Iwasawa Main Conjecture for Rankin-Selberg p-adic L-functions

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Abstract

In this paper we prove that the p-adic L-function that interpolates the Rankin-Selberg product of a general modular form and a CM form of higher weight divides the characteristic ideal of the corresponding Selmer group. This is one divisibility of the Iwasawa main conjecture for this p-adic L-function. We prove this conjecture using congruences between Klingen Eisenstein series and cusp forms on the group $\mathrm{GU}(3,1)$, following the strategy of a recent work of C. Skinner and E. Urban. The actual argument is, however, more complicated due to the need to work with general Fourier-Jacobi expansions. This theorem is used to deduce a converse of Gross-Zagier-Kolyvagin theorem and the p-adic part of the precise BSD formula in the rank one case.

1 Introduction

Let p be an odd prime. Let $\mathcal{K} \subset \mathbb{C}$ be an imaginary quadratic field such that p splits in $\mathcal{K}$ as $(p) = v_0\bar{v}_0$. We fix an isomorphism $\iota : \mathbb{C}_p \simeq \mathbb{C}$ and suppose v_0 is determined by ι. There is a unique $\mathbb{Z}_p^2$-extension $\mathcal{K}_\infty/\mathcal{K}$ unramified outside p. Let $\Gamma_\mathcal{K} := \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$. Suppose $\mathbf{f}$ is a Hida family of ordinary cuspidal eigenforms new outside p with coefficient ring $\mathbb{I}$, a normal finite extension of the power series ring $\mathbb{Z}_p[[W]]$ of one variable W. Let L be a finite extension of $\mathbb{Q}_p$ with integer ring $\mathcal{O}_L$. Suppose ξ is an L-valued Hecke character of $\mathbb{A}_\mathcal{K}^\times/\mathcal{K}^\times$ whose infinite type is $(\frac{\kappa}{2}, -\frac{\kappa}{2})$ for some even integer $\kappa \geq 6$ and such that $\mathrm{ord}_{v_0}(\mathrm{cond}(\xi_{v_0})) \leq 1$ and $\mathrm{ord}_{\bar{v}_0}(\mathrm{cond}(\xi_{\bar{v}_0})) \leq 1$. Denote by $\boldsymbol{\xi}$ the $\mathcal{O}_L[[\Gamma_\mathcal{K}]]$-adic family of Hecke characters containing ξ as some specialization (we make this precise later). In this paper we associate with $\mathbf{f}$, $\mathcal{K}$ and $\boldsymbol{\xi}$ a dual Selmer group $X_{\mathbf{f},\mathcal{K},\xi}$, which is a finite module over the ring $\mathbb{I}[[\Gamma_\mathcal{K}]]$. On the other hand, there is a p-adic L-function $\mathcal{L}_{\mathbf{f},\mathcal{K},\xi} \in F_{\mathbb{I}[[\Gamma_\mathcal{K}]]}$ interpolating the algebraic parts of the special L-values $L_\mathcal{K}(f_\phi, \xi_\phi, \frac{\kappa}{2})$, where f_ϕ and ξ_ϕ are elements in the families $\mathbf{f}$ and $\boldsymbol{\xi}$ and F_A is the fraction field of A (f_ϕ has weight 2 and ξ_ϕ has infinite type $(\kappa/2, -\kappa/2)$. We let $\mathcal{L}_{f_0,\mathcal{K},\xi}$ be the specialization of $\mathcal{L}_{\mathbf{f},\mathcal{K},\xi}$ to a single form f_0 of weight 2 and trivial character in the family $\mathbf{f}$, which we assume is defined over L. We write $\hat{\mathcal{O}}_L^{ur}$ for the completion of the maximal unramified extension of $\mathcal{O}_L^{ur}$ and $\hat{\mathbb{I}}^{ur}$ for the normalization of the ring corresponding to an irreducible component of $\mathbb{I}\hat{\otimes}_{\mathcal{O}_L}\hat{\mathcal{O}}_L^{ur}$. For a finite set of primes Σ containing all bad primes, we construct in this paper the "Σ-primitive" p-adic L-functions $\mathcal{L}^\Sigma_{\mathbf{f},\xi,\mathcal{K}} \in \hat{\mathbb{I}}^{ur}[[\Gamma_\mathcal{K}]]$, $\mathcal{L}^\Sigma_{f,\xi,\mathcal{K}} \in \hat{\mathcal{O}}_L^{ur}[[\Gamma_\mathcal{K}]]$ using the doubling method. The general case is obtained by putting back the local Euler factors at primes in Σ. In Section 7 we also recall closely related p-adic L-function constructed by Hida: $\mathcal{L}^{\Sigma,Hida}_{\mathbf{f},\xi,\mathcal{K}}$, $\mathcal{L}^{Hida}_{\mathbf{f},\xi,\mathcal{K}}$. The Iwasawa-Greenberg main conjecture for $\mathcal{L}_{\mathbf{f},\mathcal{K},\xi}$ essentially states that the characteristic ideal (to be defined later) of $X_{\mathbf{f},\mathcal{K},\xi}$ is generated by $\mathcal{L}_{\mathbf{f},\mathcal{K},\xi}$. We also associate with f, $\mathcal{K}$, ξ a dual Selmer group $X_{f,\mathcal{K},\xi}$ over $\hat{\mathcal{O}}_L^{ur}[[\Gamma_\mathcal{K}]]$. The Iwasawa-Greenberg main conjecture says that its characteristic ideal of $X_{f,\mathcal{K},\xi}$ is generated by $\mathcal{L}_{f,\mathcal{K},\xi}$.

Let $\bar{\mathbb{Q}} \subset \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$ and let $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the Galois gruop. Let $G_p \subset G_{\mathbb{Q}}$ be the decomposition group determined by the inclusion $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$ coming from ι. We write ϵ for the cyclotomic character and ω for the Techimuller character of $G_{\mathbb{Q}}$.

Let g be a cuspidal eigenform on $\mathrm{GL}_2/\mathbb{Q}$ with the associated p-adic Galois representation $\rho_g : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_L)$. We say g satisfies (**irred**) if:

- The residual representation $\bar{\rho}_g$ is absolutely irreducible.

If g is nearly ordinary at p, then $\rho_g|_{G_p}$ is equivalent to an upper triangular representation and we say it satisfies (**dist**) if:

- The characters of $\rho_g|_{G_p}$ on the diagonal are distinct modulo the maximal ideal of $\mathcal{O}_L$.

We will see later (in Section 7) that if the CM form g_ξ associated to ξ satisfies (**irred**) and (**dist**) then $\mathcal{L}_{\mathbf{f},\mathcal{K},\xi} \in \mathbb{I}[[\Gamma_{\mathcal{K}}]]$.

In this paper, under certain conditions on $\mathbf{f}, \xi, \mathcal{K}$, we prove one inclusion (or divisibility) of the Iwasawa-Greenberg main conjecture for $\mathcal{L}_{\mathbf{f},\xi,\mathcal{K}}$. Our first theorem is a three-variable result for Hida families.

Theorem 1.1. *Let* $\mathbf{f}$ *be a Hida family of ordinary eigenforms that are new outside* p *of square-free tame level* N*, and suppose* $\mathbf{f}$ *has a weight two specialization* f *that has trivial nebentypus and is the ordinary stabilization of a newform of level* N*. Let* $\bar{\rho}$ *be the* mod p *residual* $G_{\mathbb{Q}}$*-representation associated with the Hida family* $\mathbf{f}$*. Let* ξ *be a Hecke character of* $\mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times$ *with infinite type* $(\frac{\kappa}{2}, -\frac{\kappa}{2})$ *for some* $\kappa \geq 6$*. If*

(a) $p \geq 5$*;*

(b) $\xi|_{\mathbb{A}_{\mathbb{Q}}^\times} = \omega \circ \mathrm{Nm}$ *and* $\kappa \equiv 0 (\mathrm{mod}\ 2(p-1))$*;*

(c) $\bar{\rho}_{\mathbf{f}}|_{G_{\mathcal{K}}}$ *is irreducible;*

(d) there exists $q|N$ *that does not split in* $\mathcal{K}$ *and such that* $\bar{\rho}_{\mathbf{f}}$ *is ramified at* q*;*

(e) the CM eigenform g_ξ *associated to the character* ξ *satisfies* (**dist**) *and* (**irred**)*;*

(f) For each non-split prime v *of* $\mathbb{Q}$ *we have the conductor of* ξ_v *is not* (ϖ_v) *where* ϖ_v *is a uniformizer of the integer ring of* $\mathcal{K}_v$*, and that*

$$\epsilon(\pi_v, \xi_v, \frac{1}{2}) = 1.$$

(As in [18] the $\epsilon(\pi_{f,v}, \xi_v, \frac{1}{2})$ *is the local root number for the base change of* $\pi_{f,v}$ *to* $\mathcal{K}_v$ *twisted by* ξ_v*. It differs from the local root number for the Rankin-Selberg product of* $\pi_{f,v}$ *and* $g_{\xi,v}$ *by a factor* $\chi_{\mathcal{K}/\mathbb{Q},v}(-1)$*.)*

Then $\mathcal{L}^{Hida}_{\mathbf{f},\xi,\mathcal{K}} \in \hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$ *and* $(\mathcal{L}^{Hida}_{\mathbf{f},\mathcal{K},\xi}) \supseteq \mathrm{char}_{\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]}(X_{\mathbf{f},\mathcal{K},\xi})$ *as ideals of* $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$*. Here* char *means the characteristic ideal.*

We also have a two variable theorem for a single form.

Theorem 1.2. *Let N, f, κ and ξ be as before. If*

(a) $p \geq 5$;

(b) the p-adic avatar of $\xi|\cdot|^{\kappa/2}(\omega^{-1}\circ \mathrm{Nm})$ *factors through* $\Gamma_{\mathcal{K}}$ *and* $\kappa \equiv 0 (\mathrm{mod}\ 2(p-1))$;

(c) $\bar{\rho}_f|_{G_{\mathcal{K}}}$ *is irreducible;*

(d) there exists $q|N$ *that does not split in* $\mathcal{K}$ *and such that* $\bar{\rho}_f$ *is ramified at* q.

Then

$$(\mathcal{L}_{f,\mathcal{K},\xi}) \supseteq \mathrm{char}_{\hat{\mathcal{O}}_L^{ur}[[\Gamma_{\mathcal{K}}]]\otimes_{\mathcal{O}_L} L}(X_{f,\mathcal{K},\xi})$$

is true as fractional ideals of $\hat{\mathcal{O}}_L^{ur}[[\Gamma_{\mathcal{K}}]]\otimes_{\mathcal{O}_L} L$.

Hida's p-adic L-functions $\mathcal{L}^{Hida}_{\mathbf{f},\xi,\mathcal{K}}$ are more canonical than the $\mathcal{L}_{\mathbf{f},\xi,\mathcal{K}}$ in that there is a constant in $\bar{\mathbb{Q}}_p^\times$ showing up in our interpolation formula (see Proposition 7.5) that depends on some choices. Under the assumptions of Theorem 1.1 we show that Hida's p-adic L-function is integral: it belongs to $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$. Note that in the setting of Theorem 1.2 we do not know if $\mathcal{L}_{f_0,\mathcal{K},\xi}$ is actually in $\hat{\mathcal{O}}_L^{ur}[[\Gamma_{\mathcal{K}}]]$.

The assumptions on $\bar{\rho}_{\pi,\mathcal{K}}$ and the local ϵ-factors are needed to appeal to results of M. Hsieh [17], [18] in proving the non vanishing modulo p of some special L-values or vanishing of the anti-cyclotomic μ-invariant. The square-freeness of N is put at the moment for simplicity (mainly to avoid local triple product integrals for supercuspidal representations and we may come back to remove it in the future).

Hypothesis (b) of Theorem 1.2 means that $\mathcal{L}_{f,\mathcal{K},\xi}$ can be evaluated at the trivial character of $\Gamma_{\mathcal{K}}$, though it is not a point at which it interpolates classical L-values. As a result, Theorem 1.2 has interesting applications for the usual Bloch-Kato Selmer group of f.

Skinner [44] has recently been able to use Theorem 1.2 to prove a converse of the Gross-Zagier-Kolyvagin theorem: if the Mordell-Weil rank of an elliptic curve over $\mathbb{Q}$ is exactly one and the Shafarevich-Tate group is finite, then its L-function vanishes to exactly order one at the central critical point. The author has been able to prove an anti-cyclotomic main conjecture of Perrin-Riou when the root number is -1 [49] (by comparing the Selmer group in the theorem with the one studied by Perrin-Riou, using the Poitou-Tate long exact sequence and applying F. Castella's generalization [2] of a formula of Bertolini-Darmon-Prasanna relating the different p-adic L-functions). There is also an ongoing joint work of the author with Skinner and Jetchev that uses Theorem 1.2 to deduce the p-adic part of the precise BSD formula in the rank one case [50]. Finally the methods of this paper can be adapted (with some additional arguments) to the case when f is non-ordinary as well. This forms the foundation of the author's recent proof of the Iwasawa main conjecture for supersingular elliptic curves formulated by Kobayashi (see [51]). We remark that in all of the above mentioned applications one can not appeal to the main conjecture proved in [45] since the global sign of the L-functions has to be $+1$ in *loc.cit.*

Our proofs of Theorem 1.1 and Theorem 1.2 use Eisenstein congruences on the unitary group $\mathrm{U}(3,1)$, which first appeared in Hsieh's paper [19]. Recent works with a similar flavor include Skinner-Urban's [45] using the group $\mathrm{U}(2,2)$, and the work of M. Hsieh [16] for CM characters using the group $\mathrm{U}(2,1)$. The difference between our results and Skinner-Urban's is that they studied the p-adic L-function of Rankin-Selberg product of a general modular form and a CM form such that the weight of the CM form is lower, while in our case the weight of the CM form is higher. This is the very reason we work with unitary groups of different signature.

We also mention there are works establishing the other divisibility of the main conjecture using Euler systems ([49], [31]) under some more restrictions. Together with Theorem 1.1 and Theorem 1.2 these give the full equality of the main conjecture in the case when all hypotheses are satisfied.

For the reader's convenience we briefly discuss our proof of the theorems. The proof follows the main outline of Skinner-Urban's proof in [45] (which in turn followed the main outline of Wiles' proof of the Iwasawa main conjecture for totally real fields). But carrying this out requires new arguments. The main steps are: (1) constructing a p-adic family of Eisenstein series whose constant terms are essentially the p-adic Rankin-Selberg L-function $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\boldsymbol{\xi}}$; (2) proving that the Eisenstein series is coprime to p-adic L-function (that is, modulo any divisor of the p-adic L-function it is still non-zero), which shows that its congruences with cuspforms is 'measured' by the p-adic L-function; (3) the Galois argument. The main differences between our proof and that of Skinner-Urban are in steps (1) and (2). First of all we need to work with the unitary group $\mathrm{U}(3,1)$ instead of $\mathrm{U}(2,2)$ which is used in [45]. The reason is that by our assumption that the CM form has higher weight than f, the L-values interpolated by the p-adic L-function $\mathcal{L}_{\mathbf{f},\mathcal{K},\xi}$ show up in the constant terms of holomorphic Eisenstein series on the group $\mathrm{U}(3,1)$ that are induced from the Klingen parabolic subgroup with Levi $\mathrm{U}(2)\times\mathcal{K}^{\times}$. The cuspidal representations on $\mathrm{U}(2)$ is determined by the automorphic representation π_f and a Hecke character of $\mathbb{A}_{\mathcal{K}}^{\times}$ whose restriction to $\mathbb{A}_{\mathbb{Q}}^{\times}$ is the central character of π_f. As a result, the construction of the p-adic families of Eisenstein series via the pullback formula requires finding the right Siegel section at p (which turns out to be different from the one used in [45]). To have the right pullback and to make the Fourier-Jacobi coefficient computation not too hard, such choice of section is quite subtle. The idea for our choice is similar to that in [3] and is inspired by the formula for differential operators on Fourier expansions.

Step (2) is the core of the whole argument. In [45], the Klingen Eisenstein series on $\mathrm{U}(2,2)$ has a Fourier expansion $E_{Kling}=\sum_T a_T q^T$, with T running over 2×2 Hermitian matrices. By the pullback formula we have $a_T=\langle FJ_T E_{sieg},\varphi_\pi\rangle_{\mathrm{U}(1,1)}$, where E_{sieg} is a Siegel Eisenstein series on $\mathrm{U}(3,3)$, $FJ_T E_{sieg}$ is its T-Fourier-Jacobi coefficient (regarded as a form on $\mathrm{U}(1,1)$), and φ_π is a vector in the $\mathrm{U}(1,1)$ automorphic representation π considered in [45] (again determined by π_f and a Hecke character of $\mathbb{A}_{\mathcal{K}}^{\times}$). Computation tells us $FJ_T E_{sieg}$ is essentially a product of an Eisenstein series and a theta function, and thus this pairing, and hence a_T is essentially a Rankin-Selberg product. In our case, forms on $\mathrm{U}(3,1)$ only have Fourier-Jacobi expansions (instead of Fourier expansions):

$$F\mapsto\sum_{n\in\mathbb{Q}}a_n(F)q^n:=FJ(F)$$

with $a_n(F)\in H^0(\mathcal{B},\mathcal{L}(n))$, where $\mathcal{B}$ is a 2-dimensional abelian variety which is the abelian part of the universal semiabelian scheme over a point in the boundary of a toroidal compactification Shimura variety for $\mathrm{GU}(3,1)$. The $\mathcal{L}(n)$ is a line bundle on $\mathcal{B}$. We can view each $a_n(F)$ as an automorphic form on the group $\mathrm{U}(2)\cdot N$, where $\mathrm{U}(2)$ is the definite unitary group appearing as a factor of the Levi of the Klingen parabolic subgroup of $\mathrm{U}(3,1)$, and N is the unipotent radical of the parabolic, which is a Heisenberg group. It consists of matrices of the form $\begin{pmatrix} 1 & \times & \times & \times \\ & 1 & & \times \\ & & 1 & \times \\ & & & 1 \end{pmatrix}$.

To study $a_n(F)$ we use a functional l_{θ_1} on $H^0(\mathcal{B}, \mathcal{L}(n))$. This is just the pairing over N (modulo its center) with an explicit theta function θ_1 on $\mathrm{U}(2) \cdot N$. We first compute the n-th Fourier-Jacobi coefficient of a Siegel Eisenstein series, considered as a form on the Jacobi group $N' \cdot \mathrm{U}(2,2) \subseteq \mathrm{U}(3,3)$ with N' a unipotent subgroup of $\mathrm{U}(3,3)$. It consists of matrix of the form

$$\begin{pmatrix} 1 & \times & \times & \times & \times & \times \\ & 1 & & \times & & \\ & & 1 & \times & & \\ & & & 1 & & \\ & & & \times & 1 & \\ & & & \times & & 1 \end{pmatrix}.$$

This turns out to be of the form $E \cdot \Theta$, with E a Siegel Eisenstein series on $\mathrm{U}(2,2)$ and Θ a theta function on the Jacobi group. Next we restrict this n-th Fourier-Jacobi coefficient to the group

$$(N \cdot \mathrm{U}(2)) \times \mathrm{U}(2) \subset \mathrm{U}(3,1) \times \mathrm{U}(2) \cap N' \cdot \mathrm{U}(2,2).$$

Another computation shows that Θ essentially restricts to a form $\theta_2 \times \theta_3$ on $(N \cdot \mathrm{U}(2)) \times \mathrm{U}(2)$ (the actual situation is slightly more complicated, see Lemma 6.32). Applying l_{θ_1}, we show that $\langle \theta_2, \theta_1 \rangle_N$ is a constant function on $\mathrm{U}(2)$ (which we manage to make nonzero). So using the pullback formula, we get

$$l_{\theta_1}(a_n(F)) = \langle E|_{\mathrm{U}(2)\times \mathrm{U}(2)}, \varphi_\pi \cdot \theta_3 \rangle_{1 \times \mathrm{U}(2)}$$

regarded as a form on the first $\mathrm{U}(2)$, which is the $\mathrm{U}(2)$ in the Levi of the Klingen parabolic. To study its p-adic property, we pair it with an auxiliary (Hida family of) form h on $\mathrm{U}(2)$:

$$\langle \langle E|_{\mathrm{U}(2)\times\mathrm{U}(2)}, \varphi_\pi \cdot \theta_3 \rangle_{1\times \mathrm{U}(2)}, h \rangle_{\mathrm{U}(2)} = (*) \cdot \langle h, \varphi_\pi \cdot \theta_3 \rangle.$$

To obtain this formula we use the doubling method formula for $\mathrm{U}(2) \times \mathrm{U}(2) \hookrightarrow \mathrm{U}(2,2)$ applied to h. The $(*)$ is some p-adic L-function factor for h coming from this. The pairing on the right hand side is just a triple product integral $\int_{[\mathrm{U}(2)]} h(g)\theta_3(g) f(g) dg$. The fact that the θ_3 can be taken to be an eigenform follows from considering the central character (see the proof of Proposition 8.22. We use Ichino's formula to evaluate this:

$$\begin{aligned} &(\int_{[U(2)]} h(g)\theta_3(g) f(g) dg)(\int_{[U(2)]} \tilde{h}(g)\tilde{\theta_3}(g)\tilde{f}(g)dg) \\ = \langle h, \tilde{h}\rangle \langle \theta_3, \tilde{\theta_3}\rangle \langle f, \tilde{f}\rangle \cdot \quad & \frac{L^\Sigma(\frac{1}{2}, \pi \times \chi_1) L^\Sigma(\frac{1}{2}, \pi\times\chi_2)}{L^\Sigma(2, \pi_f, \mathrm{ad}) L^\Sigma(2, \pi_{\theta_3}, \mathrm{ad}) L^\Sigma(2, \pi_h, \mathrm{ad})} \prod_{v\in\Sigma} \frac{I_v(h\otimes\theta_3\otimes f, \tilde{h}\otimes\tilde{\theta_3}\otimes\tilde{f})}{\langle h_v, \tilde{h}_v\rangle \langle \theta_{3,v}, \tilde{\theta}_{3,v}\rangle \langle f_v, \tilde{f}_v\rangle}. \end{aligned}$$

Here $\tilde{}$ means some forms or vectors in the contragredient representation of the automorphic representation for h, θ_3 and f. The factor I_v is a local integral defined by Ichino, and χ_1 and χ_2 are two CM Hecke characters showing up in the computation. We interpolate everything in p-adic families and compare it to the product of several p-adic L-functions of modular forms or Hecke characters. Furthermore:

- We can choose h and θ_3 so that these p-adic L-functions are essentially units in $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]^\times$.
- The ratio of the triple product and the product of these p-adic L-functions is a product of local factors (we show that the triple product is a p-adic analytic function, so the product of these local factors is a p-adic meromorphic function). We make the local choices such that for inert or ramified primes these local factors involve only the Hida-family variable of $\mathbf{f}$ (which has nothing to do with $\Gamma^+_{\mathcal{K}}$ or $\Gamma^-_{\mathcal{K}}$). For split primes we compute these local factors explicitly.

The constructions above finally provide a nonzero element of $\mathbb{I}$, which is good enough for our use. After this we can use the same argument as in [45] to deduce our main theorem: by a geometric argument we construct a cuspidal family on $\mathrm{U}(3,1)$ congruent modulo the p-adic L-function $\mathcal{L}_{\mathbf{f},\mathcal{K},\xi}$ to the Eisenstein family constructed as above. Passing to the Galois side, we get a family of Galois representations coming from cuspidal forms that is congruent to the Galois representations coming from our Klingen Eisenstein family, but which is "more irreducible" than the Eisenstein Galois representations. Then an argument (the "lattice construction") of Urban gives the required elements in the dual Selmer group.

Remark 1.3. We emphasize here that the θ_1 is fixed throughout the whole p-adic family (instead of varying). Note also that the space of theta functions with given Archimedean kernel function and level group at finite places is finite dimensional. The space $H^0(\mathcal{B},\mathcal{L}(n))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is generated by a finite number of such theta functions. We will show in the text that by pairing the $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$-adic Fourier-Jacobi coefficient with one rational theta function (not necessarily p-integral!), we get an element in $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. We then show that by choosing the datum properly, this element is the product of a unit in $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$ and a non-zero element in $\bar{\mathbb{Q}}_p$, and proving it is prime to the p-adic L-function we study. Such strategy is notably different from the one adopted in [18], where Hsieh argued p-integrally and proved stronger result that the Fourier-Jacobi coefficient is already a unit. This is the very reason why we do not need to study the theory of p-integral theta functions.

Remark 1.4. In [54] the special L-value showing up in the Fourier-Jacobi expansion is the near central point of the Rankin-Selberg L-function, while in our case it is the central value of the triple product L-function. Moreover the Fourier-Jacobi coefficient considered in [54] is non-zero only when f is a CM form (see Theorem 4.12 in *loc.cit*). This is due to the fact that we are paring the Fourier-Jacobi coefficient with the product of a theta function and an auxiliary form h on $\mathrm{U}(2)$, while Zhang paired it with the theta function only (i.e. taking the h in our case to be the constant function). Our strategy has the advantage that these central L-values are accessible to various results of non-vanishing modulo p by Hsieh.

This rest of this paper is organized as follows. In section 2 we recall some backgrounds and formulate the main conjecture. In section 3 we discuss automorphic forms and p-adic automorphic forms on various unitary groups. In section 4 we recall the notion of theta functions which plays an important role in studying Fourier-Jacobi expansions as outlined above. In sections 5 and 6 we make the local and global calculations for Siegel and Klingen Eisenstein series using the pullback formula of Shimura. In section 7 we interpolate our previous calculations p-adically and construct the families. In section 8 we prove the co-primeness of (the Fourier-Jacobi coefficients of) the Klingen-Eisenstein series and the p-adic L-function. Finally, we deduce the main theorem in section 9.

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ideas and constant encouragement. I am also grateful to Ming-Lun Hsieh, from whose previous work [19], [16] the author has benefited a lot, and for answering me many questions. I would also like to thank Haruzo Hida, Eric Urban, and Bei Zhang for useful communications.

2 Backgrounds

We first introduce some notations. We will usually take a finite extension $L/\mathbb{Q}_p$ and write $\mathcal{O}_L$ for its integer ring and ϖ_L for a uniformizer. Let $G_\mathbb{Q}$ and $G_\mathcal{K}$ be the absolute Galois groups of $\mathbb{Q}$ and $\mathcal{K}$. Let $\Gamma_\mathcal{K}^\pm$ be the subgroups of $\Gamma_\mathcal{K}$ such that the complex conjugation acts by ± 1. We take topological generators $\gamma^\pm$ so that $\mathrm{rec}^{-1}(\gamma^+) = ((1+p)^{\frac{1}{2}}, (1+p)^{\frac{1}{2}})$ and $\mathrm{rec}^{-1}(\gamma^-) = ((1+p)^{\frac{1}{2}}, (1+p)^{-\frac{1}{2}})$ where $\mathrm{rec} : \mathbb{A}_\mathcal{K}^\times \to G_\mathcal{K}^{\mathrm{ab}}$ is the reciprocity map normalized by the geometric Frobenius. Let $\Psi_\mathcal{K}$ be the composition

$$G_\mathcal{K} \twoheadrightarrow \Gamma_\mathcal{K} \hookrightarrow \mathbb{Z}_p[[\Gamma_\mathcal{K}]]^\times.$$

Define $\Lambda_\mathcal{K} := \mathcal{O}_L[[\Gamma_\mathcal{K}]]$. Recall we defined a character ξ in the introduction. We will write σ_ξ for the Galois character corresponding to ξ via class field theory. We also let $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$ and let $\Gamma_\mathbb{Q} = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. Define $\Psi_\mathbb{Q}$ to be the composition $G_\mathbb{Q} \twoheadrightarrow \Gamma_\mathbb{Q} \hookrightarrow \mathbb{Z}_p[[\Gamma_\mathbb{Q}]]^\times$. We also define $\varepsilon_\mathcal{K}$ and $\varepsilon_\mathbb{Q}$ to be the compositions $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times \stackrel{rec}{\to} G_\mathcal{K}^{ab} \to \mathbb{Z}_p[[\Gamma_\mathcal{K}]]^\times$ and $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times \stackrel{rec}{\to} G_\mathbb{Q}^{ab} \to \mathbb{Z}_p[[\Gamma_\mathbb{Q}]]^\times$ where the second arrows are the $\Psi_\mathcal{K}$ and $\Psi_\mathbb{Q}$ defined above. Let ω and ϵ be the Techimuller character and the cyclotomic character.

2.1 p-adic Families for $\mathrm{GL}_2/\mathbb{Q}$

Let M be a positive integer prime to p and χ a character of $(\mathbb{Z}/pM\mathbb{Z})^\times$. Let $\Lambda_\mathbb{Q} := \mathbb{Z}_p[[W]]$ (we call $\Lambda_\mathbb{Q}$ the weight space). Let $\mathbb{I}$ be a domain finite over $\Lambda_\mathbb{Q}$. A point $\phi \in \mathrm{Spec}(\mathbb{I})$ is called arithmetic if the image of ϕ in $\mathrm{Spec}\Lambda(\bar{\mathbb{Q}}_p)$ is the continuous $\mathbb{Z}_p$-homomorphism sending $(1+W) \mapsto \zeta(1+p)^{\kappa-2}$ for some $\kappa \geq 2$ and ζ a p-power root of unity. We usually write κ_ϕ for this κ, called the weight of ϕ. We also define χ_ϕ to be the character of $\mathbb{Z}_p^\times \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times (1+p\mathbb{Z}_p)$ that is trivial on the first factor and given by $(1+p) \mapsto \zeta$ on the second factor.

Definition 2.1. *An $\mathbb{I}$-adic family of modular forms of tame level M and character χ is a formal q-expansion $\mathbf{f} = \sum_{n=0}^\infty a_n q^n, a_n \in \mathbb{I}$, such that for a Zariski dense set of arithmetic points $\phi \in \mathrm{Spec}(\mathbb{I})$ the specialization $f_\phi = \sum_{n=0}^\infty \phi(a_n)q^n$ of $\mathbf{f}$ at ϕ is the q-expansion of a modular form of weight κ_ϕ, character $\chi\chi_\phi\omega^{2-\kappa_\phi}$ (where ω is the Techimuller character), and level Mp^{t_ϕ} for some $t_\phi \geq 0$.*

The U_p operator is defined on both the spaces of modular forms and families. It is given by:

$$U_p(\sum_{n=0}^\infty a_n q^n) = \sum_{n=0}^\infty a_{pn} q^n.$$

Note that $(U_p \cdot \mathbf{f})_\phi = U_p \cdot \mathbf{f}_\phi$. Hida's ordinary idempotent e_p is defined by $e_p := \lim_{n\to\infty} U_p^{n!}$. A form f or family $\mathbf{f}$ is called ordinary if $e_p f = f$ or $e_p\mathbf{f} = \mathbf{f}$. (See for instance [9, Page 550].) A well known fact is that every ordinary eigenform fits into an ordinary family of eigenforms $\mathbf{f}$ ([10, Theorem II] for example). By results of Deligne, Langlands, Shimura et al there is a Galois representation $\rho_f : G_\mathbb{Q} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ for f. If the residual representation $\bar{\rho}_f$ is irreducible then one can construct a Galois representation $\rho_\mathbf{f} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I})$ such that it specializes to the Galois representation $\rho_{\mathbf{f}_\phi}$ of $\mathbf{f}_\phi$ at each arithmetic specialization $\phi \in \mathrm{Spec}(\mathbb{I})$.

2.2 The Main Conjecture

Before formulating the main conjecture, we first define the characteristic ideals and the Fitting ideals. We let A be a Noetherian ring. We write $\mathrm{Fitt}_A(X)$ for the Fitting ideal in A of a finitely generated A-module X. This is the ideal generated by the determinant of the $r \times r$ minors of the matrix giving the first arrow in a given presentation of X:

$$A^s \to A^r \to X \to 0.$$

If X is not a torsion A-module, then $\mathrm{Fitt}_A(X) = 0$.

Fitting ideals behave well with respect to base change. For $I \subset A$ an ideal

$$\mathrm{Fitt}_{A/I}(X/IX) = \mathrm{Fitt}_A(X) \bmod I.$$

Now suppose A is a Krull domain (a domain which is Noetherian and normal). Then the characteristic ideal is defined by:

$$\mathrm{char}_A(X) := \{x \in A : \mathrm{ord}_Q(x) \geq \mathrm{length}_Q(X) \text{ for any height one prime } Q \text{ of } A\}.$$

If X is not a torsion A-module, then we define $\mathrm{char}_A(X) = 0$.

