_3 \in U(3, 3)$). Now we have the following important theorem

**Theorem 3.6.** [16, Theorem 4.26] Let $A$ be as before. We have the following short exact sequence

$$0 \rightarrow \mathcal{M}_0^{\text{ord}}(K, A) \rightarrow \mathcal{M}_{\text{ord}}(K, A) \rightarrow \oplus_{g \in X(K)} \mathcal{M}_{\text{ord}}(K_{P}, A) \rightarrow 0.$$ 

We need one more theorem which gives another definition of nearly ordinary $p$-adic modular forms using Fourier-Jacobi expansions.

**Definition 3.7.** Let $A$ be a finite torsion free $\Lambda$-algebra. Let $X(K)$ be the set $\{(g, w'_3)\}$ where $g$ runs over a set of representatives of cusp labels $C(K)$. Let $N_{\text{ord}}(K, A)$ be the set of formal Fourier-Jacobi expansions:

$$F = \{ \sum_{\beta \in \pi^g} a(\beta, F) q^\beta, a(\beta, F) \in A \hat{\otimes} \hat{\otimes} A_{\text{ord}, [g]} H^0(L^g_0, \mathcal{L}(\beta)) \}_{g \in X(K)}$$

such that for a Zariski dense set $X_F$ of points $\phi \in \text{Spec}(A)$ such that the induced point in $\text{Spec}(\Lambda)$ is some arithmetic weight $k_\phi$, the specialization $F_\phi$ of $F$ is the Fourier-Jacobi expansion of a nearly ordinary modular form with prime to $p$ level $K$, weight $k$ and nebentype at $p$ given by $[k] \otimes \omega^{-k}$.
Then we have the following theorem ([16, Theorem 4.25])

**Theorem 3.8.**

\[ M_{ord}(K, A) = N_{ord}(K, A). \]

4 Backgrounds for Theta functions

Now we recall briefly the basic notions of theta functions and theta liftings, following closely to [54] with some modifications. The author claims no originality in this section.

4.1 Heisenberg Group

Let \( W \) be a finite dimensional vector space over \( \mathbb{Q}_v \) with a non-degenerate alternating form \( \langle \cdot, \cdot \rangle \).

We define:

\[ H(W) := \{(w, t) \mid w \in W, t \in \mathbb{Q}_v \} \]

with multiplication law: \((w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle)\).

4.2 Schrödinger representation

Fix an additive character \( \psi \) of \( \mathbb{Q}_v \) and a complete polarization as \( W = X \oplus Y \) of \( W \) where \( X \) and \( Y \) are maximal totally isotropic subspaces of \( W \). Let \( S(X) \) be a space of Bruhat-Schwartz functions on \( X \), and define a representation \( \rho_\psi \) of \( H(W) \) on \( S(X) \) by:

\[
\rho_\psi(x)f(z) = f(x + z), x \in X \\
\rho_\psi(y)f(z) = \psi(\langle z, y \rangle)f(z), y \in Y \\
\rho_\psi(t)f(z) = \psi(t)f(z), t \in \mathbb{Q}_v
\]

This is called the Schrödinger representation. By the theorem of Stone and von Neumann, \( \rho_\psi \) is the unique irreducible smooth representation on which \( \mathbb{Q}_v \) acts via the character \( \psi \).

4.3 Metaplectic Groups and Weil representations

Let \( Sp(W) \) be the symplectic group preserving the alternating form \( \langle \cdot, \cdot \rangle \) on \( W \). Then \( Sp(W) \) acts on \( H(W) \) by \((w, t)g = (wg, t)\) (we use row vectors for \( w \in W \) and the right action of \( Sp(W) \) instead of the left action as in [54]). By the uniqueness of \( \rho_\psi \), there is an operator \( \omega_\psi(g) \) on \( S(X) \), determined up to scalar, such that

\[
\rho_\psi(w'g, t)\omega_\psi(g) = \omega_\psi(g)\rho_\psi(w, t)
\]

for any \((w, t) \in H(W)\). Here \( g' \) is the transpose of \( g \). Define the metaplectic group \( \tilde{Sp}(W) = \{(g, \omega_\psi(g)) \mid g \in Sp(W) \} \) which we often write \( \tilde{Sp} \) for short. Thus \( \tilde{Sp}(W) \) has an action \( \omega_\psi \) on \( S(X) \) called the Weil representation.

Suppose \( \psi = \prod_v \psi_v \) is a global additive character of \( \mathbb{Q} \setminus \mathbb{A}_K \). We can put the above construction together for all \( v \)'s to get a representation of \( \tilde{Sp}(W)(\mathbb{A}_K) \) on \( S(X(\mathbb{A})) \). This can be viewed as a projective representation of \( Sp(W) \) (a representation with image in the infinite dimensional projective linear group). We now give formulas for this representation. Let \( \{e_1, \ldots, e_n; f_1, \ldots, f_n\} \) be a basis of \( W = X \oplus Y \) such that \( \langle e_i, f_j \rangle = \delta_{ij} \). With respect to this basis the projective representation of \( \tilde{Sp}(W)(\mathbb{A}_K) \) on \( \text{Proj}S(X(\mathbb{A})) \) is given by the formulas
• \(\omega_\psi(\begin{pmatrix} A & \epsilon A^{-1} \\ -\frac{1}{2} & 1 \end{pmatrix})\phi(x) = |\det A|^{\frac{1}{2}}\phi(xA)\);

• \(\omega_\psi(\begin{pmatrix} 1 & B \\ -1 & 1 \end{pmatrix})\phi(x) = \psi(\frac{xB^t}{2})\phi(x)\);

• \(\omega_\psi(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix})\phi(x) = \gamma\hat{\phi}(x)\) where \(\hat{\phi}\) means the Fourier transform of \(\phi\) with respect to the additive character \(\psi\). The \(\gamma\) is an 8-th root of unity which is called the Weil constant.

4.4 Dual Reductive Pairs

A dual reductive pair is a pair of subgroups \((G, G')\) in the symplectic group \(Sp(W)\) satisfying: (1) \(G\) is the centralizer of \(G'\) in \(Sp(W)\) and vice versa; and (2) the action of \(G\) and \(G'\) are completely reducible on \(W\). We are mainly interested in the following dual reductive pairs of unitary groups. Let \(K\) be a quadratic imaginary extension of \(\mathbb{Q}\), \((V_1, (,)_1)\) be a skew Hermitian space over \(K\) and \((V_2, (,)_2)\) a Hermitian space over \(K\). Then the unitary groups \(U(V_1)\) and \(U(V_2)\) form a dual reductive pair in \(Sp(W)\), where \(W = V_1 \otimes V_2\) is given the alternating form \(\frac{1}{2}\text{tr}_{K/\mathbb{Q}}((,)_1 \otimes (,)_2)\) over \(\mathbb{Q}\). The embedding of the dual reductive pair \((U(V_1), U(V_2))\) into \(Sp(W)\) is

\[e : U(V_1) \times U(V_2) \to Sp(W)\]

\[e(g_1, g_2) \cdot (v_1 \otimes v_2) = v_1 g_1 \otimes g_2^{-1} v_2\]

4.5 Splittings

Suppose \(\dim_K V_1 = n\) and \(\dim_K V_2 = m\). If \(\chi_1\) and \(\chi_2\) are Hecke characters of \(K^\times\) such that \(\chi_1|_{\tilde{A}_F} = \chi_1^n\) and \(\chi_2|_{\tilde{A}_F} = \chi_2^m\), then there is a splitting (see [15, Section 1])

\[s : U(V_1) \times U(V_2) \to \breve{Sp}(W)\]

determined by \(\chi_2\) and \(\chi_1\). This enables us to define the Weil representations of \(U(V_1) \times U(V_2)\) on \(S(X(\mathbb{A}))\) which we denote as \(\omega_{\chi_1, \chi_2} = \omega_{\chi_1} \otimes \omega_{\chi_2}\).

4.6 Theta Functions

Now let us define theta functions.

**Definition 4.1.** Let \(\phi \in S(X(\mathbb{A}_\mathbb{Q}))\). Define

\[\theta(\phi) = \sum_{l \in X(\mathbb{Q})} \phi(l)\]

Using the Weil representation of the dual reductive pair above (with the choices of the splitting characters) we define the theta kernel for the theta correspondence as follows:

**Definition 4.2.**

\[\theta_\phi(g_1, g_2) = \theta(\omega(g_1, g_2)\phi)\]
Let \( J = H(W) \ltimes Sp(W) \) (\( \tilde{J} = H(W) \ltimes \tilde{Sp}(W) \)) be the Jacobi group with \( Sp(W) \) acting on \( H(W) \) by \((w, t) \cdot g = (wg, t)\) (\( \tilde{Sp}(W) \) acts on \( H(W) \) by \((w, t) \cdot \tilde{g} = (wg, t)\), where \( g \) is the image of \( \tilde{g} \) in \( Sp(W) \)). We define a theta kernel on \( \tilde{J}(\mathbb{A}_Q) \).

**Definition 4.3.** Let \( \tilde{g} \in \tilde{Sp}(W) \) and \((w, t) \in H(W)\), define

\[
\theta_\phi((w, t)\tilde{g}) = \sum_{l \in X(\mathbb{Q})} \rho_\psi(w, t)(\tilde{g}) \phi(l)
\]

### 4.7 Intertwining Maps

We are going to study the intertwining maps between theta series corresponding to different polarizations \((X, Y)\) of \( W \). Suppose \( r \in Sp(W) \), then \((X_r, Y_r)\) gives another polarization of \( W \), and all polarizations are obtained this way. If \( \phi \in S(X) \) the we define an intertwining map (local or global) \( \delta_\psi : S(X) \to S(X_r) \) by

\[
\delta_\psi \phi(xr) = \omega_\psi(r) \phi(x)
\]

for \( x \in X \). It is easy to check that \( \delta_\psi \) is an isometry intertwining the action of \( \tilde{J} \).

Let \( W^\sim \) be the skew Hermitian space which is isomorphic to \( W \) as \( Q_v \)-vector spaces but equipped with the alternating pairing \(-\langle, \rangle\). For a polarization \((X, Y)\) of \( W \) we are going to study the intertwining formula for the two polarizations \((X \oplus X^-) \oplus (Y \oplus Y^-)\) and \( \{w \oplus w, w \in W\} \oplus \{w \oplus -w, w \in W\} \) of \( W \oplus W^- \). We write the formula for the map \( \delta_\psi : S(X(\mathbb{Q}_v) \oplus X^-) \to S(W(\mathbb{Q}_v)) \) and its inverse:

\[
\delta_\psi(\phi)(x_1, y) = \int \psi(\langle x_2, y \rangle) \phi\left(\frac{x_1 + x_2}{2}, \frac{x_2 - x_1}{2}\right) dx_2
\]

\[
\delta_\psi^{-1}(\phi)(x_1, x_2) = \int \psi(\langle -x_1 - x_2, y \rangle) \phi(x_1 - x_2, y) dy.
\]

Another important property is that if the two polarizations \((X, Y)\) and \((X_r, Y_r)\) are globally defined, then the theta kernels \( \Theta_\phi \) and \( \Theta_{\delta_\psi(\phi)} \) are defined and

\[
\Theta_\phi(u, ng) = \Theta_{\delta_\psi(\phi)}(u, (nr)g).
\]

### 4.8 Special Cases

#### 4.8.1 Case One

We write \( V \) for the Hermitian space for \( \zeta \) with respect to the basis \((v_1, v_2)\), \( V^- \) for the Hermitian space for \(-\zeta \) with respect to the basis \((v_1^-, v_2^-)\), and \( V_1 \) for the 1-dimensional Hermitian space with the metric \( 1 \) with respect to the basis \( v \). Let \( W = V \otimes V_1 \) and \( W^- = V^- \otimes V_1 \). We define several polarizations for the Hermitian space \( W := W \oplus W^- \) (the alternating pairing being the direct sum of those for \( W \) and \( W^- \)).

**Definition 4.4.**

\[
X := \mathbb{Q}v_1 \otimes v \oplus \mathbb{Q}v_2 \otimes v
\]

\[
X^- := \mathbb{Q}v_1^- \otimes v \oplus \mathbb{Q}v_2^- \otimes v
\]

\[
Y := \mathbb{Q}\delta v_1 \otimes v \oplus \mathbb{Q}v_2 \otimes v
\]

\[
Y^- := \mathbb{Q}v_1^- \otimes v \oplus \mathbb{Q}v_2^- \otimes v.
\]
Fix the additive character $\psi = \prod \psi_v$. Thus $W = X \oplus Y$ and $W^- = X^- \oplus Y^-$ are globally defined polarizations. For a split prime $v$ we write $v = w\bar{w}$ for its decomposition in $\mathcal{K}$. We will often use an auxiliary polarization $W_v = X'_v \oplus Y'_v$ of $W_v = W \otimes \mathcal{K}_v$ with respect to $\mathcal{K}_v \simeq \mathcal{K}_{w} \times \mathcal{K}_{\bar{w}} \simeq \mathbb{Q}_v^2$ and $W_v = X'_v \oplus Y'_v$ that is defined by $X'_v = \mathcal{K}_{w} v_1 \otimes v \oplus \mathcal{K}_{\bar{w}} v_2 \otimes v$, $Y'_v = \mathcal{K}_{w} v_1 \otimes v \oplus \mathcal{K}_{\bar{w}} v_2 \otimes v$ and similar for $X_v^-, Y_v^-$. This polarization is better suited for computing the Weil representation. For split primes $v$ let $\delta''_v : S(X'_v) \to S(X_v)$ and $\delta''_v : S(X_v^-) \to S(X_v^-)$ be the intertwining operators between Schwartz functions defined above.

Let $\mathbb{W}^d = \{w \oplus w, w \in W\}$. We denote the intertwining maps:

$$\delta_\psi : S(X_v \oplus X_v^-) \to S(\mathbb{W}_v^d)$$

and if $v$ splits,

$$\delta'_\psi : S(X'_v \oplus X'_v^-) \to S(\mathbb{W}_v^d).$$

Remark 4.5. In application in Section 6 we are going to compute the intertwining operator

$$\delta_\psi : S(X_v \oplus X_v^-) \to S(\mathbb{W}_v^d)$$

(for $\mathbb{W} = (V \oplus V^-) \otimes V_1$) in this special case and the Weil representations restricting to semidirect products $H(\mathbb{W}) \ltimes (U(V \oplus V^-) \times U(V_1))$ (recall $U(V \oplus V^-) \times U(V_1) \simeq Sp(\mathbb{W})$). We provide the matrix forms of these semidirect products that will be used in Section 6. Let $U_1$ and $U_2$ be unitary groups associated to the matrices $\begin{pmatrix} 1 & \zeta \\ -1 & -\zeta \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ 3 \\ 1 \end{pmatrix}$ respectively, and let $U'_1$ and $U'_2$ be the unitary groups associated to $\begin{pmatrix} \zeta & -\zeta \\ \zeta & \zeta \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ 2 \\ 1 \end{pmatrix}$, considered as subgroups of $U_1$ and $U_2$ respectively in the obvious way. Let $N_1$ be the subgroup of $U_1$ consisting of matrices of the form $\begin{pmatrix} 1 & x_1 & * & x_2 \\ 1 & \zeta x_1^* & 1 \\ -\zeta x_2^* & 1 \\ 1 \\ 2 \\ y^* \\ y \\ 0 \\ 1 \\ -x^* \\ \bar{I}_2 \end{pmatrix}$. The corresponding semidirect products mentioned above are $J_1 = N_1 U'_1$ and $J_2 = N_2 U'_2$.

4.8.2 Case Two

Now we discuss another special situation which will be used in the Fourier-Jacobi coefficient computations for Siegel Eisenstein series on GU(3, 3).

The local set-up.

Let $v$ be a place of $\mathbb{Q}$. Let $h \in S_1(\mathbb{Q}_v)$, $\det h \neq 0$. Let $U_h$ be the unitary group of this matrix and let $V_v$ be the corresponding one-dimensional Hermitian space. Let

$$V_{2,v} = \mathcal{K}_v^2 \oplus \mathcal{K}_v^2 = X_v \oplus Y_v$$
be the Hermitian space associated to $U_2 = U(2, 2)$ with the alternating pairing denoted as $\langle , \rangle_2$. Let $W = V_v \otimes_{\mathcal{K}_v} V_{2,v}$. Then

$$(-, -) := \text{Tr}_{\mathcal{K}_v/\mathbb{Q}_v}((-,-)_h \otimes_{\mathcal{K}_v} (-,-)_2)$$

is a $\mathbb{Q}_v$ linear pairing on $W$ that makes $W$ into an 8-dimensional symplectic space over $\mathbb{Q}_v$. The canonical embedding of $U_h \times U_2$ into $Sp(W)$ realizes the pair $(U_h, U_2)$ as a dual pair in $Sp(W)$.

Let $\lambda_v$ be a character of $\mathcal{K}_v^\times$ such that $\lambda_v|_{\mathbb{Q}_v^\times} = \chi_{\mathbb{K}/\mathbb{Q}_v}$. As noted earlier there is a splitting $U_h(Q_v) \times U_2(Q_v) \hookrightarrow \tilde{Sp}(W, \mathbb{Q}_v)$ of the metaplectic cover $\tilde{Sp}(W, \mathbb{Q}_v) \to Sp(W, \mathbb{Q}_v)$ determined by the character $\lambda_v$. This gives the Weil representation $\omega_{\lambda,1}$, which we denote here as $\omega_{h,v}(u,g)$ of $U_h(Q_v) \times U_2(Q_v)$ where $u \in U_h(Q_v)$ and $g \in U_2(Q_v)$, via the Weil representation of $\tilde{Sp}(W, \mathbb{Q}_v)$ on the space of Schwartz functions $S(V_v \otimes_{\mathcal{K}_v} X_v)$ (we use the polarization $W = V_v \otimes_{\mathcal{K}_v} X_v \otimes V_v \otimes_{\mathcal{K}_v} Y_v$). Moreover we write $\omega_{h,v}(g)$ to mean $\omega_{h,v}(1,g)$. For $X \in M_{1 \times 2}(\mathcal{K}_v)$, we define $\langle X, X \rangle_h := X h^t X$ (note that this is a $2 \times 2$ matrix). We record here some useful formulas for $\omega_{h,v}$ which are generalizations of the formulas in [45].

- $\omega_{h,v}(u,g)\Phi(X) = \omega_{h,v}(1,g)\Phi(u^{-1}X)$
- $\omega_{h,v}(\text{diag}(A, tA^{-1}))\Phi(X) = \lambda(\det A)|\det A|^{\frac{3}{2}} \Phi(XA)$,
- $\omega_{h,v}(r(S))\Phi(X) = \Phi(X)e_v(\text{tr}(X, X)_h S),$
- $\omega_{h,v}(\eta)\Phi(X) = |\det h|_v \int \Phi(Y)e_v(\text{Tr}_{\mathcal{K}_v/F_v}(\text{tr}(Y, X)_h))dY.$

Global setup:
Let $h \in S_1(\mathbb{Q}), h > 0$. We can define global versions of $U_h, GU_h, W,$ and $(-, -)$, analogous to the above. Fixing an idele class character $\lambda = \otimes \lambda_v$ of $\mathcal{A}_K^\times / \mathbb{K}^\times$ such that $\lambda|_{\mathbb{Q}_v^\times} = \chi_{\mathbb{K}/\mathbb{Q}_v}^1$, the associated local splitting described above then determines a global splitting $U_h(\mathbb{A}_\mathbb{Q}) \times U_1(\mathbb{A}_\mathbb{Q}) \hookrightarrow \tilde{Sp}(W, \mathbb{A}_\mathbb{Q})$ and hence an action $\omega_h := \otimes \omega_{h,v}$ of $U_h(\mathbb{A}_\mathbb{Q}) \times U_1(\mathbb{A}_\mathbb{Q})$ on the Schwartz space $S(V_{\mathbb{A}_\mathbb{K}} \otimes X)$. In application we require the infinite type of $\lambda$ to be $(-\frac{1}{2}, \frac{1}{2}).$

For any $\Phi \in S(V_{\mathbb{A}_\mathbb{K}} \otimes X)$ we define the theta function associated to it by:

$$\Theta_h(\Phi, u, g) = \sum_{x \in V \otimes X} \omega_h(u,g)\Phi(x).$$

4.9 Theta Functions with Complex Multiplication

We consider the situation of theta correspondences for $U(\zeta) = U(V)$ and $U(V_1)$. Let $V$ be a 2-dimensional Hermitian vector space over $\mathcal{K}$ and $L$ is a $\mathcal{O}_\mathbb{K}$ lattice. This gives an abelian variety $\mathcal{A}_L = \mathbb{C}^2/L$. Let $H$ be a Riemann form on $V$ and $\epsilon : L \to U$ be a map where $U$ is the unite circle of $\mathbb{C}$ (in application the $\epsilon$ is given by the formula after [54] (38)], there is a line bundle $\mathcal{L}_{H, \epsilon}$ on $\mathcal{A}_L$ associated to $H$ and $\epsilon$ as follows: define an analytic line bundle $\mathcal{L}_{H, \epsilon} \simeq \mathbb{C} \times \mathbb{C}^2/L$ with the action of $L$ given by

$$l \cdot (w, x) = (w + l, \epsilon(l)e(\frac{1}{2i}\overline{H(l, w + \frac{l}{2})}x), l \in L, (w, x) \in \mathbb{C}^2 \times \mathbb{C}$$

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where $e(x) = e^{2\pi ix}$. The space of global sections of this line bundle is canonically identified with the space $T(H, \epsilon, L)$ of theta functions consists of holomorphic functions $f$ on $V$ such that:

$$f(w + l) = f(w)e(l)e\left(\frac{1}{2\pi i} H(l + w + \frac{l}{2})\right), \quad w, l \in L.$$ 

There are arithmetic models for the above abelian variety and line bundle. Shimura defined subspaces $T^{ar}(H, \epsilon, L) \subset T(H, \epsilon, L)$ of arithmetic theta functions by requiring the values at all CM points are in $\hat{Q}$ which under the canonical identifiation, are identified with rational sections of the line bundle (see [28]). Also inside the $\mathbb{C}_p$-vector space $\Gamma(A_L \otimes_{\mathbb{Q}_p} \mathbb{C}_p, \mathcal{L}_H, \mathcal{L}_x \otimes_{\mathbb{Q}_p} \mathbb{C}_p)$ we have the module of $p$-integral sections which we denote as $T^{ar}(H, \epsilon, L)$.

### Adelic Theta Functions

Now we consider Theta functions for $U(3, 1)$. Let the Hermitian form on $V$ be defined by:

$$\langle v_1, v_2 \rangle = v_1^* v_2^* - v_2 v_1^*.$$ 

Let $U_f$ be some compact subgroup of $U_\zeta(A_f)$ such that the level is prime to $p$, we define the space $T_h(m, L, U_f)$ of adelic theta functions as the space of function:

$$\Theta : N(\mathbb{Q})U(\zeta)(\mathbb{Q}) \backslash N(\mathcal{A})U(\zeta)(\mathcal{A})/U(\zeta)_\infty U_f N(L)f \rightarrow \mathbb{C},$$

where $N = N_\mathcal{P} \subset U(3, 1)$ and $U(\zeta) \hookrightarrow U(3, 1)$ as before;

$$N(L)_f = \{(w, t) | x \in \hat{L}, t + \frac{w \zeta w^*}{2} \in \mu(L)\hat{\mathcal{O}_K}\},$$

where $\mu(L)$ is the ideal generated by $w \zeta w^*$ for $w \in L$ and $\Theta$ satisfies

$$\Theta((0, t)r) = e(\beta t)\Theta(r), \quad r \in N(\mathcal{A})U(\zeta)(\mathcal{A}).$$

Since $U(\zeta)$ is anisotropic $U(\zeta)\backslash U(\zeta)(\mathcal{A})/U(\zeta)_\infty U_f$ consists of finite points $\{x_1, ..., x_s\} \subset U(\zeta)(\mathcal{A}_f)$. We assume that for each $u_i$ the $p$-component is within $GL_2(\mathbb{Z}_p)$ under the first projection $U(\mathbb{Q}_p) \simeq GL_2(\mathbb{Q}_p)$. Then:

$$T_h(m, L, U_f) = \oplus_{i=1}^s T(m, x_i L)$$

such that $\Theta_i(n) = \Theta(n x_i)$ for $n \in N(\mathcal{A})$ are functions on $N(\mathcal{A})$. Then for each $i$ the function:

$$\theta_i(w_\infty) = e(-m \frac{w \zeta w^*}{2})\Theta_i((w_\infty, 0))$$

is a classical theta function in $T(H, \epsilon, x_i L)$ where $H$ and $\epsilon$ are defined using $x_i L$.

### A functional

Recall that we constructed a theta function $\theta_\phi$ on $H \times U(V)$ from some Schwartz function $\phi$. As mentioned in the introduction we only need to develop a rational theory on theta functions instead of $p$-integral theory. Upon choosing $v_1 \in V_1$ such that $\langle v_1, v_1 \rangle = 1$ we have an isomorphism $V \simeq W = V$. We consider $W^\perp = V^\perp \otimes V_1$. It is the space $W$ but with the metric being the negative of $W$. Let $H^\perp = H(W^\perp)$ be the corresponding Heisenberg group. We have an isomorphism of $H$ and $H^\perp$ (as Heisenberg groups) given by:

$$(w, t) \rightarrow (w, -t).$$

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We construct a theta functions $\theta_1 = \theta_{\phi_1}$ on $H^- \times U(V^-)$ for some Schwartz function $\phi_1$. We have chosen a set $\{x_1, \ldots, x_s\}$ above. We write

$$\langle \theta_{\phi_1}, \theta_{\phi_2} \rangle_{x_i} = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A}_\mathbb{Q})} \theta_{\omega_{\lambda-1}(x_i)(\phi_1)(n)} \theta_{\omega_{\lambda}(x_i)(\phi_2)}(n) \, dn$$

and

$$\langle \phi_1, \phi_2 \rangle_{x_i} = \int_{X(\mathbb{A}_\mathbb{Q})} \omega_{\lambda-1}(x_i)(\phi_1)(x) \omega_{\lambda}(x_i)(\phi_2)(x) \, dx.$$

Then it is easy to check

$$\langle \theta_{\phi_1}, \theta_{\phi_2} \rangle_{x_i} = \langle \phi_1, \phi_2 \rangle_{x_i}. \quad (1)$$

We first construct a functional $l'_{\theta_1}$ on the space $H^0(Z_{[g]}^\circ, \mathcal{L}(\beta))$ (we save the notation $l_{\theta_1}$ for later use) with values in $A_{[g]}$ as follows: first on $\frac{\theta_{\phi}}{\Omega_{\mathbb{K}}}$ in $H^0(Z_{[g]}^\circ, \mathcal{L}(\beta))$ for some Schwartz function $\phi$, we define:

$$l'_{\theta_1}(\frac{\theta_{\phi}}{\Omega_{\mathbb{K}}})(x_i) := \int_{N(\mathbb{Q}) \backslash N(\mathbb{A}_\mathbb{Q})} \theta_{\omega_{\lambda-1}(x_i)(\phi_1)}(n) \theta_{\omega_{\lambda}(x_i)(\phi)}(n) \, dn = \int_{X(\mathbb{A}_\mathbb{Q})} \omega_{\lambda-1}(x_i)(\phi_1)(x) \omega_{\lambda}(x_i)(\phi)(x) \, dx.$$

The last equality is easily seen and we denote the last term as $\langle \omega_{\lambda-1}(x_i)(\phi_1), \omega_{\lambda}(x_i)(\phi) \rangle$. Note that in the $N(\mathbb{Q}) \backslash N(\mathbb{A}_\mathbb{Q})$ we identified $H$ with $H^-$ using the above isomorphism. (In the literature people usually consider $\int_{N(\mathbb{Q}) \backslash N(\mathbb{A}_\mathbb{Q})} \theta_{\phi}(n) \, \theta_{\phi_1}(n) \, dn$. We use a different convention for the sake of simplicity.) The general elements in $H^0(Z_{[g]}^\circ, \mathcal{L}(\beta))$ is a linear combination of $\frac{\theta_{\phi}}{\Omega_{\mathbb{K}}}$’s with coefficients in $A_{[g]}$. Moreover all the $\phi$’s have the same $\phi_\infty$, and the $\phi_\ell$’s for $\ell < \infty$ are smooth functions taking algebraic values. (This can be seen by interpreting the theta functions defined before [54] Theorem B.2] in terms of Weil representations presented here. Note that the CM period $\Omega_{\mathbb{K}}$ is missing in [54] Theorem 2.5]. In fact in [39] the period is $h(z_0)$ for some weight $\frac{1}{2}$ form $h$ on $Sp_4$, as our Hermitian space is 2 dimensional, and $z_0$ is a CM point with $h(z_0) \neq 0$. This $h(z_0)$ is just $\Omega_{\mathbb{K}}$ up to multiplying by a non-zero algebraic number). So by taking appropriate $\phi_1$ the $l'_{\theta_1}$ is a rational functional. We extend the definition of $l'_{\theta_1}$ linearly. It is easily seen to be well defined.

**Lemma 4.6.** The $l'_{\theta_1}$ takes values in the space of constant functions on any theta function $\theta_{\phi}$ as above.

**Proof.** We note that for any $\phi$,

$$\langle \omega_{\lambda-1}(x_i)(\phi_1), \omega_{\lambda}(x_i)(\phi) \rangle = \langle \phi_1, \phi \rangle. \quad (2)$$

One way of seeing this is to consider the intertwining operator $\delta_{\psi}(\phi_1 \boxtimes \phi)(0) = \langle \phi_1, \phi \rangle$ and use the formula for the Weil representation $\omega_{\lambda}$ on $U(2, 2)$ and note also that $\gamma(g, g) = \left( g \zeta^{-1} g \zeta \right) \in U(2, 2)$ and $\omega_{\lambda-1} = \omega_{\lambda}(\lambda^{-1} \circ \det)$. The lemma follows from the above equation.

**Remark 4.7.** Later we will use this functional on Fourier-Jacobi coefficients for $U(3, 1)$. We can view it as a function on $G_P N_P(\mathbb{A})$ by $FJ_{P}(p, f) = FJ_{P}(pg, f)$ for $p \in G_P N_P(\mathbb{A})$ and thus an adelic theta function [28] has proved the following compatibility of the analytically and algebraically
defined Fourier-Jacobi expansions using the usual identification of the global sections of \(\mathcal{L}(\beta)\) and (classical or adelic) theta functions, keeping the integral and rational structures:

\[
F_{J_{[h,g]}(-,f)} = F_{J_{[g]}^h}(f)(-)
\]

Note that the period factor appearing in [16, 3.6.5] is 1 since we are in the scalar weight \(\kappa\).