We consider the Galois representation:

$$T_{\mathbf{f},\mathcal{K},\xi} := T_{\mathbf{f}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda_{\mathcal{K}}$$

with the $G_{\mathcal{K}}$ action given by $\rho_{\mathbf{f}} \sigma_{\bar{\xi}^c} \epsilon^{\frac{4-\kappa}{2}} \hat{\otimes} \Lambda_{\mathcal{K}}(\Psi_{\mathcal{K}}^{-c})$. We define the Selmer group (recall κ is assumed to be even)

$$\mathrm{Sel}_{\mathbf{f},\mathcal{K},\xi} := \ker\{H^1(\mathcal{K}, T_{\mathbf{f},\mathcal{K},\xi} \otimes_{\mathbb{I}[[\Gamma_{\mathcal{K}}]]} \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*) \to H^1(I_{\bar{v}_0}, T_{\mathbf{f},\mathcal{K},\xi} \otimes \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*) \times \prod_{v \nmid p} H^1(I_v, T_{\mathbf{f},\mathcal{K},\xi} \otimes \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*)$$

where $*$ means the Pontrayagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(-, \mathbb{Q}_p/\mathbb{Z}_p)$. We also define the Σ-primitive Selmer groups:

$$\mathrm{Sel}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi} := \ker\{H^1(\mathcal{K}, T_{\mathbf{f},\mathcal{K},\xi} \otimes \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*) \to H^1(I_{\bar{v}_0}, T_{\mathbf{f},\mathcal{K},\xi} \otimes \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*) \times \prod_{v \notin \Sigma} H^1(I_v, T_{\mathbf{f},\mathcal{K},\xi} \otimes \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*).$$

We let

$$X_{\mathbf{f},\mathcal{K},\xi} := (\mathrm{Sel}_{\mathbf{f},\mathcal{K},\xi})^*.$$
$$X^{\Sigma}_{\mathbf{f},\mathcal{K},\xi} := (\mathrm{Sel}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi})^*.$$

These are finitely generated $\mathbb{I}[[\Gamma_{\mathcal{K}}]]$-modules (see e.g. [45, Lemma 3.3]). We take the extension of scalars of them to $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$ and still denote them using the same notations. In section 7 we are going to construct p-adic L-functions $\mathcal{L}^{Hida}_{\mathbf{f},\mathcal{K},\xi}$ and $\mathcal{L}^{\Sigma,Hida}_{\mathbf{f},\mathcal{K},\xi}$ which are elements in $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$ or its fraction field. Their interpolation formulas are given in equation (7) (see also Remark 7.6). The 3-variable Iwasawa main conjecture is

Conjecture 2.2. *$X_{\mathbf{f},\mathcal{K},\xi}$ and $X^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}$ are torsion $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$-modules and*

$$\mathrm{char}_{\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]} X_{\mathbf{f},\mathcal{K},\xi} = (\mathcal{L}^{Hida}_{\mathbf{f},\mathcal{K},\xi}),$$

$$\mathrm{char}_{\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]} X^{\Sigma}_{\mathbf{f},\mathcal{K},\xi} = (\mathcal{L}^{\Sigma,Hida}_{\mathbf{f},\mathcal{K},\xi}).$$

We can also replace $\mathbf{f}$ with a single form f_0 and have the two-variable main conjecture

Conjecture 2.3. *$X_{f_0,\mathcal{K},\xi}$ and $X^{\Sigma}_{f_0,\mathcal{K},\xi}$ are torsion $\hat{\mathcal{O}}^{ur}_{L}[[\Gamma_{\mathcal{K}}]]$-modules and*

$$\mathrm{char}_{\hat{\mathcal{O}}^{ur}_{L}[[\Gamma_{\mathcal{K}}]]} X_{f_0,\mathcal{K},\xi} = (\mathcal{L}^{Hida}_{f_0,\mathcal{K},\xi}),$$

$$\mathrm{char}_{\hat{\mathcal{O}}^{ur}_{L}[[\Gamma_{\mathcal{K}}]]} X^{\Sigma}_{f_0,\mathcal{K},\xi} = (\mathcal{L}^{\Sigma,Hida}_{f_0,\mathcal{K},\xi}).$$

2.3 Control of Selmer Groups

In this subsection we prove a control theorem of Selmer groups which will be used to prove Theorem 1.2. Let $\phi_0 \in \mathrm{Spec}\mathbb{I}[[\Gamma_{\mathcal{K}}]](\bar{\mathbb{Q}}_p)$ to be the point mapping $\gamma^{\pm}$ to 1 and such that $\phi_0|_{\mathbb{I}}$ corresponds to a form f_0. Let $\wp = \ker\phi_0|_{\mathbb{I}}$ of weight two and trivial character. Then we prove the following proposition.

Proposition 2.4. *There is an exact sequence of $\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]$-modules*

$$M \to X^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}/\wp X^{\Sigma}_{\mathbf{f},\mathcal{K},\xi} \to X^{\Sigma}_{f_0,\mathcal{K},\xi} \to 0$$

where $M \otimes_{\mathcal{O}_L} L$ has support of codimension at least 2 in $\mathrm{Spec}\mathcal{O}_L[[\Gamma_{\mathcal{K}}]] \otimes L$.

Proof. We write $\mathbb{I}_{\mathcal{K}}$ for $\mathbb{I}[[\Gamma_{\mathcal{K}}]]$ for simplicity. Write $\mathbb{T} = T_{\mathbf{f},\mathcal{K},\xi}$ as a $\mathbb{I}_{\mathcal{K}}$-module. Let T be the $\Lambda_{\mathcal{K}}$-module $T_{f_0,\mathcal{K},\xi}$. Recall that $p = v_0\bar{v}_0$. We have an exact sequence

$$0 \to T \otimes_{\Lambda_{\mathcal{K}}} \Lambda^*_{\mathcal{K}} \to \mathbb{T} \otimes_{\mathbb{I}_{\mathcal{K}}} \mathbb{I}^*_{\mathcal{K}} \to \mathbb{T} \otimes_{\mathbb{I}_{\mathcal{K}}} (\wp\mathbb{I}_{\mathcal{K}})^* \to 0.$$

Write $G_{\mathcal{K}_\Sigma}$ for the Galois group over $\mathcal{K}$ of the maximal algebraic extension of $\mathcal{K}$ unramified outside Σ. From this we deduce

$$H^1(G_{\mathcal{K}_\Sigma}, T \otimes_{\Lambda} \Lambda^*_{\mathcal{K}}) \xrightarrow{\sim} H^1(G_{\mathcal{K}_\Sigma}, \mathbb{T} \otimes_{\mathbb{I}_{\mathcal{K}}} \mathbb{I}^*_{\mathcal{K}})[\wp]$$

as in [45, Proposition 3.7]. We also have an exact sequence:

$$\begin{aligned}&H^0(I_{\bar{v}_0}, \mathbb{T} \otimes_{\mathbb{I}[[\Gamma_{\mathcal{K}}]]} (\mathbb{I}[[\Gamma_{\mathcal{K}}]]^*)) \xrightarrow{s_1} H^0(I_{\bar{v}_0}, \mathbb{T} \otimes_{\mathbb{I}[[\Gamma_{\mathcal{K}}]]} (\wp[[\Gamma_{\mathcal{K}}]]^*))\\ &\to H^1(I_{\bar{v}_0}, T \otimes_{\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]} (\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]^*)) \to H^1(I_{\bar{v}_0}, \mathbb{T} \otimes_{\mathbb{I}[[\Gamma_{\mathcal{K}}]]} (\mathbb{I}[[\Gamma_{\mathcal{K}}]]^*)).\end{aligned}$$

From these we deduce an exact sequence of $\Lambda_{\mathcal{K}}$-modules

$$M := ((\mathrm{coker}s_1)^{G_{\bar{v}_0}})^*/\wp(\mathrm{coker}s_1)^{G_{\bar{v}_0}} \hookrightarrow X^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}/\wp X^{\Sigma}_{\mathbf{f},\mathcal{K},\xi} \to X^{\Sigma}_{f_0,\mathcal{K},\xi} \to 0.$$

Let $\mathcal{K}_{\infty,\bar{v}_0}$ (resp. $\mathcal{K}_{\infty,v_0}$) be the $\mathbb{Z}_p$-extension of $\mathcal{K}$ unramified outside $\bar{v}_0$ (resp. v_0) and let $\Gamma_{\bar{v}_0} = \mathrm{Gal}(\mathcal{K}_{\infty,\bar{v}_0})$ (resp. $\Gamma_{v_0} = \mathrm{Gal}(\mathcal{K}_{v_0}/\mathcal{K})$). Let $\gamma_{\bar{v}_0} \in \Gamma_{\bar{v}_0}$ and $\gamma_{v_0} \in \Gamma_{v_0}$ be topological generators. Note that we have

$$0 \to T^+ \to T \to T/T^+ \to 0$$

as $G_{\mathbb{Q}_p}$-modules. By the description of the Galois action, there is a $\gamma \in I_{\bar{v}_0}$ such that $\gamma - 1$ acts invertibly on $T^+ \otimes_{\mathbb{I}[[\Gamma_\mathcal{K}]]} (\mathbb{I}[[\Gamma_\mathcal{K}]]^*))$. We take a basis (v_1, v_2) such that v_1 generates T^+ and the action of γ on T is diagonal under this basis. Then it is not hard to see (by looking at the $I_{\bar{v}_0}$-action) that if

$$v \in H^0(I_{\bar{v}_0}, \mathbb{T} \otimes_{\mathbb{I}_\mathcal{K}} \mathbb{I}_\mathcal{K}^*)$$

we have $v \in (\mathbb{I}[[\Gamma_{v_0}]])^* \cdot v_2$ and if $v \in H^0(I_{\bar{v}_0}, \mathbb{T} \otimes_{\mathbb{I}_\mathcal{K}} (\wp \mathbb{I}_\mathcal{K})^*)$ then $v \in (\wp \mathbb{I}_\mathcal{K})^* v_2$. From the above discussion we know that $((\mathrm{coker} s_1)^{G_{\bar{v}_0}})^* / \ker \phi_0' (\mathrm{coker} s_1)^{G_{\bar{v}_0}}$ is supported in

$$\mathrm{Spec}\mathcal{O}_L[[\Gamma_v]] \otimes L.$$

Moreover by looking at the action of $\mathrm{Frob}_{\bar{v}_0}$ we see it is killed by the function $a_p^{-1}R - 1$ where a_p is the invertible function in $\mathbb{I}$ which gives the U_p-eigenvalue of $\mathbf{f}$ and R is the image in Γ_v of $\mathrm{Frob}_{\bar{v}_0}$ under class field theory. But $a_p(\phi_0) \neq 1$ and $R(\phi_0) = 1$ so $a_p^{-1}R - 1$ is non zero at ϕ_0. So the support of $M \otimes_{\mathcal{O}_L} L$ has support of dimension at most zero and this proves the proposition. □

3 Unitary Groups

In this section we introduce our notation for unitary groups and develop Hida theory on them. We define $S_n(R)$ to be the set of $n \times n$ Hermitian matrices with entries in $\mathcal{O}_\mathcal{K} \otimes_\mathbb{Z} R$. We define a map $e_\mathbb{A} = \prod_v e_v : \mathbb{A}_\mathbb{Q} \to \mathbb{C}^\times$ where for each place v of $\mathbb{Q}$ e_v is the usual exponential map at v. We refer to [16, Section 2.8] for the discussion of the CM period Ω_∞ and the p-adic period Ω_p. For two automorphic forms f_1, f_2 on U(2) we write $\langle f_1, f_2 \rangle = \int_{[\mathrm{U}(2)]} f_1(g) f_2(g) dg$. Here the Harr measure is normalized so that at finite places $\mathrm{U}(2)(\mathbb{Z}_\ell)$ has measure 1, and at ∞ the compact set $\mathrm{U}(1)(\mathbb{R}) \backslash \mathrm{U}(2)(\mathbb{R})$ has measure 1.

3.1 Groups

Let $\delta \in \mathcal{K}$ be a totally imaginary element such that $-i\delta$ is positive. Let $d = \mathrm{Nm}(\delta)$ which we assume to be a p-adic unit. Let $\mathrm{U}(2) = \mathrm{U}(2,0)$ (resp. $\mathrm{GU}(2) = \mathrm{GU}(2,0)$) be the unitary group (resp. unitary similitude group) associated to the skew-Hermitian matrix $\zeta = \begin{pmatrix} \mathfrak{s}\delta & \\ & \delta \end{pmatrix}$ for some $\mathfrak{s} \in \mathbb{Z}_+$ prime to p. More precisely $\mathrm{GU}(2)$ is the group scheme over $\mathbb{Z}$ defined by: for any $\mathbb{Z}$ algebra A,

$$\mathrm{GU}(2)(A) = \{g \in \mathrm{GL}_2(A \otimes_\mathbb{Z} \mathcal{O}_\mathcal{K}) | {}^t\bar{g}\zeta g = \lambda(g)\zeta, \ \lambda(g) \in A^\times.\}$$

The map $\mu : \mathrm{GU}(2) \to \mathbb{G}_m$, $g \mapsto \lambda(g)$ is called the similitude character and $\mathrm{U}(2) \subseteq \mathrm{GU}(2)$ is the kernel of μ. Let $G = \mathrm{GU}(3,1)$ (resp. $\mathrm{U}(3,1)$) be the similarly defined unitary similitude group (resp. unitary group) over $\mathbb{Z}$ associated to the skew-Hermitian matrix $\begin{pmatrix} & & 1 \\ & \zeta & \\ -1 & & \end{pmatrix}$. Let $P \subseteq G$ be the parabolic subgroup of $\mathrm{GU}(3,1)$ consisting of those matrices in G of the form $\begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & & & \times \end{pmatrix}$.

Let N_P be the unipotent radical of P. Then if $X_{\mathcal{K}}$ is the 1-dimensional space over $\mathcal{K}$,

$$M_P := \mathrm{GL}(X_{\mathcal{K}}) \times \mathrm{GU}(2) \hookrightarrow \mathrm{GU}(V),\ (a, g_1) \mapsto \mathrm{diag}(a, g_1, \mu(g_1)\bar{a}^{-1})$$

is the Levi subgroup. Let $G_P := \mathrm{GU}(2)(\subseteq M_P) \mapsto \mathrm{diag}(1, g_1, \mu(g))$. Let δ_P be the modulus character for P. We usually use a more convenient character δ such that $\delta^3 = \delta_P$.

Since p splits as $v_0\bar{v}_0$ in $\mathcal{K}$, $\mathrm{GL}_4(\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{v_0}}) \times \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{\bar{v}_0}})$. Here

$$\mathrm{U}(3,1)(\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{v_0}}) = \mathrm{GL}_4(\mathbb{Z}_p)$$

with the projection onto the first factor. Let B and N be the upper triangular Borel subgroup of $G(\mathbb{Q}_p)$ and its unipotent radical, respectively. Let

$$K_p = \mathrm{GU}(3,1)(\mathbb{Z}_p) \simeq \mathrm{GL}_4(\mathbb{Z}_p),$$

and for any $n \geq 1$ let K_0^n be the subgroup of K consisting of matrices upper-triangular modulo p^n. Let $K_1^n \subset K_0^n$ be the subgroup of matrices whose diagonal elements are 1 modulo p^n.

The group $\mathrm{GU}(2)$ is closely related to a division algebra. Put

$$D = \{g \in M_2(\mathcal{K}) | g^t \zeta \bar{g} = \det(g)\zeta\}.$$

Then D is a definite quaternion algebra over $\mathbb{Q}$ with local invariants $\mathrm{inv}_v(D) = (-\mathfrak{s}, -D_{\mathcal{K}/\mathbb{Q}})_v$ (the Hilbert symbol). The relation between $\mathrm{GU}(2)$ and D is explained by

$$\mathrm{GU}(2) = D^\times \times_{\mathbb{G}_m} \mathrm{Res}_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_m.$$

For each finite place v we write D_v^1 for the set of elements $g_v \in D_v^\times$ such that $|\mathrm{Nm}(g_v)|_v = 1$, where Nm is the reduced norm.

Let Σ be a finite set of primes containing all the primes at which $\mathcal{K}/\mathbb{Q}$ or ξ is ramified, the primes dividing the level of f_0 (as in the introduction), the primes dividing $\mathfrak{s}$, the primes such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is compact and the prime 2. Let Σ^1 and Σ^2, respectively be the set of non-split primes in Σ such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is non-compact, and compact. We will sometimes write $[D^\times]$ for $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_{\mathbb{Q}})$. We similarly write $[\mathrm{U}(2)]$, $[\mathrm{GU}(2,0)]$, etc.

We define $G_n = \mathrm{GU}(n,n)$ for the unitary similitude group for the skew-Hermitian matrix $\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$ and $\mathrm{U}(n,n)$ for the corresponding unitary groups.

3.2 Hermitian Spaces and Automorphic Forms

Let $(r,s) = (3,3)$ or $(3,1)$ or $(2,0)$. Then the unbounded Hermitian symmetric domain for $\mathrm{GU}(r,s)$ is

$$X^+ = X_{r,s} = \{\tau = \begin{pmatrix} x \\ y \end{pmatrix} | x \in M_s(\mathbb{C}), y \in M_{(r-s)\times s}(\mathbb{C}), i(x^* - x) > iy^*\zeta^{-1}y\}.$$

We use x_0 to denote the Hermitian symmetric domain for GU(2), which is just a point. We have the following embedding of Hermitian symmetric domains:

$$\iota : X_{3,1} \times X_{2,0} \hookrightarrow X_{3,3}$$

$$(\tau, x_0) \hookrightarrow Z_\tau,$$

where $Z_\tau = \begin{pmatrix} x & 0 \\ y & \frac{\zeta}{2} \end{pmatrix}$ for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $G_{r,s} = \mathrm{GU}(r,s)$ and $H_{r,s} = \mathrm{GL}_r \times \mathrm{GL}_s$. Let $G_{r,s}(\mathbb{R})^+$ be the subgroup of elements of $G_{r,s}(\mathbb{R})$ whose similitude factors are positive. If $s \neq 0$ we define a cocycle:

$$J : G_{r,s}(\mathbb{R})^+ \times X^+ \to H_{r,s}(\mathbb{C})$$

by $J(\alpha,\tau) = (\kappa(\alpha,\tau), \mu(\alpha,\tau))$, where for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\alpha = \begin{pmatrix} a & b & c \\ g & e & f \\ h & l & d \end{pmatrix}$ (blocks matrix with respect to the partition $(s + (r-s) + s)$),

$$\kappa(\alpha,\tau) = \begin{pmatrix} \bar{h}^t x + \bar{d} & \bar{h}^t y + l\bar{\zeta} \\ -\bar{\zeta}^{-1}(\bar{g}^t x + \bar{f}) & -\bar{\zeta}^{-1}\bar{g}^t y + \bar{\zeta}^{-1}\bar{e}\bar{\zeta} \end{pmatrix},\ \mu(\alpha,\tau) = hx + ly + d$$

in the GU(3, 1) case and

$$\kappa(\alpha,\tau) = \bar{h}^t x + \bar{d},\ \mu(\alpha,\tau) = hx + d$$

in the GU(3, 3) case. Let $i \in X^+$ be the point $\begin{pmatrix} i1_s \\ 0 \end{pmatrix}$. Let K_∞^+ be the compact subgroup of $\mathrm{U}(r,s)(\mathbb{R})$ stabilizing i and let K_∞ be the groups generated by K_∞^+ and $\mathrm{diag}(1_{r+s}, -1_s)$. Then

$$K_\infty^+ \to H(\mathbb{C}),\ k_\infty \mapsto J(k_\infty, i)$$

definesis an algebraic representation of K_∞^+.

Definition 3.1. *A weight* $\underline{k}$ *is defined to be an* $(r+s)$*-tuple*

$$\underline{k} = (c_{r+s}, ..., c_{s+1}; c_1, ..., c_s) \in \mathbb{Z}^{r+s}$$

with $c_1 > ... > c_{r+s}$*.*

We refer to [16, Section 3.1] for the definition of the algebraic representation $L_{\underline{k}}(\mathbb{C})$ of H with the action denoted by $\rho_{\underline{k}}$ (note the different index for weight) and define a model $L^{\underline{k}}(\mathbb{C})$ of the representation $H(\mathbb{C})$ with the highest weight $\underline{k}$ as follows. The underlying space of $L^{\underline{k}}(\mathbb{C})$ is $L_{\underline{k}}(\mathbb{C})$ and the group action is defined by

$$\rho^{\underline{k}}(h) = \rho_{\underline{k}}({}^t h^{-1}), h \in H(\mathbb{C}).$$

Our convention for identifying a weight with a tuple of integers is different from others in literature. For example our c_{s+i} $(1 \leq i \leq r)$ and c_j $(1 \leq j \leq s)$ corresponds to $-a_{r+1-i}$ and $b_{s+1,j}$ in [16, Section 3.1]. We also note that if each $\underline{k} = (0, ..., 0; \kappa, ..., \kappa)$ then $L^{\underline{k}}(\mathbb{C})$ is one dimensional.

For a weight $\underline{k}$, define $\|\underline{k}\|$ by:

$$\|\underline{k}\| := -c_{s+1} - ... - c_{s+r} + c_1 + ... + c_s$$

and $|\underline{k}|$ by:

$$|\underline{k}| = (c_1 + ... + c_s).\sigma - (c_{s+1} + ... + c_{s+r}).\sigma c \in \mathbb{Z}^I.$$

Here I is the set of embeddings $\mathcal{K} \hookrightarrow \mathbb{C}$ and σ is the Archimedean place of $\mathcal{K}$ determined by our fixed embedding $\mathcal{K} \hookrightarrow \mathbb{C}$. Let χ be a Hecke character of $\mathcal{K}$ with infinite type $|\underline{k}|$, i.e. the Archimedean part of χ is given by:

$$\chi_\infty(z) = (z^{c_1+...+c_s} \cdot \bar{z}^{-(c_{s+1}+...+c_{s+r})}).$$

Definition 3.2. *Let U be an open compact subgroup in $G(\mathbb{A}_f)$. We denote by $M_{\underline{k}}(U, \mathbb{C})$ the space of holomorphic $L^{\underline{k}}(\mathbb{C})$-valued functions f on $X^+ \times G(\mathbb{A}_f)$ such that for $\tau \in X^+$, $\alpha \in G(\mathbb{Q})^+$ and $u \in U$ we have*

$$f(\alpha\tau, \alpha g u) = \mu(\alpha)^{-\|\underline{k}\|}\rho^{\underline{k}}(J(\alpha, \tau))f(\tau, g).$$

Now we consider automorphic forms on unitary groups in the adelic language. The space of automorphic forms of weight $\underline{k}$ and level U with central character χ consists of smooth and slowly increasing functions $F : G(\mathbb{A}) \to L_{\underline{k}}(\mathbb{C})$ such that for every $(\alpha, k_\infty, u, z) \in G(\mathbb{Q}) \times K_\infty^+ \times U \times Z(\mathbb{A})$,

$$F(z\alpha g k_\infty u) = \rho^{\underline{k}}(J(k_\infty, \boldsymbol{i})^{-1})F(g)\chi^{-1}(z).$$

We can associate a $L_{\underline{k}}$-valued function on $X^+ \times G(\mathbb{A}_f)/U$ given by

$$f(\tau, g) := \chi_f(\mu(g))\rho^{\underline{k}}(J(g_\infty, \mathbf{i}))F((f_\infty, g))$$

where $g_\infty \in G(\mathbb{R})$ such that $g_\infty(\mathbf{i}) = \tau$. If this function is holomorphic then we say that the automorphic form F is holomorphic.

3.3 Galois representations Associated to Cuspidal Representations

In this section we follow [45, Theorem 7.1, Lemma 7.2] to discuss the Galois representations associated to cuspidal automorphic representations on $\mathrm{GU}(r, s)(\mathbb{A}_\mathbb{Q})$. Let π be an irreducible automorphic representation of $\mathrm{GU}(r, s)(\mathbb{A}_\mathbb{Q})$ generated by a holomorphic cuspidal eigenform with weight $\underline{k} = (c_{r+s}, ..., c_{s+1}; c_1, ..., c_s)$ and central character χ_π. Let $\Sigma(\pi)$ be a finite set of primes of $\mathbb{Q}$ containing all the primes at which π is unramified and all the primes dividing p. Then for some L finite over $\mathbb{Q}_p$ there is a Galois representation (see [42], [33] and [43]):

$$R_p(\pi) : G_\mathcal{K} \to \mathrm{GL}_n(L)$$

such that:

(a) $R_p(\pi)^c \simeq R_p(\pi)^\vee \otimes \rho_{p,\chi_\pi^{1+c}}\epsilon^{1-n}$ where $\rho_{p,\chi_\pi^{1+c}}$ denotes the p-adic Galois character associated to χ_π^{1+c} by class field theory and ϵ is the cyclotomic character.

(b) $R_p(\pi)$ is unramified at all finite places not above primes in $\Sigma(\pi)$, and for such a place $w \nmid p$:

$$\det(1 - R_p(\pi)(\mathrm{Frob}_w)q_w^{-s}) = L(BC(\pi)_w \otimes \chi^c_{\pi,w}, s + \frac{1-n}{2})^{-1}$$

Here the Frob_w is the geometric Frobenius and BC means the base change from $\mathrm{U}(r,s)$ to GL_{r+s}. Suppose π_v is nearly ordinary (see Subsection 3.8) and unramified at all primes v dividing p with respect to $\underline{k}$. Recall $v_0|p$ corresponds to $\iota:\mathbb{C}\simeq\mathbb{C}_p$. If we write $\kappa_i=s-i+c_i$ for $1\le i\le s$ and $\kappa_i=c_i+s+r+s-i$ for $s+1\le i\le r+s$, then

$$R_p(\pi)|G_{\mathcal{K},v_0}\simeq\begin{pmatrix}\xi_{r+s,v}\epsilon^{-\kappa_{r+s}} & * & * & * \\ & \xi_{r+s-1,v}\epsilon^{\kappa_{r+s-1}} & & * \\ & 0 & \cdots & * \\ & & & \xi_{1,v}\epsilon^{-\kappa_1}\end{pmatrix}$$

where $\xi_{i,v}$ are unramified characters. Using the fact (a) above we also know that $R_p(\pi)_{\bar{v}_0}$ is equivalent to an upper triangular representation as well (with the Hodge-Tate weight being $(-(\kappa_1+1-r-s-|\underline{k}|),\ldots,-(\kappa_{r+s}+1-r-s-|\underline{k}|))$ (in our geometric convention ϵ^{-1} has Hodge-Tate weight one).

3.4 Shimura varieties

Now we consider the group $\mathrm{GU}(3,1)$. For any open compact subgroup $K=K_pK^p$ of $\mathrm{GU}(3,1)(\mathbb{A}_f)$ whose p-component is $K_p=\mathrm{GU}(3,1)(\mathbb{Z}_p)$, we refer to [16, Section 2.1] for the definition and arithmetic models of the associated Shimura variety, which we denote as $S_G(K)_{/\mathcal{O}_{\mathcal{K},(v_0)}}$. The scheme $S_G(K)$ represents the following functor: for any $\mathcal{O}_{\mathcal{K},(v_0)}$-algebra R, $\underline{A}(R)=\{(A,\bar{\lambda},\iota,\bar{\eta}^p)\}$ where A is an abelian scheme over R of relative dimension four with CM by $\mathcal{O}_\mathcal{K}$ given by ι, $\bar{\lambda}$ is an orbit of prime-to-p polarizations and $\bar{\eta}^p$ is an orbit of prime-to-p level structures. There is also a theory of compactifications of $S_G(K)$ developed in [29]. We denote by $\bar{S}_G(K)$ a toroidal compactification and $S^*_G(K)$ the minimal compactification. We refer to [16, Section 2.7] for details. The boundary components of $S^*_G(K)$ is in one-to-one correspondence with the set of cusp labels defined below. For $K=K_pK^p$ as above we define the set of cusp labels to be:

$$C(K):=(\mathrm{GL}(X_\mathcal{K})\times G_P(\mathbb{A}_f))N_P(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K.$$

This is a finite set. We denote by $[g]$ the class represented by $g\in G(\mathbb{A}_f)$. For each such g whose p-component is 1 we define $K_P^g=G_P(\mathbb{A}_f)\cap gKg^{-1}$ and denote $S_{[g]}:=S_{G_P}(K_P^g)$ the corresponding Shimura variety for the group G_P with level group K_P^g. By strong approximation we can choose a set $\underline{C}(K)$ of representatives of $C(K)$ consisting of elements $g=pk^0$ for $p\in P(\mathbb{A}_f^{(\Sigma)})$ and $k^0\in K^0$ for K^0 the maximal compact subgroup of $G(\mathbb{A}_f)$ defined in [16, Section 1.10].

3.5 Igusa varieties and p-adic automorphic forms

Now we recall briefly the notion of Igusa varieties in [16, Section 2.3]. Let M be the standard lattice of V and $M_p=M\otimes_\mathbb{Z}\mathbb{Z}_p$. Let $\mathrm{Pol}_p=\{N^{-1},N^0\}$ be a polarization of M_p. Recall that this means that N^{-1} and N^0 are maximal isotropic $\mathcal{O}_\mathcal{K}\otimes\mathbb{Z}_p$-submodules in M_p such that they are dual to each other with respect to the Hermitian metric on V, and

$$\mathrm{rank}_{\mathbb{Z}_p}N^{-1}_{v_0}=\mathrm{rank}_{\mathbb{Z}_p}N^0_{\bar{v}_o}=3,\mathrm{rank}_{\mathbb{Z}_p}N^{-1}_{\bar{v}_0}=\mathrm{rank}_{\mathbb{Z}_p}N^0_{v_0}=1.$$

We mainly follow [16, Section 2.3] in this Subsection. The Igusa variety of level p^n is the scheme over $\mathcal{O}_{\mathcal{K},(v_0)}$ representing the quadruple $\underline{A}(R)=\{(A,\bar{\lambda},\iota,\bar{\eta}^p)\}$ for Shimura variety of $\mathrm{GU}(3,1)$ as above, together with an injection of group schemes

$$j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$$

over R which is compatible with the $\mathcal{O}_{\mathcal{K}}$-action on both hand sides. Note that the existence of j implies that A must be ordinary along the special fiber. There is also a theory of Igusa varieties over $\bar{S}_G(K)$. Let $\underline{\omega}$ be the automorphic vector bundle on $S_G(K)$ as defined in [?, 2.7.3]. As in *loc.cit* let $\bar{H}_{p-1} \in H^0(S_G(K)_{/\bar{\mathbb{F}}}, \det(\underline{\omega})^{p-1})$ be the Hasse invariant. Over the minimal compactification, some power (say the tth) of the Hasse invariant can be lifted to $\mathcal{O}_{v_0}$, by the ampleness of $\det \underline{\omega}$. We denote such a lift by E. By the Koecher principle we can regard E as in $H^0(\bar{S}_G(K), \det(\underline{\omega}^{t(p-1)}))$. Let $\mathcal{O}_m := \mathcal{O}_{\mathcal{K},v_0}/p^m\mathcal{O}_{\mathcal{K},v_0}$. Set $T_{0,m} := \bar{S}_G(K)[1/E]_{/\mathcal{O}_m}$. For any positive integer n define $T_{n,m} := I_G(K^n)_{/\mathcal{O}_m}$ and $T_{\infty,m} = \varprojlim_n T_{n,m}$. Then $T_{\infty,m}$ is a Galois cover over $T_{0,m}$ with Galois group $\mathbf{H} \simeq \mathrm{GL}_3(\mathbb{Z}_p) \times \mathrm{GL}_1(\mathbb{Z}_p)$. Let $\mathbf{N} \subset \mathbf{H}$ be the upper triangular unipotent radical. Define:

$$V_{n,m} = H^0(T_{n,m}, \mathcal{O}_{T_{n,m}}).$$

Let $V_{\infty,m} = \varinjlim_n V_{n,m}$ and $V_{\infty,\infty} = \varprojlim_m V_{\infty,m}$ be the space of p-adic automorphic forms on $\mathrm{GU}(3,1)$ with level K. We also define $W_{n,m} = V_{n,m}^{\mathbf{N}}$, $W_{\infty,m} = V_{\infty,m}^{\mathbf{N}}$ and $\mathcal{W} = \varinjlim_n \varinjlim_m W_{n,m}$. We define $V_{n,m}^0$, etc, to be the cuspidal part of the corresponding spaces.

We can make similar definitions for the definite unitary similitude groups G_P as well and define $V_{n,m,P}$,$V_{\infty,m,P}$, $V_{\infty,\infty,P}$, $V_{n,m,P}^{\mathbf{N}}$, $\mathcal{W}_P$, etc.

Let K_0^n and K_1^n be the subgroup of $\mathbf{H}$ consisting of matrices which are in $B_3 \times {}^tB_1$ or $N_3 \times {}^tN_1$ modulo p^n. (These notations are already used for level groups of automorphic forms. The reason for using the same notation here is that automorphic forms with level group $K_\bullet^n$ are p-adic automorphic forms of level group $K_\bullet^n$). We sometimes denote $I_G(K_1^n) = I_G(K^n)^{K_1^n}$ and $I_G(K_0^n) = I_G(K^n)^{K_0^n}$.

We can define the Igusa varieties for G_P as well. For $\bullet = 0, 1$ we let $K_{P,\bullet}^{g,n} := gK_\bullet^n g^{-1} \cap G_P(\mathbb{A}_f)$ and let $I_{[g]}(K_\bullet^n) := I_{G_P}(K_{P,\bullet}^{g,n})$ be the corresponding Igusa variety over $S_{[g]}$. We denote $A_{[g]}^n$ the coordinate ring of $I_{[g]}(K_1^n)$. Let $A_{[g]}^\infty = \varinjlim_n A_{[g]}^n$ and let $\hat{A}_{[g]}^\infty$ be the p-adic completion of $A_{[g]}^\infty$. This is the space of p-adic automorphic forms for the group $\mathrm{GU}(2,0)$ of level group $gKg^{-1} \cap G_P(\mathbb{A}_f)$.

<u>For Unitary Groups</u>

Assume the tame level group K is neat. For any c an element in $\mathbb{Q}_+ \backslash \mathbb{A}_{\mathbb{Q},f}^\times / \mu(K)$. We refer to [16, 2.5] for the notion of c-Igusa schemes $I_{\mathrm{U}(2)}^0(K,c)$ for the unitary groups $\mathrm{U}(2,0)$ (not the similitude group). It parameterizes quintuples $(A, \lambda, \iota, \bar{\eta}^{(p)}, j)_{/S}$ similar to the Igusa schemes for unitary similitude groups but requiring λ to be a prime to p c-polarization of A such that $(A, \bar{\lambda}, \iota, \bar{\eta}^{(p)}, j)$ is a quintuple as in the definition of Shimura varieties for $\mathrm{GU}(2)$. For g_c with $\mu(g) \in \mathbb{A}_{\mathbb{Q}}^\times$ in the class of c. Let ${}^cK = g_c K g_c^{-1} \cap U(2)(\mathbb{A}_{\mathbb{Q},f})$. Then the space $I_{\mathrm{U}(2)}^0(K,c)$ is isomorphic to the space of forms on $I_{\mathrm{U}(2)}^0({}^cK, 1)$ (see *loc.cit*).

<u>Embedding of Igusa Schemes</u>

In order to use the pullback formula geometrically we need a map from the Igusa scheme of $\mathrm{U}(3,1) \times \mathrm{U}(0,2)$ to that of $\mathrm{U}(3,3)$ (or from the Igusa scheme of $\mathrm{U}(2,0) \times \mathrm{U}(0,2)$ to that of $\mathrm{U}(2,2)$) given by:

$$i([(A_1, \lambda_1, \iota_1, \eta_1^p K_1, j_1)], [(A_2, \lambda_2, \iota_2, \eta_2^p K_2, j_2)]) = [(A_1 \times A_2, \lambda_1 \times \lambda_2, \iota_1, \iota_2, (\eta_1^p \times \eta_2^p)K_3, j_1 \times j_2)].$$

We define an element $\Upsilon \in \mathrm{U}(3,3)(\mathbb{Q}_p)$ such that $\Upsilon_{v_0} = S_{v_0}^{-1}$ and $\Upsilon'_{v_0} = S_{v_0}^{-1,'}$, where S is defined at the end of Section 6.2. Similar to [16], we know that under the complex uniformization, taking the change of polarization into consideration the above map is given by

$$i([\tau, g], [x_0, h]) = [Z_\tau, (g, h)\Upsilon]$$

(see [16, Section 2.6].)

3.6 Fourier-Jacobi expansions

Analytic Fourier-Jacobi Coefficients:
Let us go back to the Fourier-Jacobi coefficients of automorphic forms on $G := \mathrm{GU}(3,1)$. Let $\beta \in \mathbb{Q}_+$. Over $\mathbb{C}$ we have the β-analytic Fourier-Jacobi coefficient for a holomorphic automorphic form f given by:

$$FJ_\beta(f, g) = \int_{\mathbb{Q}\backslash\mathbb{A}} f(\begin{pmatrix} 1 & & n \\ & 1_2 & \\ & & 1 \end{pmatrix} g) e_{\mathbb{A}}(-\beta n) dn.$$

The Harr measure is normalized so that the set $(\mathbb{Q}\backslash\mathbb{A})$ has measure 1.
p-adic Cusps

As in [16] each pair $(g, j) \in C(K) \times \mathbf{H}$ can be regarded as a p-adic cusp, i.e. cusps of the Igusa tower. In the following we are going to give the algebraic Fourier-Jacobi expansion at p-adic cusps.

Algebraic Theory for Fourier-Jacobi expansions
We follow [16, Pages 16-17] to give some backgrounds about the algebraic theory for the group $G = \mathrm{GU}(r, 1)$. Recall $[g]$ is a cusp label corresponding to class $g \in G(\mathbb{A}_f)$. One defines $\mathcal{Z}_{[g]}$ a group scheme over $S_{[g]}$ using the universal abelian variety as in *loc.cit* and denote $\mathcal{Z}_{[g]}^\circ$ the connected component over $S_{[g]}$. It is well known that there is a Fourier-Jacobi expansion for modular forms by evaluating the form at the Mumford family $(\mathfrak{M}, \omega_{\mathfrak{M}})$ over the ring

$$\prod_\beta (\hat{A}_{[g]}^\infty \otimes_{\mathcal{O}} R) \otimes_{A_{[g]}} H^0(\mathcal{Z}_{[g]}^\circ, \mathcal{L}(\beta))$$

(we again refer to *loc.cit*).

Now let $f \in H^0(I_G(K_1^n)_{/R}, \omega_\kappa)$ be a scalar weight $\kappa \geq 6$ (i.e. of weight $(0,0,0;\kappa)$) modular form over an $\mathcal{O}$ algebra R, then by ([16, 3.6.2]) there is a Fourier-Jacobi expansion of f at the p-adic cusp (g, h) for $h \in \mathbf{H}$:

$$FJ_{[g]}^h(f) = \sum_{\beta \in \mathscr{S}_{[g]}} a_{[g]}^h(\beta, f) q^\beta$$

where

$$a_{[g]}^h(\beta, f) \in (\hat{A}_{[g]}^\infty \otimes_{\mathcal{O}} R) \otimes_{A_{[g]}} H^0(\mathcal{Z}_{[g]}^\circ, \mathcal{L}(\beta))$$

and $\mathscr{S}_{[g]}$ is a sub-lattice of $\mathbb{Q}$ determined by the level subgroup. This is given by evaluating f at the Mumford family $(\mathfrak{M}, h^{-1} j_{\mathfrak{M}}, \omega_{\mathfrak{M}})$ where $j_{\mathfrak{M}}$ is a fixed level structure (see [16, 2.7.4]). (note that we do not have the subsript N_H^1 since it is a scalar weight κ.)

Siegel Operators

We have a Siegel Φ operator at the p-adic cusp (g,h) defined by:

$$\Phi^h_{[g]} : H^0(I_G(K_1^n)_{/R}, \omega_\kappa) \to A^n_{[g]} \otimes_{\mathcal{O}} R$$

$$f \mapsto \Phi^h_{[g]}(f) := a^h_{[g]}(f) := a_{[g]}(0,f).$$

The Siegel operator at $[g]$ can be defined analytically as follows:
For any $g \in G(\mathbb{A}_f)$ we define:

$$\Phi_{P,g}(f) = \int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A}_{\mathbb{Q}})} f(ng)dn.$$

We fix the Haar measure on $N_P(\mathbb{Q})\backslash N_P(\mathbb{A}_{\mathbb{Q}})$ as in [45, Section 8.2]. The relation between the algebraic and analytic Siegel operator is given in [16, (3.12)].

3.7 Weight Space for $\mathrm{GU}(3,1)$

Let $H = \mathrm{GL}_3 \times \mathrm{GL}_1$ and $T \subseteq H$ be the diagonal torus. Then $\mathbf{H} \simeq H(\mathbb{Z}_p)$. We let $\Lambda_2 = \Lambda$ be the completed group algebra $\mathbb{Z}_p[[T(1+\mathbb{Z}_p)]]$. This is (non-canonically) isomorphic to a formal power series ring with four variables. There is an action of $T(\mathbb{Z}_p)$ on the Igusa scheme given by its action on the embedding $j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$. (See [16, 3.4]) This gives the spaces of p-adic modular forms for $\mathrm{GU}(3,1)$ a structure of Λ-algebra. A $\bar{\mathbb{Q}}_p$-point ϕ of $\mathrm{Spec}\Lambda$ is call arithmetic if it is determined by a character $[\underline{k}]\cdot[\zeta]$ of $T(1+p\mathbb{Z}_p) \simeq (1+p\mathbb{Z}_p)^4$ where $\underline{k}$ is a weight and $\zeta = (\zeta_1,\zeta_2,\zeta_3;\zeta_4)$ for $\zeta_i \in \mu_{p^\infty}$. Here $[\underline{k}]$ is the character by regarding $\underline{k}$ as a character of $T(1+\mathbb{Z}_p)$ by $[\underline{k}](t_1,t_2,t_3,t_4) = (t_1^{-c_4}t_2^{-c_3}t_3^{-c_2}t_4^{-c_1})$ and $[\zeta]$ is the finite order character given by mapping $(1+p)$ to ζ_i at the corresponding entry of $T(\mathbb{Z}_p)$. We often write this point $\underline{k}_\zeta$. We also define $\omega^{[\underline{k}]}$ a character of the torsion part of $T(\mathbb{Z}_p)$ (canonically isomorphic to $(\mathbb{F}_p^\times)^4$) given by $\omega^{[\underline{k}]}(t_1,t_2,t_3,t_4) = \omega(t_1^{-c_4}t_2^{-c_3}t_3^{-c_2}t_4^{-c_1})$.

We can define the weight ring Λ_P for the definite unitary group G_P as well.

3.8 Nearly Ordinary Forms

We refer to [16, 3.8.3, 4.3] for the notion and definition of Hida's idempotent e acting on the space $V^{\mathbf{N}}_{\infty,\infty}$ of p-adic automorphic forms on $\mathrm{GU}(3,1)$ and the nearly ordinary subspace of the space of the p-adic modular forms. The spaces of nearly ordinary automorphic forms are denoted as $\mathcal{W}_{ord}$, $\mathcal{W}^0_{ord}$, etc. For $q=0$ or $\emptyset$ we let $\mathbf{V}^q_{ord}$ be the Pontryagin dual of $\mathcal{W}^q_{ord}$. Then we have the following theorem ([16, Theorem 4.21])

Theorem 3.3. *Let $q=0$ or $\emptyset$. Then:*
(1) $\mathbf{V}^q_{ord}$ is a free Λ module of finite rank;
(2) For any $\underline{k}$ very regular we have nature isomorphisms:

$$\mathcal{M}^q_{ord}(K,\Lambda)\otimes\Lambda/P_{\underline{k}} \xrightarrow{\sim} eM^q_{\underline{k}}(K,\mathcal{O}_p)$$

where $\mathcal{M}^q_{ord}(K,\Lambda)$ is defined in Definition 3.5. Here we identify $eM^q_{\underline{k}}(K,\mathcal{O}_{v_0})$ with its image in the space of p-adic automorphic forms of weight $\underline{k}$ under $\beta_{\underline{k}}$.

Remark 3.4. If $\mathcal{K}$ is a general CM field, then the statement of the corresponding result is more complicated; see [16, Section 4.5].

3.9 Λ-adic forms

Definition 3.5. *For any finite Λ algebra A, and $q = 0$ or $\emptyset$ we define the space of A-adic ordinary forms to be:*

$$\mathcal{M}^q_{ord}(K, A) := \mathrm{Hom}_\Lambda(\mathbf{V}^q_{ord}, A).$$

Similarly, if A is a Λ_P-algebra, then we define:

$$\mathcal{M}_{ord,[g],P}(K_{P,[g]}, A) := \mathrm{Hom}_{\Lambda_P}(\mathbf{V}_{ord,P,[g]}, A).$$

Here the subscript means the prime to P level is $K_P^{[g]}$ defined before.

For any $f \in \mathcal{M}_{ord}(K, A)$ we have an A-adic Fourier-Jacobi expansion:

$$FJ^h_{[g]}(f) = \sum_{\beta \in \mathscr{S}_{[g]}} a^h_{[g]}(\beta, f) q^\beta$$

obtained from the Fourier-Jacobi expansion on $\mathcal{W}^q_{ord}$, where $a^h_{[g]}(\beta, f) \in A \hat{\otimes} \hat{A}^\infty_{[g]} \otimes_{A_{[g]}} H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta))$ (see [16, 4.6.1]). We also have a Λ-adic Siegel operator which we denote as $\hat{\Phi}^h_{[g]}$. Let $w'_3 = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}_p) \simeq \mathrm{U}(3,1)(\mathbb{Z}_p)$. (Notice that we used the place v_0 to identify $\mathrm{GL}_4(\mathbb{Z}_p)$ with $\mathrm{U}(3,1)(\mathbb{Z}_p)$ here. The matrix $\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$ itself is not in $\mathrm{U}(3,1)(\mathbb{Z}_p)$. We use w'_3 instead of w_3 as in [16, Page 35] to distinguish it from $w_3 \in \mathrm{U}(3,3)$). Now we have the following important theorem

Theorem 3.6. *[16, Theorem 4.26] Let A be as before. We have the following short exact sequence*

$$0 \to \mathcal{M}^0_{ord}(K, A) \to \mathcal{M}_{ord}(K, A) \xrightarrow{\hat{\Phi}^{w'_3} = \oplus \hat{\Phi}^{w'_3}_{[g]}} \oplus_{g \in C(K)} \mathcal{M}_{ord}(K^g_P, A) \to 0.$$

We need one more theorem which gives another definition of nearly ordinary p-adic modular forms using Fourier-Jacobi expansions.

Definition 3.7. *Let A be a finite torsion free Λ-algebra. Let $X(K)$ be the set $\{(g, w'_3)\}$ where g runs over a set of representatives of cusp labels $C(K)$. Let $\mathcal{N}_{ord}(K, A)$ be the set of formal Fourier-Jacobi expansions:*

$$F = \{ \sum_{\beta \in \mathscr{S}_{[g]}} a(\beta, F) q^\beta, a(\beta, F) \in A \hat{\otimes} \hat{A}^\infty_{[g]} \otimes H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta)) \}_{g \in X(K)}$$

such that for a Zariski dense set $\mathcal{X}_F$ of points $\phi \in \mathrm{Spec}(A)$ such that the induced point in $\mathrm{Spec}(\Lambda)$ is some arithmetic weight $\underline{k}_\zeta$, the specialization F_ϕ of F is the Fourier-Jacobi expansion of a nearly ordinary modular form with prime to p level K, weight $\underline{k}$ and nebentype at p given by $[\underline{k}][\underline{\zeta}]\omega^{-[\underline{k}]}$.

Then we have the following theorem ([16, Theorem 4.25])

Theorem 3.8.

$$\mathcal{M}_{ord}(K, A) = \mathcal{N}_{ord}(K, A).$$

4 Backgrounds for Theta functions

Now we recall briefly the basic notions of theta functions and theta liftings, following closely to [54] with some modifications. The author claims no originality in this section.

4.1 Heisenberg Group

Let W be a finite dimensional vector space over $\mathbb{Q}_v$ with a non-degenerate alternating form $\langle , \rangle$. We define:

$$H(W) := \{(w,t) | w \in W, t \in \mathbb{Q}_v\}$$

with multiplication law: $(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2\rangle)$.

4.2 Schrödinger representation

Fix an additive character ψ of $\mathbb{Q}_v$ and a complete polarization as $W = X \oplus Y$ of W where X and Y are maximal totally isotropic subspaces of W. Let $S(X)$ be a space of Bruhat-Schwartz functions on X, and define a representation ρ_ψ of $H(W)$ on $S(X)$ by:

$$\rho_\psi(x)f(z) = f(x+z), x \in X$$

$$\rho_\psi(y)f(z) = \psi(\langle z, y\rangle)f(z), y \in Y$$

$$\rho_\psi(t)f(z) = \psi(t)f(z), t \in \mathbb{Q}_v$$

This is called the Schrödinger representation. By the theorem of Stone and von Neumann, ρ_ψ is the unique irreducible smooth representation on which $\mathbb{Q}_v$ acts via the character ψ.

4.3 Metaplectic Groups and Weil representations

Let $Sp(W)$ be the symplectic group preserving the alternating form $\langle , \rangle$ on W. Then $Sp(W)$ acts on $H(W)$ by $(w,t)g = (wg, t)$ (we use row vectors for $w \in W$ and the right action of $Sp(W)$ instead of the left action as in [54]). By the uniqueness of ρ_ψ, there is an operator $\omega_\psi(g)$ on $S(X)$, determined up to scalar, such that

$$\rho_\psi(w^t g, t)\omega_\psi(g) = \omega_\psi(g)\rho_\psi(w,t)$$

for any $(w,t) \in H(W)$. Here ${}^t g$ is the transpose of g. Define the metaplectic group $\tilde{Sp}_\psi(W) = \{(g, \omega_\psi(g))$ as above $\}$ which we often write $\tilde{Sp}$ for short. Thus $\tilde{Sp}(W)$ has an action ω_ψ on $S(X)$ called the Weil representation.

Suppose $\psi = \prod_v \psi_v$ is a global additive character of $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$. We can put the above construction together for all v's to get a representation of $\tilde{Sp}(W)(\mathbb{A})$ on $S(X(\mathbb{A}))$. This can be viewed as a projective representation of $Sp(W)$ (a representation with image in the infinite dimensional projective linear group). We now give formulas for this representation. Let $\{e_1, ..., e_n; f_1, ..., f_n\}$ be a basis of $W = X \oplus Y$ such that $\langle e_i, f_j\rangle = \delta_{ij}$. With respect to this basis the projective representation of $\tilde{Sp}(W)(\mathbb{A}_\mathbb{Q})$ on $\mathrm{Proj}S(X(\mathbb{A}))$ is given by the formulas

- $\omega_\psi(\begin{pmatrix} A & \\ & {}^tA^{-1}\end{pmatrix})\phi(x) = |\det A|^{\frac{1}{2}}\phi(xA)$;

- $\omega_\psi(\begin{pmatrix} 1 & B \\ & 1\end{pmatrix})\phi(x) = \psi(\frac{xB^tx}{2})\phi(x)$;

- $\omega_\psi(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix})\phi(x) = \gamma\hat{\phi}(x)$ where $\hat{\phi}$ means the Fourier transform of ϕ with respect to the additive character ψ. The γ is an 8-th root of unity which is called the Weil constant.

4.4 Dual Reductive Pairs

A dual reductive pair is a pair of subgroups (G, G') in the symplectic group $Sp(W)$ satisfying: (1) G is the centralizer of G' in $Sp(W)$ and vice versa; and (2) the action of G and G' are completely reducible on W. We are mainly interested in the following dual reductive pairs of unitary groups. Let $\mathcal{K}$ be a quadratic imaginary extension of $\mathbb{Q}$, $(V_1, (,)_1)$ be a skew Hermitian space over $\mathcal{K}$ and $(V_2, (,)_2)$ a Hermitian space over $\mathcal{K}$. Then the unitary groups $\mathrm{U}(V_1)$ and $\mathrm{U}(V_2)$ form a dual reductive pair in $\mathrm{Sp}(W)$, where $W = V_1 \otimes V_2$ is given the alternating form $\frac{1}{2}\mathrm{tr}_{\mathcal{K}/\mathbb{Q}}((,)_1 \otimes \overline{(,)_2})$ over $\mathbb{Q}$. The embedding of the dual reductive pair $(\mathrm{U}(V_1), \mathrm{U}(V_2))$ into $\mathrm{Sp}(W)$ is

$$e : \mathrm{U}(V_1) \times \mathrm{U}(V_2) \to \mathrm{Sp}(W)$$

$$e(g_1, g_2) \cdot (v_1 \otimes v_2) = v_1 g_1 \otimes g_2^{-1} v_2.$$

4.5 Splittings

Suppose $\dim_{\mathcal{K}} V_1 = n$ and $\dim_{\mathcal{K}} V_2 = m$. If χ_1 and χ_2 are Hecke characters of $\mathcal{K}^\times$ such that $\chi_1|_{\mathbb{A}_F^\times} = \chi_{\mathcal{K}}^n$ and $\chi_2|_{\mathbb{A}_F^\times} = \chi_{\mathcal{K}}^m$, then there is a splitting (see [15, Section 1])

$$s : \mathrm{U}(V_1) \times \mathrm{U}(V_2) \to \tilde{Sp}(W)$$

determined by χ_2 and χ_1. This enables us to define the Weil representations of $U(V_1) \times U(V_2)$ on $S(X(\mathbb{A}))$ which we denote as $\omega_{\chi_1,\chi_2} = \omega_{\chi_1} \otimes \omega_{\chi_2}$.

4.6 Theta Functions

Now let us define theta functions.

Definition 4.1. *Let $\phi \in S(X(\mathbb{A}_\mathbb{Q}))$. Define*

$$\theta(\phi) = \sum_{l \in X(\mathbb{Q})} \phi(l)$$

Using the Weil representation of the dual reductive pair above (with the choices of the splitting characters) we define the theta kernel for the theta correspondence as follows:

Definition 4.2.

$$\theta_\phi(g_1, g_2) = \theta(\omega(g_1, g_2))\phi)$$

Let $J = H(W) \ltimes Sp(W)$ ($\tilde{J} = H(W) \ltimes \tilde{Sp}(W)$) be the Jacobi group with $Sp(W)$ acting on $H(W)$ by $(w,t)\cdot g = (wg,t)$ ($\tilde{Sp}(W)$ acts on $H(W)$ by $(w,t)\cdot \tilde{g} = (wg,t)$, where g is the image of $\tilde{g}$ in $Sp(W)$). We define a theta kernerl on $\tilde{J}(\mathbb{A}_{\mathbb{Q}})$.

Definition 4.3. *Let $\tilde{g} \in \tilde{Sp}(W)$ and $(w,t) \in H(W)$, define*

$$\theta_\phi((w,t)\tilde{g}) = \sum_{l \in X(\mathbb{Q})} \rho_\psi(w,t)\tilde{g}.\phi(l)$$

4.7 Intertwining Maps

We are going to study the intertwining maps between theta series corresponding to different polarizations (X,Y) of W. Suppose $r \in Sp(W)$, then (Xr, Yr) gives another polarization of W, and all polarizations are obtained this way. If $\phi \in S(X)$ the we define an intertwining map (local or global) $\delta_\psi : S(X) \to S(Xr)$ by

$$\delta_\psi \phi(xr) = \omega_\psi(r)\phi(x)$$

for $x \in X$. It is easy to check that δ_ψ is an isometry intertwining the action of $\tilde{J}$.

Let W^- be the skew Hermitian space which is isomorphic to W as Q_v-vector spaces but equipped with the alternating pairing $-\langle,\rangle$. For a polarization (X,Y) of W we are going to study the intertwining formula for the two polarizations $(X \oplus X^-) \oplus (Y \oplus Y^-)$ and $\{w \oplus w, w \in W\} \oplus \{w \oplus -w, w \in W\}$ of $W \oplus W^-$. We write the formula for the map $\delta_\psi : S(X(\mathbb{Q}_v) \oplus X^-(\mathbb{Q}_v)) \to S(W(\mathbb{Q}_v))$ and its inverse:

$$\delta_\psi(\phi)(x_1, y) = \int \psi(\langle x_2, y\rangle)\phi(\frac{x_1+x_2}{2}, \frac{x_2-x_1}{2})dx_2$$

$$\delta_\psi^{-1}(\phi)(x_1, x_2) = \int \psi(\langle -x_1 - x_2, y\rangle)\phi(x_1 - x_2, y)dy.$$

Another important property is that if the two polarizations (X,Y) and (Xr,Yr) are globally defined, then the theta kernels Θ_ϕ and $\Theta_{\delta_\psi(\phi)}$ are defined and

$$\Theta_\phi(u, ng) = \Theta_{\delta_\psi(\phi)}(u, (nr)g).$$

4.8 Special Cases

4.8.1 Case One

We write V for the Hermitian space for ζ with respect to the basis (v_1, v_2), V^- for the Hermitian space for $-\zeta$ with respect to the basis (v_1^-, v_2^-), and V_1 for the 1-dimensional Hermitian space with the metric 1 with respect to the basis v. Let $W = V \otimes V_1$ and $W^- = V^- \otimes V_1$. We define several polarizations for the Hermitian space $\mathbb{W} := W \oplus W^-$ (the alternating pairing being the direct sum of those for W and W^-).

Definition 4.4.

$$X := \mathbb{Q}v_1 \otimes v \oplus \mathbb{Q}v_2 \otimes v$$

$$X^- := \mathbb{Q}v_1^- \otimes v \oplus \mathbb{Q}v_2^- \otimes v$$

$$Y := \mathbb{Q}\delta v_1 \otimes v \oplus \mathbb{Q}\delta v_2 \otimes v$$

$$Y^- := \mathbb{Q}\delta v_1^- \otimes v \oplus \mathbb{Q}\delta v_2^- \otimes v.$$

Fix the additive character $\psi = \prod \psi_v$. Thus $W = X \oplus Y$ and $W^- = X^- \oplus Y^-$ are globally defined polarizations. For a split prime v we write $v = w\bar{w}$ for its decomposition in $\mathcal{K}$. We will often use an auxiliary polarization $W_v = X'_v \oplus Y'_v$ of $W_v = W \otimes_{\mathcal{K}} \mathcal{K}_v$ with respect to $\mathcal{K}_v \simeq \mathcal{K}_w \times \mathcal{K}_{\bar{w}} \simeq \mathbb{Q}_v^2$ and $W_v = X_v^{'-} \oplus Y_v^{'-}$ that is defined by $X'_v = \mathcal{K}_w v_1 \otimes v \oplus \mathcal{K}_w v_2 \otimes v, Y'_v = \mathcal{K}_{\bar{w}} v_1 \otimes v \oplus \mathcal{K}_{\bar{w}} v_2 \otimes v$ and similar for $X_v^{'-}, Y_v^{'-}$. This polarization is better suited for computing the Weil representation. For split primes v let $\delta''_\psi : S(X'_v) \to S(X_v)$ and $\delta_\psi^{-,''} : S(X_v^{-,'}) \to S(X_v^-)$ be the intertwining operators between Schwartz functions defined above.

Let $\mathbb{W}^d = \{w \oplus w, w \in W\}$. We denote the intertwining maps:

$$\delta_\psi : S(X_v \oplus X_v^-) \to S(\mathbb{W}_v^d)$$

and if v splits,

$$\delta'_\psi : S(X'_v \oplus X_v^{'-}) \to S(\mathbb{W}_v^d).$$

Remark 4.5. In application in Section 6 we are going to compute the intertwining operator

$$\delta_\psi : S(X_v \oplus X_v^-) \to S(\mathbb{W}_v^d)$$

(for $\mathbb{W} = (V \oplus V^-) \otimes V_1$) in this special case and the Weil representations restricting to semidirect products $H(\mathbb{W}) \ltimes (\mathrm{U}(V \oplus V^-) \times \mathrm{U}(V_1))$ (recall $\mathrm{U}(V \oplus V^-) \times \mathrm{U}(V_1) \hookrightarrow Sp(\mathbb{W})$). We provide the matrix forms of these semidirect products that will be used in Section 6. Let U_1 and U_2 be unitary groups associated to the matrices $\begin{pmatrix} & & 1 & \\ & \zeta & & \\ -1 & & & \\ & & & -\zeta \end{pmatrix}$ and $\begin{pmatrix} & 1_3 \\ -1_3 & \end{pmatrix}$ respectively, and let U'_1 and U'_2 be the unitary groups associated to $\begin{pmatrix} \zeta & \\ & -\zeta \end{pmatrix}$ and $\begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix}$, considered as subgroups of U_1 and U_2 respectively in the obvious way. Let N_1 be the subgroup of U_1 consisting of matrices of the form $\begin{pmatrix} 1 & x_1 & * & x_2 \\ & 1_2 & \zeta x_1^* & \\ & & 1 & \\ & & -\zeta x_2^* & 1_2 \end{pmatrix}$, and $N_2 \subset \mathrm{U}_2$ the subgroup consisting of matrices of the form $\begin{pmatrix} 1 & x & t + \frac{1}{2}(xy^* - yx^*) & y \\ & 1_2 & y^* & 0 \\ & & 1 & \\ & & -x^* & I_2 \end{pmatrix}$. The corresponding semidirect products mentioned above are $J_1 = N_1\mathrm{U}'_1$ and $J_2 = N_2\mathrm{U}'_2$.

4.8.2 Case Two

Now we discuss another special situation which will be used in the Fourier-Jacobi coefficient computations for Siegel Eisenstein series on $\mathrm{GU}(3,3)$.

<u>The local set-up.</u>
Let v be a place of $\mathbb{Q}$. Let $h \in S_1(\mathbb{Q}_v), \det h \neq 0$. Let U_h be the unitary group of this matrix and let V_v be the corresponding one-dimensional Hermitian space. Let

$$V_{2,v} = \mathcal{K}_v^2 \oplus \mathcal{K}_v^2 = \mathbf{X}_v \oplus \mathbf{Y}_v$$

be the Hermitian space associated to $\mathrm{U}_2 = \mathrm{U}(2,2)$ with the alternating pairing denoted as $\langle , \rangle_2$. Let $\mathbf{W} = V_v \otimes_{\mathcal{K}_v} V_{2,v}$. Then

$$(-,-) := \mathrm{Tr}_{\mathcal{K}_v/\mathbb{Q}_v}(\langle -,- \rangle_h \otimes_{\mathcal{K}_v} \langle -,- \rangle_2)$$

is a $\mathbb{Q}_v$ linear pairing on $\mathbf{W}$ that makes $\mathbf{W}$ into an 8-dimensional symplectic space over $\mathbb{Q}_v$. The canonical embedding of $\mathrm{U}_h \times \mathrm{U}_2$ into $Sp(\mathbf{W})$ realizes the pair $(\mathrm{U}_h, \mathrm{U}_2)$ as a dual pair in $Sp(\mathbf{W})$. Let λ_v be a character of $\mathcal{K}_v^\times$ such that $\lambda_v|_{\mathbb{Q}_v^\times} = \chi_{\mathcal{K}/\mathbb{Q},v}$. As noted earlier there is a splitting $\mathrm{U}_h(\mathbb{Q}_v) \times \mathrm{U}_2(\mathbb{Q}_v) \hookrightarrow \tilde{Sp}(\mathbf{W}, \mathbb{Q}_v)$ of the metaplectic cover $\tilde{Sp}(\mathbf{W}, \mathbb{Q}_v) \to Sp(\mathbf{W}, \mathbb{Q}_v)$ determined by the character λ_v. This gives the Weil representation $\omega_{\lambda_v,1}$, which we denote here as $\omega_{h,v}(u,g)$ of $\mathrm{U}_h(\mathbb{Q}_v) \times \mathrm{U}_2(\mathbb{Q}_v)$ where $u \in \mathrm{U}_h(\mathbb{Q}_v)$ and $g \in \mathrm{U}_2(\mathbb{Q}_v)$, via the Weil representation of $\tilde{Sp}(\mathbf{W}, \mathbb{Q}_v)$ on the space of Schwartz functions $\mathcal{S}(V_v \otimes_{\mathcal{K}_v} \mathbf{X}_v)$ (we use the polarization $\mathbf{W} = V_v \otimes_{\mathcal{K}_v} \mathbf{X}_v \oplus V_v \otimes_{\mathcal{K}_v} \mathbf{Y}_v$). Moreover we write $\omega_{h,v}(g)$ to mean $\omega_{h,v}(1,g)$. For $X \in M_{1\times 2}(\mathcal{K}_v)$, we define $\langle X, X \rangle_h := X h {}^t\bar{X}$ (note that this is a 2×2 matrix). We record here some useful formulas for $\omega_{h,v}$ which are generalizations of the formulas in [45].