## 5 Klingen-Eisenstein Series

### 5.1 Archimedean Picture

Let \( (\pi_\infty, V_\infty) \) be a finite dimensional representation of \( D_\infty^\times \). Let \( \psi_\infty \) and \( \tau_\infty \) be characters of \( \mathbb{C}^\times \) such that \( \psi_\infty|_{\mathbb{R}^\times} \) is the central character of \( \pi_\infty \). We assume here that \( \tau_\infty(z) = z^{-\frac{\kappa}{2}}\bar{z}^{\frac{\kappa}{2}} \) and \( \psi_\infty \) is trivial. Then there is a unique representation \( \pi_\psi \) of \( \text{GU}(2)(\mathbb{R}) \) determined by \( \pi_\infty \) and \( \psi_\infty \) such that the central character is \( \psi_\infty \). These determine a representation \( \pi_\psi \times \tau_\infty \) of \( M_P(\mathbb{R}) \simeq \text{GU}(2)(\mathbb{R}) \times \mathbb{C}^\times \). We extend this to a representation \( \rho_\infty \) of \( P(\mathbb{R}) \) by requiring \( N_P(\mathbb{R}) \) acts trivially. Let \( I(V_\infty) = \text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} \rho_\infty \) (smooth induction) and \( I(\rho_\infty) \subset I(V_\infty) \) be the subspace of \( K_\infty \)-finite vectors. (Elements of \( I(V_\infty) \) can be realized as functions on \( K_\infty \)) For any \( f \in I(V) \) and \( z \in \mathbb{C}^\times \) we define a function \( f_z \) on \( G(\mathbb{R}) \) by

\[
f_z(g) := \delta(m)\frac{3}{2} + z \rho(m)f(k), g = m nk \in P(\mathbb{R})K_\infty.
\]

There is an action \( \sigma(\rho,z) \) on \( I(V_\infty) \) by

\[
(\sigma(\rho,z)(g))(k) = f_z(kg).
\]

As in [15, Section 9.1] we let \( (\pi', V_\infty) \) be the irreducible representation of \( D_\infty^\times \) given by the same space of \( \pi \) but

\[
\pi'(x) = \pi(\text{Ad}(w) \cdot x).
\]

We let \( \rho^\prime_\infty \) and \( I(\rho^\prime_\infty) \) be the corresponding objects by replacing \( \pi_\infty, \psi_\infty, \tau_\infty \) by \( \pi_\infty \otimes (\tau_\infty \circ \text{Nm}), \psi_\infty\tau_\infty, \tau^c_\infty \). Let \( w = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \). Then there is an intertwining operator \( A(\rho_\infty, z, -) : I(\rho_\infty) \rightarrow I(\rho^\prime_\infty) \) by:

\[
A(\rho_\infty, z, f)(k) := \int_{N_P(\mathbb{R})} f_z(wnk)dn.
\]

In this paper we use the case when \( \pi_\infty \) is the trivial representation. Then by Frobenius reciprocity law there is a unique (up to scalar) vector \( \tilde{v} \in I(\rho) \) such that \( k.\tilde{v} = \det(\mu(k,i)^{-\kappa})\tilde{v} \) for any \( k \in K_\infty^+ \). We fix \( v \) and scale \( \tilde{v} \) such that \( \tilde{v}(1) = v \). In \( \pi' \), \( \pi(w)v \) (\( w \) is defined in section 3.1) has the action of \( K_\infty^+ \) given by multiplying by \( \det(\mu(k,i)^{-\kappa}) \). There is a unique vector \( \tilde{v}' \in I(\rho') \) such that the action of \( K_\infty^+ \) is given by \( \det(\mu(k,i)^{-\kappa})\tilde{v}'(w) = \pi(w)v \). Then by uniqueness there is a constant \( c(\rho, z) \) such that \( A(\rho, z, \tilde{v}) = c(\rho, z)\tilde{v}' \).

**Definition 5.1.** We define \( F_\kappa \in I(\rho) \) to be the \( \tilde{v} \) as above.

We record the following lemma proved in [48].

**Lemma 5.2.** Let \( \kappa \geq 6 \) and \( z_\kappa = \frac{\kappa^2 - 3}{2} \). Then \( c(\rho, z_\kappa) = 0 \).
5.2 \(\ell\)-adic Picture

Let \((\pi_\ell, V_\ell)\) be an irreducible admissible representation of \(D^\times(\mathbb{Q}_\ell)\) and \(\pi_\ell\) is unitary and tempered if \(D\) is split at \(\ell\). Let \(\psi\) and \(\tau\) be characters of \(K_\ell^\times\) such that \(\psi|_{\mathbb{Q}_\ell^\times}\) is the central character of \(\pi_\ell\). Then there is a unique irreducible admissible representation \(\pi_\psi\) of \(\text{GU}(2)(\mathbb{Q}_\ell)\) determined by \(\pi_\ell\) and \(\psi_\ell\). As before we have a representation \(\pi_\psi \times \tau\) of \(M_\ell(\mathbb{Q}_\ell)\) and extend it to a representation \(\rho_\ell\) of \(P(\mathbb{Q}_\ell)\) by requiring \(N_\ell(\mathbb{Q}_\ell)\) acts trivially. Let \(I(\rho_\ell) = \text{Ind}_{P(\mathbb{Q}_\ell)}^G(\rho_\ell)\) be the admissible induction. Similarly we let \(\pi^\vee\) be the \(D^\times(\mathbb{Q}_\ell)\) representation whose space is that of \(\pi\) but the action is given by

\[
\pi^\vee(x) = \pi(\text{Ad}(\omega) \cdot x).
\]

Define \(f_z\) for \(f \in I(\rho_\ell)\) and \(\rho_\ell^\vee, I(\rho_\ell^\vee), A(\rho_\ell, z, f)\) etc as before. For \(v \notin \Sigma\) we have \(D^\times(\mathbb{Q}_\ell) \simeq \text{GL}_2(\mathbb{Q}_\ell)\). Moreover we can choose isomorphism as a conjugation by elements in \(\text{GL}_2(\mathcal{O}_{K_\ell})\) (note that both groups are subgroups of \(\text{GL}_2(K_\ell)\). We have \(\pi_\ell, \psi_\ell, \tau_\ell\) are unramified and \(\varphi_\ell \in V_\ell\) is a spherical vectors then there is a unique vector \(f^0_{\varphi_\ell} \in I(\rho_\ell)\) which is invariant under \(G(\mathbb{Z}_\ell)\) and \(f^0_{\varphi_\ell}(1) = \varphi_\ell\).

5.3 Global Picture

Now let \((\pi = \otimes_v \pi_v, V)\) be an irreducible unitary cuspidal automorphic representation of \(D(\mathbb{A}_\mathbb{Q})\) we define \(I(\rho)\) to be the restricted tensor product of \(\otimes_v I(\rho_v)\) with respect to the unramified vectors \(f^0_{\varphi_v}\) for some \(\varphi = \otimes_v \phi_v \in \pi\). We can define \(f_z, I(\rho^\vee), A(\rho, z, f)\) similar to the local case. The \(f_z\) takes values in \(V\) which can be realized as automorphic forms on \(D(\mathbb{A}_\mathbb{Q})\). We also write \(f_z\) for the scalar-valued functions \(f_z(g) := f_z(g)(1)\) and define the Klingen Eisenstein series:

\[
E(f, z, g) := \sum_{\gamma \in \mathcal{P}(\mathbb{Q}) \backslash G(\mathbb{Q})} f_z(\gamma g).
\]

This is absolutely convergent if \(\text{Re} z > 0\) and has meromorphic continuation to all \(z \in \mathbb{C}\).

5.4 Constant Terms

**Definition 5.3.** For any parabolic subgroup \(R\) of \(\text{GU}(3, 1)\) and an automorphic form \(\varphi\) we define \(\varphi_R\) to be the constant term of \(\varphi\) along \(R\) given by the following:

\[
\varphi_R(g) = \int_{N_\ell(\mathbb{Q}) \backslash N_\ell(\mathbb{A}_\mathbb{Q})} \varphi(ng)dn.
\]

The following lemma is well-known (see [34 II.1.7]).

**Lemma 5.4.** Let \(R\) be a standard \(\mathbb{Q}\)-parabolic of \(\text{GU}(3, 1)\) (i.e, \(R \supseteq B\) where \(B\) is the standard Borel). Suppose \(\text{Re}(z) > \frac{3}{2}\).

(i) If \(R \neq P\) then \(E(f, z, g)_R = 0\);

(ii) \(E(f, z, -)_P = f_z + A(\rho, f, z)_{-z}\).
5.5 Galois representations

For the holomorphic Klingen Eisenstein series we can also associate a reducible Galois representation with the same recipe as in subsection 3.3. Write $\tau'$ for the restriction of $\tau$ to $A_Q$ and let $\sigma_{\tau'}$ be the corresponding Galois character of $G_Q$ via class field theory. The resulting Galois representation is easily seen to be

$$\sigma_{\tau'} \sigma_{\psi} \epsilon^{-\kappa} \oplus \sigma_{\psi} \epsilon^{-3} \oplus \rho_{\pi_f} \sigma_{\tau} \epsilon^{-\frac{\kappa+3}{2}}.$$ 

Note that $\kappa + 3$ is an odd number. However $\pi_f$ is a unitary representation whose $L$-function is the usual $L$-function for $f$ shifted by $\frac{1}{2}$. So it makes sense to write in the above way. This can be obtained in the same manner as [45, Sections 9.5, 9.6].

6 Siegel Eisenstein Series and Pullback

6.1 Generalities

Local Picture:

Our discussion in this section follows [45, 11.1-11.3] closely. Let $Q = Q_n$ be the Siegel parabolic subgroup of $GU_n$ consisting of matrices $\begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$. It consists of matrices whose lower-left $n \times n$ block is zero. For a place $v$ of $Q$ and a character $\chi$ of $K_v^\times$ we let $I_n(\chi_v)$ be the space of smooth $K_{n,v}$-finite functions (here $K_{n,v}$ means the maximal compact subgroup $G_n(Z_v)$) $f : K_{n,v} \to \mathbb{C}$ such that $f(qk) = \chi_v(\det D_q)f(k)$ for all $q \in Q_n(Q_v) \cap K_{n,v}$ (we write $q$ as block matrix $q = \begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$). For $z \in \mathbb{C}$ and $f \in I(\chi)$ we also define a function $f(z,-) : G_n(Q_v) \to \mathbb{C}$ by $f(z,qk) := \chi(\det D_q)|\det A_q D_q^{-1}|^{\frac{z+n/2}{2}} f(k)$, $q \in Q_n(Q_v)$ and $k \in K_{n,v}$.

For $f \in I_n(\chi_v)$, $z \in \mathbb{C}$, and $k \in K_{n,v}$, the intertwining integral is defined by:

$$M(z,f)(k) := \chi_v^n(\mu_n(k)) \int_{N_{Q_n}(F_v)} f(z,w_n r k) dr.$$ 

For $z$ in compact subsets of $\{\text{Re}(z) > n/2\}$ this integral converges absolutely and uniformly, with the convergence being uniform in $k$. In this case it is easy to see that $M(z,f) \in I_n(\chi_v^n)$. A standard fact from the theory of Eisenstein series says that this has a continuation to a meromorphic section on all of $\mathbb{C}$.

Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set. By a meromorphic section of $I_n(\chi_v)$ on $\mathcal{U}$ we mean a function $\varphi : \mathcal{U} \to I_n(\chi_v)$ taking values in a finite dimensional subspace $V \subset I_n(\chi_v)$ and such that $\varphi : \mathcal{U} \to V$ is meromorphic.

Global Picture

Recall that we defined $z_\kappa = \frac{\kappa+3}{2}$. We also define $z'_\kappa = \frac{\kappa-2}{2}$. For an idele class character $\chi = \otimes \chi_v$ of $A_K^\times$ we define a space $I_n(\chi)$ to be the restricted tensor product defined using the spherical vectors $f_v^{\text{ph}} \in I_n(\chi_v)$ (invariant under $K_{n,v}$) such that $f_v^{\text{ph}}(K_{n,v}) = 1$, at the finite places $v$ where $\chi_v$ is unramified.
For $f \in I_n(\chi)$ we consider the Eisenstein series

$$E(f; z, g) := \sum_{\gamma \in \mathcal{Q}(\mathbb{Q}) \setminus \mathcal{G}_n(\mathbb{Q})} f(z, \gamma g).$$

This series converges absolutely and uniformly for $(z, g)$ in compact subsets of

$$\{\text{Re}(z) > n/2\} \times \mathcal{G}_n(\mathbb{A}_\mathbb{Q}).$$

The defined automorphic form is called Siegel Eisenstein series.

The Eisenstein series $E(f; z, g)$ has a meromorphic continuation in $z$ to all of $\mathbb{C}$ in the following sense. If $\varphi : \mathcal{U} \to I_n(\chi)$ is a meromorphic section, then we put $E(\varphi; z, g) = E(\varphi(z); z, g)$. This is defined at least on the region of absolute convergence and it is well known that it can be meromorphically continued to all $z \in \mathbb{C}$.

Now for $f \in I_n(\chi), z \in \mathbb{C}$, and $k \in \prod_{v \mid \infty} K_{n,v} \prod_{v \mid \infty} K_{\infty}$ there is a similar intertwining integral $M(z, f)(k)$ as above but with the integration being over $N_{Q_n}(\mathbb{A}_F)$. This again converges absolutely and uniformly for $z$ in compact subsets of $\{\text{Re}(z) > n/2\} \times \mathcal{K}_n$. Thus $z \mapsto M(z, f)$ defines a holomorphic section $\{\text{Re}(z) > n/2\} \to I_n(\bar{\chi})$. This has a continuation to a meromorphic section on $\mathbb{C}$. For $\text{Re}(z) > n/2$, we have

$$M(z, f) = \otimes_v M(z, f_v), f = \otimes f_v.$$

The functional equation for Siegel Eisenstein series is:

$$E(f, z, g) = \chi^n(\mu(g))E(M(z, f); -z, g)$$

in the sense that both sides can be meromorphically continued to all $z \in \mathbb{C}$ and the equality is understood as of meromorphic functions of $z \in \mathbb{C}$.

### 6.2 Embeddings

We define some embeddings of a subgroup of $\text{GU}(3, 1) \times \text{GU}(0, 2)$ into some larger groups. This will be used in the doubling method. First we define $G(3, 3)'$ to be the unitary similitude group associated to:

$$\begin{pmatrix} 1 & \zeta \\ -1 & -\zeta \end{pmatrix}$$

and $G(r + s, r + s)'$ to be associated to

$$\begin{pmatrix} \zeta & \\ -\zeta \end{pmatrix}.$$

We define an embedding

$$\alpha : \{g_1 \times g_2 \in \text{GU}(3, 1) \times \text{GU}(0, 2), \mu(g_1) = \mu(g_2)\} \to G(3, 3)'$$
\[ \alpha' : \{ g_1 \times g_2 \in GU(2,0) \times GU(0,2), \mu(g_1) = \mu(g_2) \} \rightarrow GU(2,2)' \]
as \[ \alpha(g_1, g_2) = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \] and \[ \alpha'(g_1, g_2) = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \]. We also define an isomorphism:
\[ \beta : GU(3,3)' \sim GU(3,3) \]
and
\[ \beta' : GU(2,2)' \sim GU(2,2) \]
by:
\[ g \mapsto S^{-1}gS \]
or
\[ g \mapsto S'^{-1}gS' \]
where
\[ S = \begin{pmatrix} 1 & -\zeta \\ 1 & -\frac{\zeta}{2} \\ -1 & -\zeta \end{pmatrix} \]
and
\[ S' = \begin{pmatrix} 1 & -\frac{\zeta}{2} \\ -1 & -\zeta \end{pmatrix} \]
We write \( \gamma \) and \( \gamma' \) for the embeddings \( \beta \circ \alpha \) and \( \beta' \circ \alpha' \), respectively.

### 6.3 Pullback Formula

We recall the pullback formula of Shimura (see [45] for details). Let \( \chi \) be a unitary idele class character of \( \mathbb{A}_K^\times \). Given a cuspform \( \varphi \) on \( GU(2) \) we consider
\[
F_\varphi(f; z, g) := \int_{U(2)(\mathbb{A}_Q)} f(z, S^{-1} \alpha(g, g_1 h) S) \chi(\det g_1 h) \varphi(g_1 h) dg_1,
\]
\[ f \in I_2(\chi), g \in GU(3,1)(\mathbb{A}_Q), h \in GU(2)(\mathbb{A}_Q), \mu(g) = \mu(h) \]
or
\[
F'_\varphi(f'; z, g) = \int_{U(2)(\mathbb{A}_Q)} f'(z, S'^{-1} \alpha(g, g_1 h) S') \chi(\det g_1 h) \varphi(g_1 h) dg_1
\]
\[ f' \in I_2(\chi), g \in GU(2)(\mathbb{A}_Q), h \in GU(2)(\mathbb{A}_Q), \mu(g) = \mu(h) \]
This is independent of \( h \). The pullback formulas are the identities in the following proposition.

**Proposition 6.1.** Let \( \chi \) be a unitary idele class character of \( \mathbb{A}_K^\times \).

(i) If \( f' \in I_2(\chi) \), then \( F'_\varphi(f'; z, g) \) converges absolutely and uniformly for \((z, g)\) in compact sets of \( \{ \text{Re}(z) > 1 \} \times GU(2,0)(\mathbb{A}_Q) \), and for any \( h \in GU(2)(\mathbb{A}_Q) \) such that \( \mu(h) = \mu(g) \)
\[
\int_{U(2)(\mathbb{Q}) \setminus U(2)(\mathbb{A}_Q)} E(f'; z, S'^{-1} \alpha(g, g_1 h) S') \chi(\det g_1 h) \varphi(g_1 h) dg_1 = F'_\varphi(f'; z, g).
\]
(ii) If \( f \in I_3(\chi) \), then \( F_\varphi(f; z, g) \) converges absolutely and uniformly for \((z, g)\) in compact sets of \( \{\text{Re}(z) > 3/2\} \times \text{GU}(3,1)(\mathbb{A}_Q) \) such that \( \mu(h) = \mu(g) \)

\[
\int_{U(2)(\mathbb{Q}) \setminus U(2)(\mathbb{A}_Q)} E(f; z, S^{-1} \alpha(g, g_1 h) S) \chi(g_1 h) \varphi(g_1 h) dg_1 = \sum_{\gamma \in P(\mathbb{Q}) \setminus \text{GU}(3,1)(\mathbb{Q})} F_\varphi(f; z, \gamma g),
\]

with the series converging absolutely and uniformly for \((z, g)\) in compact subsets of \( \{\text{Re}(z) > 3/2\} \times \text{GU}(3,1)(\mathbb{A}_Q) \).

This is a special case of [48, Proposition 3.5].

### 6.4 Fourier-Jacobi expansion

From now on we fix a splitting character \( \lambda \) of \( \mathcal{K} \setminus \mathbb{A} \) of infinite type \((-\frac{1}{2}, \frac{1}{2})\) which is unramified at \( p \) and unramified outside \( \Sigma \) and such that \( \lambda|_{\mathcal{K}_3} = \chi_{\mathcal{K}/Q} \). The following formula is proved in [48, 3.3.1]. Let \( \tau \) be a Hecke character of \( \mathcal{K} \setminus \mathbb{A} \) of infinite type \((-\frac{\kappa}{2}, \frac{\kappa}{2})\).

**Definition 6.2.** For \( \beta \in S_n(\mathbb{Q}) \) and \( \varphi \) a holomorphic automorphic form on \( \text{GU}(n, n) \) we define the \( \beta \)-th Fourier-coefficient

\[
\varphi_\beta(g) = \int \varphi\left( \begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix} g \right) e_h(-\text{Tr}_\beta S) dS.
\]

For a prime \( v \) and \( f_v \in I_n(\tau) \) we also define the local Fourier coefficient at \( g_v \in \text{GU}(n, n)(\mathbb{Q}_v) \) as

\[
f_{v,\beta}(z, g_v) = \int_{S_n(\mathbb{Q}_v)} f_v(z, \omega_n(\begin{pmatrix} 1_n & S_v \\ 0 & 1_n \end{pmatrix} g_v) e_v(-\text{Tr}_\beta S_v) dS_v.
\]

For \( \varphi \) a holomorphic automorphic form on \( \text{GU}(3,3) \) and \( \beta \in \mathbb{Q}^+ \) we define

\[
FJ_\beta(\varphi)(g) = \int \varphi(\begin{pmatrix} 1_3 & S & 0 \\ 0 & 0 & 0 \\ 0 & 1_3 \end{pmatrix} g) e_h(-\text{Tr}_\beta S) dS.
\]

(The \( S \) has size \((1 \times 1)\). For \( E(f; z, g) \) with \( f \in I_3(\tau) \) we define

\[
FJ_\beta(f; z, g) = FJ_\beta(E(f; z, -))(g).
\]

**Proposition 6.3.** Suppose \( f \in I_3(\tau) \) and \( \beta \in S_1(\mathbb{Q}) \), \( \beta \) is positive. If \( E(f; z, g) \) is the Siegel Eisenstein Series on \( \text{GU}(3,3) \) defined by \( f \) for some \( \text{Re}(z) \) sufficiently large then the \( \beta \)-th Fourier-Jacobi coefficient \( E_\beta(f; z, g) \) satisfies:

\[
E_\beta(f; z, g) = \sum_{\gamma \in Q_2(\mathbb{Q}) \setminus \text{GU}_2(\mathbb{Q})} \sum_{y \in M_{1 \times 2}(\mathbb{K})} \int_{S_m(\mathbb{A})} f(w_3) \begin{pmatrix} 1_3 & S & y \\ t & 0 & 0 \\ 0 & 1_3 \end{pmatrix} \alpha_2(1, \gamma) g e_h(-\text{Tr}_\beta S) dS
\]

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where if \( g_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), \( g_2 = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \) then:
\[
\alpha(g_1, g_2) = \begin{pmatrix} A & B \\ D' & C' \\ C & D \\ B' & A' \end{pmatrix}.
\]

(the notation \( \alpha \) is already used before. However the meaning should be clear from context.)

**Definition 6.4.** If \( g_v \in U_2(\mathbb{Q}_v), \alpha \in GL_1(\mathbb{Q}_v) \), then define:
\[
FJ_\beta(f_v; z, y, g, x) = \int_{S_1(\mathbb{Q}_v)} f(w_3) \begin{pmatrix} 13 & S & y \\ t & 0 & 0 \\ 13 \end{pmatrix} \alpha(\text{diag}(x, t^{-1})), g_\mathbb{Q}_v(-\text{Tr}S) dS
\]
Since
\[
\begin{pmatrix} 13 & S & X \\ t & \bar{X} \\ 13 \end{pmatrix} \begin{pmatrix} 11 & \bar{A}^{-1} & 1 \\ \bar{A}^{-1} & 1 & A \\ B \bar{A}^{-1} & 1 & A \end{pmatrix} = \begin{pmatrix} 1 & X \bar{B} \bar{A}^{-1} & 1 \\ \bar{A}^{-1} & 1 & A \\ \bar{A}^{-1} & 1 & A \end{pmatrix} \begin{pmatrix} 13 & S & X \bar{A} \bar{X} \\ 1 & n \end{pmatrix},
\]

it follows that:
\[
\tau^*_v(\text{det } A)^{-1}\alpha(\text{det } A, B \bar{A}^{-1}) g, x = \tau^*_v(\text{det } A)^{-1} \alpha(\text{det } \bar{A} A, B) FJ_\beta(f_v; z, y A, g, x)
\]
Also we have:
\[
FJ_\beta(f; z, y, g, x) = \tau^*_v(\text{det } x) \alpha(\text{det } x, A, B) FJ_\beta(f; z, x^{-1} y, g, 1).
\]

We write \((x, y, t)\) for \(\begin{pmatrix} 1 & x & t + \frac{y^r - yxx}{2} & y \\ 12 & y^* & 0 \\ 1 & -x & 2 \\ 1 & 1 \end{pmatrix} \) so that it becomes a Heisenberg group if we give the pairing \(((x_1, y_1), (x_2, y_2)) = x_1 y_2^* + y_2 x_1^* - x_2 y_1^* - y_1 x_2^*\) (see [54], section 4)

**Lemma 6.5.** Suppose for some place \( v \) the local Fourier Jacobi coefficient
\[
FJ_\beta(f_v; (0, y, 0) \alpha(1, u)) = f(u, z) \omega_\lambda(u) \phi(y)
\]
for any \( u \in U(2, 2) \) and \( y \in W^d \) and some Schwartz function \( \phi \in S(W^d) \) and some \( f \in I((\tau/\lambda)_v, z) \), then we have:
\[
FJ_{\beta}^{(3)}(f_v; (x, y, t) \alpha(1, u)) = f(u, z) \omega_\lambda((x, y, t), u) \phi(0)
\]
for any \((x, y, t)\).

**Proof.** This is a consequence of:
\[
\begin{pmatrix} 1 & -x \\ 12 \end{pmatrix} = \begin{pmatrix} 1 & x & t + \frac{y^r - yxx}{2} & y \\ 12 & y^* & 0 \\ 1 & -x & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & t - \frac{y^r + yxx}{2} & y \\ 12 & y^* & 0 \\ 1 & -x & 2 \\ 1 & 1 \end{pmatrix}
\]
where we write \( x^* \) for \( t \bar{x} \). 

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6.5 Archimedean Cases

We let \( i := \left( \frac{i}{2} \right) \) or \( \frac{3}{2} \) depending on the size \( 3 \times 3 \) or \( 2 \times 2 \). The Siegel section we choose is \( f_{\text{siegel, } \infty} := f_\kappa(g, z) := J_3(g, i)^{-\kappa} |J_3(g, i)|^{\kappa - 2z - 3} \) and \( f'_\kappa(g, z) = J_2(g, i)^{-\kappa} |J_n(g, i)|^{\kappa - 2z - 2} \). For \( \varphi \in \pi_\infty \) we define the pullback sections:

\[
F_\kappa(z, g) := \int_{U(2)(\mathbb{R})} f_\kappa(z, S^{-1} \alpha(g, g_1) S) \tau(\det g_1) \pi(g_1) \varphi dg_1
\]

and

\[
F'_\kappa(z, g) := \int_{U(2)(\mathbb{R})} f'_\kappa(z, S^{-1} \alpha(g, g_1) S') \tau(\det g_1) \pi(g_1) \varphi dg_1
\]

If we define an auxiliary \( f_{\kappa, n}^\circ(z, g) = J_n(g, i1_n)^{-\kappa} |J_n(g, i1_n)|^{\kappa - 2z - n} \) for \( n = 2, 3 \), then \( f_\kappa(g, z) = f_{\kappa, 3}(gg_0) \) and \( f'_\kappa(g, z) = f_{\kappa, 2}(gg_0) \) for

\[
g_0 = \text{diag}(1, \frac{s^2 d^2}{\sqrt{2}}, \frac{d^2}{\sqrt{2}}, 1, (\frac{s^2 d^2}{\sqrt{2}})^{-1}, (\frac{d^2}{\sqrt{2}})^{-1})
\]

or

\[
g_0 = \text{diag}(\frac{s^2 d^2}{\sqrt{2}}, \frac{d^2}{\sqrt{2}}, (\frac{s^2 d^2}{\sqrt{2}})^{-1}, (\frac{d^2}{\sqrt{2}})^{-1})
\]

depending on the size.

**Lemma 6.6.** The integrals are absolutely convergent for \( \text{Re}(z) \) sufficiently large and for such \( z \), we have:

(i)

\[
F_{\text{Kling, } \infty}(z, g) := F_\kappa(z, g) = F_{\kappa, z}(g);
\]

(ii)

\[
F'_\kappa(z, g) = \pi(g) \varphi;
\]

where \( F_{\kappa, z} \) is defined in definition [5.1] using \( \varphi \) as the \( v \) there.

**Fourier Coefficients**

The following lemma is [45, Lemma 11.4].

**Lemma 6.7.** Suppose \( \beta \in S_n(\mathbb{R}) \). Then the function \( z \to f_{\kappa, \beta}(z, g) \) has a meromorphic continuation to all of \( \mathbb{C} \). Furthermore, if \( \kappa \geq n \) then \( f_{\kappa, n, \beta}(z, g) \) is holomorphic at \( z_\kappa := (\kappa - n)/2 \) and for \( y \in \text{GL}_n(\mathbb{C}) \), \( f_{\kappa, n, \beta}(z_\kappa, \text{diag}(y, t^y)) = 0 \) if \( \det \beta \leq 0 \) and if \( \det \beta > 0 \) then

\[
f_{\kappa, n, \beta}(z_\kappa, \text{diag}(y, t^y)) = \frac{(-2)^{-n}(2\pi i)^n (2/\pi)^{n(n-1)/2}}{\prod_{j=0}^{n-1} (\kappa - j - 1)!} e(i Tr(\beta y)) \det(\beta)^{\kappa-n} \det t^y.
\]

The local Fourier coefficient for \( f_\kappa \) can be easily deduced from that for \( f_{\kappa}^\circ \).

**Fourier-Jacobi Coefficients**

The following lemma can be found in [48, Lemma 4.4].
Lemma 6.8. Let \( z_\kappa = \frac{\kappa-3}{2} \), \( \beta \in S_1(\mathbb{R}) \), det \( \beta > 0 \). then:

(i) \( FJ_\beta(z_\kappa, f_{\kappa,\beta}(x, \eta, 1)) = f_{\kappa,1,\beta}(z_\kappa + 1, 1)e(i Tr(x \beta x)) \);

(ii) If \( g \in U(2,2)(\mathbb{R}) \), then
\[
FJ_{\beta,k}(z_\kappa, f^{\kappa}_{\beta,2}(x, g, 1)) = e(i Tr(\beta))c_1(\beta, \kappa)\Phi_{\beta, \kappa}(x).
\]
where \( g' = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \), \( c_1(\beta, \kappa) = \left( \frac{(-1)^{-1}(2\pi i)^{\kappa}}{(\kappa-1)!} \right) \det \beta^{\kappa-1} \) and \( \Phi_{\beta, \kappa} = e^{-2\pi Tr(x \beta x)} \).

Lemma 6.9. Suppose \( f \in U(2,2)(\mathbb{R}) \), \( x \in \mathbb{C}^2 \).

Proof. Note that
\[
f_{\kappa}(g, z_\kappa) = J(g, i)^{-\kappa} J(gg_0, i)^{-\kappa} J(g_0, i)^{\kappa}.
\]

Lemma 6.10. Let \( x_1 = (x_{11}, x_{12}), x_2 = (x_{21}, x_{22}) \) where the \( x_{ij} \in \mathbb{R} \). Then
\[
\delta^{-1}_\psi(\omega_{1,\lambda}(\eta g_0)\Phi_\infty)(x_1, x_2) = \frac{g_2^2 d_2^1}{4} e^{-2\pi \sqrt{d}(x_{11}^2 + x_{12}^2)} e^{-2\pi \sqrt{d}(x_{21}^2 + x_{22}^2)}
\]

Proof. Straightforward from the expression for \( \Phi_\infty \) and \( \delta_\psi \).