- $\omega_{h,v}(u,g)\Phi(X) = \omega_{h,v}(1,g)\Phi(u^{-1}X)$
- $\omega_{h,v}(\mathrm{diag}(A, {}^t\bar{A}^{-1}))\Phi(X) = \lambda(\det A)|\det A|_{\mathcal{K}}^{\frac{1}{2}}\Phi(XA)$,
- $\omega_{h,v}(r(S))\Phi(X) = \Phi(X)e_v(\mathrm{tr}\langle X, X\rangle_h S)$,
- $\omega_{h,v}(\eta)\Phi(X) = |\det h|_v \int \Phi(Y)e_v(\mathrm{Tr}_{\mathcal{K}_v/F_v}(\mathrm{tr}\langle Y, X\rangle_h))dY$.

Global setup:
Let $h \in S_1(\mathbb{Q}), h > 0$. We can define global versions of $\mathrm{U}_h, \mathrm{GU}_h, \mathbf{W}$, and $(-,-)$, analogous to the above. Fixing an idele class character $\lambda = \otimes \lambda_v$ of $\mathbb{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ such that $\lambda|_{\mathbb{Q}^\times} = \chi^1_{\mathcal{K}/\mathbb{Q}}$, the associated local splitting described above then determines a global splitting $\mathrm{U}_h(\mathbb{A}_\mathbb{Q}) \times \mathrm{U}_1(\mathbb{A}_\mathbb{Q}) \hookrightarrow \tilde{Sp}(\mathbf{W}, \mathbb{A}_\mathbb{Q})$ and hence an action $\omega_h := \otimes \omega_{h,v}$ of $\mathrm{U}_h(\mathbb{A}_\mathbb{Q}) \times \mathrm{U}_1(\mathbb{A}_\mathbb{Q})$ on the Schwartz space $\mathcal{S}(V_{\mathbb{A}_\mathcal{K}} \otimes \mathbf{X})$. In application we require the infinite type of λ to be $(-\frac{1}{2}, \frac{1}{2})$.

For any $\Phi \in \mathcal{S}(V_{\mathbb{A}_\mathcal{K}} \otimes \mathbf{X})$ we define the theta function associated to it by:

$$\Theta_h(\Phi, u, g) = \sum_{x \in V \otimes X} \omega_h(u,g)\Phi(x).$$

4.9 Theta Functions with Complex Multiplication

We consider the situation of theta correspondences for $\mathrm{U}(\zeta) = \mathrm{U}(V)$ and $\mathrm{U}(V_1)$. Let V be a 2-dimensional Hermitian vector space over $\mathcal{K}$ and L is a $\mathcal{O}_\mathcal{K}$ lattice. This gives an abelian variety $\mathcal{A}_L = \mathbb{C}^2/L$. Let H be a Riemann form on V and $\epsilon : L \to U$ be a map where U is the unite circle of $\mathbb{C}$ (in application the ϵ is given by the formula after [54, (38)], there is a line bundle $\mathfrak{L}_{H,\epsilon}$ on $\mathcal{A}_\mathcal{L}$ associated to H and ϵ as follows: define an analytic line bundle $\mathfrak{L}_{H,\epsilon} \simeq \mathbb{C} \times \mathbb{C}^2/L$ with the action of L given by

$$l \cdot (w, x) = (w + l, \epsilon(l)e(\frac{1}{2i}H(l, w + \frac{l}{2}))x), l \in L, (w, x) \in \mathbb{C}^2 \times \mathbb{C}$$

where $e(x) = e^{2\pi ix}$. The space of global sections of this line bundle is canonically identified with the space $T(H, \epsilon, L)$ of theta functions consists of holomorphic functions f on V such that:

$$f(w + l) = f(w)\epsilon(l)e(\frac{1}{2i}H(l + w + \frac{l}{2})), w \in V, l \in L.$$

There are arithmetic models for the above abelian variety and line bundle. Shimura defined subspaces $T^{ar}(H, \epsilon, L) \subset T(H, \epsilon, L)$ of arithmetic theta functions by requiring the values at all CM points are in $\bar{\mathbb{Q}}$ which under the canonical identification, are identified with rational sections of the line bundle (see [28]). Also inside the $\mathbb{C}_p$-vector space $\Gamma(\mathcal{A}_L \otimes_{\iota_p} \mathbb{C}_p, \mathfrak{L}_{H,\epsilon} \otimes_{\iota_p} \mathbb{C}_p)$ we have the module of p-integral sections which we denote as $T^{ar}(H, \epsilon, L)$.

Adelic Theta Functions

Now we consider Theta functions for $\mathrm{U}(3, 1)$. Let the Hermitian form on V be defined by:

$$\langle v_1, v_2 \rangle = v_1\zeta v_2^* - v_2\zeta v_1^*.$$

Let U_f be some compact subgroup of $U_\zeta(\mathbb{A}_f)$ such that the level is prime to p, we define the space $T_\mathbb{A}(m, L, U_f)$ of adelic theta functions as the space of function:

$$\Theta : N(\mathbb{Q})\mathrm{U}(\zeta)(\mathbb{Q})\backslash N(\mathbb{A})\mathrm{U}(\zeta)(\mathbb{A})/\mathrm{U}(\zeta)_\infty \mathrm{U}_f N(L)_f \to \mathbb{C},$$

where $N = N_P \subset U(3, 1)$ and $U(\zeta) \hookrightarrow U(3, 1)$ as before;

$$N(L)_f = \{(w, t)|x \in \hat{L}, t + \frac{w\zeta w^*}{2} \in \mu(L)\hat{\mathcal{O}}_\mathcal{K}\},$$

where $\mu(L)$ is the ideal generated by $w\zeta w^*$ for $w \in L$ and Θ satisfies

$$\Theta((0, t)r) = e(\beta t)\Theta(r), r \in N(\mathbb{A})U(\zeta)(\mathbb{A}).$$

Since $\mathrm{U}(\zeta)$ is anisotropic $\mathrm{U}(\zeta)\backslash U(\zeta)(\mathbb{A})/\mathrm{U}(\zeta)_\infty \mathrm{U}_f$ consists of finite points $\{x_1, ..., x_s\} \subset \mathrm{U}(\zeta)(\mathbb{A}_f)$. We assume that for each u_i the p-component is within $\mathrm{GL}_2(\mathbb{Z}_p)$ under the first projection $\mathrm{U}(\mathbb{Q}_p) \simeq \mathrm{GL}_2(\mathbb{Q}_p)$. Then:

$$T_\mathbb{A}(m, L, U_f) = \oplus_{i=1}^s T(m, x_iL)$$
$$\Theta \to (\Theta_i)$$

such that $\Theta_i(n) = \Theta(nx_i)$ for $n \in N(\mathbb{A})$ are functions on $N(\mathbb{A})$. Then for each i the function: $\theta_i(w_\infty) = e(-m\frac{w\zeta w^*}{2})\Theta_i((w_\infty, 0))$ is a classical theta function in $T(H, \epsilon, x_iL)$ where H and ϵ are defined using x_iL.

A functional

Recall that we constructed a theta function θ_ϕ on $H \rtimes U(V)$ from some Schwartz function ϕ. As mentioned in the introduction we only need to develop a rational theory on theta functions instead of p-integral theory. Upon choosing $v_1 \in V_1$ such that $< v_1, v_1 >= 1$ we have an isomorphism $V \simeq W = V$. We consider $W^- = V^- \otimes V_1$. It is the space W but with the metric being the negative of W. Let $H^- = H(W^-)$ be the corresponding Heisenberg group. We have an isomorphism of H and H^- (as Heisenberg groups) given by:

$$(w, t) \to (w, -t).$$

We construct a theta functions $\theta_1 = \theta_{\phi_1}$ on $H^- \rtimes \mathrm{U}(V^-)$ for some Schwartz function ϕ_1. We have chosen a set $\{x_1, ..., x_s\}$ above. We write

$$\langle \theta_{\phi_1}, \theta_{\phi_2} \rangle_{x_i} = \int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} \theta_{\omega_{\lambda^{-1}}(x_i)(\phi_1)}(n)\theta_{\omega_\lambda(x_i)(\phi_2)}(n)dn$$

and

$$(\phi_1, \phi_2)_{x_i} = \int_{X(\mathbb{A}_\mathbb{Q})} \omega_{\lambda^{-1}}(x_i)(\phi_1)(x)\omega_\lambda(x_i)(\phi_2)(x)dx.$$

Then it is easy to check

$$\langle \theta_{\phi_1}, \theta_{\phi_2} \rangle_{x_i} = (\phi_1, \phi_2)_\phi. \tag{1}$$

We first construct a functional l'_{θ_1} on the space $H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta))$ (we save the notation l_{θ_1} for later use) with values in $A_{[g]}$ as follows: first on $\frac{\theta_\phi}{\Omega_\mathcal{K}} \in H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta))$ for some Schwartz function ϕ, we define:

$$l'_{\theta_1}(\frac{\theta_\phi}{\Omega_\mathcal{K}})(x_i) := \int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} \theta_{\omega_{\lambda^{-1}}(x_i)(\phi_1)}(n)\theta_{\omega_\lambda(x_i)(\phi)}(n)dn = \int_{X(\mathbb{A}_\mathbb{Q})} \omega_{\lambda^{-1}}(x_i)(\phi_1)(x)\omega_\lambda(x_i)(\phi)(x)dx.$$

The last equality is easily seen and we denote the last term as $(\omega_\lambda^{-1}(x_i)(\phi_1), \omega_\lambda(x_i)(\phi))$. Note that in the $N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})$ we identified H with H^- using the above isomorphism. (In the literature people usually consider $\int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} \theta_\phi(n)\bar{\theta}_{\phi_1}(n)dn$. We use a different convention for the sake of simplicity.) The general elements in $H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta))$ is a linear combination of $\frac{\theta_\phi}{\Omega_\mathcal{K}}$'s with coefficients in $A_{[g]}$. Moreover all the ϕ's have the same ϕ_∞, and the ϕ_ℓ's for $\ell < \infty$ are smooth functions taking algebraic values .(This can be seen by interpreting the theta functions defined before [54, Theorem B.2] in terms of Weil representations presented here. Note that the CM period $\Omega_\mathcal{K}$ is missing in [54, Theorem B.2]. The algebracity follows from [39, Theorem 2.5]. In fact in [39] the period is $h(z_0)$ for some weight $\frac{1}{2}$ form h on Sp_4, as our Hermitian space is 2 dimensional, and z_0 is a CM point with $h(z_0) \neq 0$. This $h(z_0)$ is just $\Omega_\mathcal{K}$ up to multiplying by a non-zero algebraic number). So by taking appropriate ϕ_1 the l'_{θ_1} is a rational functional. We extend the definition of l'_{θ_1} linearly. It is easily seen to be well defined.

Lemma 4.6. *The l'_{θ_1} takes values in the space of constant functions on any theta function θ_ϕ as above.*

Proof. We note that for any ϕ,

$$(\omega_{\lambda^{-1}}(x_i)(\phi_1)\omega_\lambda(x_i)(\phi)) = (\phi_1, \phi). \tag{2}$$

One way of seeing this is to consider the intertwining operator $\delta_\psi(\phi_1 \boxtimes \phi)(0) = (\phi_1, \phi)$ and use the formula for the Weil representation ω_λ on $\mathrm{U}(2,2)$ and note also that $\gamma(g, g) = \begin{pmatrix} g & \\ & \zeta^{-1}g\zeta \end{pmatrix} \in \mathrm{U}(2,2)$ and $\omega_{\lambda^{-1}} = \omega_\lambda.(\lambda^{-1} \circ \det)$. The lemma follows from the above equation. $\square$

Remark 4.7. Later we will use this functional on Fourier-Jacobi coefficients for $\mathrm{U}(3,1)$. We can view it as a function on $G_PN_P(\mathbb{A})$ by $FJ_{[g]}(p, f) = FJ_P(pg, f)$ for $p \in G_PN_P(\mathbb{A})$ and thus an adelic theta function. [28] has proved the following compatibility of the analytically and algebraically defined Fourier-Jacobi expansions using the usual idenfitication of the global sections of $\mathcal{L}(\beta)$ and (classical or adelic) theta functions, keeping the integral and rational structures:

$$FJ_{[hg]}(-,f) = FJ^{h}_{[g]}(f)(-)$$

Note that the period factor appearing in [16, 3.6.5] is 1 since we are in the scalar weight κ.

5 Klingen-Eisenstein Series

5.1 Archimedean Picture

Let (π_∞, V_∞) be a finite dimensional representation of $D^\times_\infty$. Let ψ_∞ and τ_∞ be characters of $\mathbb{C}^\times$ such that $\psi_\infty|_{\mathbb{R}^\times}$ is the central character of π_∞. We assume here that $\tau_\infty(z) = z^{-\frac{\kappa}{2}}\bar{z}^{\frac{\kappa}{2}}$ and ψ_∞ is trivial. Then there is a unique representation π_ψ of $\mathrm{GU}(2)(\mathbb{R})$ determined by π_∞ and ψ_∞ such that the central character is ψ_∞. These determine a representation $\pi_\psi \times \tau_\infty$ of $M_P(\mathbb{R}) \simeq \mathrm{GU}(2)(\mathbb{R}) \times \mathbb{C}^\times$. We extend this to a representation ρ_∞ of $P(\mathbb{R})$ by requiring $N_P(\mathbb{R})$ acts trivially. Let $I(V_\infty) = \mathrm{Ind}^{G(\mathbb{R})}_{P(\mathbb{R})}\rho_\infty$ (smooth induction) and $I(\rho_\infty) \subset I(V_\infty)$ be the subspace of K_∞ -finite vectors. (Elements of $I(V_\infty)$ can be realized as functions on K_∞) For any $f \in I(V)$ and $z \in \mathbb{C}^\times$ we define a function f_z on $G(\mathbb{R})$ by

$$f_z(g) := \delta(m)^{\frac{3}{2}+z}\rho(m)f(k), g = mnk \in P(\mathbb{R})K_\infty.$$

There is an action $\sigma(\rho, z)$ on $I(V_\infty)$ by

$$(\sigma(\rho, z)(g))(k) = f_z(kg).$$

As in [45, Section 9.1] we let $(\pi^\vee, V_\infty)$ be the irreducible representation of $D^\times_\infty$ given by the same space of π but

$$\pi^\vee(x) = \pi(\mathrm{Ad}(w)\cdot x).$$

We let $\rho^\vee_\infty$ and $I(\rho^\vee_\infty)$ be the corresponding objects by replacing $\pi_\infty, \psi_\infty, \tau_\infty$ by $\pi_\infty \otimes (\tau_\infty \circ \mathrm{Nm}), \psi_\infty\tau_\infty\tau^c_\infty, \bar{\tau}^c_\infty$. Let $w = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{pmatrix}$. Then there is an intertwining operator $A(\rho_\infty, z, -)$: $I(\rho_\infty) \to I(\rho^\vee_\infty)$ by:

$$A(\rho_\infty, z, f)(k) := \int_{N_P(\mathbb{R})} f_z(wnk)dn.$$

In this paper we use the case when π_∞ is the trivial representation. Then by Frobenius reciprocity law there is a unique (up to scalar) vector $\tilde{v} \in I(\rho)$ such that $k.\tilde{v} = \det\mu(k, i)^{-\kappa}\tilde{v}$ for any $k \in K^+_\infty$. We fix v and scale $\tilde{v}$ such that $\tilde{v}(1) = v$. In $\pi^\vee$, $\pi(w)v$ (w is defined in section 3.1) has the action of K^+_∞ given by multiplying by $\det\mu(k, i)^{-\kappa}$. There is a unique vector $\tilde{v}^\vee \in I(\rho^\vee)$ such that the action of K^+_∞ is given by $\det\mu(k, i)^{-\kappa}$ and $\tilde{v}^\vee(w) = \pi(w)v$. Then by uniqueness there is a constant $c(\rho, z)$ such that $A(\rho, z, \tilde{v}) = c(\rho, z)\tilde{v}^\vee$.

Definition 5.1. *We define $F_\kappa \in I(\rho)$ to be the $\tilde{v}$ as above.*

We record the following lemma proved in [48].

Lemma 5.2. *Let $\kappa \geq 6$ and $z_\kappa = \frac{\kappa-3}{2}$. Then $c(\rho, z_\kappa) = 0$.*

5.2 ℓ-adic Picture

Let (π_ℓ, V_ℓ) be an irreducible admissible representation of $D^\times(\mathbb{Q}_\ell)$ and π_ℓ is unitary and tempered if D is split at ℓ. Let ψ and τ be characters of $\mathcal{K}_\ell^\times$ such that $\psi|_{\mathbb{Q}_\ell^\times}$ is the central character of π_ℓ. Then there is a unique irreducible admissible representation π_ψ of $\mathrm{GU}(2)(\mathbb{Q}_\ell)$ determined by π_ℓ and ψ_ℓ. As before we have a representation $\pi_\psi \times \tau$ of $M_P(\mathbb{Q}_\ell)$ and extend it to a representation ρ_ℓ of $P(\mathbb{Q}_\ell)$ by requiring $N_P(\mathbb{Q}_\ell)$ acts trivially. Let $I(\rho_\ell) = \mathrm{Ind}_{P(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}\rho_\ell$ be the admissible induction. Similarly we let $\pi^\vee$ be the $D^\times(\mathbb{Q}_\ell)$ representation whose space is that of π but the action is given by

$$\pi^\vee(x) = \pi(\mathrm{Ad}(\omega)\cdot x).$$

Define f_z for $f \in I(\rho_\ell)$ and $\rho_\ell^\vee, I(\rho_\ell^\vee), A(\rho_\ell, z, f)$ etc as before. For $v \notin \Sigma$ we have $D^\times(\mathbb{Q}_\ell) \simeq \mathrm{GL}_2(\mathbb{Q}_\ell)$. Moreover we can choose isomorphism as a conjugation by elements in $\mathrm{GL}_2(\mathcal{O}_{\mathcal{K},\ell})$ (note that both groups are subgroups of $\mathrm{GL}_2(\mathcal{K}_\ell)$. We have π_ℓ, ψ_ℓ, τ_ℓ are unramified and $\varphi_\ell \in V_\ell$ is a spherical vectors then there is a unique vector $f^0_{\varphi_\ell} \in I(\rho_\ell)$ which is invariant under $G(\mathbb{Z}_\ell)$ and $f^0_{\varphi_\ell}(1) = \varphi_\ell$.

5.3 Global Picture

Now let $(\pi = \otimes_v \pi_v, V)$ be an irreducible unitary cuspidal automorphic representation of $D(\mathbb{A}_\mathbb{Q})$ we define $I(\rho)$ to be the restricted tensor product of $\otimes_v I(\rho_v)$ with respect to the unramified vectors $f^0_{\varphi_\ell}$ for some $\varphi = \otimes_v \phi_v \in \pi$. We can define f_z, $I(\rho^\vee)$ and $A(\rho, z, f)$ similar to the local case. The f_z takes values in V which can be realized as automorphic forms on $D(\mathbb{A}_\mathbb{Q})$. We also write f_z for the scalar-valued functions $f_z(g) := f_z(g)(1)$ and define the Klingen Eisenstein series:

$$E(f, z, g) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f_z(\gamma g).$$

This is absolutely convergent if $\mathrm{Re} z >> 0$ and has meromorphic continuation to all $z \in \mathbb{C}$.

5.4 Constant Terms

Definition 5.3. *For any parabolic subgroup R of $\mathrm{GU}(3,1)$ and an automorphic form φ we define φ_R to be the constant term of φ along R given by the following:*

$$\varphi_R(g) = \int_{N_R(\mathbb{Q})\backslash N_R(\mathbb{A}_\mathbb{Q})} \varphi(ng)dn.$$

The following lemma is well-known (see [34, II.1.7]).

Lemma 5.4. *Let R be a standard $\mathbb{Q}$-parabolic of $\mathrm{GU}(3,1)$ (i.e, $R \supseteq B$ where B is the standard Borel). Suppose $\mathrm{Re}(z) > \frac{3}{2}$.*
(i) If $R \neq P$ then $E(f, z, g)_R = 0$;
(ii) $E(f, z, -)_P = f_z + A(\rho, f, z)_{-z}$.

5.5 Galois representations

For the holomorphic Klingen Eisenstein series we can also associate a reducible Galois representation with the same recipe as in subsection 3.3. Write τ' for the restriction of τ to $\mathbb{A}_{\mathbb{Q}}$ and let $\sigma_{\tau'}$ be the corresponding Galois character of $G_{\mathbb{Q}}$ via class field theory. The resulting Galois representation is easily seen to be

$$\sigma_{\tau'}\sigma_{\psi^c}\epsilon^{-\kappa} \oplus \sigma_{\psi^c}\epsilon^{-3} \oplus \rho_{\pi_f}.\sigma_{\tau^c}\epsilon^{-\frac{\kappa+3}{2}}.$$

Note that $\kappa+3$ is an odd number. However π_f is a unitary representation whose L-function is the usual L-function for f shifted by $\frac{1}{2}$. So it makes sense to write in the above way. This can be obtained in the same manner as [45, Sections 9.5, 9.6].

6 Siegel Eisenstein Series and Pullback

6.1 Generalities

Local Picture:

Our discussion in this section follows [45, 11.1-11.3] closely. Let $Q=Q_n$ be the Siegel parabolic subgroup of GU_n consisting of matrices $\begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$. It consists of matrices whose lower-left $n\times n$ block is zero. For a place v of $\mathbb{Q}$ and a character χ of $\mathcal{K}_v^\times$ we let $I_n(\chi_v)$ be the space of smooth $K_{n,v}$-finite functions (here $K_{n,v}$ means the maximal compact subgroup $G_n(\mathbb{Z}_v)$) $f: K_{n,v}\to\mathbb{C}$ such that $f(qk)=\chi_v(\det D_q)f(k)$ for all $q\in Q_n(\mathbb{Q}_v)\cap K_{n,v}$ (we write q as block matrix $q=\begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$). For $z\in\mathbb{C}$ and $f\in I(\chi)$ we also define a function $f(z,-):G_n(\mathbb{Q}_v)\to\mathbb{C}$ by $f(z,qk):=\chi(\det D_q))|\det A_qD_q^{-1}|_v^{z+n/2}f(k), q\in Q_n(\mathbb{Q}_v)$ and $k\in K_{n,v}$.

For $f\in I_n(\chi_v), z\in\mathbb{C}$, and $k\in K_{n,v}$, the intertwining integral is defined by:

$$M(z,f)(k):=\bar{\chi}_v^n(\mu_n(k))\int_{N_{Q_n}(F_v)} f(z,w_nrk)dr.$$

For z in compact subsets of $\{\mathrm{Re}(z)>n/2\}$ this integral converges absolutely and uniformly, with the convergence being uniform in k. In this case it is easy to see that $M(z,f)\in I_n(\bar{\chi}_v^c)$. A standard fact from the theory of Eisenstein series says that this has a continuation to a meromorphic section on all of $\mathbb{C}$.

Let $\mathcal{U}\subseteq\mathbb{C}$ be an open set. By a meromorphic section of $I_n(\chi_v)$ on $\mathcal{U}$ we mean a function $\varphi:\mathcal{U}\mapsto I_n(\chi_v)$ taking values in a finite dimensional subspace $V\subset I_n(\chi_v)$ and such that $\varphi:\mathcal{U}\to V$ is meromorphic.

Global Picture

Recall that we defined $z_\kappa=\frac{\kappa-3}{2}$. We also define $z'_\kappa=\frac{\kappa-2}{2}$. For an idele class character $\chi=\otimes\chi_v$ of $\mathbb{A}_{\mathcal{K}}^\times$ we define a space $I_n(\chi)$ to be the restricted tensor product defined using the spherical vectors $f_v^{sph}\in I_n(\chi_v)$ (invariant under $K_{n,v}$) such that $f_v^{sph}(K_{n,v})=1$, at the finite places v where χ_v is unramified.

For $f \in I_n(\chi)$ we consider the Eisenstein series

$$E(f; z, g) := \sum_{\gamma \in Q_n(\mathbb{Q}) \backslash G_n(\mathbb{Q})} f(z, \gamma g).$$

This series converges absolutely and uniformly for (z, g) in compact subsets of

$$\{\mathrm{Re}(z) > n/2\} \times G_n(\mathbb{A}_{\mathbb{Q}}).$$

The defined automorphic form is called Siegel Eisenstein series.

The Eisenstein series $E(f; z, g)$ has a meromorphic continuation in z to all of $\mathbb{C}$ in the following sense. If $\varphi : \mathcal{U} \to I_n(\chi)$ is a meromorphic section, then we put $E(\varphi; z, g) = E(\varphi(z); z, g)$. This is defined at least on the region of absolute convergence and it is well known that it can be meromorphically continued to all $z \in \mathbb{C}$.

Now for $f \in I_n(\chi), z \in \mathbb{C}$, and $k \in \prod_{v \nmid \infty} K_{n,v} \prod_{v | \infty} K_\infty$ there is a similar intertwining integral $M(z, f)(k)$ as above but with the integration being over $N_{Q_n}(\mathbb{A}_F)$. This again converges absolutely and uniformly for z in compact subsets of $\{\mathrm{Re}(z) > n/2\} \times K_n$. Thus $z \mapsto M(z, f)$ defines a holomorphic section $\{\mathrm{Re}(z) > n/2\} \to I_n(\bar{\chi}^c)$. This has a continuation to a meromorphic section on $\mathbb{C}$. For $Re(z) > n/2$, we have

$$M(z, f) = \otimes_v M(z, f_v), f = \otimes f_v.$$

The functional equation for Siegel Eisenstein series is:

$$E(f, z, g) = \chi^n(\mu(g)) E(M(z, f); -z, g)$$

in the sense that both sides can be meromorphically continued to all $z \in \mathbb{C}$ and the equality is understood as of meromorphic functions of $z \in \mathbb{C}$.

6.2 Embeddings

We define some embeddings of a subgroup of $\mathrm{GU}(3,1) \times \mathrm{GU}(0,2)$ into some larger groups. This will be used in the doubling method. First we define $G(3,3)'$ to be the unitary similitude group associated to:

$$\begin{pmatrix} & & 1 & \\ & \zeta & & \\ -1 & & & \\ & & & -\zeta \end{pmatrix}$$

and $G(r+s, r+s)'$ to be associated to

$$\begin{pmatrix} \zeta & \\ & -\zeta \end{pmatrix}.$$

We define an embedding

$$\alpha : \{g_1 \times g_2 \in \mathrm{GU}(3,1) \times \mathrm{GU}(0,2), \mu(g_1) = \mu(g_2)\} \to \mathrm{GU}(3,3)'$$

and

$$\alpha' : \{g_1 \times g_2 \in \mathrm{GU}(2,0) \times \mathrm{GU}(0,2), \mu(g_1) = \mu(g_2)\} \to \mathrm{GU}(2,2)'$$

as $\alpha(g_1, g_2) = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$ and $\alpha'(g_1, g_2) = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$. We also define an isomorphism:

$$\beta : \mathrm{GU}(3,3)' \xrightarrow{\sim} \mathrm{GU}(3,3)$$

and

$$\beta' : \mathrm{GU}(2,2)' \xrightarrow{\sim} \mathrm{GU}(2,2)$$

by:

$$g \mapsto S^{-1} g S$$

or

$$g \mapsto S'^{-1} g S'$$

where

$$S = \begin{pmatrix} 1 & & & \\ & 1 & & -\frac{\zeta}{2} \\ & & 1 & \\ & -1 & & -\frac{\zeta}{2} \end{pmatrix}$$

and

$$S' = \begin{pmatrix} 1 & -\frac{\zeta}{2} \\ -1 & -\frac{\zeta}{2} \end{pmatrix}$$

We write γ and γ' for the embeddings $\beta \circ \alpha$ and $\beta' \circ \alpha'$, respectively.

6.3 Pullback Formula

We recall the pullback formula of Shimura (see [48] for details). Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$. Given a cuspform φ on GU(2) we consider

$$F_{\varphi}(f; z, g) := \int_{\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})} f(z, S^{-1}\alpha(g, g_1 h) S) \bar{\chi}(\det g_1 g) \varphi(g_1 h) dg_1,$$

$$f \in I_3(\chi), g \in \mathrm{GU}(3,1)(\mathbb{A}_{\mathbb{Q}}), h \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}}), \mu(g) = \mu(h)$$

or

$$F'_{\varphi}(f'; z, g) = \int_{\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})} f'(z, S'^{-1}\alpha(g, g_1 h) S') \bar{\chi}(\det g_1 g) \varphi(g_1 h) dg_1$$

$$f' \in I_2(\chi), g \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}}), h \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}}), \mu(g) = \mu(h)$$

This is independent of h. The pullback formulas are the identities in the following proposition.

Proposition 6.1. *Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$.*
(i) If $f' \in I_2(\chi)$, then $F'_{\varphi}(f'; z, g)$ converges absolutely and uniformly for (z, g) in compact sets of $\{\mathrm{Re}(z) > 1\} \times \mathrm{GU}(2,0)(\mathbb{A}_{\mathbb{Q}})$, and for any $h \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}})$ such that $\mu(h) = \mu(g)$

$$\int_{\mathrm{U}(2)(\mathbb{Q}) \backslash \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})} E(f'; z, S'^{-1}\alpha(g, g_1 h) S') \bar{\chi}(\det g_1 h) \varphi(g_1 h) dg_1 = F'_{\varphi}(f'; z, g).$$

(ii) If $f \in I_3(\chi)$, then $F_\varphi(f;z,g)$ converges absolutely and uniformly for (z,g) in compact sets of $\{Re(z) > 3/2\} \times \mathrm{GU}(3,1)(\mathbb{A}_\mathbb{Q})$ such that $\mu(h) = \mu(g)$

$$\int_{\mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})} E(f;z,S^{-1}\alpha(g,g_1h)S)\bar{\chi}(\det g_1h)\varphi(g_1h)dg_1$$
$$= \sum_{\gamma\in P(\mathbb{Q})\backslash\mathrm{GU}(3,1)(\mathbb{Q})} F_\varphi(f;z,\gamma g),$$

with the series converging absolutely and uniformly for (z,g) in compact subsets of

$$\{\mathrm{Re}(z) > 3/2\} \times \mathrm{GU}(3,1)(\mathbb{A}_\mathbb{Q}).$$

This is a special case of [48, Proposition 3.5].

6.4 Fourier-Jacobi expansion

From now on we fix a splitting character λ of $\mathcal{K}^\times\backslash\mathbb{A}^\times$ of infinite type $(-\frac{1}{2},\frac{1}{2})$ which is unramified at p and unramified outside Σ and such that $\lambda|_{\mathbb{A}_\mathbb{Q}^\times} = \chi_{\mathcal{K}/\mathbb{Q}}$. The following formula is proved in [48, 3.3.1]. Let τ be a Hecke character of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ of infinite type $(-\frac{\kappa}{2},\frac{\kappa}{2})$.

Definition 6.2. *For $\beta \in S_n(\mathbb{Q})$ and φ a holomorphic automorphic form on $\mathrm{GU}(n,n)$ we define the β-th Fourier-coefficient*

$$\varphi_\beta(g) = \int \varphi(\begin{pmatrix} 1_n & S \\ & 1_n \end{pmatrix} g)e_\mathbb{A}(-\mathrm{Tr}\beta S)dS.$$

For a prime v and $f_v \in I_n(\tau)$ we also define the local Fourier coefficient at $g_v \in \mathrm{GU}(n,n)(\mathbb{Q}_v)$ as

$$f_{v,\beta}(z,g_v) = \int_{S_n(\mathbb{Q}_v)} f_v(z,\omega_n(\begin{pmatrix} 1_n & S_v \\ & 1_n \end{pmatrix} g_v)e_v(-\mathrm{Tr}\beta S_v)dS_v.$$

For φ a holomorphic automorphic form on $\mathrm{GU}(3,3)$ and $\beta \in \mathbb{Q}^+$ we define

$$FJ_\beta(\varphi)(g) = \int \varphi(\begin{pmatrix} 1_3 & \begin{matrix} S & 0 \\ 0 & 0 \end{matrix} \\ & 1_3 \end{pmatrix} g)e_\mathbb{A}(-\mathrm{Tr}\beta S)dS.$$

(The S has size (1×1). For $E(f;z,g)$ with $f \in I_3(\tau)$ we define

$$\mathrm{FJ}_\beta(f;z,g) = FJ_\beta(E(f;z,-))(g).$$

Proposition 6.3. *Suppose $f \in I_3(\tau)$ and $\beta \in S_1(\mathbb{Q})$, β is positive. If $E(f;z,g)$ is the Siegel Eisenstein Series on $\mathrm{GU}(3,3)$ defined by f for some $Re(z)$ sufficiently large then the β-th Fourier-Jacobi coefficient $E_\beta(f;z,g)$ satisfies:*

$$E_\beta(f;z,g) = \sum_{\gamma\in Q_2(\mathbb{Q})\backslash\mathrm{GU}_2(\mathbb{Q})} \sum_{y\in M_{1\times 2}(\mathcal{K})} \int_{S_m(\mathbb{A})} f(w_3 \begin{pmatrix} 1_3 & \begin{matrix} S & y \\ {}^t\bar{y} & 0 \end{matrix} \\ & 1_3 \end{pmatrix} \alpha_2(1,\gamma)g)e_\mathbb{A}(-\mathrm{Tr}\beta S)dS$$

where if $g_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, g_2 = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ *then:*

$$\alpha(g_1, g_2) = \begin{pmatrix} A & & B & \\ & D' & & C' \\ C & & D & \\ & B' & & A' \end{pmatrix}.$$

(the notation α *is already used before. However the meaning should be clear from context.)*

Definition 6.4. *If* $g_v \in \mathrm{U}_2(\mathbb{Q}_v), x \in \mathrm{GL}_1(\mathcal{K}_v)$*, then define:*

$$FJ_\beta(f_v; z, y, g, x) = \int_{S_1(\mathbb{Q}_v)} f(w_3 \begin{pmatrix} 1_3 & S & y \\ & {}^t\bar{y} & 0 \\ & & 1_3 \end{pmatrix} \alpha(\mathrm{diag}(x, {}^t\bar{x}^{-1}), g)) e_{\mathbb{Q}_v}(-\mathrm{Tr}\beta S) dS$$

Since

$$\begin{pmatrix} 1_3 & S & X \\ & {}^t\bar{X} & \\ & & 1_3 \end{pmatrix} \begin{pmatrix} 1_1 & & & \\ & \bar{A}^{-1} & & \\ & & 1 & \\ & B\bar{A}^{-1} & & A \end{pmatrix} = \begin{pmatrix} 1 & XB\bar{A}^{-1} & & \\ & \bar{A}^{-1} & & \\ & & 1 & \\ & B\bar{A}^{-1} & & A \end{pmatrix} \begin{pmatrix} 1_3 & & S - XB{}^t\bar{X} & XA \\ & & \bar{A}{}^t\bar{X} & \\ & & & \\ & & 1_n & \end{pmatrix}.$$

it follows that:

$$FJ_\beta(f_v; z, y, \begin{pmatrix} A & B\bar{A}^{-1} \\ & \bar{A}^{-1} \end{pmatrix} g, x) =$$

$$\tau_v^c(\det A)^{-1} |\det A\bar{A}|_v^{z+3/2} e_v(-\mathrm{Tr}({}^t\bar{x}\beta x B)) FJ_\beta(f; z, yA, g, x)$$

Also we have:

$$FJ_\beta(f; z, y, g, x) = \tau_v(\det x) |\det x\bar{x}|_{\mathbb{A}}^{-(z+\frac{1}{2})} FJ_{{}^t\bar{x}\beta x}(f; z, x^{-1}y, g, 1).$$

We write (x, y, t) for $\begin{pmatrix} 1 & x & t + \frac{xy^* - yx*}{2} & y \\ & 1_2 & y^* & 0_2 \\ & & 1 & \\ & & -x^* & 1_2 \end{pmatrix}$ so that it becomes a Heisenberg group if we give the pairing $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 y_2^* + y_2 x_1^* - x_2 y_1^* - y_1 x_2^*$. (see [54, section 4])

Lemma 6.5. *Suppose for some place* v *the local Fourier Jacobi coefficient*

$$FJ_\beta(f_v; z, (0, y, 0)\alpha(1, u)) = f(u, z)\omega_\lambda(u)\phi(y)$$

for any $u \in U(2,2)$ *and* $y \in W^d$ *and some Schwartz function* $\phi \in S(W^d)$ *and some* $f \in I((\tau/\lambda)_v, z)$*, then we have:*

$$FJ_\beta^{(3,3)}(f_v; z, (x, y, t)\alpha(1, u)) = f(u, z)\omega_\lambda((x, y, t).u)\phi(0)$$

for any (x, y, t)*.*

Proof. This is a consequence of:

$$\begin{pmatrix} 1 & -x & & \\ & 1_2 & & \\ & & 1 & \\ & & x^* & 1_2 \end{pmatrix} = \begin{pmatrix} 1 & x & t + \frac{xy^* - yx*}{2} & y \\ & 1_2 & y^* & 0_2 \\ & & 1 & \\ & & -x^* & 1_2 \end{pmatrix} \begin{pmatrix} 1 & & t - \frac{xy^* + yx^*}{2} & y \\ & 1_2 & y^* & 0_2 \\ & & 1 & \\ & & & 1_2 \end{pmatrix}$$

where we write x^* for ${}^t\bar{x}$. $\square$

6.5 Archimedean Cases

We let $\mathbf{i} := \begin{pmatrix} i & \\ & \frac{\zeta}{2} \end{pmatrix}$ or $\frac{\zeta}{2}$ depending on the size 3×3 or 2×2. The Siegel section we choose is $f_{sieg,\infty} := f_\kappa(g,z) := J_3(g,\mathbf{i})^{-\kappa}|J_3(g,\mathbf{i})|^{\kappa-2z-3}$ and $f'_\kappa(g,z) = J_2(g,\mathbf{i})^{-\kappa}|J_n(g,\mathbf{i})|^{\kappa-2z-2}$. For $\varphi \in \pi_\infty$ we define the pullback sections:

$$F_\kappa(z,g) := \int_{\mathrm{U}(2)(\mathbb{R})} f_\kappa(z, S^{-1}\alpha(g,g_1)S)\bar{\tau}(\det g_1)\pi(g_1)\varphi dg_1$$

and

$$F'_\kappa(z,g) := \int_{\mathrm{U}(2)(\mathbb{R})} f'_\kappa(z, S'^{-1}\alpha(g,g_1)S')\bar{\tau}(\det g_1)\pi(g_1)\varphi dg_1$$

If we define an auxiliary $f^\circ_{\kappa,n}(z,g) = J_n(g, i1_n)^{-\kappa}|J_n(g,i1_n)|^{\kappa-2z-n}$ for $n = 2,3$, then $f_\kappa(g,z) = f^\circ_{\kappa,3}(gg_0)$ and $f'_\kappa(g,z) = f^\circ_{\kappa,2}(gg_0)$ for

$$g_0 = \mathrm{diag}(1, \frac{\mathfrak{s}^{\frac{1}{2}}d^{\frac{1}{4}}}{\sqrt{2}}, \frac{d^{\frac{1}{4}}}{\sqrt{2}}, 1, (\frac{\mathfrak{s}^{\frac{1}{2}}d^{\frac{1}{4}}}{\sqrt{2}})^{-1}, (\frac{d^{\frac{1}{4}}}{\sqrt{2}})^{-1})$$

or

$$g_0 = \mathrm{diag}(\frac{\mathfrak{s}^{\frac{1}{2}}d^{\frac{1}{4}}}{\sqrt{2}}, \frac{d^{\frac{1}{4}}}{\sqrt{2}}, (\frac{\mathfrak{s}^{\frac{1}{2}}d^{\frac{1}{4}}}{\sqrt{2}})^{-1}, (\frac{d^{\frac{1}{4}}}{\sqrt{2}})^{-1})$$

depending on the size.

Lemma 6.6. *The integrals are absolutely convergent for* $\mathrm{Re}(z)$ *sufficiently large and for such* z*, we have:*
(i)

$$F_{Kling,\infty}(z,g) := F_\kappa(z,g) = F_{\kappa,z}(g);$$

(ii)

$$F'_\kappa(z,g) = \pi(g)\varphi;$$

where $F_{\kappa,z}$ *is defined in definition 5.1 using* φ *as the* v *there.*

Fourier Coefficients
The following lemma is [45, Lemma 11.4].

Lemma 6.7. *Suppose* $\beta \in S_n(\mathbb{R})$*. Then the function* $z \to f_{\kappa,\beta}(z,g)$ *has a meromorphic continuation to all of* $\mathbb{C}$*. Furthermore, if* $\kappa \geq n$ *then* $f_{\kappa,n,\beta}(z,g)$ *is holomorphic at* $z_\kappa := (\kappa-n)/2$ *and for* $y \in \mathrm{GL}_n(\mathbb{C})$, $f^\circ_{\kappa,n,\beta}(z_\kappa, \mathrm{diag}(y, {}^t\bar{y}^{-1})) = 0$ *if* $\det\beta \leq 0$ *and if* $\det\beta > 0$ *then*

$$f^\circ_{\kappa,n,\beta}(z_\kappa, \mathrm{diag}(y, {}^t\bar{y}^{-1})) = \frac{(-2)^{-n}(2\pi i)^{n\kappa}(2/\pi)^{n(n-1)/2}}{\prod_{j=0}^{n-1}(\kappa-j-1)!} e(i\mathrm{Tr}(\beta y {}^t\bar{y}))\det(\beta)^{\kappa-n}\det\bar{y}^\kappa.$$

The local Fourier coefficient for f_κ can be easily deduced from that for f°_κ.
Fourier-Jacobi Coefficients
The following lemma can be found in [48, Lemma 4.4].

Lemma 6.8. *Let* $z_\kappa = \frac{\kappa-3}{2}$*,* $\beta \in S_1(\mathbb{R})$*,* $\det\beta > 0$*. then:*

(i) $FJ_\beta(z_\kappa, f^\circ_{\kappa,3}, x, \eta, 1) = f^\circ_{\kappa,1,\beta}(z_\kappa+1,1)e(i\mathrm{Tr}({}^t\bar{x}\beta x))$*;*

(ii) If $g \in \mathrm{U}(2,2)(\mathbb{R})$*, then*

$$FJ_{\beta,\kappa}(z_\kappa, f^\circ_{\kappa,3}, x, g, 1) = e(i\mathrm{Tr}\beta)c_1(\beta,\kappa)f^\circ_{\kappa-1,2}(z_\kappa, g')w_\beta(g')\Phi_{\beta,\infty}(x).$$

where $g' = \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} g \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix}$, $c_1(\beta,\kappa) = \frac{(-2)^{-1}(2\pi i)^\kappa}{(\kappa-1)!}\det\beta^{\kappa-1}$ *and* $\Phi_{\beta,\infty} = e^{-2\pi\mathrm{Tr}(<x,x>_\beta)}$.

Lemma 6.9. *We have*

$$FJ_\beta(f_\kappa, x, g, 1) = e(i\mathrm{Tr}\beta)c_1(\beta,\kappa)J(g,\boldsymbol{i})^{-\kappa}\omega_\beta(g'g_0)\Phi_{\beta,\infty}(x)$$

for all $g \in \mathrm{U}(2,2)(\mathbb{R}), x \in \mathbb{C}^2$*.*

Proof. Note that

$$f_\kappa(g, z_\kappa) = J(g,\boldsymbol{i})^{-\kappa} = J(gg_0,i)^{-\kappa}J(g_0,i)^\kappa = (\frac{\sqrt{d}}{2})^{-\kappa}J(gg_0,i)^{-\kappa}.$$

□

Lemma 6.10. *Let* $x_1 = (x_{11}, x_{12}), x_2 = (x_{21}, x_{22})$ *where the* $x_{ij} \in \mathbb{R}$*. Then*

$$\delta_\psi^{-1}(\omega_{1,\lambda}(\eta g_0)\Phi_\infty)(x_1,x_2) = \frac{\mathfrak{s}^{\frac{1}{2}}d^{\frac{1}{2}}}{4}e^{-2\pi\sqrt{d}(\mathfrak{s}x_{11}^2+x_{12}^2)}e^{-2\pi\sqrt{d}(\mathfrak{s}x_{21}^2+x_{22}^2)}$$

Proof. Straightforward from the expression for Φ_∞ and δ_ψ. □

Definition 6.11.

$$\Phi_\infty = \omega_1(g_0)\Phi_{1,\infty}, \Phi''_\infty = \omega_1(\eta g_0)\Phi_\infty,$$

$$f_{2,\infty}(g) = f'_{\kappa-1}(gg_0), f''_{2,\infty}(g) = f'_{\kappa-1}(g\eta g_0),$$

$$\phi_{1,\infty}(x_1,x_2) = \phi_{2,\infty}(x_1,x_2) = \frac{\mathfrak{s}^{\frac{1}{4}}d^{\frac{1}{4}}}{2}e^{-2\pi\sqrt{d}(\mathfrak{s}x_1^2+x_2^2)}, x_1, x_2 \in \mathbb{R}.$$

6.6 Unramified Cases

Let v be a prime outside Σ (in particular $v \nmid p$). Then the Siegel sections $f_{v,sieg} = f_v^{sph}$ and $f'_{v,sieg} = f_v^{sph,'}$ is defined to be the unique section that is invariant under $\mathrm{GU}(n,n)(\mathbb{Z}_v)$ $(n = 3,2)$ and is 1 at identity.

Lemma 6.12. *Suppose* π, ψ *and* τ *are unramified and* $\phi \in \pi$ *is a newvector. If* $Re(z) > 3/2$ *then the pull back integral converges and*

$$F_\varphi(f_v^{sph}; z, g) = \frac{L(\tilde{\pi},\xi,z+1)}{\prod_{i=0}^1 L(2z+3-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}F_{\rho,z}(g)$$

where F_ρ *is the spherical section defined using* $\varphi \in \pi$*. Also:*

$$F'_\varphi(f_v^{sph,'}, z, g) = \frac{L(\tilde{\pi},\xi,z+\frac{1}{2})}{\prod_{i=0}^1 L(2z+2-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}\pi(g)\varphi.$$

Fourier Coefficients

Definition 6.13. *Let Φ_0 be the characteristic function of $\mathcal{O}_{\mathcal{K}}^2$.*

Lemma 6.14. *Let $\beta \in S_n(\mathbb{Q}_v)$ and let $r := \mathrm{rank}(\beta)$. Then for $y \in \mathrm{GL}_n(\mathcal{K}_v)$,*

$$\begin{aligned} f_{v,\beta}^{sph}(z, diag(y, {}^t\bar{y}^{-1})) = & \quad \tau(\det y)|\det y\bar{y}|_v^{-z+n/2} D_v^{-n(n-1)/4} \\ & \times \frac{\prod_{i=r}^{n-1} L(2z+i-n+1, \bar{\tau}'\chi_{\mathcal{K}}^i)}{\prod_{i=0}^{n-1} L(2z+n-i, \bar{\tau}'\chi_{\mathcal{K}}^i)} h_{v,{}^t\bar{y}\beta y}(\bar{\tau}'(q_v) q_v^{-2z-n}). \end{aligned}$$

where $h_{v,{}^t\bar{y}\beta y} \in \mathbb{Z}[X]$ is a monic polynomial depending on v and ${}^t\bar{y}\beta y$ but not on τ. If $\beta \in S_n(\mathbb{Z}_v)$ and $\det \beta \in \mathbb{Z}_v^\times$, then we say that β is v-primitive and in this case $h_{v,\beta} = 1$.

Fourier-Jacobi Coefficients

In this subsection we consider a prime $v \in \Sigma$ not dividing p.

Lemma 6.15. *Suppose v is unramified in $\mathcal{K}$. Let $\beta \in S_1(\mathbb{Q}_v)$ such that $\det \beta \neq 0$. Let $y \in \mathrm{GL}_2(\mathcal{K}_v)$ such that ${}^t\bar{y}\beta y \in S_1(\mathbb{Z}_v)$, let λ be an unramified character of $\mathcal{K}_v^\times$ such that $\lambda|_{\mathbb{Q}_v \times} = 1$. If $\beta \in \mathrm{GL}_1(\mathcal{O}_{\mathcal{K},v})$, then for $u \in \mathrm{U}_\beta(\mathbb{Q}_v)$:*

$$FJ_\beta(f_3^{sph}; z, x, g, u) = \tau(\det u)|\det u\bar{u}|_v^{-z+1/2} \frac{f_2^{sph}(z, g')(\omega_\beta(u, g')\Phi_0)(x)}{L(2z+3, \bar{\tau}')}.$$

Here $g' \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} g \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix}$.

For these primes we define $\phi_{1,v}$ and $\phi_{2,v}$ to be the Schwartz function on X_v which is the characteristic function of $\mathbb{Z}_v^2$. We define $f_{2,v} = f_v^{sph,'}$ and $\Phi_v = \Phi_0 = \Phi_v''$.

6.7 Ramified Cases

The Siegel section we choose is $I_3(\tau) \ni f_{v,sieg} = f^\dagger(g\tilde{\gamma}_v)$ where $\tilde{\gamma}_v$ is $\begin{pmatrix} 1 & & & \\ & 1_2 & & \frac{1}{y\bar{y}}1_2 \\ & & 1 & \\ & & & 1_2 \end{pmatrix}$ where $y \in \mathcal{O}_v$ is some fixed element such that the valuation is sufficiently large. We also define

$$I_2(\tau) \ni f'_{v,sieg} = f^\dagger(g\tilde{\gamma}'_v)$$

where $\tilde{\gamma}'_v = \begin{pmatrix} 1_2 & \frac{1}{y\bar{y}}1_2 \\ & 1_2 \end{pmatrix}$.

Pullback Formulas

The proofs of the following lemmas are just special cases of [48, Lemma 4.9, 4.10].

Lemma 6.16. *Let $K_v^{(2)}$ be the subgroup of $G(\mathbb{Q}_v)$ of the form $\begin{pmatrix} 1 & f & c \\ & 1_2 & g \\ & & 1 \end{pmatrix}$ where*

$$g = -\zeta {}^t\bar{f}, c - \frac{1}{2} f\zeta {}^t\bar{f} \in \mathbb{Z}_\ell, f \in (y\bar{y}), g \in (\zeta y\bar{y}), c \in \mathcal{O}_v.$$

Then $F_\phi(z; g, f)$ is supported in $PwK_v^{(2)}$ and is invariant under the action of $K_v^{(2)}$.

Now we let $\mathfrak{Y}$ be the set of matrices $A \in U(2)(\mathbb{Q}_v)$ such that $M = A - 1$ satisfies:

$$M(1 + \frac{\zeta}{2}y\bar{y} + y\bar{y}N) = \zeta y\bar{y}$$

for some $N \in M_2(\mathcal{O}_v)$.

Lemma 6.17. *Let φ be some vector invariant under the action of $\mathfrak{Y}$ defined above, then*

$$F_\varphi(z,w) = \tau(y\bar{y})|(y\bar{y})^2|_v^{-z-\frac{3}{2}}\mathrm{Vol}(\mathfrak{Y}) \cdot \varphi.$$

Also $F'_\varphi(f'_{v,sieg}; z, g) = \tau(y\bar{y})|(y\bar{y})^2|_v^{-z-1}\mathrm{Vol}(\mathfrak{Y}) \cdot \pi(g)\varphi$.

Fourier Coefficients

Lemma 6.18. *(i) Let $\beta \in S_3(\mathbb{Q}_\ell)$. Then $f_{v,\beta}(z,1) = 0$ if $\beta \notin S_3(\mathbb{Z}_\ell)^*$. If $\beta \in S_3(\mathbb{Z}_\ell)^*$ then*

$$f_{v,\beta}(z, \mathrm{diag}(A, {}^t\bar{A}^{-1})) = D_\ell^{-\frac{3}{2}}\tau(\det A)|\det A\bar{A}|_\ell^{-z+\frac{3}{2}}e_\ell(\frac{\beta_{22}+\beta_{33}}{y\bar{y}})$$

where D_ℓ is the discriminant of $\mathcal{K}_\ell$.

(ii) If $\beta \in S_2(\mathbb{Q}_\ell)$, then $f_{v,\beta}(z,1) = 0$ if $\beta \in S_2(\mathbb{Z}_\ell)^*$. If $\beta \in S_2(\mathbb{Z}_\ell)^*$ then

$$f'_{v,\beta}(z, \mathrm{diag}(A, {}^t\bar{A}^{-1})) = D_\ell^{-\frac{1}{2}}\tau(\det A)|\det A\bar{A}|_\ell^{-z+\frac{r}{2}}e_\ell(\frac{\beta_{11}+\beta_{22}}{y\bar{y}}).$$

Fourier-Jacobi Coefficients

Lemma 6.19. *If $\beta \notin S_1(\mathbb{Z}_v)^*$ then $FJ_\beta(f^\dagger; z, u, g, 1) = 0$. If $\beta \in S_1(\mathbb{Z}_v)^*$ then*

$$FJ_\beta(f^\dagger; z, u, g, 1) = f^\dagger(z, g'\eta)\omega_\beta(h, g'\eta^{-1})\Phi_0(u).\mathrm{Vol}(S_1(\mathbb{Z}_v)),$$

where $g' = \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} g \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix}$.

Let $A = \frac{1}{y\bar{y}}1_2$. So:

$$FJ_\beta(f_{sieg,v}; z, u, g, h) = f^\dagger(z, g'\begin{pmatrix} 1 & \\ -A & 1 \end{pmatrix}\eta)(\omega_\beta(h, g'\begin{pmatrix} 1 & \\ -A & 1 \end{pmatrix}\eta))\Phi_0(u)$$

for $h \in \mathrm{U}_\beta(\mathbb{Q}_\ell)$. We define $\Phi''_v := \omega_\beta(\begin{pmatrix} 1 & A \\ & 1 \end{pmatrix})\Phi_0$ and $\Phi_v = \omega_\beta(\begin{pmatrix} 1 & \\ -A & 1 \end{pmatrix}\eta)\Phi_0$. We also define $f_{2,v} = \rho(\begin{pmatrix} 1 & \\ -A & 1 \end{pmatrix}\eta)f^\dagger \in I_2(\frac{\tau}{\lambda})$.

Split Case

Suppose $v = w\bar{w}$ is a split prime. Recall we have the local polarization $X'_v \oplus Y'_v$. Now we write $x'_1 = (x'_{11}, x'_{12})$ and $x'_2 = (x'_{21}, x'_{22})$ with respect to $\mathcal{K}_v \simeq \mathcal{K}_w \times \mathcal{K}_{\bar{w}}$. The following lemma follows from a straightforward computations and will be used later. Let $\lambda_{\theta,v}$ be a character of $\mathcal{K}_v^\times$.

Lemma 6.20. *Let $\chi_{\theta,v}$ be a character of $\mathbb{Z}_v^\times$. It is possible to choose a Schwartz function ϕ_1' such that the function: $\phi_2'(x_2') := \int_{X_1'} \delta_\psi'^{-1}(\Phi_v'')(x_1', x_2')\phi_1(x_1')dx_1'$ is given by*

$$\phi_2'(x_2') = \begin{cases} \lambda_{\theta,v}\chi_{\theta,v}(x_{22}') & x_{21}' \in \mathbb{Z}_v, x_{22}' \in \mathbb{Z}_v^\times \\ 0 & \text{otherwise.} \end{cases}$$

Moreover when we are moving our datum in p-adic families, this ϕ_1' is not going to change.

We define $\phi_{1,v} = \delta_\psi^{-,''}(\phi_1')$, $\phi_{2,v} = \delta_\psi^{-,''}(\phi_2')$.

Non-split Case

Lemma 6.21. *We consider the action of the compact abelian group $\mathrm{U}(1)(\mathbb{Q}_v)$ on $\delta_\psi^{-1}(\Phi_v'')$ by the Weil representation (Weil representation using the splitting character λ) of*

$$1 \times \mathrm{U}(1)(\mathbb{Q}_v) \hookrightarrow 1 \times \mathrm{U}(2)(\mathbb{Q}_v) \hookrightarrow \mathrm{U}(2)(\mathbb{Q}_v) \times \mathrm{U}(2)(\mathbb{Q}_v) \hookrightarrow \mathrm{U}(2,2)(\mathbb{Q}_v).$$

We can write $\delta_\psi^{-1}(\Phi_v'')$ as a sum of eigenfunctions of this action. Let $m = \max\{\mathrm{ord}_v(\mathrm{cond}\lambda_v), 3\}+1$. If $\mathrm{ord}_v(y\bar{y}) > m$, then there is a such eigenfunction ϕ_2' whose eigenvalue is a character $\lambda_v^2\chi_{\theta,v}$ for $\chi_{\theta,v}$ of conductor at least ϖ_v^m. Moreover there is a p-integral valued Schwartz function ϕ_1 and a set of p-integral $C_{v,i} \in \bar{\mathbb{Q}}_p$ and $u_{v,i} \in \mathrm{U}(1)(\mathbb{Q}_v)$'s such that the function

$$\phi_2(x_2) = \int_{X_1(\mathbb{Q}_v)} \sum_i \delta_\psi^{-1}(C_{v,i}\omega_\lambda(u_{v,i}, 1)\Phi_v'')(x_1, x_2)\phi_1(x_1)dx_1$$

(here $1 \in \mathrm{U}(2,2)(\mathbb{Q}_v)$) is a non-zero multiple of ϕ_2'.

Proof. Consider the embedding $\mathrm{U}(1,1) \hookrightarrow \mathrm{U}(2,2)$ by

$$j : g \mapsto \begin{pmatrix} \mathfrak{s}^{-1} & & & \\ & 1 & & \\ & & \mathfrak{s}^{-1} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_g & & b_g & \\ & a_g & & b_g \\ d_g & & d_g & \\ & c_g & & d_g \end{pmatrix} \begin{pmatrix} \mathfrak{s} & & & \\ & 1 & & \\ & & \mathfrak{s} & \\ & & & 1 \end{pmatrix}$$

for $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$. Define $f_{\Phi_v''}(g, \frac{1}{2}) = (\omega_\lambda(j(g))\Phi_v'')(0) \in I_2(\frac{\tau}{\lambda})$. We define $i : \mathrm{U}(1) \times \mathrm{U}(1) \hookrightarrow \mathrm{U}(1,1)$ by

$$i(g_1, g_2) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\delta^{-1} & -\delta^{-1} \end{pmatrix} \begin{pmatrix} 1 & \\ & g_1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\delta}{2} \\ -1 & -\frac{\delta}{2} \end{pmatrix}.$$

For $g_1 \in \mathrm{U}(1)(\mathbb{Q}_v)$,

$$f_{\Phi_v''}(i(1, g_1), \frac{1}{2}) = (\omega_\lambda(j \circ i(1, g_1))\Phi_v'')(0) = (\delta_\psi\omega(1, g_1)\delta_\psi^{-1}\Phi_v'')(0)$$

here in the first and last expression $1 \in \mathrm{U}(2)(\mathbb{Q}_v)$ and g_1 is viewed as the element in the center of $\mathrm{U}(2)(\mathbb{Q}_v)$. Thus we are reduced to proving the following lemma. $\square$

Lemma 6.22. *Let $g_1 = 1 + \varpi_v^m.a \in \mathrm{U}(1)(\mathbb{Q}_v)$ for m as in the above lemma and $a \in \mathcal{O}_{\mathcal{K},v}$, if $n = y\bar{y}$ is such that $\mathrm{ord}_v n > m$ then $f_{\Phi_v''}(i(1, g_1); \frac{1}{2}) \neq f_{\Phi_v''}(1; \frac{1}{2})$.*

Proof. We have that:

$$\begin{aligned}
& \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\delta^{-1} & -\delta^{-1} \end{pmatrix} \begin{pmatrix} 1 & \\ & g_1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\delta}{2} \\ -1 & -\frac{\delta}{2} \end{pmatrix} \\
= & \begin{pmatrix} \frac{1}{2} & -\frac{g_1}{2} \\ -\delta^{-1} & -\delta^{-1} g_1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\delta}{2} \\ -1 & -\frac{\delta}{2} \end{pmatrix} \\
= & \begin{pmatrix} \frac{1}{2} + \frac{g_1}{2} & -\frac{\delta}{4} + \frac{\delta g_1}{4} \\ -\delta^{-1} + \delta^{-1} g_1 & \frac{1}{2} + \frac{g_1}{2} \end{pmatrix}
\end{aligned}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{n} \\ & 1 \end{pmatrix} = \begin{pmatrix} a & \frac{a}{n} + b \\ c & \frac{c}{n} + d \end{pmatrix} = \begin{pmatrix} \frac{n(ad-bc)}{c+nd} & \frac{a}{n} + b \\ 0 & \frac{c}{n} + d \end{pmatrix} \begin{pmatrix} 1 & \\ \frac{n}{1+\frac{nd}{c}} & 1 \end{pmatrix}.$$

Now the lemma follows readily. $\square$

In all cases we write $\phi_{1,v}, \phi_{2,v}$ for the ϕ_1, ϕ_2 above.

Remark 6.23. Later when we are moving the datum p-adic analytically, this choice of ϕ_1 is not going to change.

6.8 p-adic Cases

We recall some results in [48] with some modifications. Recall that we have the triple π_p, ψ_p, τ_p and $\xi_p := \psi_p/\tau_p$, χ_p is the central character of π_p and $\psi_p|_{\mathbb{Q}_p^\times} = \chi_p$. Suppose π_p is nearly ordinary in the sense that $\pi_p = \pi(\chi_{1,p}, \chi_{2,p})$ such that $\mathrm{ord}_p(\chi_{1,p}(p)) = -\frac{1}{2}$ and $\mathrm{ord}_p(\chi_{2,p}(p)) = \frac{1}{2}$. We write $\tau_p = (\tau_1, \tau_2)$ and $\xi_p = (\xi_1, \xi_2)$. We call the triple is generic if there is a $t \geq 2$ such that $\xi_{1,p}, \xi_{2,p}, \chi_p, \chi_p^{-1}\xi_{1,p}, \chi_p^{-1}\xi_{p,2}$ all have conductor p^t. Later when we are working with families, it is easily seen that these points are Zariski dense due to the fact that $p \geq 5$ (in fact this is the only place where we used the fact $p \geq 5$). Although the definition for generic points is different from [48] however the argument there goes through since the only place using this definition is lemma 4.4.3 there, which can be proved completely in the same way under our definition for generic. We define: $\xi_1^\dagger = \chi_1\bar{\xi}_2, \xi_2^\dagger = \chi_2\bar{\xi}_2$. We define f_t to be the section supported in $Q(\mathbb{Q}_v)K_t$, invariant under K_t and takes value 1 on the identity. Define

$$\begin{aligned}
f_{sieg,p}(g) \quad & = \mathfrak{g}(\tau_p')^{-3} c_3^{-1}(\bar{\tau}_p', -z) p^{-3t} \mathfrak{g}(\xi_1^\dagger)\xi_1^\dagger(-1)\mathfrak{g}(\xi_2^\dagger)\xi_2^\dagger(-1)\bar{\xi}_1^\dagger(p^t a)\bar{\xi}_2^\dagger(p^t b) \\
& \times \sum_{a,b \in p^{-t}\mathbb{Z}_p^\times/\mathbb{Z}_p, m,n \in \mathbb{Z}_p/p^t\mathbb{Z}_p} f_t(g\Upsilon \begin{pmatrix} 1 & & & a+bmn & bm & \\ & 1 & & bn & b & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}).
\end{aligned}$$

and

$$
\begin{aligned}
f^{\square}_{sieg,p}(g) \qquad & = \mathfrak{g}(\tau'_p)^{-3}c_3^{-1}(\bar{\tau}'_p,-z)p^{-3t}\mathfrak{g}(\xi_1^\dagger)\xi_1^\dagger(-1)\mathfrak{g}(\xi_2^\dagger)\xi_2^\dagger(-1)\bar{\xi}_1^\dagger(p^ta)\bar{\xi}_2^\dagger(p^tb)\\
& \times \sum_{a,b\in p^{-t}\mathbb{Z}_p^\times/\mathbb{Z}_p, n\in\mathbb{Z}_p/p^t\mathbb{Z}_p} f_t(g\Upsilon\begin{pmatrix}1 & & & & a & bm\\ & 1 & & & & b\\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1\end{pmatrix}).
\end{aligned}
$$

We also define:

$$
\begin{aligned}
f'_{sieg,p}(g) \qquad & = \mathfrak{g}(\tau'_p)^{-3}c_2^{-1}(\bar{\tau}'_p,-z)p^{-3t}\mathfrak{g}(\xi_1^\dagger)\xi_1^\dagger(-1)\mathfrak{g}(\xi_2^\dagger)\xi_2^\dagger(-1)\bar{\xi}_1^\dagger(p^ta)\bar{\xi}_2^\dagger(p^tb)\\
& \times \sum_{a,b\in p^{-t}\mathbb{Z}_p^\times/\mathbb{Z}_p, m,n\in\mathbb{Z}_p/p^t\mathbb{Z}_p} f_t(g\Upsilon\begin{pmatrix}1 & & a+bmn & bm\\ & 1 & bn & b\\ & & 1 & \\ & & & 1\end{pmatrix}).
\end{aligned}
$$

where $\Upsilon \in \mathrm{U}(3,3)(\mathbb{Q}_p)$ is such that it is $\begin{pmatrix}1 & & & \\ & \frac{1}{2}.1_2 & & -\frac{1}{2}1_2\\ & & 1 & \\ & -\zeta^{-1} & & -\zeta^{-1}\end{pmatrix}$ via the first projection $\mathrm{U}(3,3)(\mathbb{Q}_p) \simeq \mathrm{GL}_6(\mathbb{Q}_p)$ and $c_n(\tau',z) = \tau'(p^{nt})p^{2ntz-tn(n+1)/2}$.