Definition 6.11.
\[
\Phi_\infty = \omega_1(g_0)\Phi_1, \Phi'_\infty = \omega_1(\eta g_0)\Phi_\infty, \\
f_{2, \infty}(g) = f'_{\kappa-1}(gg_0), f''_{2, \infty}(g) = f'_{\kappa-1}(g \eta g_0), \\
\phi_{1, \infty}(x_1, x_2) = \phi_{2, \infty}(x_1, x_2) = \left( \frac{g_1^2 d_2^1}{2} \right) e^{-2\pi \sqrt{d}(x_{11}^2 + x_{22}^2)}, x_1, x_2 \in \mathbb{R}.
\]

6.6 Unramified Cases

Let \( v \) be a prime outside \( \Sigma \) (in particular \( v \nmid p \)). Then the Siegel sections \( f_{v, \text{sieg}} = f_v^{\text{sph}} \) and \( f'_{v, \text{sieg}} = f_v^{\text{sph}} \) defined to be the unique section that is invariant under \( GU(n, n)(\mathbb{Z}_v) \) \( n = 3, 2 \) and is 1 at identity.

Lemma 6.12. Suppose \( \pi, \psi \) and \( \tau \) are unramified and \( \phi \in \pi \) is a newvector. If \( Re(z) > 3/2 \) then the pull back integral converges and
\[
F_{\varphi}(f_v^{\text{sph}}; z, g) = \frac{L(\tilde{\tau}, \xi, z + 1)}{\prod_{i=0}^{1} L(2z + 3 - i, \tau_i \chi_{k}^i)} F_{\rho, z}(g)
\]
where \( F_{\rho} \) is the spherical section defined using \( \varphi \in \pi \). Also:
\[
F'_{\varphi}(f_v^{\text{sph}}; z, g) = \frac{L(\tilde{\tau}, \xi, z + \frac{1}{2})}{\prod_{i=0}^{1} L(2z + 2 - i, \tau' \chi_{k}^i)} \pi(g) \varphi.
\]

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Lemma 6.14. Let $\beta \in S_n(\mathbb{Q}_v)$ and let $r := \text{rank}(\beta)$. Then for $y \in \text{GL}_n(K_v)$,

$$
\tau = \left(\det y\right)\left|\det y\right|^{z+n/2}D_v^{-n(n-1)/4} \prod_{n=1}^{n-1} \frac{L(2z+i-n+1,\tau'\chi_k)}{L(2z+n-i,\tau'\chi_k)}h_{v,y\beta y}(\tau'q_vq_v^{-2z-n}).
$$

where $h_{v,y\beta y} \in \mathbb{Z}[X]$ is a monic polynomial depending on $v$ and $y\beta y$ but not on $\tau$. If $\beta \in S_n(\mathbb{Z}_v)$ and $\text{det} \beta \in \mathbb{Z}_v^\times$, then we say that $\beta$ is $v$-primitive and in this case $h_{v,\beta} = 1$.

Fourier-Jacobi Coefficients

In this subsection we consider a prime $v \in \Sigma$ not dividing $p$.

Lemma 6.15. Suppose $v$ is unramified in $K$. Let $\beta \in S_1(\mathbb{Q}_v)$ such that $\text{det} \beta \neq 0$. Let $y \in \text{GL}_2(K_v)$ such that $y\beta y \in S_1(\mathbb{Z}_v)$, let $\lambda$ be an unramified character of $K_v^\times$ such that $\lambda|_{\mathbb{Q}_v} = 1$. If $\beta \in \text{GL}_1(O_{K,v})$, then for $u \in U_\beta(\mathbb{Q}_v)$:

$$
FJ_\beta(f^{sph}_3; z, x, g, u) = \tau(\text{det} u)\left|\det u\right|^{-z+1/2}f^{sph}(z, g')(\omega_\beta(u, g')\Phi_0)(x) \frac{L(2z+3, \tau')}{L(2z+1, \tau')}. 
$$

Here $g' = \begin{pmatrix} 12 \\ -12 \end{pmatrix} g \begin{pmatrix} 12 \\ -12 \end{pmatrix}$.

For these primes we define $\phi_{1,v}$ and $\phi_{2,v}$ to be the Schwartz function on $X_v$ which is the characteristic function of $\mathbb{Z}_v^2$. We define $f_{2,v} = f^{sph}$ and $\Phi_v = \Phi_0 = \Phi''_v$.

6.7 Ramified Cases

The Siegel section we choose is $I_3(\tau) \ni f_{v,sieg} = f^1(g\tilde{\gamma}_v)$ where $\tilde{\gamma}_v$ is

$$
\begin{pmatrix}
1 \\
12 \\
12 \\
\end{pmatrix}
\begin{pmatrix}
12 \\
\frac{1}{y} \\
12 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
\end{pmatrix}
$$

where $y \in O_v$ is some fixed element such that the valuation is sufficiently large. We also define

$$
I_2(\tau) \ni f'_{v,sieg} = f^1(g\tilde{\gamma}'_v)
$$

where $\tilde{\gamma}'_v = \begin{pmatrix} 12 \\ \frac{1}{y} \\ 12 \end{pmatrix}$.

Pullback Formulas

The proofs of the following lemmas are just special cases of [38, Lemma 4.9, 4.10].

Lemma 6.16. Let $K_v^{(2)}$ be the subgroup of $G(\mathbb{Q}_v)$ of the form

$$
\begin{pmatrix}
1 \\
12 \\
\end{pmatrix}
\begin{pmatrix}
f \\
c \\
1 \\
\end{pmatrix}
$$

where

$$
g = -\zeta^f, c - \frac{1}{2} f\zeta^f \in \mathbb{Z}_v, f \in (y\bar{y}), g \in (\zeta\bar{y}), c \in O_v.
$$

Then $F_\phi(z; g, f)$ is supported in $PwK_v^{(2)}$ and is invariant under the action of $K_v^{(2)}$. 

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Lemma 6.17. Let $\varphi$ be some vector invariant under the action of $G$ defined above, then

$$F_\varphi(z,w) = \tau(y\bar{y})\frac{((y\bar{y})^2)^{\frac{1}{2}}}{z^{\frac{3}{2}}}\text{Vol}(G) \cdot \varphi.$$ 

Also $F_{\varphi}(f'_{v,siegel};z,g) = \tau(y\bar{y})\frac{((y\bar{y})^2)^{\frac{1}{2}}}{z^{\frac{3}{2}}\text{Vol}(G) \cdot \pi(g)\varphi}.$

Fourier Coefficients

Lemma 6.18. (i) Let $f_{v,\beta}(z,1) = 0$ if $\beta \notin S_3(\mathbb{Z}_v)^*$. If $\beta \in S_3(\mathbb{Z}_v)^*$ then

$$f_{v,\beta}(z,\text{diag}(A,t\bar{A}^{-1})) = D_1^{-\frac{3}{2}}\tau(\det A)\det A\bar{A}^{-\frac{3}{2}}\epsilon_3(\frac{\beta_{22} + \beta_{33}}{y\bar{y}})$$

where $D_1$ is the discriminant of $K_1$.

(ii) If $\beta \in S_2(\mathbb{Q}_v)$, then $f_{v,\beta}(z,1) = 0$ if $\beta \in S_2(\mathbb{Z}_v)^*$. If $\beta \in S_2(\mathbb{Z}_v)^*$ then

$$f'_{v,\beta}(z,\text{diag}(A,t\bar{A}^{-1})) = D_1^{-\frac{1}{2}}\tau(\det A)\det A\bar{A}^{-\frac{1}{2}}\epsilon_3(\frac{\beta_{11} + \beta_{22}}{y\bar{y}}).$$

Fourier-Jacobi Coefficients

Lemma 6.19. If $\beta \notin S_1(\mathbb{Z}_v)^*$ then $FJ_\beta(f^\dagger;z,u,g,1) = 0$. If $\beta \in S_1(\mathbb{Z}_v)^*$ then

$$FJ_\beta(f^\dagger;z,u,g,1) = f^\dagger(z,g')\omega_\beta(h,g'\eta^{-1})\Phi_0(u).\text{Vol}(S_1(\mathbb{Z}_v)),$$

where $g' = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} g \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$.

Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$. So:

$$FJ_\beta(f_{siegel,v};z,u,g,h) = f^\dagger(z,g'\begin{pmatrix} 1 & 1 \\ -A & 1 \end{pmatrix})\omega_\beta(h,g'\begin{pmatrix} 1 & 1 \\ -A & 1 \end{pmatrix})\Phi_0(u)$$

for $h \in U(1)$. We define $\Phi'_v := \omega_\beta(\begin{pmatrix} 1 & 1 \\ -A & 1 \end{pmatrix})\Phi_0$ and $\Phi_v = \omega_\beta(\begin{pmatrix} 1 & 1 \\ -A & 1 \end{pmatrix})\Phi_0$. We also define $f_{2,v} = \rho(\begin{pmatrix} 1 & 1 \\ -A & 1 \end{pmatrix})\Phi^v \in I_2(\mathbb{F}_v)$.

Split Case

Suppose $v = w\bar{w}$ is a split prime. Recall we have the local polarization $X'_v \oplus Y'_v$. Now we write $x'_1 = (x'_{11}, x'_{12})$ and $x'_2 = (x'_{21}, x'_{22})$ with respect to $K_v \simeq K_w \times K_{\bar{w}}$. The following lemma follows from a straightforward computations and will be used later. Let $\lambda_\beta,v$ be a character of $K_v^\times$. 

Now we let $G$ be the set of matrices $A \in U(2)(\mathbb{Q}_v)$ such that $M = A - 1$ satisfies:

$$M(1 + \frac{\zeta}{2}y\bar{y} + y\bar{y}N) = \zeta y\bar{y}$$

for some $N \in M_2(\mathbb{O}_v)$. 

Lemma 6.17. Let $\varphi$ be some vector invariant under the action of $G$ defined above, then

$$F_\varphi(z,w) = \tau(y\bar{y})\frac{((y\bar{y})^2)^{\frac{1}{2}}}{z^{\frac{3}{2}}}\text{Vol}(G) \cdot \varphi.$$
Lemma 6.20. Let $\chi_{\theta,v}$ be a character of $\mathbb{Z}_v^\times$. It is possible to choose a Schwartz function $\phi'_1$ such that the function: $\phi'_2(x'_2) \coloneqq \int_{X_1} \delta^{-1}_v(\Phi''_v)(x'_1, x'_2)\phi_1(x'_1)dx'_1$ is given by

$$\phi'_2(x'_2) = \begin{cases} \lambda_{\theta,v}\chi_{\theta,v}(x'_{22}) & x'_{21} \in \mathbb{Z}_v, x'_{22} \in \mathbb{Z}_v^\times \\ 0 & \text{otherwise.} \end{cases}$$

Moreover when we are moving our datum in $p$-adic families, this $\phi'_1$ is not going to change.

We define $\phi_{1,v} = \delta^{-\prime\prime}_v(\phi'_1), \phi_{2,v} = \delta^{-\prime\prime}_v(\phi'_2).

Non-split Case

Lemma 6.21. We consider the action of the compact abelian group $U(1)(\mathbb{Q}_v)$ on $\delta^{-1}_v(\Phi''_v)$ by the Weil representation (Weil representation using the splitting character $\lambda$) of

$$1 \times U(1)(\mathbb{Q}_v) \hookrightarrow 1 \times U(2)(\mathbb{Q}_v) \hookrightarrow U(2)(\mathbb{Q}_v) \times U(2)(\mathbb{Q}_v) \hookrightarrow U(2,2)(\mathbb{Q}_v).$$

We can write $\delta^{-1}_v(\Phi''_v)$ as a sum of eigenfunctions of this action. Let $m = \max \{\text{ord}_v(\text{cond}\lambda_v), 3\}+1$. If $\text{ord}_v(y\bar{y}) > m$, then there is a such eigenfunction $\phi'_2$ whose eigenvalue is a character $\lambda_{\theta,v}'\chi_{\theta,v}$ for $\chi_{\theta,v}$ of conductor at least $\omega_v^m$. Moreover there is a $p$-integral valued Schwartz function $\phi_1$ and a set of $p$-integral $C_{v,i} \in \mathbb{Q}_p$ and $u_{v,i} \in U(1)(\mathbb{Q}_v)$'s such that the function

$$\phi_2(x_2) = \int_{X_1(\mathbb{Q}_v)} \sum_i \delta^{-1}_v(C_{v,i}\omega_{\lambda}(u_{v,i}, 1)\Phi''_v)(x_1, x_2)\phi_1(x_1)dx_1$$

(here $1 \in U(2,2)(\mathbb{Q}_v)$) is a non-zero multiple of $\phi'_2$.

Proof. Consider the embedding $U(1,1) \hookrightarrow U(2,2)$ by

$$j : g \mapsto \begin{pmatrix} s^{-1} & \begin{pmatrix} a_g & b_g \\ d_g & c_g \end{pmatrix} \\ 1 & s^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}$$

for $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$. Define $f_{\Phi''_v}(g, \frac{1}{2}) = (\omega_{\lambda}(j(g))\Phi''_v)(0) \in I_2(\mathbb{Z}_v)$. We define $i : U(1) \times U(1) \hookrightarrow U(1,1)$ by

$$i(g_1, g_2) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\delta_{1} & \delta_{1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}.$$  

For $g_1 \in U(1)(\mathbb{Q}_v)$,

$$f_{\Phi''_v}(i(1, g_1), \frac{1}{2}) = (\omega_{\lambda}(j \circ i(1, g_1))\Phi''_v)(0) = (\delta_{\psi}\omega(1, g_1)\lambda^{-1}(\Phi''_v))(0)$$

here in the first and last expression $1 \in U(2)(\mathbb{Q}_v)$ and $g_1$ is viewed as the element in the center of $U(2,2)(\mathbb{Q}_v)$. Thus we are reduced to proving the following lemma.

Lemma 6.22. Let $g_1 = 1 + \omega_v^{\chi_{\theta,v}} a \in U(1)(\mathbb{Q}_v)$ for $m$ as in the above lemma and $a \in O_{K,v}$, if $n = y\bar{y}$ is such that $\text{ord}_v n > m$ then $f_{\Phi''_v}(i(1, g_1); \frac{1}{2}) \neq f_{\Phi''_v}(1; \frac{1}{2})$. 

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Proof. We have that:

\[
\begin{pmatrix}
\frac{1}{2} - \frac{1}{2} - \delta & 1 \\
\frac{1}{2} - \delta & -1 - \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
g_1 \\
-1
\end{pmatrix}
\begin{pmatrix}
1 - \delta \\
-\frac{1}{2}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{2} - g_1 \\
-1
\end{pmatrix}
\begin{pmatrix}
1 - \delta \\
-\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{2} + \frac{1}{2} g_1 \\
\frac{1}{2} + \frac{1}{2} g_1
\end{pmatrix}
\begin{pmatrix}
1 - \delta \\
-\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
1 \\
\frac{n}{1}
\end{pmatrix}
= \begin{pmatrix}
a & a + b \\
c & c + d
\end{pmatrix}
\begin{pmatrix}
\frac{n}{1} + \frac{d}{c + nd} \\
\frac{n}{c + d}
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

Now the lemma follows readily.

In all cases we write \( \phi_{1,v}, \phi_{2,v} \) for the \( \phi_1, \phi_2 \) above.

Remark 6.23. Later when we are moving the datum \( p \)-adic analytically, this choice of \( \phi_1 \) is not going to change.

6.8 \( p \)-adic Cases

We recall some results in \([48]\) with some modifications. Recall that we have the triple \( \pi_p, \psi_p, \tau_p \) and \( \xi_p := \psi_p/\tau_p, \chi_p \) is the central character of \( \pi_p \) and \( \psi_p|_{\mathbb{Q}^\times} = \chi_p \). Suppose \( \pi_p \) is nearly ordinary in the sense that \( \pi_p = \pi(\chi_1, \chi_2, p) \) such that \( \text{ord}_p(\chi_1(p)) = -\frac{1}{2} \) and \( \text{ord}_p(\chi_2(p)) = \frac{1}{2} \). We write \( \tau_p = (\tau_1, \tau_2) \) and \( \xi_p = (\xi_1, \xi_2) \). We call the triple is generic if there is a \( t \geq 2 \) such that \( \xi_1, \xi_2, \chi_p, \chi_p^{-1} \xi_1, \chi_p^{-1} \xi_2 \) all have conductor \( p^t \). Later when we are working with families, it is easily seen that these points are Zariski dense due to the fact that \( p \geq 5 \) (in fact this is the only place where we used the fact \( p \geq 5 \)). Although the definition for generic points is different from \([48]\) however the argument there goes through since the only place using this definition is lemma 4.4.3 there, which can be proved completely in the same way under our definition for generic. We define \( \xi_1^\dagger = \chi_1 \xi_2, \xi_2^\dagger = \chi_2 \xi_2 \). We define \( f_t \) to be the section supported in \( Q(\mathbb{Q}_v)K_t \), invariant under \( K_t \) and takes value 1 on the identity. Define

\[
f_{\text{sieg},p}(g) = g(\tau_p')^{-3} c_3^{-1}(\tau_p', -z)p^{-3t} g(\xi_1^\dagger)\xi_1^\dagger(-1)g(\xi_2^\dagger)\xi_2^\dagger(-1)g(\xi_1^\dagger(p^t a)\xi_2^\dagger(p^t b)
\]

\[
\times \sum_{a, b \in \mathbb{Z}_p^* / \mathbb{Z}_p, m, n \in \mathbb{Z}_p / p^t \mathbb{Z}_p} f_t(g \Upsilon \left( \begin{array}{cc}
1 & a + b \frac{m}{n} \\
1 & b
\end{array} \right)
\right).
\]

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and
\[
\begin{aligned}
f_{\text{sieg}, p}(g) &= \mathfrak{g}(\tau'_p)^{-3}c_3^{-1}(\tau'_p, -z)p^{-3H} \mathfrak{g}(\xi'_1)^{-1}\xi'_1(-1)\mathfrak{g}(\xi'_1)^{-1}\xi'_1(p' a)\xi'_1(p' b) \\
&\times \sum_{a, b \in p^{-1}Z_p / Z_p, n \in Z_p/p'Z_p} f_t(g\Upsilon) \begin{pmatrix} 1 & a & bm \\ 1 & b & 1 \\ 1 & 1 \end{pmatrix}.
\end{aligned}
\]

We also define:
\[
\begin{aligned}
f'_{\text{sieg}, p}(g) &= \mathfrak{g}(\tau'_p)^{-3}c_2^{-1}(\tau'_p, -z)p^{-3H} \mathfrak{g}(\xi'_1)^{-1}\xi'_1(-1)\mathfrak{g}(\xi'_1)^{-1}\xi'_1(p' a)\xi'_1(p' b) \\
&\times \sum_{a, b \in p^{-1}Z_p / Z_p, m \in Z_p/p'Z_p} f_t(g\Upsilon) \begin{pmatrix} 1 & a + bmn & bm \\ 1 & b & 1 \\ 1 & 1 \end{pmatrix}.
\end{aligned}
\]

where \( \Upsilon \in U(3, 3)(\mathbb{Q}_p) \) is such that it is \(
\begin{pmatrix} 1 & \frac{\tau'_1}{\tau'_2} & 1 \\
-\frac{\tau'_2}{\tau'_1} & 1 & 1 \\
-\zeta^{-1} & -\zeta^{-1} & 1
\end{pmatrix}
\) via the first projection \( U(3, 3)(\mathbb{Q}_p) \approx \text{GL}_6(\mathbb{Q}_p) \) and \( c_n(\tau', z) = \tau'(p^nt) p^{2ntz - t(n + 1)/2} \).

Pullback Formulas
We refer to [48 4.4.1] for the discussion of nearly ordinary vectors, which means the vector whose \( U_p \)-eigenvalues are \( p \)-adic units. Let \( \varphi = \varphi^{\text{ord}} \in \pi_p \) be a nearly ordinary vector. Define \( f^0 \in I_p(\rho) \) to be the nearly ordinary Klingen section supported in \( \mathcal{P}(\mathbb{Q}_p)w'_3B_t(\mathbb{Z}_p) \) where \( B_t(\mathbb{Z}_p) \) consists of matrices in \( \text{GL}_4(\mathbb{Z}_p) \) which are upper triangular modulo \( p^t \) and \( w'_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \text{GL}_4(\mathbb{Z}_p) \)

and the right action of \( B_t(\mathbb{Z}_p) \) on \( f^0 \) is given by a character (see [48 4.4.1], note the differences in the indices discussed there). Moreover we require the value of \( f^0 \) on \( w'_3 \) is given by
\[
\mathfrak{g}(\tau'_p)^{-1}\tau'_p(p^t)p^{-2}p^{(\kappa - 3)}\xi_{1, p}^{-1}\xi_{2, p}^{-1}(p^{-t})\mathfrak{g}(\xi_{1, p}\chi_{1, p}^{-1})\mathfrak{g}(\xi_{1, p}\chi_{2, p}^{-1})\cdot \varphi.
\]

The fact that it is indeed a nearly ordinary vector is explained in the discussion before [48 Definition 4.43]. Then by the computations in [48] we have the following (see the end of [48 Section 4]).

**Lemma 6.24.** (1) \( F_\varphi(f_{\text{sieg}, p}, z, g) = f^0(g) \); (2) \( F'_{\varphi}(f'_{\text{sieg}, p}, z, g) = p^{(\kappa - 3)}\xi_{1, p}^{-1}\xi_{2, p}^{-1}(p^{-t})\mathfrak{g}(\xi_{1, p}\chi_{1, p}^{-1})\mathfrak{g}(\xi_{1, p}\chi_{2, p}^{-1}). \pi(g) \cdot \varphi. \)

**Fourier Coefficients**
We define the function \( \Phi_{\xi^t} \) as the function on the set of \( (2 \times 2) \ \mathbb{Q}_p \)-matrices as follows. If \( x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is such that both its determinant and \( a \) are in \( \mathbb{Z}_p^\times \) then \( \Phi_{\xi^t}(x) = \xi_{1}^t(a)\xi_{2}^t \frac{|\det x|}{a} \). Otherwise \( \Phi_{\xi^t} = 0 \). The following lemma is proved in [48].
Lemma 6.25.
\[ f_{\text{sieg}, p, \beta} = \Phi_{\xi_1}( \begin{pmatrix} \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{pmatrix} ) \]
for \( \beta = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} \) with \( \beta_{11}, \beta_{12}, \beta_{13}, \beta_{23}, \beta_{33} \in \mathbb{Z}_p \), and 0 otherwise.

Fourier-Jacobi Coefficients
For \( \beta \in S_1(\mathbb{Q}_v) \cap \text{GL}_4(\mathbb{Z}_v) \) we are going to compute the Fourier-Jacobi coefficient for \( f_t \) at \( \beta \). We have the following ([18, lemma 4.54])

Lemma 6.26. Let \( x := \begin{pmatrix} 1 & D & 1 \end{pmatrix} \) (this is a block matrix with respect to \( (2 + 2) \)).
(a) \( F J_\beta(f_t; -z, v, x \eta^{-1}, 1) = 0 \) if \( D \notin p^4 M_2(\mathbb{Z}_p) \);
(b) if \( D \in p^4 M_2(\mathbb{Z}_p) \) then \( F J_\beta(f_t; -z, v, x \eta^{-1}, 1) = c(\beta, \tau, z) \Phi_0(v) \), where
\[ c(\beta, \tau, z) := \tilde{\tau}(- \det \beta)|\det \beta|_{v}^{2z+2} g(\tau') g(\tau_{p}')(p^{t})p^{-2tz-3t}. \]

Note the formula:
\[ g(\tau_{p}')(p^{t})p^{-2tz-3t} \]

We get:

Lemma 6.27.
\[ g(\tau_{p}')(p^{t})p^{-2tz-3t} = g(\tau_{p}')(p^{t})p^{-2zd-3t} g(\tau_{p}')(p^{2t})p^{-4zd-3t}. \]

Here \( A_x = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \). Also under the projection \( U(3, 3)(\mathbb{Q}_p) \simeq \text{GL}_6(\mathbb{Q}_p) \), the \( v_1, v_2, v_3, v_4 \) are
\[ \begin{pmatrix} 1 & v_1 \\ 1 & v_2 \\ 1 & v_3 \\ 1 & v_4 \end{pmatrix} \]. The Weil representations are the one in subsection 4.8. We use \( \rho \) to denote the right action of \( \text{GU}(3, 3)(\mathbb{Q}_p) \) on the Siegel sections. And
\[ \Phi_{\rho}(v_1, v_2, v_3, v_4) = \begin{cases} \xi_1^{\tau}(a), & v_1, v_2 \in \mathbb{Z}_p, v_3 \in p^{-t} \mathbb{Z}_p, v_3 \in \frac{a}{p^t} + \mathbb{Z}_p \text{ for some } a \in \mathbb{Z}_p^\times \\ 0, & \text{otherwise.} \end{cases} \]
Proof. First we fix \(b\) and consider the Fourier-Jacobi expansion of:

\[
\sum_{a,m} \rho(\begin{pmatrix} 1 & a & bm \\ 1 & 1 & \end{pmatrix}) f_t
\]

note that:

\[
\begin{pmatrix} 1 & -a & -bm \\ 1 & 1 & \end{pmatrix} \begin{pmatrix} \alpha(1, \eta^{-1}) \\ \end{pmatrix} = \begin{pmatrix} \alpha(1, \eta^{-1}) \\ \end{pmatrix}
\]

where \(v'_3 = v_3 + D_1, a + D_2,bm, v'_4 = v_4 + D_2, a + D_4,bm, t' = t + v_1, a + v_2, bm\). From this a simple calculation shows that the Fourier-Jacobi expansion of (3) at \(g\) is:

\[
(g(\tau'_p)^{-2}c_2^{-1}(\tau'_p, -z)p^{-t}g(\xi'_2)\rho(\eta)f_t(g')p^{-2t}\omega(g')\Phi'_p)(v_1, v_2, v_3, v_4).
\]

So the Fourier-Jacobi expansion of

\[
\sum_{a,b,m} \rho(\begin{pmatrix} 1 & a & bm \\ 1 & 1 & \end{pmatrix}) f_t
\]

is

\[
\sum_b (g(\tau'_p)^{-2}c_2^{-1}(\tau'_p, -z)p^{-t}g(\xi'_2)\rho(\begin{pmatrix} 1 \\ A_{-b} 1 \\ \end{pmatrix}) \rho(\eta)f_t(g')p^{-2t}\omega(g')\omega(g'(\begin{pmatrix} 1 \\ A_{-b} 1 \\ \end{pmatrix})\Phi'_p)(v_1, v_2, v_3, v_4)
\]

Note that \(\omega(\begin{pmatrix} 1 \\ A_{-b} 1 \\ \end{pmatrix})\Phi'_p = \Phi'_p\), we get the required Fourier-Jacobi expansion.

We define \(f_{2,p} = \sum_b g(\xi'_2)p^{-t}\xi'_2(-p^tb)g(\tau'_p)^{-2}c_2^{-1}(\tau'_p, -z)\rho(\begin{pmatrix} 1 \\ A_{-b} 1 \\ \end{pmatrix}) \eta f_t\) and \(\Phi_p = g(\xi'_4)p^{-2t}\Phi'_p\).
We record some formulas:

\[ \rho(\eta)f_{2,p} = f'_{2,p} = \frac{\tau_p}{\nu_p}(-1)g(\xi_2^\dagger) \sum_b \xi_2^\dagger(b)g(\tau'_p)^{-2}c_2^{-1}(\tau'_p, -z)\rho\left( \begin{array}{c} 1 \\ \lambda b \end{array} \right)f_t \]

\[ (\omega_\beta(\eta)\Phi_p)(v_1, v_2, v_3, v_4) = \begin{cases} \xi_2^\dagger(-v_1), & v_3, v_4 \in \mathbb{Z}_p, v_1 \in \mathbb{Z}_p^\times, v_2 \in p^t\mathbb{Z}_p \\ 0, & \text{otherwise.} \end{cases} \tag{4} \]

We define two Schwartz functions on \( X_p \) by letting \( \phi'_1,p \) to be the characteristic function of \( \mathbb{Z}_p^2 \subset \mathbb{Q}_p^2 \) and \( \phi'_2,p(\xi_2^\dagger(-x)) \) if \( x_1 \in \mathbb{Z}_p^\times, x_2 \in p^t\mathbb{Z}_p \) and is zero otherwise.

**Definition 6.28.** We define \( \phi_1,p = \delta_{\psi}^{\sim''}(\phi'_1,p), \phi_2,p = \delta_{\psi}^{\sim''}(\phi'_2,p). \)

### 6.9 Pullback Formulas Again

In this section we prove the local pullback formulas for \( U(2) \times U(2) \hookrightarrow U(2, 2) \) which will be used to decompose the restriction to \( U(2) \times U(2) \) of the Siegel Eisenstein series on \( U(2, 2) \) showing up in the Fourier-Jacobi expansion of \( E_{\text{Siegel}} \) on \( U(3, 3) \). Fortunately, the local calculations are the same as in the previous sections for \( f'_{\text{Siegel}} \) and \( F'_{\varphi} \)'s except for the case \( v = p \).

**p-adic case:**

We temporarily denote the \( p \)-component of automorphic representation \( \pi_h \) of some \( h \) on \( U(2)(\mathbb{Q}_p) \) as \( \pi(\chi_1, \chi_2) \) with \( v_p(\chi_1(p)) = -\frac{1}{2}, v_p(\chi_2(p)) = \frac{1}{2} \). We also write temporarily write \( \tau \) for \( \tau/\lambda \) in this subsection. We let \( \tau_p = (\tau_1, \tau_2) \) and require \( \chi_1\tau_2^{-1} \) and \( \chi_2\tau_2^{-1}\xi_2^\dagger \) are unramified. We let \( \varphi = \varphi^{\text{ss}} \in \pi_{h,p} \) for \( \varphi^{\text{ss}} = \pi_p\left(p^{-1}\right)\varphi^{\text{ord}} \) for some nearly ordinary vector \( \varphi^{\text{ord}} \). Define

\[ f_{2,p}(g) = g(\tau'_p)^{-2}c_2^{-1}(\tau'_p, -z/2)^{-1}p^{-t}g(\xi_2^\dagger) \sum_{b \in \mathbb{Q}_p^{\times}} \xi_2^\dagger(bp^t)\rho\left( \begin{array}{c} 1 \\ \lambda b \end{array} \right)f_t(g\tau'). \]

It is hard to evaluate the integral directly. So we use the trick of using the functional equation as in [45 Proposition 11.13]. We first do the integration for the auxiliary

\[ \mathcal{F}_\varphi(f_{2,p}^\dagger; z, g) := \int_{U(2)(\mathbb{Q}_p)} f_{2,p}^\dagger(z, S^{-1}\alpha(g_1, g)S)\tilde{\tau}(\det g)\pi_h(g_1)\varphi dg_1 \]

at \( g = w \) where \( \varphi \in \pi_{h,p} \) and

\[ f_{2,p}^\dagger(g) := p^{-t}g(\xi_2^\dagger) \sum_b \xi_2^\dagger(bp^t)\rho\left( \begin{array}{c} 1 \\ \lambda b \end{array} \right)f_t(g\tau'), \quad f^{(2,2)}(g) \in \tilde{I}(\tau'). \]

For \( A \in U(2)(\mathbb{Q}_p) \simeq \text{GL}_2(\mathbb{Q}_p) \) note that

\[ S^{-1}\text{diag}(A, 1) \left( \begin{array}{cc} 1 & A \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & -A \\ -A & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & -A \\ -A & 1 \end{array} \right)w. \]
So in order for this to be in supp $f^1$ we must have $A^{-1} + A_b \in M_2(\mathbb{Z}_p)$. So $A^{-1}$ can be written as
\[
\begin{pmatrix}
1 & m \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
u & u \\
m & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\text{ for } v \in \mathbb{Z}_p \setminus \{0\}, u \in -b + \mathbb{Z}_p \text{ and } m, n \in p^{\prime} \mathbb{Z}_p. \text{ Thus}
\[
A = \begin{pmatrix}1 & m \\ 1 & 1\end{pmatrix} \begin{pmatrix}v^{-1} & u^{-1} \\ 1 & 1\end{pmatrix} \begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix}.
\]

A direct computation gives the integral equals:
\[
\chi_2(-1) \chi_1 \chi_2 (p^1) \mu^{(z+1)} \bar{\tau} c((-p^1, -1)) \sum_{i=1}^{\infty} (\chi_1(p^{-1}) \bar{\tau} c((p^{-1}, 1)) p^{-(z+\frac{1}{2})})^i \varphi_{ord}
= p^{(z+1)} \bar{\tau} c((-p^1, -1)) L_p(\pi, \tau, z + \frac{1}{2}) \chi_1 \chi_2 (p^1) \chi_2 (-1).
\]

The $\pi$ in the $L$-factor means the base change of $\pi$ from $U(2)$ to $GL_2$. Note that it is not convergent at $z = -z_\kappa$ and is defined by analytic continuation at that point.

Now we apply the functional equation trick to evaluate the pullback integral for $f_{2, p}$. As in [45 Proposition 11.28] the local constant showing up when applying the intertwining operator at $z = -z_\kappa$ is
\[
\epsilon(\pi, \tau, -z_\kappa + \frac{1}{2}) = \varphi_\tau(\bar{\tau} \chi_1) \tau_1 \chi_1 (p^1) \varphi_\tau(\bar{\tau} \chi_2) \tau_1 \chi_2 (p^1) \varphi_\tau(\bar{\tau} \chi_2) \tau_2 \chi_2 (p^1) p^{\frac{3}{2} \kappa - 6}.
\]

To sum up our original local integral equals
\[
L_p(\pi_h, \bar{\tau} c, z_\kappa + \frac{1}{2}) \varphi_\tau(\bar{\tau} \chi_1) \tau_1 \chi_1 (p^1) \varphi_\tau(\bar{\tau} \chi_2) \tau_1 \chi_2 (p^1) p^{(2 \kappa - 5)} \chi_1 (p^1) p^{\frac{1}{2} \varphi_{ord}}.
\]

Note that $< \varphi_{low}, \varphi^{ss} > < \varphi_{ord}, \varphi_{low} > \chi_1 (p^1) p^{\frac{1}{2}}$ where we define $\varphi_{low} = \pi_p \left( \begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix}\right) \varphi_{ord}$. Thus if we replace $\varphi^{ss}$ by $\varphi_{low}$ in the definition for pullback integral then it equals
\[
L_p(\pi_h, \bar{\tau} c, z_\kappa + \frac{1}{2}) \varphi_\tau(\bar{\tau} \chi_1) \tau_1 \chi_1 (p^1) \varphi_\tau(\bar{\tau} \chi_2) \tau_1 \chi_2 (p^1) p^{(2 \kappa - 5)} \chi_1 (p^1) p^{\frac{1}{2} \varphi_{ord}}.
\]

**Remark 6.29.** Later when we are defining $E_{sieg, 2}$ the $\tau$ here should be $\bar{\tau}$.

### 6.10 Global Computations

We first define two normalization factors as in [48 5.3.1]
\[
B_D : = \frac{\Omega_{\pi_\kappa}^{2 \kappa} \xi_{\kappa}}{\Omega_{\pi_\kappa}^{2 \kappa} \xi_{\kappa}} \left((-2)^{-3} (2 \pi i)^{3 \kappa} (2/\pi)^{3} \Pi_{j=0}^{\kappa-j-1} \right)^{-1} \Pi_{i=0}^{1} L_s(2 z_\kappa + 3 - i, \bar{\tau} \chi_1^i),
\]
\[
B_D' : = \frac{\Omega_{\pi_\kappa}^{2 \kappa} \xi_{\kappa}}{\Omega_{\pi_\kappa}^{2 \kappa} \xi_{\kappa}} \left((-2)^{-2} (2 \pi i)^{2 \kappa} (2/\pi)^{2} \Pi_{j=0}^{\kappa-j-1} \right)^{-1} \Pi_{i=0}^{1} L_s(2 z_\kappa + 2 - i, \bar{\tau} \chi_1^i).
\]

The $z_\kappa \equiv \frac{3-\kappa}{2}$ and $z_\kappa' \equiv \frac{\kappa-2}{2}$. We define $E_{sieg} (z, g) = E_{sieg} (z, f_{sieg, g})$ on $GU(3, 3)$ for $f_{sieg} = B_D \Pi_{v} f_{sieg, v}$ and $E_{sieg}' (z, g) = E_{sieg}' (z, f_{sieg, g}')$ on $GU(3, 3)$ for $f_{sieg}' = B_D' \Pi_{v} f_{sieg, v}'$ (Note that compared to [48], the normalization factors at $p$ here are already included in our definitions of
Lemma 6.30. Consider the model for the unramified principal series representation $\pi$ at $\Phi = \text{Fourier-Jacobi Coefficients}$ $D, U, \psi$. The period factor showing up is to make it rational (see [16, 2.8]). The $\varphi$ is defined as follows. First recall that given a CM character $\psi$ and a form on $D^\times$ whose central character is $\psi|_{\mathbb{A}_{\mathbb{Q}}}$ we can produce a form on $U(2)$ whose central character is the restriction of $\psi$. So we often construct forms on $D^\times$ and get forms on $U(2)$ this way. In section 8.2 we are going to construct a Dirichlet character $\vartheta$. We define

$$f_\Sigma = \prod_{v \in \Sigma, v \not\in N} \pi\left(\frac{1}{w_v}\right) - \chi_{1,v}(w_v)q_{1,v}^{1/2}f_{\text{new}}$$

where $f_{\text{new}}$ is a new vector in $\pi$ and define $f_{\vartheta} \in \pi$ by

$$f_{\vartheta}(g) = \prod_{v \in \Sigma \text{ split }, v \not\in N} \sum_{a_v \in \mathbb{Z}, \varphi_v \in \mathbb{Z}} \vartheta\left(\frac{-a_v}{w_v}\right)f_\Sigma(g\prod_{v} \left(\frac{1}{a_v} \varphi_v\right)\left(\frac{w_v^{-s_v}}{1}\right)_v)$$

where the $s_v$ above is the order of the conductor of $\vartheta$ at $v$ and $t_v$ is the order of the conductor of $\pi$ at $v$. Define $\varphi = \prod_{v} \varphi_v = \pi(g_1)f_{\vartheta}$ for some $g_1$ defined in section 8. We record the following easy lemma, which explains the motivation for the definition of $f_\Sigma$: to pick up a certain Iwahari-invariant vector from the unramified representation $\pi_v$ for $v \in \Sigma \setminus \{v, v|N\}$.

**Lemma 6.30.** Consider the model for the unramified principal series representation

$$\pi(\chi_{1,v}, \chi_{2,v}) = \{f : K_v \to \mathbb{C}, f(qk) = \chi_{1,v}(a)\chi_{2,v}(d)\delta_B(q)f(k), q = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in B(\mathbb{Z}_v)\}.\]$$

Let $f_{\text{ur}}$ be the constant function 1 on $K_v$, $f_0$ be the function supported and takes value 1 on $K_1$ for $K_1 = \{(a b \mid c d) \mid w_v|c\}$. Then

$$(\chi_{2,v}(w_v)q_{1,v}^{1/2} - \chi_{1,v}(w_v)q_{1,v}^{1/2})f_0 = \pi\left(\frac{1}{w_v}\right)f_{\text{ur}} - \chi_{1,v}(w_v)q_{1,v}^{1/2}f_{\text{ur}}.$$  

**Fourier-Jacobi Coefficients**

**Proposition 6.31.** The Fourier-Jacobi coefficient for $\beta = 1$ at $[1]$ is given by:

$$FJ_1(E_\text{sig}(z, \text{diag}(u, 1, 2, 1, 2)n')g) = \sum_{\frac{m}{n} \in \mathbb{Z}_p/p^2\mathbb{Z}_p} E_{\text{sig}, 2}(z, f_2, g\gamma(1, \left(\begin{array}{cc} 1 \\ n \end{array}\right)_p))\Theta(u, n'g\gamma(1, \left(\begin{array}{cc} 1 \\ n \end{array}\right)_p)$$

for $n' \in N_2$ ($N_2$ defined in subsection 4.3), $g \in U(2, 2)A_Q$ and $u \in U(1)A_Q$. Here $f_2 = \prod f_{2,v}$ and $\Phi = \prod\Phi_v$ are given in previous subsections. $E_{\text{sig}, 2}$ is considered as a function on $U(2, 2)A_Q$.
Proof. This follows from our computations for Fourier-Jacobi coefficients and lemma \[6.5\] \(\square\)

So far we have used the embedding \(\alpha(1, g) = \begin{pmatrix} D_g & C_g \\ B_g & A_g \end{pmatrix}\) for \(g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\), to keep accordance with the convention of \[26\]. In our actual applications later on we will use another embedding \(\alpha''\) as below. Now that we have

\[FJ_1(\square_{\text{sieg}}, \kappa, g, x, u) = E_{\text{sieg},2}(\kappa, f_2, g')\Theta_{\Phi}(u, g').\]

Now we consider another embedding \(\alpha''(1, g) = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}\). Let \(\Phi'' = \omega_{\beta}(\eta)\Phi\) and \(f''_2 = \rho(\eta)f_2\) and we let \(FJ''_\beta\) be defined as \(FJ_\beta\) but replacing \(\alpha\) by \(\alpha''\). By observing that \(E_{\text{sieg},2}\) and \(\Theta\) are automorphic forms and thus invariant under left multiplication by \(\eta^{-1}\), we get:

\[FJ''_\beta(\square_{\text{sieg}}, \kappa, g, x, u) = E_{\text{sieg},2}(\kappa, f''_2, g)\Theta_{\Phi''}(u, g).\] (5)

Lemma 6.32. Suppose \(\delta^{-1}_\psi(\Phi''_\infty) = \phi_{1,\infty} \boxtimes \phi_{2,\infty}\), and for each \(v < \infty\)

\[\phi_{2,v}(x) = \int_{X_v} \delta^{-1}_\psi(\Phi''_v)(x', x)\phi_1(x')dx'.\]

Then

\[l'_{\theta_{\phi_1}}(\gamma^{-1}(\Theta_{\Phi}))(x) = \theta_{\phi_2}(x).\]

Here we consider \(\gamma^{-1}(\Theta_{\Phi})\) as a function on \((NU(2)) \times U(2) \hookrightarrow U(3, 1) \times U(2)\) and apply \(l'_{\theta_{\phi_1}}\) to it on the NU(2) part.

The lemma follows easily from writing \(\delta^{-1}_\psi\Phi\) into a finite sum of expressions of the form \(\phi_1 \boxtimes \phi_2\) and then applying \([1]\) and \([2]\).

Corollary 6.33.

\[l'_{\theta_1}(FJ_1(\prod_v \sum_i \rho(\text{diag}(u_{1,v}, 1, u_{1,v})))E_{\text{Kling}}(g))\]

\[= \langle \frac{1}{\Omega_{\infty}^{2n-2}} \sum_{n \in \mathbb{Z}_p/p' \mathbb{Z}_p} E_{\text{sieg},2}(\alpha(g, -(1 \ n \ 1) \ p) \cdot \frac{1}{\Omega_{\infty}} \theta_2(-(1 \ n \ 1) \ p), \varphi(-)) \rangle\]

where \(\theta_1\) and \(\theta_2\) are the theta function on \(U(-\zeta)(\mathbb{A}_\mathbb{Q})\) defined using the kernel \(\phi_1 = \prod_v \phi_{1,v}\) or \(\phi_2 = \prod_v \phi_{2,v}\) for \(\phi_{1,v}\) and \(\phi_{2,v}\)'s defined as before. Note that the \(\phi_2\) is defined using \(\phi_1\), which explains the dependence of the right hand side on \(\phi_1\). The inner product is over the group \(1 \times U(2) \hookrightarrow U(3, 3)\). Moreover suppose \(\varphi_p \in \pi_p\) is chosen such that \(\varphi_p\) is the ordinary vector, then the above expression is:

\[\langle E(\alpha(g, -)) \cdot \theta_2(-), \varphi(-)\rangle.\]

Proof. It follows from the above proposition and our discussions for intertwining maps in subsection 4.7. The last sentence follows from a description of the paring between \(\pi_p\) and \(\pi_p'\). \(\square\)
7 \ p\text{-adic Interpolation}

7.1 Eisenstein Datum

**Definition 7.1.** An Eisenstein datum \( \mathcal{D} = (L, \mathcal{I}, f, \xi) \) consists of:

- A finite extension \( L/\mathbb{Q}_p \);
- a finite normal \( \mathcal{O}_L[[W]] \)-algebra \( \mathcal{I} \);
- an \( \mathcal{I} \)-adic Hida family \( f \) of cuspidal ordinary eigenforms new outside \( p \), of square-free conductor \( N \) such that some weight 2 specialization \( f_0 \) has trivial character; Assume moreover that for some odd prime \( q \) non-split in \( K \) we have \( q|N \).
- a finite order \( L \)-valued Hecke character \( \xi \) of \( K^\times \backslash \mathbb{A}_K^\times \) whose \( p \)-part conductor divides \( p \).

Now we need to modify our \( \mathcal{I} \). By taking an irreducible component of the normalization of a series of quadratic extension of \( \mathcal{I} \) we may assume that for each \( v \in \Sigma \) not dividing \( N \), we can find two functions \( \alpha, \beta \in \mathcal{I} \) interpolating the Satake parameters of \( \pi_v \). This enables us to do the constructions in the global computations in section 6 in families. We still denote \( \mathcal{I} \) for simplicity. At the end of this paper we will see how to deduce the main conjecture for the original \( \mathcal{I} \) from that for the new \( \mathcal{I} \).

Let \( \mathcal{I}_K := \mathcal{I}[[\Gamma_K]] \). We define \( \alpha : \mathcal{O}_L[[\Gamma_K]] \to \mathcal{I}[[\Gamma_K]] \) and \( \beta : \mathcal{O}_L[[\Gamma_K]] \to \mathcal{I}[[\Gamma_K]] \) by:

\[
\alpha(\gamma_+) = (1 + W)^\frac{1}{2}, \quad \alpha(\gamma_-) = \gamma_-, \quad \beta(\gamma_+) = \gamma_+, \quad \beta(\gamma_-) = \gamma_-
\]

We let \( \psi = \alpha \circ \Psi_K, \xi = (\beta \circ \Psi_K) \cdot \xi \). Define: \( \psi_\phi = \phi \circ \psi \) and \( \xi_\phi = \phi \circ \xi \). Let \( \Lambda_\mathcal{D} = \mathcal{I}[[\Gamma_K]][[\Gamma_K^{-}]] \). We give \( \Lambda_\mathcal{D} \) a \( \Lambda_2 \)-algebra structure by first give a homomorphism \( \Gamma_2 = (1 + p\mathbb{Z}_p)^4 \to \Gamma_K \times \Gamma_K \) given by:

\[
(a, b, c, d) \to \text{rec}_K(d b, a^{-1} c^{-1}) \times \text{rec}_K(d^{-1}, c)
\]

and then compose with \( \alpha \otimes \beta \).

We remark here that only the quotient \( \mathcal{I}[[\Gamma_K]] \) of \( \Lambda_\mathcal{D} \) really matters: the \( \Gamma_K^{-} \) variable corresponds to twisting everything by the same character and does not affect the \( p \)-adic \( L \)-functions and the Selmer groups.

Now we consider \( \Lambda_\mathcal{D} \)-adic forms. For \( \mathcal{X} \) a Zariski dense set of arithmetic points. Suppose \( U_f \subset D(\mathbb{A}_f) \) be a compact open subgroup such that the level is prime to \( p \). A point \( \phi \in \text{Spec} \Lambda_\mathcal{D} \) is called arithmetic if \( \phi(1 + W), \phi(\gamma_+), \phi(\gamma_-) \) for \( \gamma^{-} \in \Gamma_K \) and \( \Gamma_K^{-} \) are all \( p \)-power roots of unity. We call it generic if \( (f_\phi, \psi_\phi, \xi_\phi) \) is generic in the sense defined in section 5. We let \( \mathcal{X} \) be the set of arithmetic points and \( \mathcal{X}^{\text{gen}} \) be the set of generic arithmetic points.

7.2 Congruence Module and the Canonical Period

We now discuss the theory of congruences of modular forms on \( \text{GL}_2(\mathbb{Q}) \). Let \( R \) be a finite extension of \( \mathbb{Z}_p \) and \( \varepsilon \) a finite order character of \( \mathbb{Z}_p^\times \) whose \( p \)-component has conductor dividing \( p \). Let \( M^{\text{ord}}(Mp^r, \varepsilon; R) \) be the space of ordinary modular forms on \( \text{GL}_2/\mathbb{Q} \) with level \( N = Mp^r \), character \( \varepsilon \) and coefficient \( R \). Let \( S^{\text{ord}}(Mp^r, \varepsilon; R) \) be the subspace of cusp forms. Suppose \( R \) is a finite extension of \( \mathbb{Z}_p \) we let \( T^{\text{ord}}(N, \varepsilon; R) \) (\( T^{0,\text{ord}}(Mp^r, \varepsilon; R) \)) be the \( R \)-sub-algebra of \( \text{End}_R(M^{\text{ord}}_K(Mp^r, \varepsilon; R)) \)}
For any complete local 

(a nearly ordinary eigenform. Then we have 1

Let \( \mathfrak{m}_f \) be the maximal ideal of the Hecke algebra corresponding to \( f \). Suppose that the localization of the Hecke algebra at \( \mathfrak{m}_f \) satisfies the Gorenstein property. Then \( T^{ord,0}(M,\varepsilon;R)\mathfrak{m}_f \) is a Gorenstein \( R \)-algebra, so \( T^{ord,0}(M,\varepsilon;R) \cap (0 \otimes FR) \) is a rank one \( R \)-module. We let \( \ell_f \) be a generator; so \( \ell_f = \eta_f 1_f \) for some \( \eta_f \in R \). This \( \eta_f \) is called the congruence number of \( f \).

Now let \( \mathbb{I} \) be as at the beginning of this section. Suppose \( f \in \mathcal{M}^{ord}(M,\varepsilon;\mathbb{I}) \) is an ordinary \( \mathbb{I} \)-adic cuspidal eigenform. Then as above \( T^{ord,0}(M,\varepsilon;\mathbb{I}) \otimes FR \simeq T' \times F_1 \), \( F_1 \) being the fraction field of \( \mathbb{I} \) where projection onto the second factor gives the eigenvalues for the actions on \( f \). Again let \( 1_f \) be the idempotent corresponding to projection onto the second factor. Then for an \( g \in S^{ord}(M,\varepsilon;\mathbb{I}) \otimes F_1 \), \( 1_fg = cf \) for some \( c \in F_1 \). As above, under the Gorenstein property for \( T_f \), we can define \( \ell_f \) and \( \eta_f \).

From now on we will define \( D^\times \) to be the unique quaternion algebra ramified exactly at \( \infty \) and the \( q \) in our main theorems in the introduction. We choose the group \( U(2) \) with \( D^\times \) being its associated quaternion algebra. We also make the following definition for \( p \)-adic families of forms on \( D^\times \).

**Definition 7.2.** For any complete local \( \mathcal{O}_L[[W]] \)-aglgebra \( R \) (need not be finite over \( \mathcal{O}_L[[W]] \)) we define the space of \( R \)-adic families on \( D^\times \) with level group \( K_D \subset D^\times(A_f) \) which is \( GL_2(\mathbb{Z}_p) \) at \( p \) to be

\[
\left( \lim_{m} \lim_{n} \mathbb{M}(K_D(p^n),\mathcal{O}_L/p^m\mathcal{O}_L) \otimes R \right)^{\mathbb{Z}_p[[W]]}
\]

where \( K_D(p^n) \) is obtained by replacing the \( p \)-component of \( K_D \) by \( \Gamma_1(p^n) \). The action of \( \mathbb{Z}_p[[W]] \) is given by the usual action as nebentypus on the first factor, and by \( (t \mapsto t^{-1}) \) on the second factor. The \( \mathbb{M}(K_D(p^n),\mathcal{O}_L/p^m\mathcal{O}_L) \) is the space of forms on \( D^\times(A_Q) \) with level group \( K_D(p^n) \) and coefficient ring \( \mathcal{O}_L/p^m\mathcal{O}_L \). It is not hard to see that each \( R \)-adic family gives a continuous \( R \)-valued function on \( D^\times(Q) \backslash D^\times(A_Q)/K^{(p)} \) where we give the last set the topology induced from the \( p \)-adic Lie-group \( GL_2(\mathbb{Z}_p) \).

The Ordinary Family \( f \) on \( D^\times \)

Let \( f \) be a Hida family of ordinary eigenforms new outside \( p \) as in the main theorem. Suppose \( T_{mf} \) is Gorenstein. Thus we have the integral projector \( \ell_f \). We are going to construct from it a Hida family of ordinary forms on \( D^\times(A_Q) \), also denoted as \( f \). We refer to [10] for the definition and theory of ordinary forms on quaternion algebras. By our assumption we may choose \( f_0 \) a form on \( D^\times(A_Q) \) which is in the Jacquet-Langlands correspondence of \( f_0 \) in \( GL_2(A_Q) \) with values in \( \mathcal{O}_L \) and \( f_0 \) is not divisible by \( \pi_L \). If we are under the assumption the main theorem in the introduction we can choose some \( g_0 \in S^{ord}(K_{f_0},\mathcal{O}_L) \) such that \( \ell_{f_0}g_0 \) is a \( p \)-adic unit times \( f_0 \) as forms on \( D^\times(A_Q) \). Note that \( \ell_{f_0}f_0 \) is not a \( p \)-adic unit times \( f_0 \). (This is possible by our assumptions in Theorem 4.1 on \( f \). The local Hecke algebras for \( f_0 \) on \( GL_2 \) and \( D^\times \) are the same, thus they have the same congruence numbers. On the other hand such congruence number is generated by the Petersson inner product of \( f_0 \) by result of Pollack-Weston [36]). Since the space of ordinary \( \mathbb{I} \)-adic forms is finite and free over \( \mathbb{I} \) by [10] section 10], and, by loc.cit section 9 we can choose an ordinary \( \mathbb{I} \)-adic form \( g \)
such that some weight 2 specialization is \( g_0 \). Thus we can define \( f := \ell_{fg} \). We make the following important remark: we can find a point (still denote as \( g_0 \)) on \( U(2)({\mathbb{A}}_Q) \) such that the value of \( \varphi \) is a \( p \)-adic unit. We only need to see that there is \( g_0 \in D^\times({\mathbb{A}}_Q) \) with \( \det g_0 \in {\mathbb{Q}}^\times \cdot \Nm({\mathbb{A}}_K^\times/{\mathbb{A}}_Q^\times) \) with \( f(g_0) \) being a \( p \)-adic unit. This follows easily from the assumption that \( f \) has square-free conductor.

### 7.3 Siegel Eisenstein Measure

**Proposition 7.3.** There are \( \Lambda_2 \)-adic formal Fourier expansions \( E_{\mathcal{D},\text{sieg}} \) and \( E'_{\mathcal{D},\text{sieg}} \) which when specializing to an arithmetic point \( \phi \) are the Siegel Eisenstein series we constructed from the datum \((f_\phi, \xi_\phi, \psi_\phi)\) at \( \phi \) and our choices of Siegel Eisenstein sections \( f_{\text{sieg},v} \).

**Proof.** It is a special case of \([48, \text{Lemma 5.7}]\) and follows from our computations of the local Fourier coefficients for Siegel Eisenstein series. For example the \( \beta \)-th Fourier coefficient of

\[
\left( \frac{\Omega_p\kappa_\infty}{\Omega_\infty^{2\kappa}} \right)^{-1} E_{\text{sieg}}(f_{\text{sieg}}; z_\kappa, g)
\]

at \( \text{diag}(y, \bar{y}) \) for \( y \in \text{GL}_3({\mathbb{A}}_{K,f}) \) is given by

\[
\prod_{\ell \in \Sigma, \ell \nmid p} D_\ell^{-\frac{3}{2}} \tau(\det y_\ell) \det y_\ell \bar{y}_\ell \frac{\beta_{22} + \beta_{33}}{\beta_{22} + \beta_{33}} \\
\times \prod_{\ell \in \Sigma} h_\ell, y_\ell \beta_\ell \left( \frac{\lambda_\ell}{\lambda_\ell} \right)^{-n} \\
\times \Phi_\ell \left( \begin{pmatrix} \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{pmatrix} \right) \cdot \text{char}(Z_p, \beta_{11}) \text{char}(Z_p, \beta_{12}) \text{char}(Z_p, \beta_{13}) \text{char}(Z_p, \beta_{23}) \text{char}(Z_p, \beta_{33})
\]

where \( \text{char}(Z_p, x) \) is the characteristic function for \( Z_p \) for the variable \( x \). This expression is clearly interpolated by an element in the Iwasawa algebra. The case for

\[
\left( \frac{\Omega_p\kappa_\infty}{\Omega_\infty^{2\kappa}} \right)^{-1} E'_{\text{sieg}}(f'_{\text{sieg}}; z, g)
\]

is similar. We finally remark there that the CM and \( p \)-adic periods will show up when pulling back the forms on \( U(3,3) \) (\( U(2,2) \)) to \( U(3,1) \times U(2) \) (\( U(2) \times U(2) \), respectively). 

This formal Fourier expansion gives measures on \( \Gamma_K \times Z_p \) with values in the space of \( p \)-adic automorphic forms on \( GU(3,3) \), which we denote as \( \mathcal{E}_{\mathcal{D},\text{sieg}} \) and \( \mathcal{E}'_{\mathcal{D},\text{sieg}} \), respectively.

We make an additional construction for interpolating Petersson inner products of forms on definite unitary groups. Write \( N^- \) for the lower triangular unipotent subgroup of \( \text{GL}_2 \). For a compact open subgroup \( K \) of \( U(2)({\mathbb{A}}_Q) \) which is \( U(2)({\mathbb{Z}}_p) \) at \( p \) we take \( \{g_i\} \), a set of representatives for \( U(2)(\mathbb{Q}) \backslash U(2)({\mathbb{A}}_Q) / K_0(p) \) (\( K_0(p) \) is obtained by replacing the \( p \)-component of \( K \) by the \( \Gamma_0(p) \) level subgroup). Suppose \( K = \prod_v K_v \) is sufficiently small so that for all \( i \) we have \( U(2)(\mathbb{Q}) \cap g_i K g_i^{-1} = 1 \).

Let \( \chi \) be the central character of \( f \) we write \( f^\vee \) for the family \( \pi(\begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix})(f \otimes \chi^{-1}(\det -)) \). For the Hida family \( f^\vee \) of eigenforms (with lower triangular level group at \( p \)) we construct a set of
bounded $R$-valued measure $\mu_i$ on $N^-(p\mathbb{Z}_p)$ as follows. Let $T^-$ be the set of elements $\begin{pmatrix} p^{t_1} \\ p^{t_2} \end{pmatrix}$ with $t_2 > t_1$. We only need to specify the measure for sets of the form $nt^{-1} N((p\mathbb{Z}_p)/(t^-)^{-1}$ where $n \in N^-(\mathbb{Z}_p)$ and $t^- \in T^-$. We assign $\mathbf{f}^\vee(g nt^-) \lambda(t^-)^{-1}$ where $\lambda(t^-)$ is the Hecke eigenvalue of $\mathbf{f}^\vee$ for $U_{t^-}$. This measure is well defined by the expression for Hecke operators $U_{t^-}$.

Proposition 7.4. If we define

$$\langle \mathbf{f}, \mathbf{f}^\vee \rangle := \sum_i \int_{n \in N^-(p\mathbb{Z}_p)} \mathbf{f}(gn) \, d\mu_i \in \mathbb{I}.$$ 

Then up to a constant depending only on the quaternion algebra $D$ and the level group $K$, we have for all $\phi \in \mathcal{X}^{gen}$ the specialization of $\langle \mathbf{f}, \mathbf{f}^\vee \rangle$ to $\phi$ is $\langle \mathbf{f}_\phi, \mathbf{f}^\vee_\phi \rangle \cdot \text{Vol}(K_\phi)^{-1}$ where $K_\phi = K_0(p^\phi)$ for $p^\phi$ the conductor at $\phi$.

Proof. For each $\phi \in \mathcal{X}^{gen}$, we choose $t^-$ such that $t^- N^-(p\mathbb{Z}_p)/(t^-)^{-1} \subseteq K_\phi$. We consider

$$\langle \mathbf{f}_\phi, \pi_{f_\phi}(t^-) \mathbf{f}^\vee_\phi \rangle.$$ 

Unfolding the definitions, note that $\chi^{-1}_\phi(t^-) \delta_B(t^-)$ gives the Hecke eigenvalue $\lambda(t^-)$ acting on $\mathbf{f}^\vee$. This gives $\text{Vol}(K_\phi) \cdot \delta_B(t^-) \chi^{-1}_\phi(t^-) \sum_i \int_{n \in N^-(p\mathbb{Z}_p)} \mathbf{f}(gn) \, d\mu_i$. On the other hand, using the model of $\pi_{f_\phi,p}$ and $\pi_{f^\vee_\phi,p}$ as the induced representation $\pi(\chi_1, \phi, \ldots, \chi_r, \phi)$ and $\pi(\chi^{-1}_1, \phi, \ldots, \chi^{-1}_r, \phi)$ of $\text{GL}_r(\mathbb{Q}_p)$, we get

$$\langle \mathbf{f}_\phi, \pi_{f_\phi}(t^-) \mathbf{f}^\vee_\phi \rangle = \delta_B(t^-) \chi^{-1}_\phi(t^-) \langle \mathbf{f}_\phi, \mathbf{f}^\vee_\phi \rangle.$$ 

This proves the specialization of $\langle \mathbf{f}, \mathbf{f}^\vee \rangle$ to $\phi$ is $\langle \mathbf{f}_\phi, \mathbf{f}^\vee_\phi \rangle \cdot \text{Vol}(K_\phi)^{-1}$. \hfill $\square$

The above set $\{\mu_i\}_i$ can be viewed as a measure on $U(\mathbb{Q}) \setminus U(\mathbb{A}_\mathbb{Q})$, which we denote as $\mu_f$.