<u>Pullback Formulas</u>
We refer to [48, 4.4.1] for the discussion of nearly ordinary vectors, which means the vector whose U_p-eigenvalues are p-adic units. Let $\varphi = \varphi^{ord} \in \pi_p$ be a nearly ordinary vector. Define $f^0 \in I_p(\rho)$ to be the nearly ordinary Klingen section supported in $P(\mathbb{Q}_p)w'_3B_t(\mathbb{Z}_p)$ where $B_t(\mathbb{Z}_p)$ consists of matrices in $\mathrm{GL}_4(\mathbb{Z}_p)$ which are upper triangular modulo p^t and $w'_3 = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1\end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}_p)$ and the right action of $B_t(\mathbb{Z}_p)$ on f^0 is given by a character (see [48, section 4], note the differences in the indices discussed there). Moreover we require the value of f^0 on w'_3 is given by

$$\mathfrak{g}(\tau'_p)^{-1}\tau'_p(p^{-t})p^{\kappa-2}p^{(\kappa-3)t}\xi_{1,p}^2\chi_{1,p}^{-1}\chi_{2,p}^{-1}(p^{-t})\mathfrak{g}(\xi_{1,p}\chi_{1,p}^{-1})\mathfrak{g}(\xi_{1,p}\chi_{2,p}^{-1})\varphi.$$

The fact that it is indeed a nearly ordinary vector is explained in the discussion before [48, Definition 4.43]. Then by the computations in [48] we have the following (see the end of [48, Section 4].

Lemma 6.24. *(1)* $F_\varphi(f_{sieg,p}, z_\kappa, g) = f^0(g) := F_{Kling,p}$*;*
(2) $F'_\varphi(f'_{sieg,p}, z_\kappa, g) = p^{(\kappa-3)t}\xi_{1,p}^2\chi_{1,p}^{-1}\chi_{2,p}^{-1}(p^{-t})\mathfrak{g}(\xi_{1,p}\chi_{1,p}^{-1})\mathfrak{g}(\xi_{1,p}\chi_{2,p}^{-1}).\pi(g)\varphi.$

<u>Fourier Coefficients</u>
We define the function $\Phi_{\xi^\dagger}$ as the function on the set of (2×2) $\mathbb{Q}_p$-matrices as follows. If $x = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is such that both its determinant and a are in $\mathbb{Z}_p^\times$ then $\Phi_{\xi^\dagger}(x) = \xi_1^\dagger(a)\xi_2^\dagger(\frac{\det x}{a})$. Otherwise $\Phi_{\xi^\dagger} = 0$. The following lemma is proved in [48].

Lemma 6.25.

$$f_{sieg,p,\beta} = \Phi_{\xi^\dagger}(\begin{pmatrix}\beta_{21} & \beta_{22}\\ \beta_{31} & \beta_{32}\end{pmatrix})$$

for $\beta = \begin{pmatrix}\beta_{11} & \beta_{12} & \beta_{13}\\ \beta_{21} & \beta_{22} & \beta_{23}\\ \beta_{31} & \beta_{32} & \beta_{33}\end{pmatrix}$ *with* $\beta_{11}, \beta_{12}, \beta_{13}, \beta_{23}, \beta_{33} \in \mathbb{Z}_p$*, and is* 0 *otherwise.*

<u>Fourier-Jacobi Coefficients</u>

For $\beta \in S_1(\mathbb{Q}_v) \cap \mathrm{GL}_1(\mathbb{Z}_v)$ we are going to compute the Fourier-Jacobi coefficient for f_t at β. We have the following ([48, lemma 4.54])

Lemma 6.26. *Let* $x := \begin{pmatrix}1 & \\ D & 1\end{pmatrix}$ *(this is a block matrix with respect to* $(2+2)$*).*
(a) $FJ_\beta(f_t; -z, v, x\eta^{-1}, 1) = 0$ *if* $D \notin p^t M_2(\mathbb{Z}_p)$*;*
(b) if $D \in p^t M_2(\mathbb{Z}_p)$ *then* $FJ_\beta(f_t; -z, v, x\eta^{-1}, 1) = c(\beta, \tau, z)\Phi_0(v)$*, where*

$$c(\beta, \tau, z) := \bar{\tau}(-\det\beta)|\det\beta|_v^{2z+2}\mathfrak{g}(\tau')\mathfrak{g}(\tau'_p)\tau'_p(p^t)p^{-2tz-3t}.$$

Note the formula:

$$\mathfrak{g}(\tau'_p)^3\tau'_p(p^{3t})p^{-6tz-6t} = \mathfrak{g}(\tau'_p)\tau'_p(p^t)p^{-2tz-3t}\mathfrak{g}(\tau'_p)^2\tau'_p(p^{2t})p^{-4tz-3t}.$$

We get:

Lemma 6.27.

$$\mathfrak{g}(\tau'_p)^{-3}c_3(\bar{\tau}'_p, -z_\kappa)p^{-3t}\sum_{a,b,m} FJ_1(\rho\begin{pmatrix} & & a & bm\\ 1_3 & & & b\\ & & & \\ & & 1_3 & \end{pmatrix}f_{t,p}; z_\kappa, v, g)\bar{\xi}_1^\dagger(-p^t a)\bar{\xi}_2^\dagger(-p^t b)\mathfrak{g}(\xi_1^\dagger)\mathfrak{g}(\xi_2^\dagger)$$

$$= \quad (\mathfrak{g}(\tau'_p)^{-2}c_2^{-1}(\bar{\tau}'_p, -z)p^{-t}\mathfrak{g}(\xi_2^\dagger)\sum_b \bar{\xi}_2^\dagger(p^t b)f_t(g\begin{pmatrix}1 & \\ A_{-b} & 1\end{pmatrix}\eta))(\mathfrak{g}(\xi_1^\dagger)p^{-2t}\omega_\beta(g)\Phi'_p(v_1, v_2, v_3, v_4)).$$

Here $A_x = \begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}$*. Also under the projection* $\mathrm{U}(3,3)(\mathbb{Q}_p) \simeq \mathrm{GL}_6(\mathbb{Q}_p)$*, the* v_1, v_2, v_3, v_4 *are* $\begin{pmatrix}1 & & & & v_3 & v_4\\ & 1 & & v_1 & & \\ & & 1 & v_2 & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1\end{pmatrix}$*. The Weil representations are the one in subsection 4.8. We use* ρ *to denote the right action of* $\mathrm{GU}(3,3)(\mathbb{Q}_p)$ *on the Siegel sections. And*

$$\Phi'_p(v_1, v_2, v_3, v_4) = \begin{cases} \bar{\xi}_1^\dagger(a), & v_1, v_2 \in \mathbb{Z}_p, v_4 \in p^{-t}\mathbb{Z}_p, v_3 \in \frac{a}{p^t} + \mathbb{Z}_p \text{ for some } a \in \mathbb{Z}_p^\times\\ 0, & \text{otherwise.}\end{cases}$$

Proof. First we fix b and consider the Fourier-Jacobi expansion of:

$$\sum_{a,m} \rho(\begin{pmatrix} 1 & & & & a & bm \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}) f_t \tag{3}$$

note that:

$$w_3 \begin{pmatrix} & t & v_3 & v_4 \\ 1_3 & v_1 & D_1 & D_2 \\ & v_2 & D_3 & D_4 \\ & & 1_3 & \end{pmatrix} \alpha(1,\eta^{-1}) \begin{pmatrix} 1 & & & & a & bm \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

$$= \quad w_3 \begin{pmatrix} 1 & -a & -bm & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} & t' & v_3' & v_4' \\ 1_3 & v_1 & D_1 & D_2 \\ & v_2 & D_3 & D_4 \\ & & 1_3 & \end{pmatrix} \alpha(1,\eta^{-1})$$

where $v_3' = v_3 + D_1.a + D_3.bm, v_4' = v_4 + D_2.a + D_4.bm, t' = t + v_1.a + v_2.bm$. From this a simple calculation shows that the Fourier-Jacobi expansion of (3) at g is:

$$(\mathfrak{g}(\tau_p')^{-2} c_2^{-1}(\bar{\tau}_p', -z) p^{-t} \mathfrak{g}(\xi_2^\dagger) \rho(\eta) f_t(g') (\mathfrak{g}(\xi_1^\dagger) p^{-2t} \omega(g') \Phi_p')(v_1, v_2, v_3, v_4).$$

So the Fourier-Jacobi expansion of

$$\sum_{a,b,m} \rho(\begin{pmatrix} 1 & & & & a & bm \\ & 1 & & & & b \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}) f_t$$

is

$$\sum_b (\mathfrak{g}(\tau_p')^{-2} c_2^{-1}(\bar{\tau}_p', -z) p^{-t} \mathfrak{g}(\xi_2) \rho(\begin{pmatrix} 1 & \\ A_{-b} & 1 \end{pmatrix}) \rho(\eta) f_t(g') (\mathfrak{g}(\xi_2) p^{-2t} \omega(g') \omega(g' \begin{pmatrix} 1 & \\ A_{-b} & 1 \end{pmatrix}) \Phi_p')(v_1, v_2, v_3, v_4)$$

Note that $\omega(\begin{pmatrix} 1 & \\ A_{-b} & 1 \end{pmatrix}) \Phi_p' = \Phi_p'$, we get the required Fourier-Jacobi expansion. $\square$

We define $f_{2,p} = \sum_b \mathfrak{g}(\xi_2^\dagger) p^{-t} \bar{\xi}_2^\dagger(-p^t b) \mathfrak{g}(\tau_p')^{-2} c_2^{-1}(\bar{\tau}_p', -z) \rho(\begin{pmatrix} 1 & \\ A_{-b} & 1 \end{pmatrix} \eta) f_t$ and $\Phi_p = \mathfrak{g}(\xi_1^\dagger) p^{-2t} \Phi_p'$.

We record some formulas:

$$\rho(\eta)f_{2,p}=f''_{2,p}=\frac{\tau_p}{\lambda_p}(-1)\mathfrak{g}(\xi_2^\dagger)\sum_b\bar{\xi}_2^\dagger(b)\mathfrak{g}(\tau'_p)^{-2}c_2^{-1}(\bar{\tau}'_p,-z)\rho(\begin{pmatrix}1&A_b\\&1\end{pmatrix})f_t$$

$$(\omega_\beta(\eta)\Phi_p)(v_1,v_2,v_3,v_4)=\begin{cases}\xi_1^\dagger(-v_1), & v_3,v_4\in\mathbb{Z}_p, v_1\in\mathbb{Z}_p^\times, v_2\in p^t\mathbb{Z}_p\\ 0, & \text{otherwise.}\end{cases}\tag{4}$$

We define two Schwartz functions on $X_p^{-,'}$ by letting $\phi'_{1,p}$ to be the characteristic function of $\mathbb{Z}_p^2\subset\mathbb{Q}_p^2$ and $\phi'_{2,p}(\xi_1^\dagger(-x_1))$ if $x_1\in\mathbb{Z}_p^\times, x_2\in p^t\mathbb{Z}_p$ and is zero otherwise.

Definition 6.28. *We define* $\phi_{1,p}=\delta_\psi^{-,''}(\phi'_{1,p})$, $\phi_{2,p}=\delta_\psi^{-,''}(\phi'_{2,p})$.

6.9 Pullback Formulas Again

In this section we prove the local pullback formulas for $\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$ which will be used to decompose the restriction to $\mathrm{U}(2)\times\mathrm{U}(2)$ of the Siegel Eisenstein series on $\mathrm{U}(2,2)$ showing up in the Fourier-Jacobi expansion of E_{sieg} on $\mathrm{U}(3,3)$. Fortunately, the local calculations are the same as in the previous sections for f'_{sieg} and F'_φ's except for the case $v=p$.

p-adic case:

We temporarily denote the p-component of automorphic representation π_h of some h on $\mathrm{U}(2)(\mathbb{Q}_p)$ as $\pi(\chi_1,\chi_2)$ with $v_p(\chi_1(p))=-\frac{1}{2}, v_p(\chi_2(p))=\frac{1}{2}$. We also write temporarily write τ for τ/λ in this subsection. We let $\tau_p=(\tau_1,\tau_2)$ and require $\chi_1\tau_2^{-1}$ and $\chi_2\tau_2^{-1}\xi_2^\dagger$ are unramified. We let $\varphi=\varphi^{ss}\in\pi_{h,p}$ for $\varphi^{ss}=\pi_p(\begin{pmatrix}&1\\p^t&\end{pmatrix})\varphi^{ord}$ for some nearly ordinary vector φ^{ord}. Define

$$f_{2,p}(g)=\mathfrak{g}(\tau'_p)^{-2}c_2(\bar{\tau}'_p,-z_\kappa/2)^{-1}p^{-t}\mathfrak{g}(\xi_2^\dagger)\sum_{b\in\frac{p^{-t}\mathbb{Z}_p^\times}{\mathbb{Z}_p}}\bar{\xi}_2^\dagger(bp^t)\rho(\begin{pmatrix}1&A_b\\&1\end{pmatrix})f_t(g\Upsilon').$$

It is hard to evaluate the integral directly. So we use the trick of using the functional equation as in [45, Proposition 11.13]. We first do the integration for the auxilliary

$$F_\varphi(f_{2,p}^\dagger;z,g):=\int_{\mathrm{U}(2)(\mathbb{Q}_p)}f_{2,p}^\dagger(z,S^{-1}\alpha(g_1,g)S)\bar{\tau}(\det g)\pi_h(g_1)\varphi dg_1$$

at $g=w$ where $\varphi\in\pi_{h,p}$ and

$$f_{2,p}^\dagger(g):=p^{-t}\mathfrak{g}(\xi_2^\dagger)\sum_b\bar{\xi}_2^\dagger(bp^t)\rho(\begin{pmatrix}1&A_b\\&1\end{pmatrix})f^\dagger(g\Upsilon),\ f^{(2,2),\dagger}\in I(\bar{\tau}^c).$$

For $A\in\mathrm{U}(2)(\mathbb{Q}_p)\simeq\mathrm{GL}_2(\mathbb{Q}_p)$ note that

$$S^{-1}\mathrm{diag}(A,1)\begin{pmatrix}1&A_b\\&1\end{pmatrix}=\begin{pmatrix}-1&-A\\&-A\end{pmatrix}\begin{pmatrix}&1\\-A_b-A^{-1}&1\end{pmatrix}w.$$

So in order for this to be in supp$f^\dagger$ we must have $A^{-1}+A_b\in M_2(\mathbb{Z}_p)$. So A^{-1} can be written as $\begin{pmatrix}1 & \\ n & 1\end{pmatrix}\begin{pmatrix} & u\\ v & \end{pmatrix}\begin{pmatrix}1 & \\ m & 1\end{pmatrix}$ for $v\in\mathbb{Z}_p\backslash\{0\}$, $u\in -b+\mathbb{Z}_p$ and $m,n\in p^t\mathbb{Z}_p$. Thus

$$A=\begin{pmatrix}1 & m\\ & 1\end{pmatrix}\begin{pmatrix}v^{-1} & \\ & u^{-1}\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}1 & \\ n & 1\end{pmatrix}.$$

A direct computation gives the integral equals:

$$\begin{aligned}&\chi_2(-1)\chi_1\chi_2(p^t)p^{t(z+1)}\bar{\tau}^c((-p^t,-1))\sum_{i=1}^{\infty}(\chi_1(p^{-1})\bar{\tau}^c((p^{-1},1))p^{-z-\frac{1}{2}})^i\phi^{ord}\\ = \quad & p^{t(z+1)}\bar{\tau}^c((-p^t,-1))L_p(\pi,\tau,z+\frac{1}{2})\chi_1\chi_2(p^t)\chi_2(-1).\end{aligned}$$

The π in the L-factor means the base change of π from $\mathrm{U}(2)$ to GL_2. Note that it is not convergent at $z=-z_\kappa$ and is defined by analytic continuation at that point.

Now we apply the functional equation trick to evaluate the pullback integral for $f_{2,p}$. As in [45, Proposition 11.28] the local constant showing up when applying the intertwining operator at $z=-z_\kappa$ is

$$\epsilon(\pi,\tau,-z_\kappa+\frac{1}{2})=\mathfrak{g}(\bar{\tau}_1\bar{\chi}_1)\tau_1\chi_1(p^t)\mathfrak{g}(\bar{\tau}_1\bar{\chi}_2)\tau_1\chi_2(p^t)\mathfrak{g}(\bar{\tau}_2\chi_2)\tau_2\chi_2^{-1}(p^t)p^{\frac{3}{2}\kappa-6}.$$

To sum up our original local integral equals

$$L_p(\pi_h,\bar{\tau}^c,z_\kappa+\frac{1}{2})\mathfrak{g}(\bar{\tau}_1\bar{\chi}_1)\tau_1\chi_1(p^t)\mathfrak{g}(\bar{\tau}_1\bar{\chi}_2)\tau_1\chi_2(p^t)p^{(2\kappa-5)t}\chi_1(p^t)p^{\frac{t}{2}}\varphi^{ord}.$$

Note that $<\tilde{\varphi}^{ord},\varphi^{ss}>=<\tilde{\phi}^{ord},\varphi_{low}>.\chi_1(p^t)p^{\frac{t}{2}}$ where we define $\varphi_{low}=\pi_p(\begin{pmatrix} & 1\\ 1 & \end{pmatrix})\varphi^{ord}$. Thus if we replace φ^{ss} by φ_{low} in the definition for pullback integral then it equals

$$L_p(\pi_h,\bar{\tau}^c,z_\kappa+\frac{1}{2})\mathfrak{g}(\bar{\tau}_1\bar{\chi}_1)\tau_1\chi_1(p^t)\mathfrak{g}(\bar{\tau}_1\bar{\chi}_2)\tau_1\chi_2(p^t)p^{(2\kappa-5)t}\varphi^{ord}.$$

Remark 6.29. Later when we are defining $E_{sieg,2}$ the τ here should be $\frac{\tau}{\lambda}$.

6.10 Global Computations

We first define two normalization factors as in [48, 5.3.1]

$$B_{\mathcal{D}}:\ =\frac{\Omega_p^{2\kappa\Sigma_\infty}}{\Omega_\infty^{2\kappa\Sigma_\infty}}(\frac{(-2)^{-3}(2\pi i)^{3\kappa}(2/\pi)^3}{\prod_{j=0}^{2}(\kappa-j-1)!})^{-1}\prod_{i=0}^{2}L^\Sigma(2z_\kappa+3-i,\bar{\tau}'\chi_{\mathcal{K}}^i),$$

$$B'_{\mathcal{D}}:\ =\frac{\Omega_p^{2\kappa\Sigma_\infty}}{\Omega_\infty^{2\kappa\Sigma_\infty}}(\frac{(-2)^{-2}(2\pi i)^{2\kappa}(2/\pi)}{\prod_{j=0}^{1}(\kappa-j-1)!})^{-1}\prod_{i=0}^{1}L^\Sigma(2z_\kappa+2-i,\bar{\tau}'\chi_{\mathcal{K}}^i).$$

The $z_\kappa=\frac{\kappa-3}{2}$ and $z'_\kappa=\frac{\kappa-2}{2}$. We define $E_{sieg}(z,g)=E_{sieg}(z,f_{sieg},g)$ on $\mathrm{GU}(3,3)$ for $f_{sieg}=B_{\mathcal{D}}\prod_v f_{sieg,v}$ and $E'_{sieg}(z,g)=E'_{sieg}(z,f'_{sieg},g)$ on $\mathrm{GU}(3,3)$ for $f'_{sieg}=B'_{\mathcal{D}}\prod_v f'_{sieg,v}$. (Note that compared to [48], the normalization factors at p here are already included in our definitions of

p-adic Siegel sections.) We also define $E^{\square}_{sieg}(g)=E(z,f^{\square}_{sieg},g)$ where $f^{\square}_{sieg}$ is the same as f_{sieg} at all primes not dividing p and is $f^{\square}_{sieg,p}$ at p.

For some $g_1\in \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})$ (which we are going to specify in section 8) we define E_{Kling} by:

$$E_{Kling}(z,g)=\frac{1}{\Omega_\infty^{2\kappa}}\int_{[\mathrm{U}(2)]}E_{sieg}(z,\gamma(g,hg'))\bar{\tau}(\det g')\varphi(g'g_1)dg'.$$

This is the Klingen Eisenstein series constructed using the Klingen section $F_{Kling}=\prod_v F_{Kling,v}$. The period factor showing up is to make it rational (see [16, 2.8]). The φ is defined as follows. First recall that given a CM character ψ and a form on $D^\times$ whose central character is $\psi|_{\mathbb{A}_{\mathbb{Q}}}$ we can produce a form on $U(2)$ whose central character is the restriction of ψ. So we often construct forms on $D^\times$ and get forms on $U(2)$ this way. In section 8.2 we are going to construct a Dirichlet character ϑ. We define

$$f_\Sigma=(\prod_{v\in\Sigma,v\nmid N}\pi(\begin{pmatrix}&1\\ \varpi_v&\end{pmatrix})-\chi_{1,v}(\varpi_v)q_v^{\frac{1}{2}})f_{new}$$

where f_{new} is a new vector in π and define $f_\vartheta\in\pi$ by

$$f_\vartheta(g)=\prod_{v\in\Sigma\text{ split },v\nmid p}\ \sum_{\{a_v\in\frac{\varpi_v\mathbb{Z}_v^\times}{\varpi_v^{1+s_v}\mathbb{Z}_v}\}_v}\vartheta(\frac{-a_v}{\varpi_v})f_\Sigma(g\prod_v\begin{pmatrix}1&\\ a_v&1\end{pmatrix}_v\begin{pmatrix}\varpi_v^{-s_v}&\\ &1\end{pmatrix}_v)$$

where the s_v above is the order of the conductor of ϑ at v and t_v is the order of the conductor of π at v. Define $\varphi=\prod_v\varphi_v=\pi(g_1)f_\vartheta$ for some g_1 defined in section 8. We record the following easy lemma, which explains the motivation for the definition of f_Σ: to pick up a certain Iwahari-invariant vector from the unramified representation π_v for $v\in\Sigma\backslash\{v,v|N\}$.

Lemma 6.30. *Consider the model for the unramified principal series representation*

$$\pi(\chi_{1,v},\chi_{2,v})=\{f:K_v\to\mathbb{C},f(qk)=\chi_{1,v}(a)\chi_{2,v}(d)\delta_B(q)f(k),q=\begin{pmatrix}a&b\\ &d\end{pmatrix}\in B(\mathbb{Z}_v)\}.$$

Let f_{ur} be the constant function 1 on K_v, f_0 be the function supported and takes value 1 on K_1 for $K_1=\{\begin{pmatrix}a&b\\ c&d\end{pmatrix}|\varpi_v|c\}$. Then

$$(\chi_{2,v}(\varpi_v)q_v^{-\frac{1}{2}}-\chi_1(\varpi_v)q_v^{\frac{1}{2}})f_0=\pi(\begin{pmatrix}&1\\ \varpi_v&\end{pmatrix})f_{ur}-\chi_{1,v}(\varpi_v)q_v^{\frac{1}{2}}f_{ur}.$$

Fourier-Jacobi Coefficients

Proposition 6.31. *The Fourier-Jacobi coefficient for $\beta=1$ at [1] is given by:*

$$FJ_1(E_{sieg})(z_\kappa,\mathrm{diag}(u,1_2,u,1_2)n'g)=\sum_{n\in\mathbb{Z}_p/p^t\mathbb{Z}_p}E_{sieg,2}(z_\kappa,f_2,g\gamma(1,\begin{pmatrix}1&\\ n&1\end{pmatrix}_p))\Theta_\Phi(u,n'g\gamma(1,\begin{pmatrix}1&\\ n&1\end{pmatrix}_p))$$

for $n'\in N_2$ (N_2 defined in subsection 4.7), $g\in\mathrm{U}(2,2)(\mathbb{A}_{\mathbb{Q}})$ and $u\in\mathrm{U}(1)(\mathbb{A}_{\mathbb{Q}})$. Here $f_2=\prod_v f_{2,v}$ and $\Phi=\prod_v\Phi_v$ are given in previous subsections. $E_{sieg,2}$ is considered as a function on $\mathrm{U}(2,2)(\mathbb{A}_{\mathbb{Q}})$.

Proof. This follows from our computations for Fourier-Jacobi coefficients and lemma 6.5. $\square$

So far we have used the embedding $\alpha(1,g) = \begin{pmatrix} 1 & & & \\ & D_g & & C_g \\ & & 1 & \\ & B_g & & A_g \end{pmatrix}$ for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, to keep accordance with the convention of [26]. In our actual applications later on we will use another embedding α'' as below. Now that we have

$$FJ_1(E^{\square}_{sieg}, z_\kappa, g, x, u) = E_{sieg,2}(z_\kappa, f_2, g')\Theta_\Phi(u, g').$$

Now we consider another embedding $\alpha''(1,g) = \begin{pmatrix} 1 & & & \\ & A_g & & B_g \\ & & 1 & \\ & C_g & & D_g \end{pmatrix}$ Let $\Phi'' = \omega_\beta(\eta)\Phi$ and $f''_2 = \rho(\eta) f_2$ and we let FJ''_β be defined as FJ_β but replacing α by α''. By observing that $E_{sieg,2}$ and Θ are automorphic forms and thus invariant under left multiplication by η^{-1}, we get:

$$FJ''_\beta(E^{\square}_{sieg}, z_\kappa, g, x, u) = E_{sieg,2}(z_\kappa, f''_2, g)\Theta_{\Phi''}(u, g). \tag{5}$$

Lemma 6.32. *Suppose $\delta_\psi^{-1}(\Phi''_\infty) = \phi_{1,\infty} \boxtimes \phi_{2,\infty}$, and for each $v < \infty$*

$$\phi_{2,v}(x) = \int_{X_v} \delta_\psi^{-1}(\Phi''_v)(x', x)\phi_1(x')dx'.$$

Then

$$l'_{\theta_{\phi_1}}(\gamma^{-1}(\Theta_\Phi))(x) = \theta_{\phi_2}(x).$$

Here we consider $\gamma^{-1}(\Theta_\Phi)$ as a function on $(N\mathrm{U}(2)) \times \mathrm{U}(2) \hookrightarrow \mathrm{U}(3,1) \times \mathrm{U}(2)$ and apply $l'_{\theta_{\phi_1}}$ to it on the $N\mathrm{U}(2)$ part.

The lemma follows easily from writing $\delta_\psi^{-1}\Phi$ into a finite sum of expressions of the form $\phi_1 \boxtimes \phi_2$ and then applying (1) and (2).

Corollary 6.33.

$$\begin{aligned} & l'_{\theta_1}(FJ_1(\prod_v \sum_i \rho(\mathrm{diag}(u_{1,v}, 1_2, u_{1,v})))E_{Kling})(g) \\ = \quad & \langle \frac{1}{\Omega_\infty^{2\kappa-2}} \sum_{n \in \mathbb{Z}_p/p^t\mathbb{Z}_p} E_{sieg,2}(\alpha(g, -\begin{pmatrix} 1 & \\ n & 1 \end{pmatrix}_p) \cdot \frac{1}{\Omega_\infty}\theta_2(-\begin{pmatrix} 1 & \\ n & 1 \end{pmatrix}_p, \varphi(-)\rangle \end{aligned}$$

where θ_1 and θ_2 are the theta function on $U(-\zeta)(\mathbb{A}_\mathbb{Q})$ defined using the kernel $\phi_1 = \prod_v \phi_{1,v}$ or $\phi_2 = \prod_v \phi_{2,v}$ for $\phi_{1,v}$ and $\phi_{2,v}$'s defined as before. Note that the ϕ_2 is defined using ϕ_1, which explains the dependence of the right hand side on ϕ_1. The inner product is over the group $1 \times \mathrm{U}(2) \hookrightarrow \mathrm{U}(3,3)$. Moreover suppose $\varphi_p \in \pi_p$ is chosen such that φ_p is the ordinary vector, then the above expression is:

$$\langle E(\alpha(g, -)) \cdot \theta_2(-), \varphi(-)\rangle.$$

Proof. It follows from the above proposition and our discussions for intertwining maps in subsection 4.7. The last sentence follows from a description of the paring between π_p and $\pi_p^\vee$. $\square$

7 p-adic Interpolation

7.1 Eisenstein Datum

Definition 7.1. *An Eisenstein datum $\mathcal{D} = (L, \mathbb{I}, \mathbf{f}, \xi)$ consists of:*

- *A finite extension $L/\mathbb{Q}_p$;*
- *a finite normal $\mathcal{O}_L[[W]]$-algebra $\mathbb{I}$;*
- *an $\mathbb{I}$-adic Hida family $\mathbf{f}$ of cuspidal ordinary eigenforms new outside p, of square-free conductor N such that some weight 2 specialization f_0 has trivial character; Assume moreover that for some odd prime q non-split in $\mathcal{K}$ we have $q \| N$.*
- *a finite order L-valued Hecke character ξ of $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times$ whose p-part conductor divides p.*

Now we need to modify our $\mathbb{I}$. By taking an irreducible component of the normalization of a series of quadratic extension of $\mathbb{I}$ we may assume that for each $v \in \Sigma$ not dividing N, we can find two functions $\alpha_v, \beta_v \in \mathbb{I}$ interpolating the Satake parameters of π_v. This enables us to do the constructions in the global computations in section 6 in families. We still denote $\mathbb{I}$ for the new ring for simplicity. At the end of this paper we will see how to deduce the main conjecture for the original $\mathbb{I}$ from that for the new $\mathbb{I}$.

Let $\mathbb{I}_\mathcal{K} := \mathbb{I}[[\Gamma_\mathcal{K}]]$. We define $\alpha : \mathcal{O}_L[[\Gamma_\mathcal{K}]] \to \mathbb{I}[[\Gamma_\mathcal{K}^-]]$ and $\beta : \mathcal{O}_L[[\Gamma_\mathcal{K}]] \to \mathbb{I}[[\Gamma_\mathcal{K}]]$ by:

$$\alpha(\gamma_+) = (1+W)^{\frac{1}{2}}, \alpha(\gamma_-) = \gamma_-, \beta(\gamma_+) = \gamma_+, \beta(\gamma_-) = \gamma_-.$$

We let $\boldsymbol{\psi} = \alpha \circ \Psi_\mathcal{K}$, $\boldsymbol{\xi} = (\beta \circ \Psi_\mathcal{K}) \cdot \xi$. Define: $\psi_\phi = \phi \circ \boldsymbol{\psi}$ and $\xi_\phi = \phi \circ \boldsymbol{\xi}$. Let $\Lambda_\mathcal{D} = \mathbb{I}[[\Gamma_\mathcal{K}]][[\Gamma_\mathcal{K}^-]]$. We give $\Lambda_\mathcal{D}$ a Λ_2-algebra structure by first give a homomorphism $\Gamma_2 = (1 + p\mathbb{Z}_p)^4 \to \Gamma_\mathcal{K} \times \Gamma_\mathcal{K}$ given by:

$$(a, b, c, d) \to \mathrm{rec}_\mathcal{K}(db, a^{-1}c^{-1}) \times \mathrm{rec}_\mathcal{K}(d^{-1}, c)$$

and then compose with $\alpha \otimes \beta$.

We remark here that only the quotient $\mathbb{I}[[\Gamma_\mathcal{K}]]$ of $\Lambda_\mathcal{D}$ really matters: the $\Gamma_\mathcal{K}^-$ variable corresponds to twisting everything by the same character and does not affect the p-adic L-functions and the Selmer groups.

Now we consider $\Lambda_\mathcal{D}$-adic forms. For $\mathcal{X}$ a Zariski dense set of arithmetic points. Suppose $U_f \subset D(\mathbb{A}_f)$ be a compact open subgroup such that the level is prime to p. A point $\phi \in \mathrm{Spec}\Lambda_\mathcal{D}$ is called arithmetic if $\phi(1+W), \phi(\gamma^+), \phi(\gamma^-)$ for $\gamma^- \in \Gamma_\mathcal{K}$ and $\Gamma_\mathcal{K}^-$ are all p-power roots of unity. We call it generic if $(f_\phi, \psi_\phi, \xi_\phi)$ is generic in the sense defined in section 5. We let $\mathcal{X}$ be the set of arithmetic points and $\mathcal{X}^{gen}$ be the set of generic arithmetic points.