### 7.4 $p$-adic $L$-functions

We have the following proposition for $p$-adic $L$-functions:

Proposition 7.5. Notations as before. There is an element $\mathcal{L}_{f,\xi,K}^{\Sigma} \in \hat{\mathbb{Q}}_p^\times$ and a $p$-integral element $C_{f,\xi,K}^{\Sigma} \in \hat{\mathbb{Q}}_p^\times$ such that for any generic arithmetic point $\phi$ we have:

$$\phi(\mathcal{L}_{f,\xi,K}^{\Sigma}) = C_{f,\xi,K}^{\Sigma} \frac{p^{(k-3)\frac{1}{2}}}{{\xi_1}_p \chi^{-1}_1 \chi^{-1}_2 p^{-1}(p-t)g(\xi_1, \chi^{-1}_1)g(\xi_2, \chi^{-1}_2)} \text{Vol}(\mathcal{K}, \pi_{f_\phi, \chi_\phi}, \chi_\phi, \chi^{-1}_2, \chi^{-1}_1, \chi^{-1}_2, \chi^{-1}_1) \cdot \text{Vol}(K_\phi)^{-1}.$$ 

Here $\chi_1, \chi_2$ is such that the unitary representation $\pi_{f_\phi} \simeq \pi(\chi_1, \chi_2)$ with $\text{val}_p(\chi_1(p)) = -\frac{1}{2}$, $\text{val}_p(\chi_2(p)) = \frac{1}{2}$.

Proof. If we are under the assumption of Theorem 1.1, take $g_0$ to be a point on the Igusa scheme for $\text{GU}(2)$ defined over $\hat{\mathbb{Q}}_p$ such that $\mathbf{f}(g_0)$ is a unit in $\hat{\mathbb{Q}}_p$ and take $h_0 \in \text{GU}(2)(\mathbb{A}_\mathbb{Q})$ such that $\mu(g_0) = \mu(h_0)$. It is noted in [16] Section 2.8 that

$$I_{\text{GU}(2)}(K_1^{\mathbb{Q}}) = \text{GU}(2)(\mathbb{Q}) \setminus \text{GU}(2)(\mathbb{A})/K_1^{\mathbb{Q}}.$$
We define $\mathcal{L}^\Sigma_{f,\xi,K}$ such that

$$\mathcal{L}^\Sigma_{f,\xi,K} = f(g_0)^{-1} \int_{g_2} e\nu_{g_0}(\gamma_{h_0}^{-1}(\mathcal{L}'_{D,\text{sieg}}(-\tau))\mathcal{T}(\det g_2))d\mu_{\pi(h_0)}f$$

Here $e\nu_{g_0}$ means evaluation at $g_0$. By $\gamma_{h_0}^{-1}(\mathcal{L}'_{D,\text{sieg}})$ we mean pullback a form on $\{(g_1, g_2) \in \text{GU}(2, 0) \times \text{GU}(0, 2), \mu(g_1) = \mu(g_2) = \mu(h_0)\}$. Let the character $\tau = \psi/\xi$. The function $\mathcal{T}(\det g_2)$ means the function taking value $\mathcal{T}(\det g_2)$ at the point $(g_1, g_2)$ in the above set. The integration is in the sense of subsection 7.3 with respect to the local group $h_0^{-1}(K_D \cap (1 \times \text{GU}(2)(\mathbb{A}_f))h_0$. (In fact by pullback we get a measure of forms on the $h_0$-Igusa schemes, see the last part of subsection 3.5). This $\mathcal{L}^\Sigma_{f,\xi,K}$ satisfies the proposition.

If we are under assumption of Theorem 1.2 then we just pick up a $g_0$ such that the specialization of $f$ at $g_0$ has non-zero specialization at $g_0$. Note that the period factor $\Omega_{2\kappa}$ and $\Omega_p^{2\kappa}$ comes from the pullback as discussed in [16, 2.8].

**Definition 7.6.** Now we define Hida’s $p$-adic $L$-function $\mathcal{L}^{\text{Hida}}_{f,\xi,K}$. First take the $p$-adic $L$-function constructed in [14, Theorem 1] choosing $f$ there to be the Hida family of nearly ordinary CM eigen forms $g_\xi$ associated to $\xi$ and $g$ there to be the nearly ordinary eigenforms of our $f$. The period factors in Hida’s construction are the Petersson inner product of $g_\xi$’s instead of the CM period $\Omega_{\infty}$. There is a Katz $p$-adic $L$-function $\mathcal{L}^{\text{Katz}}_{K,\xi} \in \hat{O}_L[[\Gamma_K]]$ interpolating near central $L$-values of $\xi^{1-c}$ twisted by characters of $\Gamma_K^{-}$ (i.e. it interpolates the values

$$\xi_{\phi,1,p}(p)\frac{p^{\lambda}(\xi_{\phi,1,c}^{-1-c}, 1)\Omega_{2\kappa}}{G(\xi_{\phi,1,c}^{-1-c})\Omega_{\infty}}$$

where $\xi_\phi$ is $\xi$ multiplied by some finite order character of $\Gamma_K$ of conductor $p^j$). We have $\text{Cl}_K \cdot \mathcal{L}^{\text{Katz}}_{K,\xi}$ interpolates the ratio of the Petersson inner product of $g_\xi$’s and $\Omega$ (Cl$_K$ is the class number for $K$. See e.g. [21, Section 7] for a detailed discussion). Then we multiply Hida’s $p$-adic $L$-function by $\text{Cl}_K \cdot \mathcal{L}^{\text{Katz}}_{K,\xi}$ and denote this new $p$-adic $L$-function as $\mathcal{L}^{\text{Hida}}_{f,\xi,K}$. The interpolation formula for it is

$$\phi(\mathcal{L}^{\text{Hida}}_{f,\xi,K}) = p^{(\kappa-3)\xi_1,1,p\chi_1^{-1}\chi_2^{-1}(p^{-1})g(\xi_1,1,p\chi_1^{-1})g(\xi_1,1,p\chi_2^{-1})L(\kappa, \pi_{f_0}, \bar{\chi}_0\xi_\phi, \frac{\kappa}{2} - \frac{1}{2})(\kappa - 1)!(\kappa - 2)!\Omega_{2\kappa}.$$

(Here we removed the product of prime to p root numbers in loc.cit which are $p$-units and moves $p$-adic analytically.) If $\xi$ is such that $g_\xi$ satisfies the (dist) and (irred) in the introduction, then the local Hecke algebra for $g_\xi$ is Gorenstein. By the main conjecture proved in [21], [22] and [13] (see the discussion after the Conjecture in [21, page 192], the $\text{Cl}_K \cdot \mathcal{L}^{\text{Katz}}_{K,\xi}$ generates the congruence module for $g$ and our $\mathcal{L}^{\text{Hida}}_{f,\xi,K}$ is integral (i.e. in $\Lambda_D$).

Given a finite set of primes $\Sigma$ we can define the $\Sigma$-primitive Hida $p$-adic $L$-function $\mathcal{L}^{\Sigma,\text{Hida}}_{f,\xi,K}$ by removing local Euler factors at $\Sigma$. Note that Hida proved the interpolation formula for general arithmetic points. We may compare [6] [7]. If we write $\mathcal{L}^{\Sigma}_{f_0,\xi,K}$ for the specialization of $\mathcal{L}^{\Sigma}_{f,\xi,K}$ to $\hat{O}_L[[\Gamma_K]]$ at $f_0$, then we get the interpolation formula

$$\phi(\mathcal{L}^{\Sigma}_{f_0,\xi,K}) = C_{\mathcal{L}^{\Sigma}_{f,\xi,K}} \frac{p^{(\kappa-3)\xi_1,1,p\chi_1^{-1}\chi_2^{-1}(p^{-1})g(\xi_1,1,p\chi_1^{-1})g(\xi_1,1,p\chi_2^{-1})L(\kappa, \pi_{f_0}, \bar{\chi}_0\xi_\phi, \frac{\kappa}{2} - \frac{1}{2})(\kappa - 1)!(\kappa - 2)!\Omega_{2\kappa}}}{(2\pi i)^{2\kappa-1}\Omega_{\infty}}.$$

(8)
Anti-cyclotomic $\mu$-invariants:

Now assume we are under assumption of Theorem 1.1 in the introduction. We define $\phi_0$ to be the arithmetic point in $\text{Spec}[[\Gamma_K]]$ sending $\gamma^\pm$ to 1 and such that $\phi_0[1]$ correspond to $f_0$. Our assumptions on $\xi$ and $\kappa$ ensures that it is in the component of our $p$-adic $L$-function. (This is not an arithmetic point by our definition, however it still interpolates the algebraic part of the special $L$-value by [11].) Consider the 1-dimensional subspace of of $\text{Spec}[[\Gamma_K]]$ of anti-cyclotomic twists by characters of order and conductor powers of $p$ that passes through $\phi_0$. The specialization of Hida’s $p$-adic $L$-function to this subspace is nothing but the anti-cyclotomic $p$-adic $L$-function considered by [18]. Thus the anti-cyclotomic $p$-adic $L$-function has $\mu$-invariant 0. As in [45] 12.3 the corresponding $\mu$-invariant for the $\Sigma$-primitive $p$-adic $L$-function is 0 as well. Thus it is easy to see that any height 1 prime $P$ of $[[\Gamma_K]]$ containing $L_{\Sigma,Hida}^{\xi}$ can not be the pullback of a height 1 prime of $[[\Gamma_K]]$. Therefore for any height 1 prime containing $L_{\Sigma,Hida}^{\xi}$,

$$\text{ord}_P(L_{\Sigma,Hida}^{\xi}) = \text{ord}_P(L_{\Sigma,Hida}^{\xi})$$

and $\text{ord}_P(L_{\chi,t\xi}^{\Sigma}) = 0$.

7.5 $p$-adic Eisenstein Series

**Proposition 7.7.** There is a $[[\Gamma_K]]$-adic formal Fourier-Jacobi expansion $E_{\text{D},\text{Kl}}$ such that for each generic arithmetic point $\phi \in \text{Spec}[[\Gamma_K]]$, the specialization $E_{\text{D},\text{Kl},\phi}$ is the Fourier-Jacobi expansion of the nearly ordinary Klingen Eisenstein series $E_{\text{Kl}}$ we constructed in section 7 using the Eisenstein datum at $\phi$. Moreover, the constant terms are divisible by $L_{\xi,K}^\Sigma L_{\chi,\xi}$. Where $L_{\chi,\xi}$ is the element in $[[\Gamma_K]]$ which is the Dirichlet $p$-adic $L$-function interpolating the algebraic part of the special values $L^\Sigma(\chi_{\phi}^\xi,\kappa_{\phi} - 2)$.

**Proof.** It is a special case of [48] Theorem 1.1 (3). In our cases the local choices are slightly different but the arguments are the same. We take a basis $(\theta'_1, \cdots, \theta'_m)$ of the $\Omega_L$-dual space of $H^0(B, L(\beta))$ consisting of theta functions. Suppose $\theta$ is one of the $\theta'_i$s. For any $g \in \text{GU}(2)(\Lambda_Q)$ we take $h \in \text{GU}(2)(\Lambda_Q)$ such that $\mu(g) = \mu(h)$. We define

$$FJ_{\beta,g,\theta}(E_{\text{D},\text{Kl}}) := \int_{g_2} \text{ev}_g l\theta(\gamma^{-1}_h(E_{\text{D},\text{Kl}})(-\Upsilon)) \tau(\det g_2) d\mu_{\Upsilon}(h) \varphi.$$  

Here by $\gamma^{-1}_h(E_{\text{D},\text{Kl}})$ we mean the form on $\{ (g_1, g_2) \in \text{GU}(3,1) \times \text{GU}(0,2), \mu(g_1) = \mu(g_2) = \mu(h) \}$. The integration is in the sense of subsection 7.3 with respect to the level group $h^{-1}(K_D \cap (1 \times \text{GU}(2)(\Lambda_f))h$ (In fact by pullback we get a measure of forms on the $h$-Igusa schemes).

The family $\varphi$ will be chosen in the last paragraph of Section 8.4. It follows that the formal Fourier-Jacobi expansion $E_{\text{D},\text{Kl}}$ comes from a family in $\mathcal{M}^{\text{ord}}(K^p\Lambda_D)$, which we still denote as $E_{\text{D},\text{Kl}}$. (In fact Theorem 8.8 is still true after replacing $A = [[\Gamma_K]]$ by $[[\Gamma_K]]$.)

8 $p$-adic Properties of Fourier-Jacobi Coefficients

The purpose of this section is to prove proposition 8.25.
8.1 Preliminaries

Some Local Representation Theory

Let \( v \in \Sigma \) then \( D_v^\times \cong \text{GL}_2(\mathbb{Q}_v) \). For some irreducible admissible representation \( \pi^{U(2)_v} \) of \( U(2)_v \) we can find a representation \( \pi^{GU(2)_v} \) of \( GU(2)_v \) such that \( \pi^{U(2)_v} \) is a summand of \( \pi^{GU(2)_v} \) restricting to \( U(2)_v \). The cases that we are interested in are those such that the restriction \( \pi^{GU(2)_v}|_{D_v^{\times}} \) is supercuspidal of level at least \( \varepsilon_v^\times \) and higher than the conductor of the central character of it. In this case the \( SL_2 \) \( L \)-packet in [30] has two elements. Thus by the discussion in [30] for the local \( L \)-packets for \( SL_2 \), we have:

\[
\pi^{GU(2)_v}|_{U(2)_v} = \pi^{U(2)_v} \oplus \alpha\pi^{U(2)_v}.
\]

for irreducible representations \( \pi^{U(2)_v}(\mathbb{Q}_v) \) of \( U(2)(\mathbb{Q}_v) \). Here \( \alpha = \left( \begin{array}{c} \overline{\epsilon} \\ 1 \end{array} \right) \) or \( \alpha = \left( \begin{array}{c} \epsilon \\ 1 \end{array} \right) \) for some \( \epsilon_v \in \mathbb{Z}_v^\times / (\mathbb{Z}_v^\times)^2 \) depending on whether it is an unramified or ramified supercuspidal representation in the sense of [30]. The \( \alpha \) means the representation composed with the automorphism given by conjugation by \( \alpha \). Also the restriction of \( \pi^{GU(2)_v} \) to \( D_v^{\times} \) is clearly irreducible. There is a newform \( v_{\text{new}} \) (up to scalar) of \( \pi^{D_v^{\times}} \). Note that the conditions on the conductors there are satisfied. We define the “new vector in our sense” \( v \in \pi^{U(2)_v} \) (up to scalar) to be the new vector of \( \pi^{U(2)_v}|_{SL_2(\mathbb{Q}_v)} \).

Thus \( v_{\text{new}} = \pi(\alpha)v \) or \( v_{\text{new}} = v + \pi(\alpha)v \) (see [30, Proposition 3.3.3, 3.3.7]).

If \( D_v^{\times} \) modulo center is compact then we let \( \alpha \) be some element such that \( \text{Nm}(\alpha) \notin \text{Nm}(\mathcal{K}_v/\mathbb{Q}_v) \). For \( \pi^{U(2)_v} \) we similarly have \( \pi^{GU(2)_v} \subseteq \pi^{D_v^{\times}} \). These can all be considered as finite dimensional representations of finite groups. \( \pi^{GU(2)_v} = \pi^{D_v^{\times}} \) as vector spaces and \( \pi^{GU(2)_v}|_{U(2)_v} = \pi^{U(2)_v} \) or \( \pi^{U(2)_v} \oplus \alpha\pi^{U(2)_v} \).

Forms on \( D^{\times} \) and \( U(2) \)

We first define \( D^{\times}(\mathbb{A}_\mathbb{Q}) \subset D^{\times}(\mathbb{A}_\mathbb{Q}) \) as the index 2 subgroup consisting of elements whose determinants are in \( \mathbb{Q}^{\times}\text{Nm}(\mathbb{A}_\mathbb{Q}) \) and \( D^{\times}(\mathbb{Q}_v) \) be the set of elements whose determinant is in \( \text{Nm}(\mathbb{Q}_v) \). Suppose \( \varphi \) is a form on \( U(2)(\mathbb{Q}) \setminus U(2)(\mathbb{A}_\mathbb{Q}) \), \( \chi \) is a Hecke character of \( \mathcal{K}^{\times} \setminus \mathbb{A}_\mathbb{Q}^{\times} \). Suppose the central action of \( U(1)(\mathbb{Z}_p) \) on \( \varphi \) is given by \( \chi|_{U(1)(\mathbb{Z}_p)} \), we can define a form \( \varphi^D_\chi \) on \( D^{\times}(\mathbb{A}_\mathbb{Q}) \) as follows. We first define \( \varphi^\chi_\chi \) on \( U(2)(\mathbb{A}_\mathbb{Q}) \) such that, consider the action of the center \( U(1)(\mathbb{A}_\mathbb{Q}) \) of \( U(2)(\mathbb{A}_\mathbb{Q}) \), \( \varphi^\chi_\chi \) is the \( \chi \)-eigen part of \( \varphi \) under this action. Let \( C \) be the cardinality of

\[
U(1)(\mathbb{Q}) \setminus U(1)(\mathbb{A}_\mathbb{Q})/U(1)(\mathbb{Z}_p) \times (\text{prime to } p \text{ level of } \chi).
\]

We define a form on \( D^{\times}(\mathbb{A}_\mathbb{Q}) \). First for \( g \) such that \( \text{det} \ g \) is in the image of \( \text{Nm}(\mathbb{A}_\mathbb{Q}^{\times}/\mathbb{A}_\mathbb{Q}^{\times}) \) then we can write \( g = ag' \) where \( a \in \mathbb{A}_\mathbb{Q}^{\times} \) and \( g' \in U(2)(\mathbb{A}_\mathbb{Q}) \). Define \( \varphi^D_\chi(g) = \chi(a)C\varphi^\chi_\chi(g) \). For general \( g \in D(\mathbb{A}_\mathbb{Q}) \) we can find a central element \( b \in D^{\times}(\mathbb{Q}) \) such that \( \text{det}(b^{-1}g) \in \text{Nm}(\mathbb{A}_\mathbb{Q}^{\times}/\mathbb{A}_\mathbb{Q}^{\times}) \), we define \( \varphi^D_\chi(g) := \varphi^D_\chi(b^{-1}g) \). Note that this is well defined since \( \mathbb{Q}^{\times} \cap \text{Nm}(\mathbb{A}_\mathbb{Q}^{\times}/\mathbb{A}_\mathbb{Q}^{\times}) = \text{Nm}(\mathcal{K}^{\times}/\mathbb{Q}^{\times}) \). For \( g \) outside \( D^{\times}(\mathbb{A}_\mathbb{Q}) \) we define \( \varphi^D_\chi(g) = 0 \). When \( \chi \) is clear from the context we simply drop the subscript \( \chi \).

**Lemma 8.1.** If \( \varphi \) is in the irreducible automorphic representation on \( U(2) \) whose restriction to \( SL_2 \) is the restriction of the \( GL_2 \) automorphic representation \( \pi_\xi \) associated to a CM character \( \xi \) over \( \mathcal{K} \). Then \( \varphi^D_\xi \) itself is in \( \pi_\xi \).
Proof. Clearly the $\varphi^D_\ell$ and $\pi_\ell$ have the same Hecke eigenvalues outside a finite set of primes: this is obvious for split $v$’s. For inert $v$’s the Hecke eigenvalues for $T_v$ on $\varphi_D$ is 0 since it is a CM form. 

We relate the integrals over $[U(2)]$ to that over a subset of $[Q^\times \backslash D^\times](A_Q)$. It is elementary to check that there is a constant $C^D_{U(2)}$ depending only on the groups $D^\times$ and $U(2)$ such that if $\chi = 1$ then:

$$
\int_{[U(2)]} \varphi_{U(2)}(g) dg = C^D_{U(2)} \int_{D^\times(Q)\backslash D^\times(A_Q)} \varphi_\chi^D(g) dg.
$$

Here we normalize the Haar measure so that the measure $U(1)\backslash [U(2)] = 1$ and the measure of $[D^\times]$ modulo center is also 1. 

8.2 Choosing some characters

We first give a result of Pin-Chi Hung (a student of M-L Hsieh) [23]. Let $\chi$ be a finite order Hecke character of $K^\times \backslash A_K^\times$ of conductor $Z = MO_K$ for some $M > 0$. Let $f \in S_k(\Gamma_0(N))$ be an elliptic cusp form of weight $k$, level $\Gamma_0(N)$ with $q$-expansion

$$
f(q) = \sum_{n\geq 0} a_n(f)q^n.
$$

We decompose $N = N^+N^-$, where $N^+$ is a product of primes split in $K$ and $N^-$ is a product of primes ramified or inert in $K$. Suppose $N^-$ is square-free and $N^- = N_f^-N_C^-$ where $N_f^-$ is a product of an odd number of primes coprime to $M$ and $N_C^-$ is a divisor of $M$. Let $\ell$ be a rational prime split in $K$. Let $\mathcal{K}_\ell^-$ be the unique abelian anticyclotomic $\mathbb{Z}_\ell$-extension of $K$ and $\Gamma^-$ be the Galois group $\text{Gal}(\mathcal{K}_\ell^-/K)$.

Theorem 8.2. Suppose $\ell^2 \nmid N$. Let $p$ be a rational prime such that

- $p \nmid \ell ND_K$ and $p \geq k - 2$,
- for every non-split $q | M$, $q + 1$ is not divisible by $p$,
- for every $q | N_f^-$ ramified in $K$, $a_q(f) = \chi(q)(= \pm 1)$, where $q = q^2$,
- the residual Galois representation $\bar{\rho}_{f,\lambda}|_{\text{Gal}(Q/K)}$ is absolutely irreducible.

Then there is a finite extension $L/Q_p$ with integer ring $\mathcal{O}_L$ and uniformizer $\lambda$. We have for all but finitely many characters $\nu : \Gamma^- \to \mu_{\ell^\infty}$, we have

$$
\frac{L(f/K, \chi_{\nu, \frac{k}{2}})}{\Omega_{f,N^-}^{\ell, \nu}} \neq 0(\text{mod } \lambda).
$$

Here the $\Omega_{f,N^-}$ is a period factor defined in loc. cit.

Now we choose the characters needed. From now on we fix once for all a split prime $\ell$ outside $\Sigma$ and write a new “$\Sigma$” for $\Sigma \cup \{\ell\}$. We choose $\chi_\theta$ a Hecke character of $K^\times \backslash A_K^\times$ as follows: $\chi_{\theta, \infty}$ is trivial. At $p$ we require that $\chi$ is unramified. For $v \in \Sigma$ non-split in $K/Q$, then we let $\chi_{\theta,v}|_{U(1)}$ to be the character chosen in section 6. For split $v \in \Sigma, v \nmid p, \ell, v = w\bar{w}$, suppose $\text{cond}(\pi_v) = (\pi_v^{t_1,v})$, we require that $\chi_{\theta,w}$ is unramified and $\text{cond}(\chi_{\theta,w}) = (\pi_v^{t_2,v})$ for $t_{2,v} \geq 2t_{1,v} + 2$. We choose a character
\( \chi_{\text{aux}} \) of \( \mathcal{K}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \) as follows: \( \chi_{\text{aux}}|_{\mathbb{A}_\mathbb{Q}^\times} = 1 \), it is trivial at \( \infty \), and is only ramified at primes in \( \Sigma \) not dividing \( p \) such that \( \U(2)(\mathbb{Q}_v) \) is not compact. For split such \( v \) we require that

\[
\text{cond}(\chi_{\text{aux}}) = \begin{cases} 
(\varpi_v^{t_{v,u} - t_{1,u}}) & \text{if } t_{1,v} \neq 0 \\
(\varpi_v^{t_{v,u} - 1}) & \text{if } t_{1,v} = 0 
\end{cases}
\]

At non-split such primes we require that

\[
\text{cond}(\chi_{\text{aux},v}) > \text{cond}(\pi_v), \text{cond}_v(\lambda^2 \chi_\theta \chi_{\text{aux}}) > \text{cond}(\pi_v)
\]

and \( \chi_{\text{aux},v}|_{\mathbb{Q}_v^\times} \) has smaller conductor than \( \chi_{\text{aux},v} \) (here \( > \) means the conductor of the former is of higher power of the uniformizer than the latter). Also for each prime \( q \) such that \( \U(2)(\mathbb{Q}_q) \) is compact and \( q \) is ramified as \( w^2 \) in \( \mathcal{K} \), suppose \( \pi_q \simeq \text{steinberg} \otimes \chi_{q,1} \) for some unramified quadratic character \( \chi_{q,1} \) we require that \( \chi_{q,1}(q) = \chi_{\text{aux}}(\pi_w) \) (These are used in the next paragraph to make sure that the special \( L \)-values are of the correct local signs when applying Theorem 8.2). Let \( \chi_h = \chi_\theta \chi_{\text{aux}} \).

We further require that \( \frac{L(\pi_f, \chi_\theta \chi_h, \frac{1}{2})}{\pi^{1/2} \Omega_f} \text{Eul}_p(\pi_f, \lambda^2 \chi_\theta \chi_h, \frac{1}{2}) \), \( \frac{\Gamma(\kappa - 1)L(\chi_{\text{aux}}, \lambda^{-c}, \kappa, \frac{1}{2})}{\Omega_f} \text{Eul}_p(\chi_{\text{aux}}, \lambda^{-c}, \kappa, \frac{1}{2}) \) and \( \frac{\Gamma(\kappa - 2)L(\chi_\theta \chi_{\text{aux}}, \lambda^{-c}, \kappa, \frac{1}{2})}{\Omega_f} \text{Eul}_p(\lambda^2 \chi_\theta \chi_{\text{aux}}, \lambda^{-c}, \kappa, \frac{1}{2}) \) are \( p \)-adic units where the \( \text{Eul}_p \) are the local Euler factors for the corresponding \( p \)-adic \( L \)-functions at \( p \) when everything is unramified at \( p \) (we refer to [18 (0.2)] [20 4.16] for their precise definitions). The first uses [18]. Our assumptions above on conductors imply that at all non-split primes the local root numbers in Theorem A of loc.cit are all \( +1 \). Then we take a split prime \( \ell \nmid Np \) and apply that theorem to see that there exists a twist by anticyclotomic character of \( \ell \)-power conductor which satisfies the requirements. The second and third uses [17] (we are in his residually non self-dual case) and again we can achieve the requirements by twisting by an appropriate anti-cyclotomic character of conductor powers of \( \ell \). Moreover we assume that \( 1 - a_p(f)^{-1} \chi_{\theta, p, 2}\chi_{h, p, 1}(p), 1 - a_p(f)\chi_{\theta, p, 1}\chi_{h, p, 2}(p)^{-1} \) and \( 1 - \chi_{h, p, 2}\chi_{h, p, 2}\chi_{\theta, p, 2}\chi_{h, p, 2}(p)^{-1} \) are \( p \)-adic units. At each prime \( v \) of \( \mathcal{K} \) above a prime where \( \U(2)(\mathbb{Q}_v) \) is compact, we require \( 1 - \chi_{\text{aux}} \chi_{h}^{-c}(q_v)q_v^{-\frac{c}{2}} \) is a \( p \)-adic unit. We also require that \( \frac{L(\pi_f, \chi_\theta \chi_h, \frac{1}{2})}{\pi^{1/2} \Omega_f} \text{Eul}_p(\chi_{\text{aux}}, \lambda^{-c}, \kappa, \frac{1}{2}) \) is non-zero (do not need to be non-zero modulo \( p! \)) using the Theorem 8.2 recalled above (by choosing a different “\( p \)” and prove non-vanishing the new \( p \)).

Now we define some characters \( \theta \) of \( \mathbb{Q}_v^\times \backslash \mathbb{A}_\mathbb{Q}^\times \) (the reason for doing so is just a cheap way to use the newform theory at split primes to pick different vectors inside an antomorphic representation of \( \U(2) \)). We require that these are ramified only at split primes in \( \Sigma \backslash \{p\} \). At such \( v = wv \) and require that \( \theta|_{\mathbb{Z}_v^\times} = \chi_h|_{\mathcal{O}_{\mathcal{K}, w}} \). These uniquely determine the character \( \theta \).

### 8.3 Constructing Auxiliary Families of Theta Functions

From now on we usually do the computations at a generic arithmetic point \( \phi \), but drop the subscript \( \phi \) for simplicity. Define \( \Lambda \)-adic families of characters \( \chi_\theta \) and \( \chi_h \) as follows: let \( \chi_\theta \) be such that the specialization to \( \phi_0 \) (\( \phi_0 \) is defined in section 7.4) is \( \chi_\theta \) defined before and the specialization to \( \phi \) satisfies \( (\chi_\theta)_{\phi, p, 1}|_{\mathbb{Z}_p^\times} = 1 \) and \( (\chi_\theta)_{\phi, p, 2}|_{\mathbb{Z}_p^\times} = \chi_\theta|_{\mathbb{Z}_p^\times} \). Let \( \chi_h \) be such that the specialization to \( \phi_0 \) is \( \chi_h \) and \( (\chi_h)_{\phi, p, 2}|_{\mathbb{Z}_p^\times} = 1 \), \( (\chi_h)_{\phi, p, 1}|_{\mathbb{Z}_p^\times} = \chi_h^{-1}|_{\mathbb{Z}_p^\times} \). So from now on our \( \chi_\theta \) and \( \chi_h \) might
be ramified at $p$.

Rallis Inner Product Formula

$$
\begin{array}{c}
U(1,1)(\omega_{\lambda^2}) & U(2)(\omega_{\lambda}) \times U(2)(\omega_{\lambda}) \\
U(1)(\omega_{\lambda^2}) \times U(1)(\omega_{\lambda^2}) & U(2)(\omega_{\lambda^2}) \\
U(1)(\omega_{\lambda^2}) & U(2)(\omega_{\lambda^2})
\end{array}
$$

We will use the results in [15] freely. We consider the seesaw pair above. The $U(2)$ above is for the Hermitian matrix $\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}$ and the $U(1)$'s are for the skew-Hermitian matrices $\delta$ and $-\delta$. The embedding $U(1) \times U(1) \hookrightarrow U(1,1)$ are given by the $i$ defined in the proof of lemma 6.21. In fact the theta functions showing up in the Fourier-Jacobi expansions arise from theta liftings from $U(V_1)$ to $U(\zeta)$ and $U(-\zeta)$. The Weil representations are the same as explained in [15] if we identify $U(V_1)$ with $U(1)$ above and $U(\pm \zeta)$ with $U(2)$ above. The splittings used are indicated in the bracket beside the groups. We want to consider the component of theta correspondence such that the first $U(1)$ on the lower left corner acts by $\lambda^2\chi_\theta$ and the second $U(1)$ acts by $\chi_{-1}\chi_\theta$. We consider a theta function on $U(2,2)$ by the dual reductive pair $U(2,2) \times U(V_1)$ and some Schwartz function $\phi$ such that $\phi = \delta_{\psi}(\phi_3 \boxtimes \phi_2)$ for some $\phi_3$ and $\phi_2$ (recall the notion of intertwining operators in section 4). Here $V_1$ is the 1-dimensional hermitian space where $\langle x, y \rangle = \overline{xy}$. The characters used in the splittings are: $\lambda$ for $U(2,2)$ and 1 for $V_1$. We consider:

$$
\int_{[U(2)]} \int_{[U(1)] \times [U(1)]} \theta_\phi(u_1, u_2, g) \lambda^{-2} \chi_{-1}(u_1) \chi_\theta(u_2) \tilde{\lambda}(\det g) du_1 du_2 dg
$$

Here $\tilde{\lambda}(\det g)$ showing up is due to the splitting $\omega_{\lambda^2}$ on $U(2)$. On one hand, one can check that this is nothing but the inner product of the theta liftings $\theta_{\phi_3, \lambda}(\lambda \chi_\theta)$ and $\theta_{\phi_2, \lambda}(\lambda^{-1} \chi_{-1}) \cdot (\lambda \circ \det)$ (by writing $\theta_{\phi, \lambda}$ we take the splitting character for $U(1)$ is trivial and for $U(2)$ is $\lambda$. (We need to notice the different choices of splitting characters). On the other hand if we change the order of integration using the Siegel-Weil formula for $U(1,1) \times U(2)$ as proved by Ichino ([25]). This equals:

$$
\int_{[U(1)] \times [U(1)]} E(f_{\delta_{\psi}(\phi)}, \{ \frac{1}{2}, i(u_1, u_2) \lambda^{-2} \chi_{-1}(u_1) \chi_\theta(u_2) du_1 du_2
$$

Here $i$ is defined right before lemma 6.22 and $f_{\delta_{\psi}(\phi)}$ is the Siegel section defined by:

$$
f(g) := \omega_{\lambda^2}(j(g)) \delta_{\psi}(\phi)(0), g \in U(1,1)
$$

where $j$ is defined in the proof of lemma 6.21 Thus we reduced the Petersson inner product of theta liftings to the pullback formula of Siegel Eisenstein series on $U(1,1)$.