7.2 Congruence Module and the Canonical Period

We now discuss the theory of congruences of modular forms on $\mathrm{GL}_2(\mathbb{Q})$. Let R be a finite extension of $\mathbb{Z}_p$ and ε a finite order character of $\mathbb{Z}^\times$ whose p-component has conductor dividing p. Let $M_\kappa^{ord}(Mp^r, \varepsilon; R)$ be the space of ordinary modular forms on $\mathrm{GL}_2/\mathbb{Q}$ with level $N = Mp^r$, character ε and coefficient R. Let $S_\kappa^{ord}(Mp^r, \varepsilon; R)$ be the subspace of cusp forms. Suppose R is a finite extension of $\mathbb{Z}_p$ we let $\mathbb{T}_\kappa^{ord}(N, \varepsilon; R)$ ($\mathbb{T}_\kappa^{0,ord}(Mp^r, \varepsilon; R)$) be the R-sub-algebra of $\mathrm{End}_R(M_\kappa^{ord}(Mp^r, \varepsilon; R))$

(respectively, $\mathrm{End}_R(S_\kappa^{ord}(Mp^r,\varepsilon;R)))$ generated by the Hecke operators T_v (these are Hecke operators defined using the double coset $\Gamma_1(N)_v \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} \Gamma_1(N)_v$ for the v's). For any $f \in S_\kappa^{ord}(N,\varepsilon;R)$ a nearly ordinary eigenform. Then we have $1_f \in \mathbb{T}_\kappa^{0,ord}(N,\varepsilon;R)\otimes_R F_R = \mathbb{T}'_\kappa \times F_R$ the projection onto the second factor.

Let $\mathfrak{m}_f$ be the maximal ideal of the Hecke algebra corresponding to f. Suppose that the localization of the Hecke algebra at $\mathfrak{m}_f$ satisfies the Gorenstein property. Then $\mathbb{T}^{ord,0}(M,\varepsilon;R)_{\mathfrak{m}_f}$ is a Gorenstein R-algebra, so $\mathbb{T}^{ord,0}(M,\varepsilon;R)\cap(0\otimes F_R)$ is a rank one R-module. We let ℓ_f be a generator; so $\ell_f = \eta_f 1_f$ for some $\eta_f \in R$. This η_f is called the congruence number of f.

Now let $\mathbb{I}$ be as at the beginning of this section. Suppose $\mathbf{f} \in \mathcal{M}^{ord}(M,\varepsilon;\mathbb{I})$ is an ordinary $\mathbb{I}$-adic cuspidal eigenform. Then as above $\mathbb{T}^{ord,0}(M,\varepsilon;\mathbb{I})\otimes F_{\mathbb{I}} \simeq \mathbb{T}'\times F_{\mathbb{I}}$, $F_{\mathbb{I}}$ being the fraction field of $\mathbb{I}$ where projection onto the second factor gives the eigenvalues for the actions on $\mathbf{f}$. Again let $1_{\mathbf{f}}$ be the idempotent corresponding to projection onto the second factor. Then for an $\mathbf{g} \in S^{ord}(M,\varepsilon;\mathbb{I})\otimes_{\mathbb{I}} F_{\mathbb{I}}$, $1_{\mathbf{f}}\mathbf{g} = c\mathbf{f}$ for some $c \in F_{\mathbb{I}}$. As above, under the Gorenstein property for $\mathbb{T}_{\mathbf{f}}$, we can define $\ell_{\mathbf{f}}$ and $\eta_{\mathbf{f}}$.

From now on we will define $D^\times$ to be the unique quaternion algebra ramified exactly at ∞ and the q in our main theorems in the introduction. We choose the group $\mathrm{U}(2)$ with $D^\times$ being its associated quaternion algebra. We also make the following definition for p-adic families of forms on $D^\times$.

Definition 7.2. *For any complete local $\mathcal{O}_L[[W]]$-aglebra R (need not be finite over $\mathcal{O}_L[[W]]$) we define the space of R-adic families on $D^\times$ with level group $K_D \subset D^\times(\mathbb{A}_f)$ which is $\mathrm{GL}_2(\mathbb{Z}_p)$ at p to be*

$$(\varprojlim_m \varinjlim_n M(K_D(p^n),\mathcal{O}_L/p^m\mathcal{O}_L)\hat{\otimes}R)^{\mathbb{Z}_p[[W]]}$$

where $K_D(p^n)$ is obtained by replacing the p-component of K_D by $\Gamma_1(p^n)$. The action of $\mathbb{Z}_p[[W]]$ is given by the usual action as nebentypus on the first factor, and by $t\mapsto t^{-1}$ on the second factor. The $M(K_D(p^n),\mathcal{O}_L/p^m\mathcal{O}_L)$ is the space of forms on $D^\times(\mathbb{A}_\mathbb{Q})$ with level group $K_D(p^n)$ and coefficient ring $\mathcal{O}_L/p^m\mathcal{O}_L$. It is not hard to see that each R-adic family gives a continuous R-valued function on $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_\mathbb{Q})/K^{(p)}$ where we give the last set the topology induced from the p-adic Lie-group $\mathrm{GL}_2(\mathbb{Z}_p)$.

The Ordinary Family $\mathbf{f}$ on $D^\times$

Let $\mathbf{f}$ be a Hida family of ordinary eigenforms new outside p as in the main theorem. Suppose $\mathbb{T}_{\mathfrak{m}_\mathbf{f}}$ is Gorenstein. Thus we have the integral projector $\ell_{\mathbf{f}}$. We are going to construct from it a Hida family of ordinary forms on $D^\times(\mathbb{A}_\mathbb{Q})$, also denoted as $\mathbf{f}$. We refer to [10] for the definition and theory of ordinary forms on quaternion algebras. By our assumption we may choose f_0 a form on $D^\times(\mathbb{A}_\mathbb{Q})$ which is in the Jacquet-Langlands correspondence of f_0 in $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with values in $\mathcal{O}_L$ and f_0 is not divisible by π_L. If we are under the assumption the main theorem in the introduction we can choose some $g_0 \in S_2^{ord}(K_{f_0},\mathcal{O}_L)$ such that $\ell_{f_0}g_0$ is a p-adic unit times f_0 as forms on $D^\times(\mathbb{A}_\mathbb{Q})$. Note that $\ell_{f_0}f_0$ is not a p-adic unit times f_0. (This is possible by our assumptions in Theorem 1.1 on $\mathbf{f}$. The local Hecke algebras for f_0 on GL_2 and $D^\times$ are the same, thus they have the same congruence numbers. On the other hand such congruence number is generated by the Petersson inner product of f_0 by result of Pollack-Weston [36]). Since the space of ordinary $\mathbb{I}$-adic forms is finite and free over $\mathbb{I}$ by [10, section 10], and, by *loc.cit* section 9 we can choose an ordinary $\mathbb{I}$-adic form $\mathbf{g}$

such that some weight 2 specialization is g_0. Thus we can define $\mathbf{f} := \ell_{\mathbf{f}}\mathbf{g}$. We make the following important remark: we can find a point (still denote as g_0) on $U(2)(\mathbb{A}_{\mathbb{Q}})$ such that the value of $\boldsymbol{\varphi}$ is a p-adic unit. We only need to see that there is $g_0 \in D^\times(\mathbb{A}_{\mathbb{Q}})$ with $\det g_0 \in \mathbb{Q}^\times \cdot \mathrm{Nm}(\mathbb{A}_{\mathcal{K}}^\times/\mathbb{A}_{\mathbb{Q}}^\times)$ with $\mathbf{f}(g_0)$ being a p-adic unit. This follows easily from the assumption that $\mathbf{f}$ has square-free conductor.

7.3 Siegel Eisenstein Measure

Proposition 7.3. *There are Λ_2-adic formal Fourier expansions $\mathbf{E}_{\mathcal{D},sieg}$ and $\mathbf{E}'_{\mathcal{D},sieg}$ which when specializing to an arithmetic point ϕ are the Siegel Eisenstein series we constructed from the datum $(f_\phi, \xi_\phi, \psi_\phi)$ at ϕ and our choices of Siegel Eisenstein sections $f_{sieg,v}$.*

Proof. It is a special case of [48, Lemma 5.7] and follows from our computations of the local Fourier coefficients for Siegel Eisenstein series. For example the β-th Fourier coefficient of

$$(\frac{\Omega_p^{2\kappa\Sigma_\infty}}{\Omega_\infty^{2\kappa\Sigma_\infty}})^{-1}E_{sieg}(f_{sieg}; z_\kappa, g)$$

at $\mathrm{diag}(y, {}^t\bar{y})$ for $y \in \mathrm{GL}_3(\mathbb{A}_{\mathcal{K},f})$ is given by

$$\begin{aligned}
&\prod_{\ell\in\Sigma,\ell\nmid p} D_\ell^{-\frac{3}{2}}\tau(\det y_\ell)|\det y_\ell\bar{y}_\ell|_\ell^{-\frac{\kappa}{2}} e_\ell(\frac{\beta_{22}+\beta_{33}}{y_\ell\bar{y}_\ell})\\
\times\quad &\prod_{\ell\notin\Sigma} h_{\ell,{}^t\bar{y}_\ell\beta y_\ell}(\bar{\tau}'(\ell)\ell^{-\kappa})\\
\times\quad &\Phi_{\xi^\dagger}(\begin{pmatrix}\beta_{21} & \beta_{22}\\ \beta_{31} & \beta_{32}\end{pmatrix})\cdot \mathrm{char}(\mathbb{Z}_p,\beta_{11})\mathrm{char}(\mathbb{Z}_p,\beta_{12})\mathrm{char}(\mathbb{Z}_p,\beta_{13})\mathrm{char}(\mathbb{Z}_p,\beta_{23})\mathrm{char}(\mathbb{Z}_p,\beta_{33})
\end{aligned}$$

where $\mathrm{char}(\mathbb{Z}_p, x)$ is the characteristic function for $\mathbb{Z}_p$ for the variable x. This expression is clearly interpolated by an element in the Iwasawa algebra. The case for

$$(\frac{\Omega_p^{2\kappa\Sigma_\infty}}{\Omega_\infty^{2\kappa\Sigma_\infty}})^{-1}E'_{sieg}(f'_{sieg}; z, g)$$

is similar. We finally remark there that the CM and p-adic periods will show up when pulling back the forms on $\mathrm{U}(3,3)$ ($\mathrm{U}(2,2)$) to $\mathrm{U}(3,1)\times\mathrm{U}(2)$ ($\mathrm{U}(2)\times\mathrm{U}(2)$, respectively). $\square$

This formal Fourier expansion gives measures on $\Gamma_{\mathcal{K}}\times\mathbb{Z}_p$ with values in the space of p-adic automorphic forms on $\mathrm{GU}(3,3)$, which we denote as $\mathcal{E}_{\mathcal{D},sieg}$ and $\mathcal{E}'_{\mathcal{D},sieg}$, respectively.

We make an additional construction for interpolating Petersson inner products of forms on definite unitary groups. Write N^- for the lower triangular unipotent subgroup of GL_2. For a compact open subgroup K of $\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})$ which is $\mathrm{U}(2)(\mathbb{Z}_p)$ at p we take $\{g_i\}_i$ a set of representatives for $\mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})/K_0(p)$ ($K_0(p)$ is obtained by replacing the p-component of K by the $\Gamma_0(p)$ level subgroup). Suppose $K = \prod_v K_v$ is sufficiently small so that for all i we have $\mathrm{U}(2)(\mathbb{Q})\cap g_iKg_i^{-1} = 1$. Let $\boldsymbol{\chi}$ be the central character of $\mathbf{f}$ we write $\mathbf{f}^\vee$ for the family $\pi(\begin{pmatrix} & 1\\ 1 & \end{pmatrix}_p)(\mathbf{f}\otimes\chi^{-1}(\det -))$. For the Hida family $\mathbf{f}^\vee$ of eigenforms (with lower triangular level group at p) we construct a set of bounded R-valued measure μ_i on $N^-(p\mathbb{Z}_p)$ as follows. Let T^- be the set of elements $\begin{pmatrix} p^{t_1} & \\ & p^{t_2} \end{pmatrix}$ with $t_2 > t_1$. We only need to specify the measure for sets of the form $nt^-N^-(\mathbb{Z}_p)(t^-)^{-1}$ where $n \in N^-(\mathbb{Z}_p)$ and $t^- \in T^-$. We assign $\mathbf{f}^\vee(g_i nt^-)\lambda(t^-)^{-1}$ where $\lambda(t^-)$ is the Hecke eigenvalue of $\mathbf{f}^\vee$ for U_{t^-}. This measure is well defined by the expression for Hecke operators U_{t^-}.

Proposition 7.4. *If we define*

$$\langle \mathbf{f}, \mathbf{f}^\vee \rangle := \sum_i \int_{n \in N^-(p\mathbb{Z}_p)} \mathbf{f}(g_i n) d\mu_i \in \mathbb{I}.$$

Then up to a constant depending only on the quaternion algebra D and the level group K, we have for all $\phi \in \mathcal{X}^{gen}$ the specialization of $< \mathbf{f}, \mathbf{f}^\vee >$ to ϕ is $\langle \mathbf{f}_\phi, \mathbf{f}_\phi^\vee \rangle \cdot \mathrm{Vol}(K_\phi)^{-1}$ where $K_\phi = K_0(p^{t_\phi})$ for p^{t_ϕ} the conductor at ϕ.

Proof. For each $\phi \in \mathcal{X}^{gen}$, we choose t^- such that $t^-N^-(p\mathbb{Z}_p)(t^-)^{-1} \subseteq \tilde{K}_\phi$. We consider

$$\langle \mathbf{f}_\phi, \pi^\vee_{\mathbf{f}_\phi}(t^-)\mathbf{f}^\vee_\phi \rangle.$$

Unfolding the definitions, note that $\chi_\phi^{-1}(t^-)\delta_B(t^-)$ gives the Hecke eigenvalue $\lambda(t^-)$ acting on $\mathbf{f}^\vee$. This gives $\mathrm{Vol}(K_\phi) \cdot \delta_B(t^-)\chi_\phi^{-1}(t^-) \sum_i \int_{n \in N^-(p\mathbb{Z}_p)} \mathbf{f}(g_i n) d\mu_i$. On the other hand, using the model of $\pi_{\mathbf{f}_\phi, p}$ and $\pi_{\mathbf{f}^\vee_\phi, p}$ as the induced representation $\pi(\chi_{1,\phi}, ..., \chi_{r,\phi})$ and $\pi(\chi_{1,\phi}^{-1}, ..., \chi_{r,\phi}^{-1})$ of $\mathrm{GL}_r(\mathbb{Q}_p)$, we get that

$$\langle \mathbf{f}_\phi, \pi^\vee_{\mathbf{f}_\phi}(t^-)\mathbf{f}^\vee_\phi \rangle = \delta_B(t^-)\chi_\phi^{-1}(t^-)\langle \mathbf{f}_\phi, \mathbf{f}^\vee_\phi \rangle.$$

This proves that the specialization of $\langle \mathbf{f}, \mathbf{f}^\vee \rangle$ to ϕ is $\langle \mathbf{f}_\phi, \mathbf{f}^\vee_\phi \rangle \cdot \mathrm{Vol}(K_\phi)^{-1}$. $\square$

The above set $\{\mu_i\}_i$ can be viewed as a measure on $\mathrm{U}(\mathbb{Q})\backslash\mathrm{U}(\mathbb{A}_\mathbb{Q})$, which we denote as $\mu_\mathbf{f}$.

7.4 p-adic L-functions

We have the following proposition for p-adic L-functions:

Proposition 7.5. *Notations as before. There is an element $\mathcal{L}^\Sigma_{\mathbf{f},\xi,\mathcal{K}}$ in $\hat{\mathbb{I}}^{ur}[[\Gamma_\mathcal{K}]]$, and a p-integral element $C^\Sigma_{\mathbf{f},\xi,\mathcal{K}} \in \bar{\mathbb{Q}}_p^\times$ such that for any generic arithmetic point ϕ we have:*

$$\phi(\mathcal{L}^\Sigma_{\mathbf{f},\xi,\mathcal{K}}) = C^\Sigma_{\mathbf{f},\xi,\mathcal{K}} \frac{p^{(\kappa-3)t}\xi^2_{1,p}\chi^{-1}_{1,p}\chi^{-1}_{2,p}(p^{-t})\mathfrak{g}(\xi_{1,p}\chi^{-1}_{1,p})\mathfrak{g}(\xi_{1,p}\chi^{-1}_{2,p})L^\Sigma(\mathcal{K}, \pi_{f_\phi}, \bar{\chi}_\phi\xi_\phi, \frac{\kappa}{2} - \frac{1}{2})(\kappa-1)!(\kappa-2)!\Omega_p^{2\kappa}}{(2\pi i)^{2\kappa-1}\Omega_\infty^{2\kappa}}. \tag{6}$$

Here $\chi_{1,p}, \chi_{2,p}$ is such that the unitary representation $\pi_{f_\phi} \simeq \pi(\chi_{1,p}, \chi_{2,p})$ with $\mathrm{val}_p(\chi_{1,p}(p)) = -\frac{1}{2}$, $\mathrm{val}(\chi_{2,p}(p)) = \frac{1}{2}$.

Proof. If we are under the assumption of Theorem 1.1. Take g_0 to be a point on the Igusa scheme for $\mathrm{GU}(2)$ defined over $\hat{\mathcal{O}}^{ur}_L$ such that $\mathbf{f}(g_0)$ is a unit in $\hat{\mathbb{I}}^{ur}$ and take a $h_0 \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q})$ such that $\mu(g_0) = \mu(h_0)$. It is noted in [16, Section 2.8] that

$$I_{\mathrm{GU}(2)}(K_1^n)(\hat{\mathcal{O}}^{ur}_L) = \mathrm{GU}(2)(\mathbb{Q})^+\backslash\mathrm{GU}(2)(\mathbb{A}_f)/K_1^n.$$

We define $\mathcal{L}^{\Sigma}_{\mathbf{f},\xi,\mathcal{K}}$ such that

$$\mathcal{L}^{\Sigma}_{\mathbf{f},\xi,\mathcal{K}} = \mathbf{f}(g_0)^{-1}\int_{g_2} \mathrm{ev}_{g_0}(\gamma^{-1}_{h_0}(\mathcal{E}'_{\mathcal{D},sieg}(-\Upsilon)\bar{\boldsymbol{\tau}}(\det g_2))d\mu_{\pi(h_0)\mathbf{f}}$$

Here ev_{g_0} means evaluation at g_0. By $\gamma^{-1}_{h_0}(\mathcal{E}'_{\mathcal{D},sieg})$ we mean pullback a form on $\{(g_1,g_2)\in \mathrm{GU}(2,0)\times\mathrm{GU}(0,2), \mu(g_1)=\mu(g_2)=\mu(h_0)\}$. Let the character $\boldsymbol{\tau}=\boldsymbol{\psi}/\boldsymbol{\xi}$. The function $\boldsymbol{\tau}(\det g_2)$ means the function taking value $\boldsymbol{\tau}(\det g_2)$ at the point (g_1,g_2) in the above set. The integration is in the sense of subsection 7.3 with respect to the level group $h_0^{-1}(K_{\mathcal{D}}\cap(1\times\mathrm{GU}(2)(\mathbb{A}_f))h_0$ (In fact by pullback we get a measure of forms on the h_0-Igusa schemes, see the last part of subsection 3.5). This $\mathcal{L}^{\Sigma}_{\mathbf{f},\xi,\mathcal{K}}$ satisfies the proposition.

If we are under assumption of Theorem 1.2 then we just pick up a g_0 such that the specialization of $\mathbf{f}$ at g_0 has non-zero specialization at g_0. Note that the period factor $\Omega_\infty^{2\kappa}$ and $\Omega_p^{2\kappa}$ comes from the pullback as discussed in [16, 2.8]. □

Definition 7.6. *Now we define Hida's p-adic L-function $\mathcal{L}^{Hida}_{\mathbf{f},\mathcal{K},\xi}$. First take the p-adic L-function constructed in [11, Theorem I] choosing $\mathbf{f}$ there to be the Hida family of nearly ordinary CM eigen forms $g_{\boldsymbol{\xi}}$ associated to ξ and $\mathbf{g}$ there to be the nearly ordinary eigenforms of our $\mathbf{f}$. The period factors in Hida's construction are the Petersson inner product of $\mathbf{g}_\phi$'s instead of the CM period Ω_∞. There is a Katz p-adic L-function $\mathcal{L}^{Katz}_{\mathcal{K},\xi}\in\hat{\mathcal{O}}^{ur}_L[[\Gamma_{\mathcal{K}}]]$ interpolating near central L-values of ξ^{1-c} twisted by characters of $\Gamma^-_{\mathcal{K}}$ (i.e. it interpolates the values*

$$\xi_{\phi,1,p}(p)\frac{p^tL(\xi_\phi\xi_\phi^{1-c},1)\Omega_p^{2\kappa}}{G(\xi_\phi\xi_\phi^{1-c})\Omega_\infty^{2\kappa}}$$

where ξ_ϕ is ξ multiplied by some finite order character of $\Gamma_{\mathcal{K}}$ of conductor p^t.). We have $\mathrm{Cl}_{\mathcal{K}}\cdot\mathcal{L}^{Katz}_{\mathcal{K},\xi}$ interpolates the ratio of the Petersson inner product of $\mathbf{g}_\phi$'s and Ω ($\mathrm{Cl}_{\mathcal{K}}$ is the class number for $\mathcal{K}$. See e.g. [21, Section 7] for a detailed discussion). Then we multiply Hida's p-adic L-function by $\mathrm{Cl}_{\mathcal{K}}\cdot\mathcal{L}^{Katz}_{\mathcal{K},\xi}$ and denote this new p-adic L-function as $\mathcal{L}^{Hida}_{\mathbf{f},\xi,\mathcal{K}}$. The interpolation formula for it is

$$\phi(\mathcal{L}^{Hida}_{\mathbf{f},\xi,\mathcal{K}}) = \frac{p^{(\kappa-3)t}\xi^2_{1,p}\chi^{-1}_{1,p}\chi^{-1}_{2,p}(p^{-t})\mathfrak{g}(\xi_{1,p}\chi^{-1}_{1,p})\mathfrak{g}(\xi_{1,p}\chi^{-1}_{2,p})L(\mathcal{K},\pi_{f_\phi},\bar{\chi}_\phi\xi_\phi,\frac{\kappa}{2}-\frac{1}{2})(\kappa-1)!(\kappa-2)!\Omega_p^{2\kappa}}{(2\pi i)^{2\kappa-1}\Omega_\infty^{2\kappa}}. \tag{7}$$

(Here we removed the product of prime to p root numbers in loc.cit *which are p-units and moves p-adic analytically.) If ξ is such that $g_{\boldsymbol{\xi}}$ satisfies the (dist) and (irred) in the introduction, then the local Hecke algebra for $g_{\boldsymbol{\xi}}$ is Gorenstein. By the main conjecture proved in [21], [22] and [13] (see the discussion after the Conjecture in [21, page 192], the $\mathrm{Cl}_{\mathcal{K}}\cdot\mathcal{L}^{Katz}_{\mathcal{K},\xi}$ generates the congruence module for $\mathbf{g}$ and our $\mathcal{L}^{Hida}_{\mathbf{f},\xi,\mathcal{K}}$ is integral (i.e. in $\Lambda_{\mathcal{D}}$).*

Given a finite set of primes Σ we can define the Σ-primitive Hida p-adic L-function $\mathcal{L}^{\Sigma,Hida}_{\mathbf{f},\mathcal{K},\xi}$ by removing local Euler factors at Σ. Note that Hida proved the interpolation formula for general arithmetic points. We may compare (6) (7). If we write $\mathcal{L}^{\Sigma}_{f_0,\xi,\mathcal{K}}$ for the specialization of $\mathcal{L}^{\Sigma}_{\mathbf{f},\xi,\mathcal{K}}$ to $\hat{\mathcal{O}}^{ur}_L[[\Gamma_{\mathcal{K}}]]$ at f_0, then we get the interpolation formula

$$\phi(\mathcal{L}^{\Sigma}_{f_0,\xi,\mathcal{K}}) = C^{\Sigma}_{\mathbf{f},\xi,\mathcal{K}}\frac{p^{(\kappa-3)t}\xi^2_{1,p}\chi^{-1}_{1,p}\chi^{-1}_{2,p}(p^{-t})\mathfrak{g}(\xi_{1,p}\chi^{-1}_{1,p})\mathfrak{g}(\xi_{1,p}\chi^{-1}_{2,p})L^{\Sigma}(\mathcal{K},\pi_{f_0},\xi_\phi,\frac{\kappa}{2}-\frac{1}{2})(\kappa-1)!(\kappa-2)!\Omega_p^{2\kappa}}{(2\pi i)^{2\kappa-1}\Omega_\infty^{2\kappa}}. \tag{8}$$

for ξ_ϕ's of conductor (p^t, p^t) at p.
Anti-cyclotomic μ-invariants:
Now assume we are under assumption of Theorem 1.1 in the introduction. We define ϕ_0 to be the arithmetic point in $\mathrm{Spec}\mathbb{I}[[\Gamma_\mathcal{K}]]$ sending $\gamma^\pm$ to 1 and such that $\phi_0|_\mathbb{I}$ correspond to f_0. Our assumptions on ξ and κ ensures that it is in the component of our p-adic L-function. (This is not an arithmetic point by our definition, however it still interpolates the algebraic part of the special L-value by [11]). Consider the 1-dimensional subspace of of $\mathrm{Spec}\mathbb{I}[[\Gamma_\mathcal{K}]]$ of anti-cyclotomic twists by characters of order and conductor powers of p that passes through ϕ_0. The specialization of Hida's p-adic L-function to this subspace is nothing but the anti-cyclotomic p-adic L-function considered by [18]. Thus the anti-cyclotomic p-adic L-function has μ-invariant 0. As in [45, 12.3] the corresponding μ-invariant for the Σ-primitive p-adic L-function is 0 as well. Thus it is easy to see that any height 1 prime P of $\mathbb{I}[[\Gamma_\mathcal{K}]]$ containing $\mathcal{L}^{\Sigma,Hida}_{\mathbf{f},\mathcal{K},\xi}$ can not be the pullback of a height 1 prime of $\mathbb{I}[[\Gamma^+_\mathcal{K}]]$. Therefore for any height 1 prime containing $\mathcal{L}^{\Sigma,Hida}_{\mathbf{f},\mathcal{K},\xi}$,

$$\mathrm{ord}_P(\mathcal{L}^{\Sigma,Hida}_{\mathbf{f},\mathcal{K},\xi}) = \mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi})$$

and $\mathrm{ord}_P(\mathcal{L}^\Sigma_{\bar{\chi}_\mathbf{f}\xi}) = 0$.

7.5 p-adic Eisenstein Series

Proposition 7.7. *There is a $\hat{\mathbb{I}}^{ur}[[\Gamma_\mathcal{K}]]$-adic formal Fourier-Jacobi expansion $\mathbf{E}_{\mathcal{D},Kling}$ such that for each generic arithmetic point $\phi \in \mathrm{Spec}\hat{\mathbb{I}}^{ur}[[\Gamma_\mathcal{K}]]$, the specialization $\mathbf{E}_{\mathcal{D},Kling,\phi}$ is the Fourier-Jacobi expansion of the nearly ordinary Klingen Eisenstein series E_{Kling} we constructed in section 6 using the Eiseinstein datum at ϕ. Moreover, the constant terms are divisible by $\mathcal{L}^\Sigma_{\mathbf{f},\xi,\mathcal{K}}.\mathcal{L}^\Sigma_{\bar{\chi}\xi'}$. Where $\mathcal{L}^\Sigma_{\bar{\chi}\xi'}$ is the element in $\mathbb{I}[[\Gamma^+_\mathcal{K}]]$ which is the Dirichlet p-adic L-function interpolating the algebraic part of the special values $L^\Sigma(\bar{\chi}_\phi\xi'_\phi, \kappa_\phi - 2)$.*

Proof. It is a special case of [48, Theorem 1.1 (3)]. In our cases the local choices are slightly different but the arguments are the same. We take a basis $(\theta'_1, \cdots, \theta'_m)$ of the $\mathcal{O}_L$-dual space of $H^0(\mathcal{B}, \mathcal{L}(\beta))$ consisting of theta functions. Suppose θ is one of the θ'_is. For any $g \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q})$ we take $h \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q})$ such that $\mu(g) = \mu(h)$. We define

$$FJ_{\beta,g,\theta}(\mathbf{E}_{\mathcal{D},Kling}) := \int_{g_2} \mathrm{ev}_g l_\theta(\gamma_h^{-1}(\mathcal{E}_{\mathcal{D},sieg}(-\Upsilon))\bar{\boldsymbol{\tau}}(\det g_2)) d\mu_{\pi(h)\boldsymbol{\varphi}}.$$

Here by $\gamma_h^{-1}(\mathcal{E}_{\mathcal{D},sieg})$ we mean the form on $\{(g_1, g_2) \in \mathrm{GU}(3,1) \times \mathrm{GU}(0,2), \mu(g_1) = \mu(g_2) = \mu(h)\}$. The integration is in the sense of subsection 7.3 with respect to the level group $h^{-1}(K_\mathcal{D} \cap (1 \times \mathrm{GU}(2)(\mathbb{A}_f))h$ (In fact by pullback we get a measure of forms on the h-Igusa schemes). ☐

The family $\boldsymbol{\varphi}$ will be chosen in the last paragraph of Section 8.4. It follows that the formal Fourier-Jacobi expansion $\mathbf{E}_{\mathcal{D},Kling}$ comes from a family in $\mathcal{M}^{ord}(K^p\Lambda_\mathcal{D})$, which we still denote as $\mathbf{E}_{\mathcal{D},Kling}$. (In fact Theorem 3.8 is still true after replacing $A = \mathbb{I}[[\Gamma_\mathcal{K}]]$ by $\hat{\mathbb{I}}^{ur}[[\Gamma_\mathcal{K}]]$.)

8 p-adic Properties of Fourier-Jacobi Coefficients

The purpose of this section is to prove proposition 8.25 .

8.1 Preliminaries

Some Local Representation Theory

Let $v \in \Sigma^1$ then $D_v^\times \simeq \mathrm{GL}_2(\mathbb{Q}_v)$. For some irreducible admissible representation $\pi^{\mathrm{U}(2)_v}$ of $\mathrm{U}(2)_v$ we can find a representation $\pi^{\mathrm{GU}(2)_v}$ of $\mathrm{GU}(2)_v$ such that $\pi^{\mathrm{U}(2)_v}$ is a summand of $\pi^{\mathrm{GU}(2)_v}$ restricting to $\mathrm{U}(2)_v$. The cases that we are interested in are those such that the restriction $\pi^{\mathrm{GU}(2)_v}|_{D_v^\times}$ is supercuspidal of level at least ϖ_v^3 and higher than the conductor of the central character of it. In this case the SL_2 L-packet in [30] has two elements. Thus by the discussion in [30] for the local L-packets for SL_2, we have:

$$\pi^{\mathrm{GU}(2)_v}|_{\mathrm{U}(2)_v} = \pi^{\mathrm{U}(2)_v} \oplus {}^{\alpha}\pi^{\mathrm{U}(2)_v}.$$

for irreducible representations $\pi^{\mathrm{U}(2)(\mathbb{Q}_v)}$ of $\mathrm{U}(2)(\mathbb{Q}_v)$. Here $\alpha = \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix}$ or $\alpha = \begin{pmatrix} \epsilon_v & \\ & 1 \end{pmatrix}$ for some $\epsilon_v \in \mathbb{Z}_v^\times/(\mathbb{Z}_v^\times)^2$ depending on whether it is an unramified or ramified supercuspidal representation in the sense of [30]. The ${}^{\alpha}$ means the representation composed with the automorphism given by conjugation by α. Also the restriction of $\pi^{\mathrm{GU}(2)_v}$ to $D_v^\times$ is clearly irreducible. There is a newform v_{new} (up to scalar) of $\pi^{D_v^\times}$. Note that the conditions on the conductors there are satisfied. We define the "new vector in our sense" $v \in \pi^{\mathrm{U}(2)_v}$ (up to scalar) to be the new vector of $\pi^{\mathrm{U}(2)_v}|_{\mathrm{SL}_2(\mathbb{Q}_v)}$. Thus $v_{new} = \pi(\alpha)v$ or $v_{new} = v + \pi(\alpha)v$ (see [30, Proposition 3.3.3, 3.3.7]).

If $D_v^\times$ modulo center is compact then we let α be some element such that $\mathrm{Nm}(\alpha) \notin \mathrm{Nm}(\mathcal{K}_v/\mathbb{Q}_v)$. For $\pi^{\mathrm{U}(2)_v}$ we similarly have $\pi^{\mathrm{GU}(2)_v}$, $\pi^{D_v^\times}$. These can all be considered as finite dimensional representations of finite groups. $\pi^{\mathrm{GU}(2)_v} = \pi^{D_v^\times}$ as vector spaces and $\pi^{\mathrm{GU}(2)_v}|_{\mathrm{U}(2)_v} = \pi^{\mathrm{U}(2)_v}$ or $\pi^{\mathrm{U}(2)_v} \oplus {}^{\alpha}\pi^{\mathrm{U}(2)_v}$.