Functorial Properties of Theta Liftings

For any Hecke character $\chi$ of $U(1)$ (in application $\chi(z_\infty) = z_\infty^{\pm 1}$ for $z_\infty \in U(\mathbb{R})$), we describe the $L$-packet of theta correspondence $\Theta_\lambda(\chi)$ (possibly zero) of $\chi$ to $U(2)$ where $\lambda$ is a Hecke character of $\mathbb{A}_K^\times$ such that $\lambda|_{\mathbb{A}_Q^\times} = \omega_K/Q$. We pick a Hecke character $\tilde{\chi}$ such that $\tilde{\chi}|_{U(1)(\mathbb{A}_Q)} = \chi^{-1}$. Let
\( \pi_\chi \) be the Jacquet-Langlands correspondence on \( D^\times \) of the automorphic representation of \( GL_2/\mathbb{Q} \) generated by the CM form \( \theta_\chi \). We form an automorphic representation in the way we introduced before: \( \pi_\chi \cong \chi^{-1} \lambda \) of \( GU(2) \). Then by looking at the local \( L \)-packets (see [15, section 7]) \( \theta_\lambda(\chi) \) is a subspace of the restriction of this representation to \( U(2) \). (This restriction is not necessarily irreducible. In fact for any \( v \) inert in \( K \) the local representation at \( v \) splits into 2 irreducible pieces. see [30]. The representation at split primes are irreducible. So we still have not specified the automorphic representation on \( U(2) \).)

Constructing Families of Theta Liftings

Let \( v \) be a prime inert or ramified in \( K \). Thanks to the recent work [7] we know Howe duality conjecture is true for any characteristic. Consider the theta lifting from \( U(1) \) to \( U(2) \) at \( v \) (the \( U(2)(\mathbb{Q}_v) \) might be \( U(1,1) \) or compact). Write \( S(X_v, \chi_{\theta,v}^{-1}) \) for the summand of \( S(X_v) \) such that \( U(1) \) acts by \( \chi_{\theta,v}^{-1} \). Given a theta kernel \( \phi^v \) on \( \otimes_{w \neq v} S(X_w) \) we consider the map \( S(X_v, \chi_{\theta,v}^{-1}) \to \pi_\theta \) by \( \iota_v : \phi_v \to \Theta_{\phi^v, \phi_v}(\chi_{\theta,v}^{-1}) \). By the Howe duality conjecture we know that there is a maximal proper subrepresentation \( V_v \) of \( S(X_v, \chi_{\theta,v}^{-1}) \) such that \( S(X_v, \chi_{\theta,v}^{-1})/V_v \) is irreducible and isomorphic to the local theta correspondence \( \pi_{\theta,v} \) of \( \chi_{\theta,v}^{-1} \) by the local and global compatibility of theta lifting (see [38, Theorem 8.5]). Suppose there is some \( \phi_v \) so that \( \iota_v(\phi_v) \neq 0 \), it is a finite sum of pure tensors in \( \pi_\theta \). We consider the representation of \( U(2)(\mathbb{Q}_v) / \pi_\theta(U(2)(\mathbb{Q}_v)) \), \( \iota(\phi_v) \), \( \iota(\phi_v) \to \otimes \pi_{\theta,v} \). Note that the automorphism group of the representation \( \pi_{\theta,v} \) consists of scalar multiplications. Thus it is easy to see that the kernel of the above embedding is exactly \( V_v \) and we have the following lemma:

**Lemma 8.3.** Fix the \( \phi^v \) as above. Let \( \nu_{\phi_v} \) be the image of \( \phi_v \) in \( \pi_{\theta,v} \) under the Howe duality isomorphism to \( S(X_v, \chi_{\theta,v}^{-1})/V_v \). Then \( \iota_v(\phi_v) \in \pi_\theta \) can be written as a finite sum of pure tensors of the form

\[
\nu_{\phi_v} \otimes \left( \sum_i \prod_{w \neq v} \phi_{w,i} \right)
\]

for \( \phi_{w,i} \in \pi_w \).

**Proposition 8.4.** We can construct a \( \Lambda_2 \otimes \mathbb{Z}_p \mathbb{Q}_p \)-adic family of theta functions on \( U(2) \) interpolating the forms

\[
\theta(g) := \sum_{i=1}^{k_k} \xi_\theta(\tilde{u}_i)^{-1} \frac{\theta_2(g\tilde{u}_j)}{\Omega_K}
\]

where \( \theta_2 \) is the one appearing in Corollary 6.33.

**Proof.** We are going to construct \( p \)-adic families of eigenforms which are theta functions of \( U(2) \) as subgroups of \( U(2,2) \) under \( U(2) \times 1 \hookrightarrow U(2,2) \) and \( 1 \times U(2) \hookrightarrow U(2,2) \), using the theta liftings from \( U(1) \) to \( U(2) \). For an eigenform \( \theta \) such constructed we sometimes write \( \pi_\theta \) for the automorphic representation of \( U(2) \) of \( \theta \). First we discuss the choices for the Schwartz function \( \phi \) for the construction using the embedding \( 1 \times U(2) \hookrightarrow U(2,2) \).

Local Computations
Case 0:
At finite places outside \( \Sigma \) we choose the obvious spherical kernel functions.

Case 1:
If \( v = \infty \), we let \( \phi_v = \omega_\lambda(\eta g_0)\Phi_\infty \). Recall that \( g_0 = \text{diag}(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^{-1} \) and \( \Phi_\infty = e^{-2\pi \text{Tr}(x,x)_{11}} \). Let \( \phi_{3,v} = \frac{\eta d_3}{2} e^{-2\pi \sqrt{a_3(s_3^2 + s_3^2)}} \) and \( \phi_{2,v} = \frac{\eta d_2}{2} e^{-2\pi \sqrt{a_2(s_2^2 + s_2^2)}} \). By our computation in section 6.5 we have: \( \delta(\phi_{3,v} \boxtimes \phi_{2,v}) = \phi_v \).

Case 2:
If \( v \in S \) is split and \( v \nmid p \) we recall that we have two different polarizations \( W = X_v \oplus Y_v = X'_v \oplus Y'_v \) where the first one is globally defined which we use to define theta function and the second is defined using \( K_v \equiv \mathbb{Q}_v \times \mathbb{Q}_v \) which is more convenient for computing the actions of level groups. We have defined intertwining operators \( \delta'_\psi \) between \( S(X_v) \) and \( S(X'_v) \) intertwining the corresponding Weil representations. Consider the theta correspondence of \( U(1) \to U(2) \) on \( S(X_v) \) and \( S(X'_v) \). We write \( X_v \ni x'_3 \equiv (x'''_3, x''_3, x''_2, x''''_2) \) and \( X'_v \ni x'_2 \equiv (x''_2, x'''_2) \). We define
\[
\phi''_3, v(x'''_3, x''_3, x''_2, x''''_2) = \left\{ \begin{array}{cl}
(\lambda \chi_\theta)_v^{-1}(x''_3, v) & x'''_3 \in \mathbb{Z}_v^*, x''_3, v \in \omega^*_v \mathbb{Z}_v \\
0 & \text{otherwise.}
\end{array} \right.
\]
where \( \omega^*_v \) is the conductor of \( \chi_\theta, v \) and
\[
\phi''_2, v(x''_3, x''_2, x''''_2) = \left\{ \begin{array}{cl}
(\lambda \chi_\theta)_v(x''_3, v) & x''_3 \in \mathbb{Z}_v^*, x''_2, v \in \mathbb{Z}_v \\
0 & \text{otherwise.}
\end{array} \right.
\]
We define \( \phi_v \in S(\mathbb{W}_v^d) \) by \( \phi_v = \delta'_\psi(\phi_{3,v} \boxtimes \phi_{2,v}) \) and define \( \phi_{2,v} = \delta''_\psi(\phi''_3,v, \phi''_2,v) \). Then if \( f_v \in I_v(\lambda^2) \) is the Siegel section corresponding to \( \phi_v \) in the Rallis inner product formula, we have
\[
f(i(1, u_2)) = \langle \phi_{3,v} \boxtimes \omega(u_2, 1) \phi_{2,v} \rangle
\]
by the formula for the intertwining operator. This is zero unless \( u_2 \in \mathbb{Z}_v^* \) \((U(1)(\mathbb{Q}_v) \cong \mathbb{Q}_v^*)\) and equals \( \lambda^2 \chi_\theta(u_2) \) for those \( u_2 \)'s. To sum up, for such \( v \) the local integral is a nonzero constant \( c_v \). Later when we are moving things \( p \)-adically, this constant is not going to change.

Case 3:
For \( v \in S \) ramified or inert such that \( U(2)(\mathbb{Q}_v) \) is not compact. In this case \( U(1)(\mathbb{Q}_v) \) is a compact abelian groups. We let \( \phi_{2,v} \) be the Schwartz function on \( S(X_v) \) constructed in section 5. Let \( \phi_{3,v} \) be a \( p \)-integral valued Schwartz function on \( S(X_v) \) such that \( \langle \phi_{3,v}, \phi_{2,v} \rangle = \int_{X_v} \phi_{3,v}(x) \phi_{2,v}(x) dx \neq 0 \) and that the action of \( U(1)(\mathbb{Q}_v) \) via the Weil representation is given by a certain character. (It is easy to see that this character is \( \lambda^2 \chi_\theta, v \), thus the action of the center of \( U(2)(\mathbb{Q}_v) \) via \( U(2) \times 1 \) is given by \( \chi_\theta^{-1} \)). We define \( \phi_v = \delta_\psi(\phi_{3,v} \boxtimes \phi_{2,v}) \in S(\mathbb{W}_v^d) \).

Case 4:
For \( v \) such that \( U(2)(\mathbb{Q}_v) \) is compact. Note that the local representation \( \pi_{\theta,v} \) is finite dimensional with some level group \( K_v \). We write \( v_1 \) for the image of \( \phi_{2,v} \) in \( \pi_{\theta,v} \) under Howe duality. We fix an \( U(2)(\mathbb{Q}_v)/K_v \)-invariant measure of \( \pi_{\theta,v} \) and extend \( v_1 \) to \( \{ v_1, \ldots, v_{d_v} \} \) an orthonormal basis of \( \pi_{\theta,v} \). Let \( (\tilde{v}_1, \ldots) \) be the dual basis. Let \( \phi_{3,v} \) be a \( p \)-integral valued Schwartz function on \( S(X_v) \) such that
the action of $U(1)(\mathbb{Q}_v)$ via the Weil representation is given by a certain character. (It is easy to see that this character is $\chi^2_{\psi,\theta,v}$, thus the action of the center of $U(2)(\mathbb{Q}_v)$ via $U(2) \times 1$ is given by $\chi_{\theta,v}^{-1}$.) We require also that the image of $\phi_3$ in the representation of $U(2)(\mathbb{Q}_v)$ (which is the dual of $\pi_{\theta,v}$) is $\tilde{v}_1$. We define $\phi_{\delta} = \delta_{\psi}(\phi_{3,v} \boxtimes \phi_{2,v}) \in S(\mathbb{W}_1^\dagger)$.

Case 5:
We will often write $\chi_{\theta,p}$ for $\chi_{p} \mid_{\mathbb{Q}_p^\times}$. For $v = p$, $W_p = X_{p} \oplus Y_{p}$, we write elements

$$x'_{p} = (x'_{p,1}, x'_{p,2}) \in X_{p}', y'_{p} = (y'_{p,1}, y'_{p,2}) \in Y_{p}' .$$

We define $\phi_p(x'_{p}, y'_{p}) = \chi_{\theta,p}(y'_{p,1})$ if $x'_{p,2} \in \mathbb{Z}_p^\times$ and $x'_{p,1}, y'_{p,1}, y'_{p,2} \in \mathbb{Z}_p$ and $\phi_p(x'_{p}, y'_{p}) = 0$ otherwise. We also write $x''_{3,p} \in X_{p}', x'''_{2,p} = (x_{2,p}, x'''_{2,p}) \in X_{p}'$ (note that we use $x_{2,p}$ to distinguish from $x'_{2,p}$ above). A straightforward computation gives

$$\delta_{\psi,p}(\omega(\lambda)(\phi_p))(x'_{3,p}, x''_{2,p}) = \frac{g(\chi_{\theta,p})}{p^{t}} \chi_{\theta,p}(-x_{2,p}p^{t})$$

if $x''_{2,p} \in p^{-t} \mathbb{Z}_p^\times$ and $x'''_{3,p}, x'''_{2,p} \in \mathbb{Z}_p$ and equals 0 otherwise. We write $\phi_{3,p}$ to be the characteristic function of $\mathbb{Z}_p^\times$ on $X_{p}'$ and define $\phi_{3,p}(x'_{3,p}) = \frac{g(\chi_{\theta,p})}{p^{t}} \chi_{\theta,p}(-x_{2,p}p^{t})$ if $x_{2,p} \in p^{-t} \mathbb{Z}_p^\times$ and $x'''_{2,p} \in \mathbb{Z}_p$ and is 0 otherwise. Here we write $\chi_{\theta,p}$ for its restriction to $\mathbb{Q}_p^\times$ as well. We define $\phi_{3,p}$ and $\phi_{2,p}$ as their images under $\delta_{\psi,p}^{-1} \circ \delta_{\psi,p}$.

Global Case:
Now by looking at the $q$-expansion we find that when we fix the Schwartz functions at $v \neq p$ and let $\lambda_{\theta}$ move in $p$-adic families according to $\chi_{\theta,p}$ at $p$. Our corresponding theta kernel functions $\Theta$ on $U(2, 2)$ move in a $p$-adic $\Lambda_2$-adic analytic family whose $q$-expansion interpolates

$$\sum_{x \in \mathbb{Q}_p^2} \prod_{v} \phi_{v}(x)q^{(x,x)}$$

as in [45, 10.3]. (We note that the $\phi_{\infty}$ here is $\Phi_{1,\infty}$ right translated by $g_0$. On the other hand our distinguished point $i$ is $\zeta \in X_{2,2}$ is chosen as $\zeta$.) As in the doubling method for $U(2) \times U(2) \hookrightarrow U(2, 2)$, the $\Theta(-\lambda)$ pulls back to $p$-adic analytic family of forms on $U(2) \times U(2)$. Moreover it is in fact of the form $\varphi_{\alpha} \boxtimes \varphi_{\beta}$ where $\varphi_{3}$ is fixed along the family by definition. By the Rallis inner products formula and the non-vanishing of the Petersson inner product (ensured by our choices) that for some $u_{aux} \in U(2)(\mathbb{A}_\mathbb{Q})$, $\theta_{\phi_3}(u_{aux}) \neq 0$. When we move $\chi_{\theta}$ $p$-adically our $\theta_{\phi_3}$ is fixed so we know that $\varphi_{\alpha}(u_{aux}) \varphi_{\beta}$ is a $\Lambda_2$-adic family of forms on $1 \times U(2) \hookrightarrow U(2, 2)$. Now we take a representative $(\tilde{u}_1, ..., \tilde{u}_{h_K})$ of $U(1)(\mathbb{Q})U_1(\mathbb{R}) \setminus U(1)(\mathbb{A}_\mathbb{Q})/U(1)(\mathbb{Z})$ considered as elements of the center of $U(2)$.

Definition 8.5. We define:

$$\theta := \sum_{i=1}^{h_K} \chi_{\theta}(\tilde{u}_j)^{-1} \omega_{\lambda^{-1}}(\tilde{u}_j)(\theta_{\phi_3}, \lambda).$$

We denote $\Theta$ for the $\Lambda_2$-adic family constructed this way (we omit the corresponding map of the weight ring of $U(2)$ into $\Lambda^\dagger$).
The property required by the proposition follows from comparing our choices of theta kernel function with the computations for \( \theta_2 \) in Section 6.

We can do the same thing to construct a \( \Lambda_D \)-adic family of forms on \( U(2) \times 1 \rightarrow U(2, 2) \). This time we define \( \phi \) such that for \( v \neq p \) the local components are as before. If \( v = p \) recall that \( W_p = X'_p \oplus Y'_p \). If \( x'_p = (x'_{p,1}, x'_{p,2}) \) and \( y'_p = (y'_{p,1}, y'_{p,2}) \) we define \( \phi_p(x'_p, y'_p) = \chi_{\theta,p}(x'_{1,p}) \) for \( x'_{1,p} \in \mathbb{Z}_p^\times \) and \( x'_{2,p}, y'_{1,p}, y'_{2,p} \in \mathbb{Z}_p \) and \( \phi_p(x'_p, y'_p) = 0 \) otherwise. Direct computation by plugging in the intertwining operator gives, if we write \( x'_{1,p} = (x''_{1,p}, x'''_{1,p}) \) and \( x'_{2,p} = (x''_{2,p}, x'''_{2,p}) \),

\[
\delta^{-1}_{\psi,p}(\omega(Y) \phi_p)(x'_{1,p}, x'_{2,p}) = \chi_{\theta,p}(x''_{1,p}) \text{ if } x''_{1,p} \in \mathbb{Z}_p^\times \text{ and } x'_{1,p}, x''_{2,p}, x'''_{2,p} \in \mathbb{Z}_p \text{ and equals 0 otherwise.}
\]

Then as before we move \( \chi_\theta \) \( p \)-adically our \( \theta_{\phi_2} \) is fixed and non zero at some point \( u'_{\text{aux}} \in U(2)(\overline{\mathbb{Q}}_p) \) and \( \theta_{\phi_3} \) is moving \( p \)-adic analytically. Thus \( \frac{\theta_{\phi_2}(u'_{\text{aux}})}{\Omega^2_{\infty}} \theta_{\phi_3} \) is a \( \Lambda_D \)-adic form on \( U(2) \times 1 \subset U(2, 2) \).

As before we define \( \tilde{\theta}_3 \) (or \( \tilde{\theta}_3 \) for the family) by

\[
\sum_{i=1}^{h_K} \theta_{\phi_3}(\tilde{u}_i) \chi_{\theta}(\tilde{u}_i).
\]

**Definition 8.6.** We define forms \( \theta_D \) and \( \tilde{\theta}_3 \) on \( D^\times(\overline{\mathbb{Q}}_p) \) using \( \theta \) and \( \tilde{\theta}_3 \) and characters \( \chi_\theta \) and \( \tilde{\chi}_\theta \), using the procedure at the beginning of subsection 8.1. Sometimes we drop the superscript \( D \) when it is clear from the context. The key functorial property of it (and some other forms constructed) is summarized in Remark 8.3.

As before we let \( L_{\chi_\theta}^{\text{Katz}} \) be the Katz \( p \)-adic \( L \)-function interpolating the values

\[
\frac{L(\lambda^2 \lambda_\theta, \phi \chi_{\theta, \phi}^{-c}, 1)\Omega^2_p}{(\lambda^2 \lambda_\theta, \phi \chi_{\theta, \phi}^{-c})_{2,p}G(\lambda^2 \chi_{\theta, \phi}^{-c})\Omega^2_{\infty}}.
\]

We are going to compute the Petersson inner product of \( \theta \) and \( \tilde{\theta}_3 \) at a generic arithmetic point.

**Proposition 8.7.** Up to multiplying by a non-zero element of \( \overline{\mathbb{Q}}_p \) which is fixed throughout the family (i.e. independent of the arithmetic point \( \phi \)), we have

\[
\int_{[U(2)]} \theta(g) \tilde{\theta}_3(g) \, dg = \phi(L_{\chi_\theta}^{\text{Katz}}).
\]

(Here \( p^f \) is the conductor of the point \( \phi \). Note that the \( \theta \) is the specialization of the \( \theta \) to \( \phi \) and similarly for \( \tilde{\theta}_3 \).)

**Proof.** For \( v \nmid p \) the corresponding local integrals are non-zero constants which are fixed along the \( p \)-adic families. For \( v = p \). We construct the Schwartz function in \( S(W_p) = S(X'_p \oplus Y'_p) \) first and apply the intertwining operators \( \delta^{-1}_{\psi} \). We write \( x'_p = (x'_{p,1}, x'_{p,2}) \in X'_p \) and \( y'_p = (y'_{p,1}, y'_{p,2}) \in Y'_p \).

We define: \( \phi_p(x_p, y_p) = \chi_{\theta,p}(x'_{p,1}, y'_{p,1}) \text{ if } x'_{p,1}, y'_{p,1} \in \mathbb{Z}_p^\times \) and \( x'_{p,2}, y'_{p,2} \in \mathbb{Z}_p \) and equals 0 otherwise.

Now as before we can compute that \( \delta^{-1}_{\psi}(\phi_p) = \phi_2 \times \phi_2 \in S(X'_{1,p}) \times S(X'_{2,p}) \) where: \( \phi_3(x''_{1,p}, x'''_{1,p}) = \chi_{\theta,p}(x''_{1,p}) \text{ if } x''_{1,p} \in \mathbb{Z}_p, x'''_{1,p} \in \mathbb{Z}_p \) and equals 0 otherwise,

\[
\phi_2(x'''_{2,p}, x'''_{2,p}) = \frac{\phi_2(x''_{2,p}, x'''_{2,p})}{p^f} \chi_{\theta,p}(x''_{2,p}).
\]

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if $x_{2,p}'' \in p^{-t}Z_p$ and $x_{2,p}''' \in Z_p$ and equals 0 otherwise. So these are exactly the theta functions whose inner product we want to compute. Now we compute the Siegel section $f \in I_1(\lambda^2)$ on $U(1,1)(\mathbb{Q}_p)$.

From the form of $\phi$ it is easy to see that $f_{\phi_p} = f_{1,p}f_{2,p}$ for $f_{2,p} \in I_1(\lambda)$ to be the spherical which takes value 1 on the identity and $f_{1,p} \in I_1(\lambda)$ is the Siegel Weil section on $U(1,1)$ of $U(1,1) \times U(1)$ for the Schwartz function $\phi_p \in S(K_p)$ which with respect to $K_p \equiv \mathbb{Q}_p \times \mathbb{Q}_p$, is $\phi(x_1, x_2) = \chi_{\theta,p}(x_1, x_2)$ for $x_1, x_2 \in \mathbb{Z}_p$ and $\phi(x_1, x_2) = 0$ otherwise. But this section is nothing but the Siegel section $f^\dagger$ we constructed in [48, section 4] for the 1-dimensional unitary group case. So the local integral is easily computed to be:

$$
\frac{g(\chi_{\theta,p})}{p^t} \cdot \lambda_p^{-2}((p^{-t}, p^t)) p^t \lambda_p^2(p^{-t}, 1) = \lambda_p^2((1, p^t)) g(\chi_{\theta,p}).
$$

To sum up, up to multiplying by certain fixed constants in $\mathbb{Q}_p$, this Petersson inner product is the value interpolated by the Katz $p$-adic $L$-function $L_{\text{Katz}}^{\chi_h}$. \qed

**Definition 8.8.** (Constructing $h$) We will repeat the above process to construct another family $h$ which will be used in computing the Fourier-Jacobi expansion as well. First we construct a family $h'$ and $h_3$ using theta liftings similar to $\theta$ and $\tilde{\theta}_3$, with $\chi_h$ in place of $\chi_\theta$ and slightly different theta kernels described as follows. For the theta kernel functions $\phi_2$ and $\phi_3$ as above, at each primes $v$ in $\Sigma^2$, note that $\pi_{h',v}$ is the dual of $\pi_{h,v}$, we choose the local theta kernel $\phi_{2,v}$ at $v$ so that the image in $\pi_{h',v}$ is $v_1$ (notations as before) and the theta kernel $\phi_3$ whose image is $v_i$. At each prime $v$ in $\Sigma^1$, we take some theta kernel $\phi_{2,v}$ and $\phi_{3,v}$ such that the images in the local theta lifting are the new vector for U(2) in our sense (subsection 8.1). We make the choices at split primes including $p$ similar to the $\theta$ case. Again we define $h^D$ on $D^\times(\mathbb{A}_Q)$ using the character $\chi_h$ using the procedure at the beginning of Section 8.1.

**Remark 8.9.** The automorphic representation for $h^D$ ($\theta^D$) is the Jacquet-Langlands correspondence of the CM form associated to $\lambda\chi_h$ ($\lambda\chi_\theta$ respectively). We define $h$ on $\text{GU}(2)(\mathbb{A}_Q)$ using $h^D$ and the character $\chi_{\theta}^{-1}$ (not $\chi_h^{-1}$) as the central character. This is just a twist of $h'$. We write $\pi_h$ for the corresponding automorphic representation. Also the automorphic representation $\pi_{\theta}^D$ on $D^\times$ is associated with the CM character $\lambda\theta$.

**Convention:** sometimes when we constructed $\theta \in \pi_\theta$ then by $\tilde{\theta} \in \pi_{\tilde{\theta}}$ we mean $\theta \cdot (\chi_{\tilde{\theta}}^{-1} \circ \det)$. If we have constructed $\tilde{\theta}_3$, then by $\theta_3$ we mean $\tilde{\theta}_3 \cdot (\chi_{\theta} \circ \det)$. We use the same conventions for $h$’s as well. We write $h, \theta$ for the corresponding $p$-adic families thus constructed.

We have the following immediate corollary:

**Corollary 8.10.** The $\theta, \tilde{\theta}_3, \theta^D, \tilde{\theta}_3^D, h, \tilde{\pi}_h, h^D, \tilde{\pi}_h^D$ constructed before are pure tensors in the corresponding automorphic representations.

### 8.4 Triple product formula

**Backgrounds for Ichino’s Formula**

Let $\pi_1, \pi_2, \pi_3$ be three irreducible cuspidal automorphic representations for $\text{GL}_2(\mathbb{Q})$ such that the product of their central characters is trivial and the archimedean components are holomorphic discrete series of weight 2. Let $\pi_i^D$ be the Jacquet-Langlands correspondence of them to $D^\times$ (assume
they do exist). Let \( \phi_i \in \pi^D_i \) and \( \tilde{\phi} \in \tilde{\pi}^D_i \). Write \( \Pi = \prod_{i=1}^2 \pi_i \), \( \phi = \prod_i \phi_i \in \Pi \), \( \tilde{\phi} = \prod_i \tilde{\phi}_i \in \tilde{\Pi} \), and \( r \) the natural 8-dimensional representation of \( \text{GL}_2 \times \text{GL}_2 \times \text{GL}_2 \). We write

\[
I(\phi \otimes \tilde{\phi}) = \left( \int_{[D]} \phi_1(g)\phi_2(g)\phi_3(g)dg \right) \left( \int_{[\tilde{D}]} \tilde{\phi}_1(g)\tilde{\phi}_2(g)\tilde{\phi}_3(g)d\tilde{g} \right).
\]

Now look at the local picture. Suppose \( \phi_i = \otimes_v \phi_{i,v} \) and \( \tilde{\phi}_i = \otimes_v \tilde{\phi}_{i,v} \). We fix \( \langle , \rangle \) a \( D_v^\times \) invariant pairing between \( \pi^D_i \) and \( \tilde{\pi}^D_i \). Let \( \zeta \) be the Riemann zeta-function. Define:

\[
I_v(\phi_v \otimes \tilde{\phi}_v) = \zeta_v(2)^{-2} \frac{L_v(1, \Pi_v, Ad)}{L_v(1/2, \Pi_v, r)} \cdot \int_{Q_v^\times \setminus D_v^\times(Q_v)} \prod_i \langle \pi^D_i(\phi_v(x_v), \tilde{\phi}_v) \rangle d^\times x_v.
\]

Note that this depends on the choice of the pairing.

Let \( \Sigma \) be a finite set of primes including all bad primes, then we have the following formula of Ichino [24]:

\[
\frac{I(\phi \otimes \tilde{\phi})}{\prod_i \langle \phi_i, \phi_i \rangle} = C \frac{\zeta^2(2)}{8} \frac{L^\Sigma(1/2, \Pi, r)}{L^\Sigma(1, \Pi, Ad)} \prod_{v \in \Sigma} \frac{I_v(\phi_v \otimes \tilde{\phi}_v)}{\langle \phi_v, \phi_v \rangle}
\]

where \( C \) is the Tamagawa number for \( D_v^\times \). This does not depend on the choice of the pairing.

In application our \( \langle \phi, \tilde{\phi} \rangle \) is usually 0, thus we need a slight variant of the above formula. Suppose we have elements \( g_v = \prod_v g_{i,v} \) such that \( \langle \phi_i, \pi(g_i)\tilde{\phi}_i \rangle \neq 0 \) for \( i = 1, 2, 3 \), where \( g_{i,v} \) are elements in the group algebra \( \mathbb{Q}_p[D_v^\times(Q_v)] \). Then:

\[
\frac{I(\phi \otimes \tilde{\phi})}{\prod_i \langle \phi_i, \pi(g_i)\tilde{\phi}_i \rangle} = C \frac{\zeta^2(2)}{8} \frac{L^\Sigma(1/2, \Pi, r)}{L^\Sigma(1, \Pi, Ad)} \prod_{v \in \Sigma} \frac{I_v(\phi_v \otimes \tilde{\phi}_v)}{\langle \phi_v, \pi(g_v)\tilde{\phi}_v \rangle}
\]

with \( g_v = \prod_v g_{i,v} \).

Local Triple Product Computations

Split Case Principal Series

Suppose \( v \) is a split prime of \( K/\mathbb{Q} \). We assume \( \pi_{1,v} \) and \( \pi_{2,v} \) are principal series representation and \( \pi_{3,v} \) is either principal series representation or special representation with square-free conductor. For \( K = \text{GL}_2(\mathbb{Z}_v) \) the maximal compact subgroup. We use the realizations of induced representations as functions on \( K \):

\[
\text{Ind}_{B(\mathbb{Q}_v)}^{\text{GL}_2(\mathbb{Q}_v)}(\chi_{1,v}, \chi_{2,v}) = \{ v : K \to \mathbb{C}, v(qk) = \chi(q)v(K), q \in B(\mathbb{Q}_v) \cap K \}
\]

where \( \chi(q) = \chi_{1,v}(a)\chi_{2,v}(d)\delta_B(q) \) for \( q = \left( \begin{array}{cc} a & b \\ d & \end{array} \right) \). We realize the inner products as

\[
\langle v_1, v_2 \rangle = \int_K v_1(k)v_2(k)dk
\]

for \( v_1 \in \text{Ind}_{B}^{\text{GL}_2}(\chi_{1,v}, \chi_{2,v}) \), \( v_2 \in \text{Ind}_{B}^{\text{GL}_2}(\chi_{1,v}^{-1}, \chi_{2,v}^{-1}) \). For a positive integer \( t \) let \( K_t \subset K \) consist of matrices in \( B(\mathbb{Z}_v) \) modulo \( \mathfrak{m}_v^t \). For \( f \in \pi(\chi_{1,v}, \chi_{2,v}) \), \( \tilde{f} \in \pi(\chi_{1,v}^{-1}, \chi_{2,v}^{-1}) \), we define the matrix coefficient \( \Phi_{f,\tilde{f}}(g) = \langle \pi(g)f, \tilde{f} \rangle \). Let \( \sigma_n = \left( \begin{array}{c} a \\ \mathfrak{m}_v \end{array} \right) \).
Lemma 8.11. Suppose \( t \geq 1 \), \( \text{cond}(\chi_1 \chi_2^{-1}) = (\varpi_v^t) \). Let \( w = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) in this lemma. If
\[
f_{\chi}(k_v) = \begin{cases} 
\chi_1(a)\chi_2(d), & k_v \in K_t \\
0, & \text{otherwise}
\end{cases}
\]
and \( \tilde{f}_{\chi^{-1}} \) is defined similar to \( f_{\chi} \) but with \( \chi \) replaced by \( \chi^{-1} \). Then \( \Phi_{f_{\chi}, \tilde{f}_{\chi^{-1}}} (g) = 0 \) on \( \bigcup \sigma_1 \bigcup \sigma_2 \bigcup \sigma_3 \bigcup \sigma_4 \).

On \( \bigcup \sigma_1 \bigcup \sigma_2 \bigcup \sigma_3 \bigcup \sigma_4 \), it is supported in
\[
\bigcup \left( \frac{1}{w_v} \right) \left( \frac{1}{w_v^{-n}Z_v} \right)_{b} \bigcup \left( \frac{1}{w_v} \right)_{1} \bigcup \left( \frac{1}{w_v^{-n}Z_v} \right)_{b} \bigcup \left( \frac{1}{w_v} \right)_{1} K_t.
\]
The corresponding values are: \( \text{Vol}(K_t)\alpha_{n} \varpi_v^{-\frac{n}{2}} \) and \( \text{Vol}(K_t)\alpha_{1} \varpi_v^{-\frac{1}{2}} \) where \( \alpha_i = \chi_i(\varpi_v) \) for \( i = 1, 2 \).

Proof. It is easy to check by considering the supports of \( f_{\chi} \) and \( \tilde{f}_{\chi} \) that \( \Phi_{f_{\chi}, \tilde{f}_{\chi}} (g) = 0 \) on \( \bigcup \sigma_1 \bigcup \sigma_2 \bigcup \sigma_3 \bigcup \sigma_4 \).

Now suppose \( g \in K_{1w}\sigma_{1}K_{1} \) for \( n \geq 1 \), without loss of generality we assume
\[
g = \begin{pmatrix} 1 \\ c \end{pmatrix} w_{\sigma_{1}} \begin{pmatrix} 1 \\ b \end{pmatrix} = g = \begin{pmatrix} 1 \\ w_v \end{pmatrix} \begin{pmatrix} 1 \\ c \end{pmatrix} (1 \ 1) w
\]
for \( w_v | b, w_v | c \). Plugging in the formula for matrix coefficients, write \( K_t \ni g' = \begin{pmatrix} a' \\ b' \\ d' \end{pmatrix} \begin{pmatrix} 1 \\ c' \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ c' \end{pmatrix} \begin{pmatrix} 1 \\ c' \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)
for \( a', b', d', c', \in \varpi_v \bigcup \sigma_{1} \bigcup \sigma_{2} \bigcup \sigma_{3} \bigcup \sigma_{4} \).

Here we write \( c'' = \frac{c' + c}{\varpi_v^n} \). We need to fix \( g \) and do the integration for \( g' \). The first observation is that we only need to consider integration with respect to \( c' \). Next we can integrate for those \( c' \) such that \( p'|b + \frac{1}{c'} \). We divide the problem into four cases according to whether \( \varpi_v | c \) and whether \( \varpi_v | b \). In any case it is not hard to check that the integration is 0 since \( \text{cond}(\chi_1 \chi_2^{-1}) = (\varpi_v^t) \). We leave the verification for \( g \in K_{1w}\sigma_{1}K_{1} \) and \( K_{1w}\sigma_{1}wK_{1} \) to the reader.

Lemma 8.12. Suppose \( \text{cond}(\chi_1 \chi_2^{-1}) = (\varpi_v^t) \), \( t > 0 \) and \( \vartheta \) is a character with conductor \( (\varpi_v^t) \).

Define
\[
f_{\chi, \vartheta}(k_v) = \begin{cases} 
\chi_1(a)\chi_2(d)\vartheta(\frac{c}{\varpi_v^n}), & a, d \in O_v^\times, c \in \varpi_v \bigcup \sigma_{1} \bigcup \sigma_{2} \bigcup \sigma_{3} \bigcup \sigma_{4} \\
0, & \text{otherwise}
\end{cases}
\]
for \( k_v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then on \( \cup_n K_1 \sigma_n K_1 \cup_n K_1 w \sigma_n w K_1 \), \( \Phi_{f,f}(g) \) is supported in \( K_{t+1} \). More over if \( g = \begin{pmatrix} 1 \\ c \\ 1 \end{pmatrix} \) with \( w_v^{t+1} || c \) then \( \Phi(g) = -q^{-t-1} \). If \( w_v^{t+1} || c \) then \( \Phi(g) = (q-1)q^{-t-1} \).

Proof. Suppose \( \Phi_{f_{\chi,0}, f_{\lambda,0}, \sigma}(g) \neq 0 \). If \( g \in K_1 \begin{pmatrix} w_v^n & 1 \\ 1 & 1 \end{pmatrix} K_1 \) for some \( n \geq 0 \), then as before we have \( g \in \begin{pmatrix} w_v^n & 1 \\ 1 & 1 \end{pmatrix} K_t \). For \( g' \in K_t \), \( g' = \begin{pmatrix} a' & b' \\ d' & c' \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Without loss ofgenerality assume \( g = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \). Then on \( 1 \) we have: if \( p^{t+1} || c \) then \( \Phi(g) = -q^{-t-1} \); if \( p^{t+1} || c \) then \( \Phi(g) = q^{-t-1} (q-1) \). In other cases \( \Phi(g) = 0 \). If \( g \in K_1 w \begin{pmatrix} w_v^n & 1 \\ 1 & 1 \end{pmatrix} \) again if \( \Phi(g) = 0 \) then \( g \in \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} K_t \).

Without loss of generality write \( g \in \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) for \( c \in w_v^{t+1} \mathbb{Z}_v, K_t \) then \( g' = \begin{pmatrix} a' & b' \\ d' & c' \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), \( g' = \begin{pmatrix} a' & b' \\ d' & c' \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). If \( n \geq 1 \), then one can check that the integration is again 0.

Suppose \( \chi^{-1} \chi_2 \) is unramified and \( \vartheta \) has conductor \( (\mathfrak{w}_v^n) \) then we define \( f_{\chi, \vartheta} \in \pi \) by:

\[
f_{\chi, \vartheta}(k_v) = \begin{cases} 
\chi_1(a) \chi_2(d) \vartheta(\mathfrak{w}_v^n) \left( \begin{array}{c} a \ b \\ d \ c \end{array} \right), & g = \left( \begin{array}{c} a \ b \\ d \ c \end{array} \right), \left( \begin{array}{c} a \ b \\ d \ c \end{array} \right) \in B(\mathbb{Z}_v), c \in \mathbb{Z}_v \times \mathbb{Z}_v^\times \\
0, & \text{otherwise}
\end{cases}
\]

We define similarly \( \tilde{f}_{\chi, \vartheta} \in \tilde{\pi} \) by replacing \( \chi, \vartheta \) by \( \chi^{-1}, \vartheta^{-1} \).

**Lemma 8.13.** Write \( f = f_{\chi, \vartheta}, \tilde{f} = \tilde{f}_{\chi, \vartheta} \in \tilde{\pi} \) then on \( \cup_n K_0 \sigma_n K_0 \cup_n K_0 w \sigma_n w K_0 \) it is supported in \( K_s \) and:

\[
\Phi_{f,f}(g) = \begin{cases} 
-q_v^{-1} \text{Vol}(K_0(\mathfrak{w}_v^n)) & \text{if } g = \begin{pmatrix} 1 \\ 1 \\ c \end{pmatrix}, \mathfrak{w}_v^n || c \\
-q_v^{-1} \text{Vol}(K_0(\mathfrak{w}_v^n)) & \text{if } g = \begin{pmatrix} 1 \\ 1 \\ c \end{pmatrix}, \mathfrak{w}_v^{n+1} || c;
\end{cases}
\]

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\[
\sum_{a \in \frac{\mathbb{Z}_v}{\mathbb{Z}_v^{+} \mathbb{Z}_v}} \vartheta_v \left( -\frac{a}{\varpi_v} \right) \pi \left( \frac{1}{a} \right) \left( \frac{\varpi_v^{-s}}{a} \right) \left( 1 - q_v \chi_1 / \chi_2 (\varpi_v) \right)^{-1} \left( 1 - q_v^{-\frac{1}{2}} \chi_2 (\varpi_v) \pi_v \right) \left( \frac{\varpi_v^{-s}}{a} \right) f^{\text{sph}} \]
eq \chi_1 (\varpi_v^{-s}) q_v^{-\frac{1}{2}} f_{\vartheta}.
\]
Proof. Similar to the above lemma.

Lemma 8.15. Let \( \chi_{f,1}, \chi_{f,2}, \chi_{\theta,1}, \chi_{\theta,2}, \chi_{h,1}, \chi_{h,2}, \vartheta \) are characters of \( \mathbb{Q}_v^\times \) and \( t_1 < s < t_2 \) are non-negative integers such that if \( t_1 \neq 0 \) then \( t_1 + s = t_2 \), if \( t_1 = 0 \) then \( s + 1 = t_2 \). Suppose \( \text{cond}(\chi_{f,1}, \chi_{f,2}) = (\varpi_v^{t_1}) \) and \( \text{cond}(\vartheta) = (\varpi_v)^s \) and \( \text{cond}(\chi_{\theta,1}, \chi_{\theta,2}) = (\varpi_v^{t_2}) \). Assume: \( \chi_{f,1}, \chi_{\theta,1}, \chi_{h,1}, \vartheta = 1 \) and \( \chi_{f,2}, \chi_{\theta,2}, \chi_{h,2}, \vartheta^{-1} = 1 \). We also define \( f_{\chi_{f,1}, \vartheta} \in \pi(\chi_{f,1}, \chi_{f,2}), f_{\chi_{\theta}} \in \pi(\chi_{\theta,1}, \chi_{\theta,2}), f_{\chi_{h}} \in \pi(\chi_{h,1}, \chi_{h,2}) \) as above. Similarly for \( \tilde{f}_{\chi_{f,1}, \vartheta}, \tilde{f}_{\chi_{\theta}}, \tilde{f}_{\chi_{h}} \). Then Ichino’s local triple product:
\[
I_v (f_{\chi_{f,1}, \vartheta} \otimes f_{\chi_{\theta}} \otimes f_{\chi_{h}}, \tilde{f}_{\chi_{f,1}, \vartheta} \otimes \tilde{f}_{\chi_{\theta}} \otimes \tilde{f}_{\chi_{h}}) = \frac{q_v}{q_v - 1} \text{Vol}(K_{t_2}).
\]
(In this lemma the \( \chi_{h}, \chi_{\theta} \) are defined using \( \chi_{h,1}, \chi_{h,2}, \chi_{\theta,1}, \chi_{\theta,2} \) similarly as in lemma 8.11.)

Special Representations
We consider the induced representation \( \pi(\chi_1, \chi_2) = \{ f : K_v \to K, f(qk) = \chi_1(a) \chi_2(d) \delta_B(q), q = \left( \begin{array}{cc} a & b \\ d & c \end{array} \right) \in B(\mathbb{Z}_v) \} \) where \( \chi_2 = \chi_1 | \cdot | \). The special representation \( \sigma(\chi_1, \chi_2) \subset \pi(\chi_1, \chi_2) \) consisting of functions \( f \) such that \( \int_k f(k) dk = 0 \). We consider the case when \( \pi_3 \) is the special representation \( \sigma(\chi_{v,1}, \chi_{v,2}) \subset \text{Ind}_{B(\mathbb{Q}_v)}^{GL_2(\mathbb{Q}_v)} (\chi_{v,1}, \chi_{v,2}) \) at \( v \) with square free conductor. Here \( \chi_{v,i} \) are unramified characters. Similar to the unramified principal series case, we use the model of induced representations. It is easy to see that the \( f_{\chi_{f,1}, \vartheta} \) defined above is inside \( \sigma(\chi_{v,1}, \chi_{v,2}) \). The inner product of \( \sigma(\chi_{v,1}, \chi_{v,2}) \) is still given by \( \langle v_1, v_2 \rangle = \int_K v_1(k) v_2(k) dk \). The formula for the triple product integral is the same as the one in the case of principal series representations.
Let \( f_{\text{new}} \in \sigma(\chi_1, \chi_2) \) be such that \( f(k) = q_v \) for \( k \in K_1 \) and \( f(k) = -1 \) otherwise. Then we have the following lemma

Lemma 8.16.  
\[
\sum_{a \in \frac{\mathbb{Q}_v^{+} \mathbb{Z}_v}{\mathbb{Z}_v^{+} \mathbb{Z}_v}} \vartheta_v \left( -\frac{a}{\varpi_v} \right) \pi \left( \frac{1}{a} \right) \left( \frac{\varpi_v^{-s}}{a} \right) f_{\text{new}} \]
is \( \chi_1 (\varpi_v^{-s}) (q_v^{1-s} + q_v^{-\frac{s}{2}}) f_{\chi_{f,1}, \vartheta} \) where \( f_{\chi_{f,1}, \vartheta} \) is defined above.

Proof. Let \( f_0 \) and \( f_1 \) be the characteristic functions on \( K_1 \) and \( K_1 \backslash K \). Then \( f_{\text{new}} = q_v f_0 - f_1 \). A computation shows that
\[
\sum_{a \in \frac{\mathbb{Q}_v^{+} \mathbb{Z}_v}{\mathbb{Z}_v^{+} \mathbb{Z}_v}} \vartheta_v \left( -\frac{a}{\varpi_v} \right) \pi \left( \frac{1}{a} \right) \left( \frac{\varpi_v^{-s}}{a} \right) f_0 = \chi_1 (\varpi_v^{-s}) q_v^{\frac{s}{2}} f_{\vartheta},
\]

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\[
\sum_{a \in \mathbb{Q}_v} \vartheta_v \left( -\frac{a}{\varpi_v} \right) \pi \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \left( \frac{\varpi_v^{-s}}{1} \right) f_1 = -\chi_2 (\varpi_v^{-s}) q_v^{-\frac{s}{2}} f_0.
\]

The lemma follows. \qed

Remark 8.17. The reason why the local integrals showing up in the triple product formula later on are the ones considered in this subsection is a consequence of the computations in subsection 6.7 and the remarks at the end of subsection 4.3.

Non-split Case 1

If \( U(2)(Q_v) \) is compact. This case is easier since we are in the representation theory for finite groups. By our assumptions on \( \chi_{\theta,v}, \chi_{h,v} \) and \( \pi_{f,v} \) and our chosen vectors \( h, \theta, \tilde{h}, \tilde{\theta} \), and that \( \pi_{D,v} \) is 1-dimensional, we conclude that: by the inner product formula of matrix coefficients of representations of finite groups the local triple product integral is \( \frac{1}{d_{\pi_{h,v}}} \) where \( d_{\pi_{h,v}} \) is the dimension of the representation \( \pi_{h,v} \). (See e.g. [5].) When we are moving our datum \( p \)-adic analytically, this integration is not going to change.

Non-split Case 2

Recall that we fix a generic arithmetic point. By [37] we know that there are \( g_{1,v}, g_{2,v}, g_{3,v}, g_{4,v} \in D^\times(Q_v) \subset GU(2)(Q_v) \) such that

\[
I_v(\pi_{h,v}(g_{2,v}) h_{D} \otimes v_{\phi,v}^D \otimes \pi_{f,v}(g_{1,v}) f_{D} \otimes \tilde{\pi}_{h,v}(g_{4,v}) \tilde{h}_{D} \otimes \tilde{v}_{\phi,v}^D \otimes \tilde{\pi}_{f,v}(g_{3,v}) \tilde{f}_{D} \neq 0.
\]

(We write \( v_{\phi,v}^D \) for the image of \( v_{\phi,v} \) in the corresponding \( D^\times(Q_v) \) representation and similarly for \( \tilde{v}_{\phi,v}^D \). Note that the local signs are correct by our choices.)

Definition 8.18. We define

\[
g_i = \prod_v g_{i,v}
\]

for \( i = 1, 2, 3, 4 \). (We take \( g_{i,v} = 1 \) if \( v \) is non-split in \( K/Q \).

The local triple product integrals are also non-zero by our computations. By Ichino’s formula and that the special \( L \)-values are non-vanishing (by our choices of characters, note that the product of the central characters for \( f, \theta, h \) is obviously trivial) the global trilinear form is also non-zero. So by our definitions for \( h_{D}, \theta_{D} \), etc, we know that \( \prod_v g_{2,v} \) is in \( D^\times(A_Q) \). We can find

\[
U(2)(A_Q) \ni g'_2 = a_2 g_2
\]

for \( a_2 \in A_Q^\times \). So

\[
\int_{D^\times} (\pi(g_2) h_{D})(g) \theta_{D}(g)(\pi(g_1) f_{D})(g) dg = \frac{1}{C_D} \chi_h(a) \int_{U(2)} (\pi(g'_2) h)(g) \theta(g)(\pi(g_1) f_{D})(g) dg.
\]

Similarly we have \( \prod_v g_{4,v} \in D^\times(A_Q) \) and can find \( U(2)(A_Q) \ni g'_4 = a_4 g_4 \) for \( a_4 \in A_Q^\times \).
Remark 8.19. These $g_{k,v}$ and $\tilde{g}_{i,v}$ are chosen only at the arithmetic point $\phi$. We are going to fix them when moving the Eisenstein datum in $p$-adic families. Then for the arithmetic points $\phi$ with fixed $\phi|_1$, the $\pi_{h,v}^D$ and $\pi_{g,v}^D$ only move by twisting by unramified characters which are converse to each other. (In fact the Weil representations on unitary groups are unchanged since $\chi_{\phi}|_{U(1)}$ are. The difference only comes from extending a form on $U(2)$ to $GU(2)$). Thus if the theta kernel we used to define $h$ and $\theta$ are fixed then the local triple product integrals $I_\nu(\pi_{h,v}(g_{k,v})h_{D,v}^D \otimes \pi_{f,v}(g_{l,v})/D_{\phi,v}) \tilde{h}_{D,v}^D \otimes \tilde{\pi}_{f,v}(g_{l,v})/D_{\phi,v})$ does not change (note that since $\pi$ has square-free conductor, the $\langle f_{D,v}, f_{D,v}' \rangle$ has non-zero inner product). Later on we will see that certain product of such local factors move $p$-adic analytically. Thus this is an analytic function depending only on the Hida family variable of $f$. This observation is crucial in studying the $p$-adic properties of the Fourier-Jacobi coefficients.

Interpolation of $f_{\phi}$’s

We will interpolate $f_{\phi}$’s as $\mathbb{I}$-adic families from the family $f$ using Lemma [8.14] and [8.16]. However in order to do so we make have to replace $\mathbb{I}$ by a larger normal domain finite over $\mathbb{I}$, so that the $\chi_{1,v}(\varpi_v)$ and $\chi_{2,v}(\varpi_v)$’s at primes in $\Sigma$ where $\pi_f$ is unramified will be elements of this newly defined $\mathbb{I}$. We choose this family as the $\varphi$ in Proposition 7.7.

8.5 Evaluating the Integral

Now we construct: for any $\Lambda_\mathcal{D}$-adic family $F$,

$$l_{\theta_1}(F) := l_{\theta_1}FJ_1(\prod_{v \in \Sigma^1 \cup \Sigma^2} \sum_i C_{v,i} \rho \left[ \begin{array}{c} u_{v,i} \\ I_2 \\ u_{v,i} \end{array} \right] ) (F)$$

whose value is a $\Lambda_\mathcal{D}$-adic family on $U(2)$. Here $C_{v,i}$ are defined in lemma [6.21]

Lemma 8.20. The $l_{\theta_1}(F)$ is a $\Lambda_\mathcal{D} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-adic form on $U(2)$.

Proof. The proof is simple: as before we take a basis of $\mathcal{O}_L$-dual space $(\theta_1, \cdots, \theta_m)$ of the finite dimensional space $H^1(B, \mathcal{L}(\beta))$. Pairing the $\Lambda_\mathcal{D}$-adic Fourier-Jacobi coefficient of $F$ with these $\theta_i$ we get a $\Lambda_\mathcal{D}$-adic family in $\Lambda$. But our $\theta_1$ is in the $L$-linear combination of the $\theta_i$’s. Thus we get a $\Lambda_\mathcal{D} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-adic family on $U(2)$. \qed

We also define

$$B_1 = \int l_{\theta_1}(E_{D,Kling})d\mu_{\pi(g_2')}h.$$ We can evaluate this expression by Ichino’s formula for triple products.

Definition 8.21. For any nearly ordinary form $f$ or family $f$ (we use the same notations for $h$, $\theta$, etc) we define $f_{low}(g) := f(g \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix})$ (under the identification $D_{p}^c \simeq GL_2(\mathbb{Q}_p)$ given by the $v_0$ projection). Also, if $\chi$ is the (family of) central characters of $f$ we define $\tilde{f} = f.(\chi^{-1} \circ Nm)$. We also define $f^{ss}(g) = f(g \begin{pmatrix} 0^t \\ 1 \end{pmatrix})$.

We have the following
Proposition 8.22. Up to multiplying by a constant in $\mathbb{Q}_p^\times$ which is fixed along the family and a unit in $\mathbb{F}_p^{ur}$, we have for any generic arithmetic point $\phi$,\[
\phi(B_1) = p^i C_{U(2)}^D L_5 L_6 \times \left( \int_{\mathbb{A}_q^\times D(\mathbb{Q}) \backslash D(\mathbb{A}_q)} \pi(g_2) h_\phi^D(g) \pi(g_1) f_{\phi,\theta}(g) \theta_{\phi,\theta,\text{low}}(g) dg \right)\]
where $L_5$ and $L_6$ are certain $p$-adic $L$-functions interpolating the the $L$-values
\[
\text{Eul}_p \cdot \frac{\Gamma(\kappa - 1) \Omega_\kappa}{\Omega_\kappa} L(\chi_{aux} \tau^{-c}, \frac{\kappa - 2}{2})
\]
and
\[
\text{Eul}_p \cdot \frac{\Gamma(\kappa - 2) \Omega_{\kappa}^{-2}}{\Omega_{\kappa}^{2-2}} L(\lambda^2 \chi_\theta^{-c} \chi_{aux} \tau^{-1}, \frac{\kappa - 2}{2}).
\]
(We refer the corresponding normalization factors $\text{Eul}_p$ at $p$ to [20, (4.16)]) are the Euler factors defined).

Proof. This is a consequence of the pullback formula. We first apply (5) and then further factorize the $E_{\text{Siegel,2}}$ via the embedding $U(2) \times U(2) \hookrightarrow U(2,2)$ of $E_{\text{Siegel,2}}$ and pair with $h$. The $L_5$ and $L_6$ come from the factors show up here. Also note that the $\theta$ part appearing in the first integral above should be $\theta_{\phi_2} \otimes \lambda$ constructed before (not an eigenform). However in view of the central character of $h$ and $f$, only the component $\theta$ of $\theta_2$ with the correct central character matters. Also by the construction this $\theta$ part is a multiple of $\theta^{ss}$ which, after applying the operator $\sum_{n \in \mathbb{Z}_p/p^n \mathbb{Z}_p} \pi_\theta \left( \frac{1}{n}, 1 \right)$ is $\theta_{\text{low}}$ from the definition of $\phi_{2,p}$. Comparing $L_5 \cdot L_6$ with the factor coming from the doubling method for $h$ under $U(2) \times U(2) \hookrightarrow U(2,2)$, it is easy to see that the ratio comes from the local Euler factors at $p$ computed in Subsection 6.9, Euler factor for $L_5$ at primes in $\Sigma^2$ and Euler factors for $L_6$ at $p$, which are units by our choice of characters (See [20, (4.16)] for the Euler factor at $p$). Then the lemma follows. 

By our choices for characters the corresponding non-$\Sigma$-primitive $p$-adic $L$-functions $L_5$ and $L_6$ are units in $\Lambda^\times_{\mathbb{F}_p}$.

In the following we often omit the superscript $D$ for simplicity. We construct a form $\tilde{f}_3 \in \tilde{\pi}$ in a similar way to $f_3$ at primes outside $p$ and is the nearly ordinary vector (with respect to the upper triangular Borel) at $p$. Up to a constant in $\mathbb{Q}_p^\times$ (which does not change along the family) we have
\[
p^{i}B_1 \cdot \left( \int (\pi(g_4^*) \tilde{h}_3)(g) (\pi_f(g_3) \tilde{f}_3)(g) \tilde{\theta}_{3,\text{low}}(g) dg \right)
\]
\[
= (\lambda_{\eta,2} \chi_{\Omega})^{-t}(p)(\chi_{\theta,1} \lambda_{\eta,1}^t)(p) p^{3t} \left( \int (\pi(g_2) h)(g) (\pi_f(g_1) f_3)(g) \theta^{ss}(g) dg \right)
\]
\[
\times \left( \int (\pi_\eta(g_4^*) \tilde{h}_3)(g) (\pi_f(g_3) \tilde{f}_3)(g) \tilde{\theta}_{3}^{ss}(g) dg \right)
\]
\[
= \lambda_{\eta,2}^{-2t}(p) \chi_{\Omega}^{-2t}(p) p^{3t} \left( \int (\pi(g_2) h)(g) (\pi_f(g_1) f_3)(g) \theta^{ss}(g) dg \right)
\]
\[
\times \left( \int (\pi(g_4^*) \tilde{h}_3)^{ss}(g) (\pi_f(g_3) \tilde{f}_3)^{ss}(g) \tilde{\theta}_3(g) dg \right)
\]
We refer to [18], [20] for the justification of their interpolation formulas. These values are interpolated by an element in \( \hat{\mathbb{V}}_{\text{ur}}[[\Gamma_K]] \).

**Definition 8.23.** Suppose \( \pi_{f,p} \) is the principal series representation \( \pi(\chi_{f,s}, \chi_{f,o}) \) where \( \text{val}_p(\chi_{f,s}) = \frac{1}{2} \) and \( \text{val}_p(\chi_{f,o}) = -\frac{1}{2} \).

We write the \( p \)-component of \( \lambda^2 \chi_{h} \chi_{h} \) as \( (\lambda^2 \chi_{h} \chi_{h})_1, (\lambda^2 \chi_{h} \chi_{h})_2 \) and \( f \) the normalized \( \text{GL}_2 \) ordinary form new outside \( p \). We also define the following (to make notations less cumbersome in the next few formulas we omit the subscript \( \phi \), e.g. \( \chi_{\theta,2} \) stands for \( \chi_{\theta,2,\phi} \) for an arithmetic point \( \phi \)): the \( p \)-adic \( L \)-function \( \mathcal{L}_1 \) interpolates values

\[
\frac{g(\chi_{\theta,2}) g(\chi_{h}^{-1}) L(f, \lambda^2 \chi_{h} \chi_{h}, \frac{1}{2})(\chi_{f,s}(p) \chi_{f,o}(p))^{-t}((\lambda^2 \chi_{h} \chi_{h})_2(p), p) - 2t p^t}{\pi \Omega_{\infty}^2};
\]

the \( p \)-adic \( L \)-function \( \mathcal{L}_2 \) interpolates values

\[
\frac{(1 - a_p(f)^{-1}) \chi_{\theta, p,1} \chi_{h, p,2}(p^{-1})) (1 - a_p(f) \chi_{\theta, p,2} \chi_{h, p,1}(p^{-1}) p^{-1})}{(1 - a_p(f)^{-1} \chi_{\theta, p,2} \chi_{h, p,1}(p))};
\]

\[
\times \frac{\bar{g}(\chi_{\theta}^{-1}) L(f, \chi_{h}^{-1} \chi_{h}, \frac{1}{2})(\chi_{f,o}(p) p^t (\chi_{h} \chi_{h})_1(p))^{-t}}{\pi^2(f, f) \left( N^{-1} \right)^{\Gamma_0(N)}};
\]

the \( p \)-adic \( L \)-function \( \mathcal{L}_3 \) interpolates values

\[
\frac{\zeta_{\chi_{\theta}}(1) g(\chi_{\theta}) L(\lambda^2 \chi_{h} \chi_{h}^{-c}, 1)((\lambda^2 \chi_{h} \chi_{h}^{-c})_2(p), p) - 2t p^t}{\pi \Omega_{\infty}};
\]

the \( p \)-adic \( L \)-function \( \mathcal{L}_4 \) interpolates values

\[
\frac{\zeta_{\chi_{\theta}}(1) g(\chi_{h}^{-1}) L(\lambda^2 \chi_{h} \chi_{h}^{-c}, 1)((\lambda^2 \chi_{h} \chi_{h}^{-c})_2(p), p) - t, p^t}{\pi \Omega_{\infty}}.
\]

We refer to [18], [20] for the justification of their interpolation formulas. These values are interpolated by some \( p \)-adic \( L \)-functions in \( \hat{\mathbb{V}}_{\text{ur}}^p \). Note that

\[
\frac{g(\chi_{f}^{-1}) L(\text{ad}, f, 1) M_{\varphi}(M)(p - 1)(\chi_{f,o} \chi_{f,s}^{-1}(p), p) - t}{2^4 \pi^3(f, f) \left( N^{-1} \right)^{\Gamma_0(N)}} = 1.
\]

Note also that \( \mathcal{L}_2 \) is in fact a non-zero element in \( \text{Frac}(\mathbb{L}) \) (i.e. does not depend on the variable \( \Gamma_K \)) by checking the \( p \)-components of \( \chi_{\phi} \chi_{h} \) and nonzero by our choice of characters in subsection [8.2]. It can actually be written as the ratio of two elements whose specializations to all but finitely

}\]
many generic arithmetic points are non-zero. By our choices for \( \chi \) and \( \chi_h \) we know that \( \mathcal{L}_1 \) is in \( \hat{I}^\text{ur}[\Gamma_K]^\times \). We consider the expression:

\[
C_{U(2)}^D \frac{\langle f, \tilde{f}^s \rangle \langle h, \tilde{h}^s \rangle \langle \tilde{\theta}, \theta^s \rangle \chi_{\tilde{\theta},1}^{-1}(p^t)\mathcal{L}_1\mathcal{L}_2\mathcal{L}_5\mathcal{L}_6}{\mathcal{L}_3\mathcal{L}_4} (\chi_{f,s}(p)\chi_{\tilde{\theta},1}^{-1}(p)\chi_{h,1}(p)p^\frac{3}{2})p^{3t} \tag{9}
\]

\[
= C_{U(2)}^D \frac{\langle f, \tilde{f}_{\text{low}} \rangle \langle h, \tilde{h}_{\text{low}} \rangle \langle \tilde{\theta}, \theta_{\text{low}} \rangle \mathcal{L}_1\mathcal{L}_2\mathcal{L}_5\mathcal{L}_6}{\mathcal{L}_3\mathcal{L}_4} p^{3t}. \tag{10}
\]

We first give the following lemma for the local triple product integral at \( p \).

**Lemma 8.24.** At a generic point the local triple product integral at \( p \) is given by:

\[
p^{-1}(1 - p) \frac{1}{1 + p} \frac{1}{1 - a_p(f)\chi_{\theta,1}(p)p^{-1}} \frac{1}{1 - a_p(f)^{-1}\chi_{\theta,1}(p)p^{-1}}.
\]

**Proof.** This follows from Lemma 8.11. \( \square \)

We examine the ratio between (9) and the expression for

\[
p^t B_1(\int (\pi(g_4)\tilde{h}_3)(g)(\pi_f(g_3)\tilde{f})(g)\tilde{\theta}_{3,\text{low}}(g)dg) \tag{11}
\]

using Ichino’s formula. By our calculations for the local triple product integrals, and the Petersson inner product of \( <\theta, \tilde{\theta}_3> \) and \( <h, \tilde{h}_3> \), the ratio is a product of:

- Euler factors for \( \zeta_{\chi_K}(1) \) at \( \Sigma \);
- the local Euler factors for \( L(ad, f, 1) \) at \( \Sigma \);
- \( p^t \) times the local triple product integral for \( v = p \), which are units times a constant in \( \hat{Q}_p \) by our choices;
- \( \langle f^D, \tilde{f}_{\theta,\text{low}}^D \rangle / \langle f^D, \tilde{f}_{\text{low}}^D \rangle \) which is interpolated by a non-zero element in \( \mathbb{I} \);
- The local Euler factors of \( \mathcal{L}_5 \) and \( \mathcal{L}_6 \) at \( \Sigma^2 \) and \( p \) which are units by our choices;
- The local integrals showing up in the Rallis inner product formula at \( \Sigma \), which is fixed along the family;
- The local Euler factors at \( \Sigma^2 \) of \( L(f, \chi^\sigma_h, \frac{1}{2}) \) which are non-zero elements in \( \mathbb{I} \).
- The local triple product integrals for \( v \nmid p \);

Thus we have the following proposition:

**Proposition 8.25.** Any height 1 prime of \( \hat{I}^\text{ur}[\Gamma_K] \) containing \( \int l_{\tilde{\theta}_1}(E_{K\text{ling},D})d\mu_{\theta}(g_4^\theta)h \) must be the pullback of a height 1 prime of \( \hat{I}^\text{ur} \).
Proof. The last item we listed above has two parts: at split primes and non-split primes. The integrals at split primes are non-zero numbers in $\mathbb{Q}_p^\times$ which are fixed throughout the family. At non-split primes we do not know much about it. We only know that at the generic arithmetic point $\phi$ at which we choose our $g_1, g_2, g_3, g_4$, this integral is not zero. We may assume that at this $\phi$ the expression (9) is non-zero (in fact just need $\mathcal{L}_2$ to be non-zero finite number here). Thus the expression (11) is not identically zero. So the ratio of (11) over (9) is a non-zero element of $\text{Frac}(\mathbb{F}^ur[[\Gamma_K]])$. On the other hand the integral in the last item above only depends on $\phi|_I$ as observed in Remark 8.19. So if we evaluate this ratio at the generic arithmetic points where (9) is not zero, it depends only on $\phi|_I$. From this it is not hard to prove that (say using the following lemma) the ratio is a non-zero element of $\text{Frac}(\mathbb{F}^ur)$ and the proposition is true. 

Lemma 8.26. Suppose $A$ is an element in $\hat{\mathbb{F}}[[\Gamma_K]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If for any generic arithmetic points $\phi, \phi' \in \hat{\mathbb{F}}[[\Gamma_K]]$ such that $\phi|_I = \phi'|_I$, we have $\phi(A) = \phi'(A)$. Then $A \in \hat{\mathbb{F}}$. 

Proof. This lemma is easily proved by observing that if $\zeta_1, \zeta_2$ are $p^t$-roots of unity and $\phi$ is a generic arithmetic point with conductors being $p^t$ such that $t' > t$, then the composition $\phi'$ of $\phi$ with the ring automorphism $\iota_{\zeta_1, \zeta_2} : \hat{\mathbb{F}}[[\Gamma_K]] \to \hat{\mathbb{F}}[[\Gamma_K]]$ given by identity on $\hat{\mathbb{F}}$ and $\gamma^+ \mapsto \gamma^+ \zeta_1$, $\gamma^- \mapsto \gamma^- \zeta_2$ is still a generic arithmetic point. Let $F$ be the element considered in the lemma. Then $F - F \circ \iota_{\zeta_1, \zeta_2}$ is 0 at a Zariski dense set of points, and is thus identically zero. The arbitrariness of $\zeta_1, \zeta_2$ implies the lemma. In the proof of the above proposition we apply this lemma to $B_2 \cdot \mathcal{L}_2$ times (11) over (9), where $B_2$ interpolates $p^t \langle f, \hat{f}_{\text{low}} \rangle$ (which are indeed interpolated by an Iwasawa algebra element by our previous discussion). 

9 Proof of the Theorems

9.1 Hecke Operators

Let $K' = K'_{\Sigma, (p)} K_{\Sigma} \subset G(\mathbb{A}_f^p)$ be an open compact subgroup with $K_{\Sigma} = G(\hat{\mathbb{Z}}^\Sigma)$ and such that $K := K' K_0$ is neat. For each $v$ outside $\Sigma$ we have $\text{GU}(3, 1)(\mathbb{Q}_v) \simeq \text{GU}(2, 2)(\mathbb{Q}_v)$ with the isomorphism given by conjugation by some elements in $\text{GL}_4(\mathbb{O}_{K_v})$. So we only need to study the unramified Hecke operators for $\text{GU}(2, 2)$ with respect to $\text{GU}(2, 2)(\mathbb{Z}_v)$. We follow closely to [45, 9.5,9.6].

Unramified Inert Case

Let $v$ be a prime of $\mathbb{Q}$ inert in $\mathcal{K}$. Recall that as in [45, 9.5.2] that $Z_{v, 0}$ is the Hecke operator associated to the matrix $z_0 := \text{diag}(\varpi_v, \varpi_v, \varpi_v, \varpi_v)$ by the double coset $K z_0 K$ where $K$ is the maximal compact subgroup of $G(\mathbb{Z}_v)$. Let $t_0 := \text{diag}(\varpi_v, \varpi_v, 1, 1)$, $t_1 := \text{diag}(1, \varpi_v, 1, \varpi_v^{-1})$ and $t_2 := \text{diag}(\varpi_v, 1, \varpi_v^{-1}, 1)$. As in [45, 9.5.2] we define

$$R_v := \mathbb{Z}[X_v, q^{1/2}, q^{-1/2}]$$

where $X_v$ is $T(\mathbb{Q}_v)/T(\mathbb{Z}_v)$ and write $[t]$ for the image of $t$ in $X_v$. Let $\mathcal{H}_K$ be the abstract Hecke ring with respect to the level group $K$. There is a Satake map: $\mathcal{S}_K : \mathcal{H}_K \to R_v$ given by

$$\mathcal{S}_K(KgK) = \sum \delta_B^{1/2}(t_i)[t_i]$$

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if $KgK = \cup t_i n_i K$ for $t_i \in T(\mathbb{Q}_v)$, $n_i \in N_B(\mathbb{Q}_v)$ and extend linearly. We define the Hecke operators $T_i$ for $i = 1, 2, 3, 4$ by requiring that

$$1 + \sum_{i=1}^{4} S(T_i)X^i = \prod_{i=1}^{2}(1 - q^3 t_i X)(1 - q^3 [t_i]^{-1} X)$$

is an equality of polynomials of the variable $X$. We also define:

$$Q_v(X) := 1 + \sum_{i=1}^{4} T_i(Z_0X)^i.$$  

Unramified Split Case

Suppose $v$ is a prime of $F$ split in $K$. In this case we define $z_0(1)$ and $z_0(2)$ to be $(\text{diag}(w_v, w_v, w_v, w_v), 1)$ and $(1, \text{diag}(w_v, w_v, w_v, w_v))$ and define the Hecke operators $Z_0(1)$ and $Z_0(2)$ as above but replacing $z_0$ by $z_0(1)$ and $z_0(2)$. Let $t_i(1) := \text{diag}(1, (w_v, 1), 1, (w_v^{-1}), 1), t_i(2) := \text{diag}((w_v, 1), 1, (w_v^{-1}), 1)$. Define $t_i := t_i(1)^{2}$ and $t_i = t_i(1)^{2}$ for $i = 1, 2$. We define $R_v$ and $S_K$ in the same way as the inert case. Then we define Hecke operators $T_i(j)$ for $i = 1, 2, 3, 4$ and $j = 1, 2$ by requiring that the following

$$1 + \sum_{i=1}^{4} S_K(T_i(j)^i)X^i = \prod_{i=1}^{2}(1 - q^3 t_i(j) X)(1 - q^3 [t_i(j)]^{-1} X)$$

to be equalities of polynomials of the variable $X$. Here $j' = 3 - j$ and $[t_i(j)]$'s are defined similarly to the inert case. Now let $v = w\bar{w}$ for $w$ a place of $K$. Define $i_w = 1$ and $i_{\bar{w}} = 2$. Then we define:

$$Q_w(X) := 1 + \sum_{i=1}^{4} T_i(i_w)(Z_0(3-i_w)X)^i,$$

$$Q_{\bar{w}}(X) := 1 + \sum_{i=1}^{4} T_i(i_{\bar{w}})(Z_0(3-i_{\bar{w}})X)^i.$$

9.2 Eisenstein Ideals

Let $K_D$ be an open compact subgroup of $\text{GU}(3,1)(\mathbb{A}_{\mathbb{Q}})$ maximal at $p$ and all primes outside $\Sigma$ such that the Klingen Eisenstein series we construct is invariant under $K_D$. We consider the ring $T_D$ of reduced Hecke algebras acting on the space of $\Lambda^D$-adic nearly ordinary cuspidal forms with level group $K_D$. It is generated by the Hecke operators $Z_{v,0}$, $Z_{v,0}$, $T_{i,v}$, $T_{i,v}$ defined above and then taking the maximal reduced quotient. It is well known that one can interpolate the Galois representations attached to nearly ordinary cusp forms to get a pseudo-character $R_D$ of $G_K$ with values in $T_D$. We define the ideal $I_D$ of $T_D$ to be generated by $\{t - \lambda(t)\}$ for $t$'s in the abstract Hecke algebra and $\lambda(t)$ is the Hecke eigenvalue of $t$ on $E_{D,\text{Kling}}$. Then it is easy to see that the structure map $\hat{\Lambda}^D \rightarrow T_D/I_D$ is surjective. Suppose the inverse image of $I_D$ in $\hat{\Lambda}^D$ is $\mathcal{E}_D$. We call it the Eisenstein ideal. It measures the congruences between the Hecke eigenvalues of cusp forms and Klingen Eisenstein series. We have:

$$R_D(\text{mod} I_D) \equiv \text{tr}\rho_{E_{D,\text{Kling}}}(\text{mod} \mathcal{E}_D).$$

Now we prove the following lemma:
Lemma 9.1. Let \( P \) be a height 1 prime of \( \mathbb{F}_{1,2}[[\Gamma_1]] \) which is not the pullback of a height 1 prime of \( \mathbb{F}_{1,2} \). Then

\[
\text{ord}_P(L_{\ell, \kappa}^v) \leq \text{ord}_P(\mathcal{E}_D).
\]

Proof. Suppose \( t := \text{ord}_P(L_{\ell, \kappa}^v) > 0 \). By the fundamental exact sequence Theorem 3.6 there is an \( H = E_{D, \text{Kling}} - L_{\ell, \kappa}^v F \) for some \( \Lambda_D \)-adic form \( F \) such that \( H \) is a cuspidal family. We write \( \ell \) for the functional \( \ell(G) = \int l_{\ell} (G) d\mu_{\pi(g)} \) constructed in subsection 8.5 on the space of \( \Lambda_D \)-adic forms. By our assumption on \( P \) we have proved that \( \ell(H) \not\equiv 0(\text{mod} P) \). Consider the \( \Lambda_D \)-linear map:

\[
\mu : \mathbb{T}_D \to \Lambda_{D,P}/P^t \Lambda_{D,P}
\]

given by: \( \mu(t) = \ell(tH)/\ell(H) \) for \( t \) in the Hecke algebra. Then:

\[
\ell(tH) \equiv \ell(tE_D) \equiv \lambda(t) \ell(E_D) \equiv \lambda(t) \ell(H)(\text{mod} P^t)
\]

so \( I_D \) is contained in the kernel of \( \mu \). Thus it induces: \( \Lambda_{D,P}/\mathcal{E}_D \Lambda_{D,P} \to \Lambda_{D,P}/P^t \Lambda_{D,P} \) which proves the lemma. \( \square \)

9.3 Galois Theoretic Argument

In this section, for ease of reference we repeat the set-up and certain results from [45, Chapter 4] with some modifications, which are used to construct elements in the Selmer group.

Let \( G \) be a group and \( C \) a ring. \( r : \text{Aut}_C(V) \) a representation of \( G \) with \( V \simeq C^n \). This can be extended to \( r : C[G] \to \text{End}_C(V) \). For any \( x \in C[G] \), define: \( \text{Ch}(r,x,T) := \det(id - r(x)T) \in C[T] \).

Let \((V_1, \sigma_1)\) and \((V_2, \sigma_2)\) be two \( C \)-representations of \( G \). Assume both are defined over a local henselian subring \( B \subseteq C \), we say \( \sigma_1 \) and \( \sigma_2 \) are residually disjoint modulo the maximal ideal \( \mathfrak{m}_B \) if there exists \( x \in B[G] \) such that \( \text{Ch}(\sigma_1, x, T) \mod \mathfrak{m}_B \) and \( \text{Ch}(\sigma_2, x, , T) \mod \mathfrak{m}_B \) are relatively prime in \( \kappa_B[T] \), where \( \kappa_B := B/\mathfrak{m}_B \).

Let \( H \) be a group with a decomposition \( H = G \times \{1,c\} \) with \( c \in H \) an element of order two normalizing \( G \). For any \( C \) representations \((V,r)\) of \( G \) we write \( r^c \) for the representation defined by \( r^c(g) = r(egc) \) for all \( g \in G \).

Polarizations:

Let \( \theta : G \to \text{GL}_L(V) \) be a representation of \( G \) on a vector space \( V \) over field \( L \) and let \( \psi : H \to L^\times \) be a character. We assume that \( \theta \) satisfies the \( \psi \)-polarization condition:

\[
\theta^c \simeq \psi \otimes \theta^\vee.
\]

By a \( \psi \)-polarization of \( \theta \) we mean an \( L \)-bilinear pairing \( \Phi_\theta : V \times V \to L \) such that

\[
\Phi_\theta(\theta(g)v,v') = \psi(g)\Phi_\theta(v,\theta^c(g)^{-1}v').
\]

Let \( \Phi'_\theta(v,v') := \Phi_\theta(v',v) \), which is another \( \psi \)-polarization. We say that \( \psi \) is compatible with the polarization \( \Phi_\theta \) if

\[
\Phi'_\theta = -\psi(c)\Phi_\theta.
\]
Suppose that:
(1) $A_0$ is a pro-finite $\mathbb{Z}_p$ algebra and a Krull domain;
(2) $P \subset A_0$ is a height one prime and $A = \hat{A}_{0,P}$ is the completion of the localization of $A_0$ at $P$. This is a discrete valuation ring.
(3) $R_0$ is local reduced finite $A_0$-algebra;
(4) $Q \subset R_0$ is prime such that $Q \cap A_0 = P$ and $R = \hat{R}_{0,Q}$;
(5) there exist ideals $J_0 \subset A_0$ and $I_0 \subset R_0$ such that $I_0 \cap A_0 = J_0, A_0/J_0 = R_0/I_0, J = J_0 A, I = I_0 R, J_0 = J \cap A_0$ and $I_0 = I \cap R_0$;
(6) $G$ and $H$ are pro-finite groups; we have subgroups $D_i \subset G$ for $i = 1, \cdots , d$.

Set Up:
suppose we have the following data:
(1) a continuous character $\nu : H \to A_0^\times$;
(2) a continuous character $\xi : G \to A_0^\times$ such that $\bar{\chi} \neq \bar{\nu}\bar{\chi}^{-c}$; Let $\chi' := \nu\chi^{-c}$;
(3) a representation $\rho : G \to \text{Aut}_{A(V), V \simeq A^n}$, which is a base change from a representation over $A_0$, such that:
\[ a. \rho^c \simeq \rho^\nu \otimes \nu, \]
\[ \bar{\rho} \text{ is absolutely irreducible }, \]
\[ \rho \text{ is residually disjoint from } \chi \text{ and } \chi'; \]
(4) a representation $\sigma : G \to \text{Aut}_{R \otimes_A F}(M), M \simeq (R \otimes_A F)^m$ with $m = n + 2$, which is defined over the image of $R_0$ in $R$, such that:
\[ a. \sigma^c \simeq \sigma^\nu \otimes \nu, \]
\[ b. \text{tr}\sigma(g) \in R \text{ for all } g \in G, \]
\[ c. \text{for any } v \in M, \sigma(R[G])v \text{ is a finitely-generated } R\text{-module } \]
(5) a proper ideal $I \subset R$ such that $J := A \cap I \neq 0$, the natural map $A/J \to R/I$ is an isomorphism, and
\[ \text{tr}\sigma(g) \equiv \chi'(g) + \text{tr}\rho(g) + \chi(g) \mod I \]
for all $g \in G$.
(6) $\rho$ is irreducible and $\nu$ is compatible with $\rho$.
(7) (local conditions for $\sigma$) For $p = v\bar{v}$ there is a $G_{\bar{v}}$-stable sub-$R \otimes_A F$-module $M^+_v \subseteq M$ such that $M^+_v$ and $M^-_v := M/M^+_v$ are free $R \otimes_A F$ modules.
(8) (compatibility with the congruence condition) For $p = v\bar{v}$ Assume that for all $x \in R[G_{\bar{v}}]$, we have congruence relation:
\[ \text{Ch}(M^+_v, x, T) \equiv (1 - T\chi(x)) \mod I \]
then we automatically have:
\[ \text{Ch}(M^-_v, x, T) \equiv \text{Ch}(V_v, x, T)(1 - T\chi'(x)) \mod I) \]
(9) For each $F$-algebra homomorphism $\lambda : R \otimes_A F \to K$, $K$ a finite field extension of $F$, the representation $\sigma_\lambda : G \to \text{GL}_m(M \otimes_{R \otimes A} F)$ obtained from $\sigma$ via $\lambda$ is either absolutely irreducible or contains an absolutely irreducible two-dimensional sub $K$-representation $\sigma'_\lambda$ such that $\text{tr}(\sigma'_\lambda(g)) \equiv \chi(g) + \chi'(g) \mod I$.

One defines the Selmer groups $X_H(\chi'/\chi) := \ker\{H^1(G, A_0^*(\chi'/\chi)) \to H^1(D, A_0^*(\chi(\chi)))^*\}$ and $X_G(\rho_0 \otimes \chi^{-1}) := \ker\{H^1(G, V_0 \otimes_{A_0} A_0^*(\chi^{-1})) \to H^1(D, V_0^{-} \otimes_{A_0} A_0^*(\chi^{-1}))^*\}$. Let $\text{Ch}_H(\chi'/\chi)$ and $\text{Ch}_G(\rho_0 \otimes \chi^{-1})$ be their characteristic ideals as $A_0$-modules.

**Proposition 9.2.** Under the above assumptions, if $\text{ord}_P(\text{Ch}_H(\chi'/\chi)) = 0$ then

$$\text{ord}_P(\text{Ch}_G(\rho_0 \otimes \chi^{-1})) \geq \text{ord}_P(J).$$

**Proof.** This can be proved in the same way as [45, Corollary 4.16]. The only difference is the Selmer condition at $p$ where we use the description of Section 3.3 to take care of. Note that the part corresponding to $\rho_f$ corresponds to the upper-left two by two block here while in [45] the $\rho_f$ contains the highest and the lowest Hodge-Tate weights.

Before proving the main theorem we first prove a useful lemma, which appears in an earlier version of [45].

**Lemma 9.3.** Let $Q \subset \mathbb{I}[[\Gamma_K]]$ be a height one prime such that $\text{ord}_Q(\mathcal{L}^\Sigma_{f, K, \xi}) \geq 1$ and $\text{ord}_Q(\mathcal{L}_{\chi_0, \xi}) = 0$, then $\text{ord}_Q(\mathcal{L}^\Sigma_{\chi_0, \xi}) = 0$.

**Proof.** Let $\theta = \chi_0 \xi^\ell$. If $\text{ord}_Q(\mathcal{L}^\Sigma_{\chi_0, \xi^\ell}) \geq 1$. Then for some $\ell \in \Sigma \setminus \{p\}$,

$$\prod_{\ell \in \Sigma \setminus \{p\}} (1 - \theta^{-1}(\gamma_+^{-1}(1 + W))^e \ell^{2-\kappa}) \in Q,$$

where $e \in \mathbb{Z}_p$ be such that $\ell = \omega^{-1}(\ell)(1 + p)^e$.

$$\theta(\ell) \equiv \gamma_+^{-e} \omega(\ell)^{\kappa-2}(1 + p)^{e(2-\kappa)} \pmod{Q}.$$ 

Thus there is some integer $f$ such that:

$$1 \equiv (\gamma_+ + W)^{-1}(1 + p)^{\kappa-2} - f^e \pmod{Q}$$

which implies that for some $p$-power root of unity $\zeta_+$, $Q$ is contained in the kernel of any $\phi'$ such that $\phi'(\gamma_+(1 + W)^{-1}) = \zeta_+(1 + p)^{2-\kappa}$. This implies, by the work of [11] for the interpolation formula, that at the central critical point $L_K(f, \theta_1, 1) = 0$ where $\theta_1$ is some fixed CM character of infinite type $(\frac{1}{2}, -\frac{1}{2})$ and $\phi$ any arithmetic point. But then we can specialize $f$ to some point $\phi''$ of weight 4 (this is not an arithmetic point in our definition, but is an interpolation point, by loc.cit).

By temperedness the specialization is not 0. This is a contradiction. 

Now we apply the above result to prove the theorem.

- $H := G_{Q, \Sigma}$, $G = G_{K, \Sigma}$, $c$ is complex conjugation.
- $A_0 = \hat{A}_D^w$, $A := \hat{A}_{D, p}$.  

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\[ J_0 := \mathcal{E}_D, J := \mathcal{E}_D A. \]
\[ R_0 := \mathbb{T}_D, I_0 := I_D. \]
\[ Q \subset R_0 \] is the inverse image of \( P \) modulo \( \mathcal{E}_D \) under \( \mathbb{T}_D \rightarrow \mathbb{T}_D/I_D = \Lambda_D/\mathcal{E}_D. \)
\[ R := T_\xi \otimes \Lambda_D, \rho_0 := \rho_\xi \sigma_\psi/\xi e^{-3/2}. \]
\[ V = V_0 \otimes_{A_0} A, \rho = \rho_0 \otimes_{A_0} A. \]
\[ \chi = \sigma_\psi, \chi' = \sigma_\psi\sigma_\psi/\xi e^{-3}. \]
\[ M := (R \otimes A F_A)^4, F_A \] is the fraction field of \( A. \)
\[ \sigma \] is the representation on \( M \) obtained as the pseudo-representation associated to \( \mathbb{T}_D \), as in \([45, Proposition 7.2.1]\).  

Now we are ready to prove the main theorem in the introduction.

**Proof.** We first remark that we need only to prove the corresponding inclusion for the \( \Sigma \)-primitive Selmer groups and \( L \)-functions since locally the size of the unramified extensions at primes outside \( p \) are controlled by the local Euler factors of the \( p \)-adic \( L \)-functions since \( \mathbb{Q}_\infty \subseteq \mathcal{K}_\infty. \) (See \([8, Proposition 2.4]\)).

Recall that we have enlarged our \( I \) at the end of Subsection \([8,4]\) which we denote as \( \mathcal{I} \) in this proof. We first prove the main theorem with \( \hat{\mathcal{I}}^{ur} \) replaced by \( \hat{\mathcal{J}}^{ur} \). Under the assumption of Theorem \([1,1]\) as in \([45]\) we know that by the discussion for the anticyclotomic \( \mu \)-invariant, \( \mathcal{L}_{\Sigma,K,\xi} \) is not contained in any height 1 prime which is the pullback of a prime in \( \hat{\mathcal{I}}^{ur}[\mathcal{K}] \), and so is \( \mathcal{L}_{\Sigma,K,\xi} \). Thus by lemma \([9,1]\) for any such height one prime \( P \) of \( \hat{\mathcal{I}}^{ur}[\mathcal{K}] \),

\[ \text{ord}_P(\mathcal{L}_{\Sigma,K,\xi}) = \text{ord}_P(\mathcal{L}_{\Sigma,K,\xi}) \leq \text{ord}_P(\mathcal{E}_D). \]

Applying proposition \([9,2]\) we prove the first part of the theorem for \( \hat{\mathcal{J}}^{ur} \) in place of \( \mathcal{I} \).

We replace \( \hat{\mathcal{J}}^{ur} \) by \( \hat{\mathcal{I}}^{ur} \). We write \( \mathcal{L} \) for \( \mathcal{L}_{\Sigma,K,\xi} \). Thus \( \text{Fitt}(X \otimes \hat{\mathcal{I}}^{ur} \hat{\mathcal{J}}^{ur}) = \text{Fitt}(X) \otimes \hat{\mathcal{I}}^{ur} \hat{\mathcal{J}}^{ur} \). We claim that for any \( x \in \text{Fitt}(X), x \mathcal{L}^{-1} \in \hat{\mathcal{I}}^{ur}[\mathcal{K}] \). In fact from what we proved for \( \hat{\mathcal{J}}^{ur} \) we have \( \text{Fitt}(V \otimes \hat{\mathcal{I}}^{ur} \hat{\mathcal{J}}^{ur}) \subset (\mathcal{L}) \) as ideals of \( \hat{\mathcal{J}}^{ur}[\mathcal{K}] \). Note \( x \mathcal{L}^{-1} \in \hat{\mathcal{I}}^{ur}[\mathcal{K}] \cap F^{ur}[\mathcal{K}] \) where \( F^{ur}[\mathcal{K}] \) is the fraction field of \( \hat{\mathcal{J}}^{ur}[\mathcal{K}] \). Since \( \hat{\mathcal{I}}^{ur} \) is normal and \( \hat{\mathcal{J}}^{ur} \) is finite over \( \hat{\mathcal{I}}^{ur} \). We have \( x \mathcal{L}^{-1} \in \hat{\mathcal{I}}^{ur}[\mathcal{K}] \). Thus \( \text{Fitt}(X) \subset (\mathcal{L}) \), this in turn implies that \( \text{char}(X) \subset (\mathcal{L}) \). This proves Theorem \([1,1]\).

Now assume we are under the assumption of Theorem \([1,2]\). Note that in this case \( \mathcal{L}^{\xi}_{\mathcal{K}} = 1 \). Thus by the last lemma \( \mathcal{L}_{\Sigma,K,\xi}^{\xi} \) is coprime to \( \mathcal{L}_{\Sigma,K,\xi}^{\xi} \). Suppose \( P_1, \ldots, P_t \) are the height 1 primes of \( \mathcal{L}_{\Sigma,K,\xi}^{\xi} \) that are pullbacks of height 1 primes in \( \hat{\mathcal{I}}^{ur} \). Note that none of the primes passes through \( \phi_0 \) since the 2 variable \( p \)-adic \( L \)-function is not identically 0. We consider the ring \( \hat{\mathcal{I}}^{ur}[[\mathcal{K}][/P_1, \ldots, P_t]] \) where the subscripts denote localizations. Then the argument as in (1) proves that \( \mathcal{L}_{\Sigma,K,\xi}^{\xi} \supset \text{Fitt}_{\hat{\mathcal{I}}^{ur}[[\mathcal{K}]/P_1, \ldots, P_t]}(X_{\phi_0}) \) as ideals of \( \hat{\mathcal{I}}^{ur}[[\mathcal{K}]/P_1, \ldots, P_t] \). Specialize with respect to \( \phi_0 \) we find:

\[ \mathcal{L}_{\Sigma,K,\xi}^{\xi} \supset \text{Fitt}_{\hat{\mathcal{I}}^{ur}[[\mathcal{K}]/\xi P_1, \ldots, P_t]}(X_{\phi_0}). \]
Thus

\[
\text{Fitt}_{L_\Sigma[[\Gamma_K]]\otimes \mathbb{Q}_p} (X_\Sigma^\Gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/(\ker \phi_0') X_\Sigma^\Gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \subseteq (L_{\Sigma f_0, K, \xi}).
\]

Thus

\[
\text{char}_{L_\Sigma[[\Gamma_K]]\otimes \mathbb{Q}_p} (X_\Sigma^\Gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/(\ker \phi_0') X_\Sigma^\Gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \subseteq (L_{\Sigma f_0, K, \xi}).
\]

By Proposition 2.4 (which is also true if we replace the ring \( \hat{\mathcal{O}}_{\text{ur}}[[\Gamma_K]] \) by \( \hat{\mathcal{O}}_{\text{ur}}[[\Gamma_K], p, P_1, \ldots, P_t] \)),

\[
\text{char}_{\hat{\mathcal{O}}_{\text{ur}}[[\Gamma_K]]\otimes \mathbb{Q}_p} (X_\Sigma^\Gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/(\ker \phi_0') X_\Sigma^\Gamma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \text{char}_{\hat{\mathcal{O}}_{\text{ur}}[[\Gamma_K]]\otimes \mathbb{Q}_p} (X_0^\Sigma \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)
\]

and we conclude Theorem 1.2.

References


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