Forms on $D^\times$ and $\mathrm{U}(2)$

We first define $\breve{D}^\times(\mathbb{A}_\mathbb{Q}) \subset D^\times(\mathbb{A}_\mathbb{Q})$ as the index 2 subgroup consisting of elements whose determinants are in $\mathbb{Q}^\times \mathrm{Nm}(\mathbb{A}_\mathcal{K}^\times)$ and $\breve{D}^\times(\mathbb{Q}_v)$ be the set of elements whose determinant is in $\mathrm{Nm}(\mathbb{Q}_v)$. Suppose φ is a form on $\mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})$, χ is a Hecke character of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$. Suppose the central action of $\mathrm{U}(1)(\mathbb{Z}_p)$ on φ is given by $\chi|_{\mathrm{U}(1)(\mathbb{Z}_p)}$, we can define a form φ_χ^D on $\breve{D}^\times(\mathbb{A}_\mathbb{Q})$ as follows. We first define φ'_χ on $\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})$ such that, consider the action of the center $\mathrm{U}(1)(\mathbb{A}_\mathbb{Q})$ of $\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})$, φ'_χ is the χ-eigen part of φ under this action. Let $\mathcal{C}$ be the cardinality of

$$\mathrm{U}(1)(\mathbb{Q})\backslash\mathrm{U}(1)(\mathbb{A}_\mathbb{Q})/\mathrm{U}(1)(\mathbb{Z}_p) \times (\text{prime to p level of } \chi).$$

We define a form on $D^\times(\mathbb{A}_\mathbb{Q})$. First for g such that $\det g$ is in the image of $\mathrm{Nm}(\mathbb{A}_\mathcal{K}^\times/\mathbb{A}_\mathbb{Q}^\times)$ then we can write $g = ag'$ where $a \in \mathbb{A}_\mathbb{Q}^\times$ and $g' \in \mathrm{U}(2)(\mathbb{A}_\mathbb{Q})$. Define $\varphi_\chi^D(g) = \chi(a)\mathcal{C}\varphi'_\chi(g)$. For general $g \in \breve{D}(\mathbb{A}_\mathbb{Q})$ we can find a central element $b \in D^\times(\mathbb{Q})$ such that $\det(b^{-1}g) \in \mathrm{Nm}(\mathbb{A}_\mathcal{K}^\times/\mathbb{A}_\mathbb{Q}^\times)$, we define $\varphi_\chi^D(g) := \varphi_\chi^D(b^{-1}g)$. Note that this is well defined since $\mathbb{Q}^\times \cap \mathrm{Nm}(\mathbb{A}_\mathcal{K}^\times/\mathbb{A}_\mathbb{Q}^\times) = \mathrm{Nm}(\mathcal{K}^\times/\mathbb{Q}^\times)$. For g outside $\breve{D}^\times(\mathbb{A}_\mathbb{Q})$ we define $\varphi_\chi^D(g) = 0$. When χ is clear from the context we simply drop the subscript χ.

Lemma 8.1. *If φ is in the irreducible automorphic representation on* $\mathrm{U}(2)$ *whose restriction to* SL_2 *is the rstriction of the* GL_2 *automorphic representation π_ξ associated to a CM character ξ over $\mathcal{K}$. Then φ_ξ^D itself is in π_ξ.*

Proof. Clearly the φ_ξ^D and π_ξ have the same Hecke eigenvalues outside a finite set of primes: this is obvious for split v's. For inert v's the Hecke eigenvalues for T_v on φ_D is 0 since it is a CM form. $\square$

We relate the integrals over $[\mathrm{U}(2)]$ to that over a subset of $[\mathbb{Q}^\times\backslash D^\times](\mathbb{A}_\mathbb{Q})$. It is elementary to check that there is a constant $C^D_{\mathrm{U}(2)}$ depending only on the groups $D^\times$ and $\mathrm{U}(2)$ such that if $\chi = 1$ then:

$$\int_{[\mathrm{U}(2)]} \varphi_{\mathrm{U}(2)}(g)dg = C^D_{\mathrm{U}(2)} \int_{D^\times(\mathbb{Q})\mathbb{A}_\mathbb{Q}^\times\backslash \check{D}^\times(\mathbb{A}_\mathbb{Q})} \varphi_\chi^D(g)dg.$$

Here we normalize the Haar measure so that the measure $\mathrm{U}(1)\backslash[\mathrm{U}(2)] = 1$ and the measure of $[D^\times]$ modulo center is also 1.

8.2 Choosing some characters

We first give a result of Pin-Chi Hung (a student of M-L Hsieh) [23]. Let χ be a finite order Hecke character of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ of conductor $\mathbb{Z} = M\mathcal{O}_\mathcal{K}$ for some $M > 0$. Let $f \in S_k(\Gamma_0(N))$ be an elliptic cusp form of weight k, level $\Gamma_0(N)$ with q-expansion

$$f(q) = \sum_{n\geq 0} a_n(f)q^n.$$

We decompose $N = N^+N^-$, where N^+ is a product of primes split in $\mathcal{K}$ and N^- is a product of primes ramified or inert in $\mathcal{K}$. Suppose N^- is square-free and $N^- = N_f^- N_\chi^-$ where N_f^- is a product of an odd number of primes coprime to M and N_χ^- is a divisor of M. Let ℓ be a rational prime split in $\mathcal{K}$. Let $\mathcal{K}_\ell^-$ be the unique abelian anticyclotomic $\mathbb{Z}_\ell$-extension of $\mathcal{K}$ and Γ^- be the Galois group $\mathrm{Gal}(\mathcal{K}_\ell^-/\mathcal{K})$.

Theorem 8.2. *Suppose $\ell^2 \nmid N$. Let p be a rational prime such that*

- *$p \nmid \ell N D_\mathcal{K}$ and $p \geq k-2$,*
- *for every non-split $q|M$, $q+1$ is not divisible by p,*
- *for every $q|N_f^-$ ramified in $\mathcal{K}$, $a_q(f) = \chi(\mathfrak{q})(=\pm 1)$, where $q = \mathfrak{q}^2$,*
- *the residual Galois representation $\bar{\rho}_{f,\lambda}|_{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathcal{K})}$ is absolutely irreducible.*

Then there is a finite extension $L/\mathbb{Q}_p$ with integer ring $\mathcal{O}_L$ and uniformizer λ. We have for all but finitely many characters $\nu : \Gamma^- \to \mu_{\ell^\infty}$, we have

$$\frac{L(f/\mathcal{K}, \chi\nu, \frac{k}{2})}{\Omega_{f,N^-}} \not\equiv 0 (\mathrm{mod} \lambda).$$

Here the Ω_{f,N^-} is a period factor defined in loc. cit.

Now we choose the characters needed. From now on we fix once for all a split prime ℓ outside Σ and write a new "Σ" for $\Sigma \cup \{\ell\}$. We choose χ_θ a Hecke character of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ as follows: $\chi_{\theta,\infty}$ is trivial. At p we require that χ is unramified. For $v \in \Sigma$ non-split in $\mathcal{K}/\mathbb{Q}$, then we let $\chi_{\theta,v}|_{\mathrm{U}(1)}$ to be the character chosen in section 6. For split $v \in \Sigma, v \nmid p, \ell$, $v = w\bar{w}$, suppose $\mathrm{cond}(\pi_v) = (\varpi_v^{t_{1,v}})$, we require that $\chi_{\theta,w}$ is unramified and $\mathrm{cond}(\chi_{\theta,\bar{w}}) = (\varpi_v^{t_{2,v}})$ for $t_{2,v} \geq 2t_{1,v} + 2$. We choose a character

χ_{aux} of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ as follows: $\chi_{aux}|_{\mathbb{A}_\mathbb{Q}^\times}=1$, it is trivial at ∞, and is only ramified at primes in Σ not dividing p such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is not compact. For split such v we require that

$$\mathrm{cond}(\chi_{aux})=\begin{cases}(\varpi_v^{t_{2,v}-t_{1,v}}) & \text{if } t_{1,v}\neq 0\\ (\varpi_v^{t_{2,v}-1}) & \text{if } t_{1,v}=0\end{cases}$$

At non-split such primes we require that

$$\mathrm{cond}(\chi_{aux,v})>\mathrm{cond}(\pi_v),\mathrm{cond}_v(\lambda^2\chi_\theta^{-c}\chi_\theta\chi_{aux})>\mathrm{cond}(\pi_v)$$

and $\chi_{aux,v}|_{\mathbb{Q}_v^\times}$ has smaller conductor than $\chi_{aux,v}$ (here $>$ means the conductor of the former is of higher power of the uniformizer than the latter). Also for each prime q such that $\mathrm{U}(2)(\mathbb{Q}_q)$ is compact and q is ramified as w^2 in $\mathcal{K}$, suppose $\pi_q\simeq \mathrm{steinberg}\otimes\chi_{q,1}$ for some unramified quadratic character $\chi_{q,1}$ we require that $\chi_{q,1}(q)=\chi_{aux}(\varpi_w)$ (These are used in the next paragraph to make sure that the special L-values are of the correct local signs when applying Theorem 8.2). Let $\chi_h=\chi_\theta^{-c}\chi_{aux}$.

We further require that $\frac{L(\pi_f,\lambda^2\chi_\theta\chi_h,\frac{1}{2})}{\pi\Omega_\infty^2}\mathrm{Eul}_p(\pi_f,\lambda^2\chi_\theta\chi_h,\frac{1}{2})$, $\frac{\Gamma(\kappa-1)L(\chi_{aux}\tau^{-c},\frac{\kappa-2}{2})}{\Omega_\infty^\kappa}\mathrm{Eul}_p(\chi_{aux}\tau^{-c},\frac{\kappa-2}{2})$ and $\frac{\Gamma(\kappa-2)L(\lambda^2\chi_\theta^{-c}\chi_\theta\chi_{aux}\tau^{-1},\frac{\kappa-2}{2})}{\Omega_\infty^{\kappa-2}}\mathrm{Eul}_p(\lambda^2\chi_\theta^{-c}\chi_\theta\chi_{aux}\tau^{-1},\frac{\kappa-2}{2})$ are p-adic units where the Eul_p are the local Euler factors for the corresponding p-adic L-functions at p when everything is unramified at p (we refer to [18, (0.2)], [20, 4.16] for their precise definitions). The first uses [18]. Our assumptions above on conductors imply that at all non-split primes the local root numbers in Theorem A of *loc.cit* are all +1. Then we take a split prime $\ell\nmid Np$ and apply that theorem to see that there exists a twist by anticyclotomic character of ℓ-power conductor which satisfies the requirements. The second and third uses [17] (we are in his residually non self-dual case) and again we can achieve the requirements by twisting by an appropriate anti-cyclotomic character of conductor powers of ℓ. Moreover we assume that $1-a_p(f)^{-1}\chi_{\theta,p,2}\chi_{h,p,1}(p)$, $1-a_p(f)\chi_{\theta,p,1}\chi_{h,p,2}(p)^{-1}$ and $1-\lambda_{p,2}^2\chi_{h,p,2}\chi_{\theta,p,2}\tau_{p,2}^{-1}(p)p^{-\frac{\kappa-2}{2}}$ are p-adic units. At each prime v of $\mathcal{K}$ above a prime where $\mathrm{U}(2)(\mathbb{Q}_v)$ is compact, we require $1-\chi_{aux}\tau^{-c}(q_v)q_v^{-\frac{\kappa-2}{2}}$ is a p-adic unit. We also require that $\frac{L(\pi_f,\chi_\theta^c\chi_h,\frac{1}{2})}{\pi^2\Omega_f^+\Omega_f^-}$ is non-zero (do not need to be non-zero modulo p!) using the Theorem 8.2 recalled above (by choosing a different "p" and prove non-vanishing the new p).

Now we define some characters ϑ of $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times$ (the reason for doing so is just a cheap way to use the newform theory at split primes to pick different vectors inside an antomorphic representation of $\mathrm{U}(2)$). We require that these are ramified only at split primes in $\Sigma\backslash\{p\}$. At such $v=w\bar{w}$ and require that $\vartheta|_{\mathbb{Z}_v^\times}=\chi_h|_{\mathcal{O}_{\mathcal{K},w}^\times}$. These uniquely determine the character ϑ.

8.3 Constructing Auxiliary Families of Theta Functions

From now on we usually do the computations at a generic arithmetic point ϕ, but drop the subscript ϕ for simplicity. Define $\Lambda_\mathcal{D}$-adic families of characters $\boldsymbol{\chi_\theta}$ and $\boldsymbol{\chi_h}$ as follows: let $\boldsymbol{\chi_\theta}$ be such that the specialization to ϕ_0 (ϕ_0 is defined in section 7.4) is χ_θ defined before and the specialization to ϕ satisfies $(\boldsymbol{\chi_\theta})_{\phi,p,1}|_{\mathbb{Z}_p^\times}=1$ and $(\boldsymbol{\chi_\theta})_{\phi,p,2}|_{\mathbb{Z}_p^\times}=\bar{\xi}^\dagger_{\phi,p,1}|_{\mathbb{Z}_p^\times}$. Let $\boldsymbol{\chi_h}$ be such that the specialization to ϕ_0 is χ_h and $(\boldsymbol{\chi_h})_{\phi,p,2}|_{\mathbb{Z}_p^\times}=1$, $(\boldsymbol{\chi_h})_{\phi,p,1}|_{\mathbb{Z}_p^\times}=\chi_{f,\phi}^{-1}\bar{\xi}^\dagger_{\phi,p,1}|_{\mathbb{Z}_p^\times}$. So from now on our χ_θ and χ_h might be ramified at p.

Rallis Inner Product Formula

$$\begin{tikzcd} \mathrm{U}(1,1)(\omega_{\lambda^2}) & \mathrm{U}(2)(\omega_\lambda)\times \mathrm{U}(2)(\omega_\lambda) \\ \mathrm{U}(1)(\omega_{\lambda^2})\times \mathrm{U}(1)(\omega_{\lambda^2}) \arrow[u] & \mathrm{U}(2)(\omega_{\lambda^2}) \arrow[u] \end{tikzcd}$$

We will use the results in [15] freely. We consider the seesaw pair above. The $\mathrm{U}(2)$ above is for the Hermitian matrix $\begin{pmatrix}\mathfrak{s} & \\ & 1\end{pmatrix}$ and the $\mathrm{U}(1)$'s are for the skew-Hermitian matrices δ and $-\delta$. The embedding $\mathrm{U}(1)\times\mathrm{U}(1)\hookrightarrow \mathrm{U}(1,1)$ are given by the i defined in the proof of lemma 6.21. In fact the theta functions showing up in the Fourier-Jacobi expansions arise from theta liftings from $\mathrm{U}(V_1)$ to $\mathrm{U}(\zeta)$ and $\mathrm{U}(\zeta)$. The Weil representations are the same as explained in [15] if we identify $\mathrm{U}(V_1)$ with $\mathrm{U}(1)$ above and $\mathrm{U}(\pm\zeta)$ with $\mathrm{U}(2)$ above. The splittings used are indicated in the bracket beside the groups. We want to consider the component of theta correspondence such that the first $\mathrm{U}(1)$ on the lower left corner acts by $\lambda^2\chi_\theta$ and the second $\mathrm{U}(1)$ acts by χ_θ^{-1}. We consider a theta function on $\mathrm{U}(2,2)$ by the dual reductive pair $\mathrm{U}(2,2)\times\mathrm{U}(V_1)$ and some Schwartz function ϕ such that $\phi=\delta_\psi(\phi_3\boxtimes\phi_2)$ for some ϕ_3 and ϕ_2 (recall the notion of intertwining operators in section 4. Here V_1 is the 1-dimensional hermitian space where $\langle x,y\rangle=\bar{x}y$. The characters used in the splittings are: λ for $\mathrm{U}(2,2)$ and 1 for V_1. We consider:

$$\int_{[\mathrm{U}(2)]}\int_{[\mathrm{U}(1)]\times[\mathrm{U}(1)]}\theta_\phi(u_1,u_2,g)\lambda^{-2}\chi_\theta^{-1}(u_1)\chi_\theta(u_2)\bar{\lambda}(\det g)du_1du_2dg$$

here $\bar{\lambda}(\det g)$ showing up is due to the splitting ω_{λ^2} on $\mathrm{U}(2)$. On one hand, one can check that this is nothing but the inner product of the theta liftings $\theta_{\phi_3,\lambda}(\lambda\chi_\theta)$ and $\theta_{\phi_2,\lambda}(\lambda^{-1}\chi_\theta^{-1})\cdot(\bar{\lambda}\circ\det)$ (by writing $\theta_{\phi,\lambda}$ we take the splitting character for $\mathrm{U}(1)$ is trivial and for $\mathrm{U}(2)$ is λ. (We need to notice the different choices of splitting characters). On the other hand if we change the order of integration using the Siegel-Weil formula for $\mathrm{U}(1,1)\times\mathrm{U}(2)$ as proved by Ichino ([25]). This equals:

$$\int_{[\mathrm{U}(1)]\times[\mathrm{U}(1)]}E(f_{\delta_\psi(\phi)},\frac{1}{2},i(u_1,u_2))\lambda^{-2}\chi_\theta^{-1}(u_1)\chi_\theta(u_2)du_1du_2$$

Here i is defined right before lemma 6.22 and $f_{\delta_\psi(\phi)}$ is the Siegel section defined by:

$$f(g):=\omega_{\lambda^2}(j(g))\delta_\psi(\phi)(0), g\in\mathrm{U}(1,1)$$

where j is defined in the proof of lemma 6.21. Thus we reduced the Petersson inner product of theta liftings to the pullback formula of Siegel Eisenstein series on $\mathrm{U}(1,1)$.

Functorial Properties of Theta Liftings

For any Hecke character χ of $\mathrm{U}(1)$ (in application $\chi(z_\infty)=z_\infty^{\pm1}$ for $z_\infty\in\mathrm{U}(\mathbb{R})$), we describe the L-packet of theta correspondence $\Theta_\lambda(\chi)$ (possibly zero) of χ to $\mathrm{U}(2)$ where λ is a Hecke character of $\mathbb{A}_\mathcal{K}^\times$ such that $\lambda|_{\mathbb{A}_\mathbb{Q}^\times}=\omega_{\mathcal{K}/\mathbb{Q}}$. We pick a Hecke character $\breve{\chi}$ such that $\breve{\chi}|_{\mathrm{U}(1)(\mathbb{A}_\mathbb{Q})}=\chi^{-1}$. Let

$\pi_{\check{\chi}}$ be the Jacquet-Langlands correspondence on $D^\times$ of the automorphic representation of $\mathrm{GL}_2/\mathbb{Q}$ generated by the CM form $\theta_{\check{\chi}}$. We form an automorphic representation in the way we introduced before: $\pi_{\check{\chi}} \boxtimes \chi^{-1}\lambda$ of GU(2). Then by looking at the local L-packets (see [15, section 7]) $\theta_\lambda(\chi)$ is a subspace of the restriction of this representation to U(2). (This restriction is not necessarily irreducible. In fact for any v inert in $\mathcal{K}$ the local representation at v splits into 2 irreducible pieces. see [30] . The representation at split primes are irreducible. So we still have not specified the automorphic representation on U(2).)

Constructing Families of Theta Liftings

Let v be a prime inert or ramified in $\mathcal{K}$. Thanks to the recent work [7] we know Howe duality conjecture is true for any characteristic. Consider the theta lifting from U(1) to U(2) at v (the $\mathrm{U}(2)(\mathbb{Q}_v)$ might be U(1,1) or compact). Write $S(X_v, \chi_{\theta,v}^{-1})$ for the summand of $S(X_v)$ such that U(1) acts by $\chi_{\theta,v}^{-1}$. Given a theta kernel ϕ^v on $\otimes_{w\neq v} S(X_w)$ we consider the map $S(X_v, \chi_{\theta,v}^{-1}) \to \pi_\theta$ by $\iota_v : \phi_v \to \Theta_{\phi^v\otimes\phi_v}(\chi_\theta^{-1})$. By the Howe duality conjecture we know that there is a maximal proper subrepresentation V_v of $S(X_v, \chi_{\theta,v}^{-1})$ such that $S(X_v, \chi_{\theta,v}^{-1})$ such that $S(X_v, \chi_{\theta,v}^{-1})/V_v$ is irreducible and isomorphic to the local theta correspondence $\pi_{\theta,v}$ of $\chi_{\theta,v}^{-1}$ by the local and global compatibility of theta lifting (see [38, Theorem 8.5]). Suppose there is some ϕ_v so that $\iota_v(\phi_v)\neq 0$, it is a finite sum of pure tensors in π_θ. We consider the representation of $\mathrm{U}(2)(\mathbb{Q}_v)$ on $\pi_\theta(\mathrm{U}(2)(\mathbb{Q}_v)(\iota(\phi_v))$. This is a subrepresentation of a direct sum of finite number of $\pi_{\theta,v}$'s. ι gives a homomorphism of representations of $\mathrm{U}(2)(\mathbb{Q}_v)$ from $S(X_v)$ to $\pi(\mathrm{U}(2)(\mathbb{Q}_v))(\iota(\phi_v)) \hookrightarrow \oplus\pi_{\theta,v}$. Note that the automorphism group of the representation $\pi_{\theta,v}$ consists of scalar multiplications. Thus it is easy to see that the kernel of the above embedding is exactly V_v and we have the following lemma:

Lemma 8.3. *Fix the ϕ^v as above. Let v_{ϕ_v} be the image of ϕ_v in $\pi_{\theta,v}$ under the Howe duality isomorphism to $S(X_v, \chi_{\theta,v}^{-1})/V_v$. Then $\iota_v(\phi_v)\in\pi_\theta$ can be written as a finite sum of pure tensors of the form*

$$v_{\phi_v}\otimes(\sum_i\prod_{w\neq v}\phi_{w,i})$$

for $\phi_{w,i}\in\pi_w$.

Proposition 8.4. *We can construct a $\Lambda_2\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$-adic family of theta functions on* U(2) *interpolating the forms*

$$\theta(g) := \sum_{i=1}^{h_{\mathcal{K}}} \xi_\theta(\breve{u}_i)^{-1}\frac{\theta_2(g\breve{u}_j)}{\Omega_{\mathcal{K}}}$$

where θ_2 is the one appearing in Corollary 6.33.

Proof. We are going to construct p-adic families of eigenforms which are theta functions of U(2) as subgroups of U(2,2) under $\mathrm{U}(2)\times 1\hookrightarrow \mathrm{U}(2,2)$ and $1\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$, using the theta liftings from U(1) to U(2). For an eigenform θ such constructed we sometimes write π_θ for the automorphic representation of U(2) of θ. First we discuss the choices for the Schwartz function ϕ for the construction using the embedding $1\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$.

Local Computations

Case 0:
At finite places outside Σ we choose the obvious spherical kernel functions.

Case 1:
If $v = \infty$, we let $\phi_v = \omega_\lambda(\eta g_0)\Phi_\infty$. Recall that $g_0 = \mathrm{diag}(\frac{\mathfrak{s}^{\frac{1}{2}} d^{\frac{1}{4}}}{\sqrt{2}}, \frac{d^{\frac{1}{4}}}{\sqrt{2}}, (\frac{\mathfrak{s}^{\frac{1}{2}} d^{\frac{1}{4}}}{\sqrt{2}})^{-1}, (\frac{d^{\frac{1}{4}}}{\sqrt{2}})^{-1})$ and $\Phi_\infty = e^{-2\pi \mathrm{Tr}(\langle x,x\rangle_1)}$. Let $\phi_{3,v} = \frac{(\mathfrak{s}d)^{\frac{1}{4}}}{2} e^{-2\pi\sqrt{d}(\mathfrak{s}x_{11}^2 + x_{12}^2)}$ and $\phi_{2,v} = \frac{(\mathfrak{s}d)^{\frac{1}{4}}}{2} e^{-2\pi\sqrt{d}(\mathfrak{s}x_{21}^2 + x_{22}^2)}$. By our computation in section 6.5 we have: $\delta(\phi_{3,v} \boxtimes \phi_{2,v}) = \phi_v$.

Case 2:
If $v \in S$ is split and $v \nmid p$ we recall that we have two different polarizations $W = X_v \oplus Y_v = X_v' \oplus Y_v'$ where the first one is globally defined which we use to define theta function and the second is defined using $\mathcal{K}_v \equiv \mathbb{Q}_v \times \mathbb{Q}_v$ which is more convenient for computing the actions of level groups. We have defined intertwining operators δ_ψ'' between $S(X_v)$ and $S(X_v')$ intertwining the corresponding Weil representations. Consider the theta correspondence of U(1) to U(2) on $S(X_v)$ and $S(X_v^-)$. We write $X_v \ni x_{3,v}' = (x_{3,v}'', x_{3,v}''')$ and $X_v'^- \ni x_{2,v}' = (x_{2,v}'', x_{2,v}''')$. We define

$$\phi_{3,v}'(x_{3,v}'', x_{3,v}''') = \begin{cases} (\lambda\chi_\theta)_v^{-1}(x_{3,v}'') & x_{3,v}'' \in \mathbb{Z}_v^\times, x_{3,v}''' \in \varpi_v^{t_v}\mathbb{Z}_v \\ 0 & \text{otherwise.} \end{cases}$$

where $\varpi_v^{t_v}$ is the conductor of $\chi_{\theta,v}$ and

$$\phi_{2,v}'(x_{2,v}'', x_{2,v}''') = \begin{cases} (\lambda\chi_\theta)_v(x_{2,v}'') & x_{2,v}'' \in \mathbb{Z}_v^\times, x_{2,v}''' \in \mathbb{Z}_v \\ 0 & \text{otherwise.} \end{cases}$$

We define $\phi_v \in S(\mathbb{W}^d)$ by $\phi_v = \delta_\psi'(\phi_{3,v} \boxtimes \phi_{2,v})$ and define $\phi_{2,v} = \delta_\psi^{-,''}(\phi_{2,v}')$, $\phi_{3,v} = \delta_\psi''(\phi_{3,v}')$. Then if $f_v \in I_v(\lambda^2)$ is the Siegel section corresponding to ϕ_v in the Rallis inner product formula, we have

$$f(i(1,u_2)) = \langle \phi_{3,v}, \omega(u_2,1)\phi_{2,v}\rangle$$

by the formula for the intertwining operator. This is zero unless $u_2 \in \mathbb{Z}_v^\times$ ($\mathrm{U}(1)(\mathbb{Q}_v) \simeq \mathbb{Q}_v^\times$) and equals $\lambda^2\chi_\theta(u_2)$ for those u_2's. To sum up, for such v the local integral is a nonzero constant c_v. Later when we are moving things p-adically, this constant is not going to change.

Case 3:
For $v \in S$ ramified or inert such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is not compact. In this case $\mathrm{U}(1)(\mathbb{Q}_v)$ is a compact abelian groups. We let $\phi_{2,v}$ be the Schwartz function on $S(X_v^-)$ constructed in section 5. Let $\phi_{3,v}$ be a p-integral valued Schwartz function on $S(X_v)$ such that $(\phi_{3,v}, \phi_{2,v}) = \int_{X_v \simeq X_v^-} \phi_{3,v}(x)\phi_{2,v}(x)dx \neq 0$ and that the action of $\mathrm{U}(1)(\mathbb{Q}_v)$ via the Weil representation is given by a certain character. (It is easy to see that this character is $\lambda_v^2\chi_{\theta,v}$, thus the action of the center of $\mathrm{U}(2)(\mathbb{Q}_v)$ via $\mathrm{U}(2) \times 1$ is given by $\chi_{\theta,v}^{-1}$). We define $\phi_v = \delta_\psi(\phi_{3,v} \boxtimes \phi_{2,v}) \in S(\mathbb{W}_v^d)$.

Case 4:
For v such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is compact. Note that the local representation $\pi_{\theta,v}$ is finite dimensional with some level group K_v. We write v_1 for the image of $\phi_{2,v}$ in $\pi_{\theta,v}$ under Howe duality. We fix an $\mathrm{U}(2)(\mathbb{Q}_v)/K_v$-invariant measure of $\pi_{\theta,v}$ and extend v_1 to $\{v_1, .., v_{d_v}\}$ an orthonormal basis of $\pi_{\theta,v}$. Let $(\tilde{v}_1, ...)$ be the dual basis. Let $\phi_{3,v}$ be a p-integral valued Schwartz function on $S(X_v)$ such that

the action of $\mathrm{U}(1)(\mathbb{Q}_v)$ via the Weil representation is given by a certain character. (It is easy to see that this character is $\lambda_v^2\chi_{\theta,v}$, thus the action of the center of $\mathrm{U}(2)(\mathbb{Q}_v)$ via $\mathrm{U}(2)\times 1$) is given by $\chi_{\theta,v}^{-1}$). We require also that the image of ϕ_3 in the representation of $\mathrm{U}(2)(\mathbb{Q}_v)$ (which is the dual of $\pi_{\theta,v}$) is $\tilde{v}_1$. We define $\phi_v = \delta_\psi(\phi_{3,v}\boxtimes\phi_{2,v})\in S(\mathbb{W}_v^d)$.

<u>Case 5</u>:
We will often write $\chi_{\theta,p}$ for $\chi_p|_{\mathbb{Q}_p^\times}$. For $v=p$, $W_p = X_p'\oplus Y_p'$, we write elements

$$x_p' = (x_{p,1}', x_{p,2}')\in X_p', y_p' = (y_{p,1}', y_{p,2}')\in Y_p'.$$

We define $\phi_p(x_p', y_p') = \chi_{\theta,p}(y_{p,1}')$ if $x_{p,2}'\in\mathbb{Z}_p^\times$ and $x_{p,1}', y_{p,1}', y_{p,2}'\in\mathbb{Z}_p$ and $\phi_p(x_p', y_p') = 0$ otherwise. We also write $x_{3,p}' = (x_{3,p}'', x_{3,p}''')\in X_p', x_{2,p}' = (x_{2,p}'', x_{2,p}''')\in X_p^{'-}$ (note that we use $x_{2,p}'$ to distinguish from $x_{p,2}'$ above). A straightforward computation gives

$$\delta_{\psi,p}^{',-1}(\omega_\lambda(\Upsilon)(\phi_p))(x_{3,p}', x_{2,p}') = \frac{\mathfrak{g}(\chi_{\theta,p})}{p^t}\chi_{\theta,p}^{-1}(-x_{2,p}''p^t)$$

if $x_{2,p}''\in p^{-t}\mathbb{Z}_p^\times$ and $x_{3,p}'', x_{3,p}''', x_{2,p}'''\in\mathbb{Z}_p$ and equals 0 otherwise. We write $\phi_{3,p}'$ to be the characteristic function of $\mathbb{Z}_p^2$ on X_p' and define $\phi_{2,p}'(x_{2,p}') = \frac{\mathfrak{g}(\chi_{\theta,p})}{p^t}\chi_{\theta,p}^{-1}(-x_{2,p}''p^t)$ if $x_{2,p}''\in p^{-t}\mathbb{Z}_p^\times$ and $x_{2,p}'''\in\mathbb{Z}_p$ and is 0 otherwise. Here we write $\chi_{\theta,p}$ for its restriction to $\mathbb{Q}_p^\times$ as well. We define $\phi_{3,p}$ and $\phi_{2,p}$ as their images under $\delta_{\psi,p}^{-1}\circ\delta_{\psi,p}'$.

<u>Global Case</u>:
Now by looking at the q-expansion we find that when we fix the Schwartz functions at $v\neq p$ and let χ_θ move in p-adic families according to $\chi_{\theta,p}$ at p. Our corresponding theta kernel functions $\boldsymbol{\Theta}$ on $\mathrm{U}(2,2)$ move in a p-adic Λ_2-adic analytic family whose q-expansion interpolates

$$\sum_{x\in\mathbb{Q}^2}\prod_v\phi_v(x)q^{\langle x,x\rangle}$$

as in [45, 10.3]. (We note that the ϕ_∞ here is $\Phi_{1,\infty}$ right translated by g_0. On the other hand our distinguished point $\boldsymbol{i}$ is $\zeta\in X_{2,2}$ is chosen as ζ). As in the doubling method for $\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$, the $\boldsymbol{\Theta}(-\Upsilon)$ pulls back to p-adic analytic family of forms on $\mathrm{U}(2)\times\mathrm{U}(2)$. Moreover it is in fact of the form $\theta_{\phi_3}\boxtimes\theta_{\phi_2}$ where ϕ_3 is fixed along the family by definition. By the Rallis inner products formula and the non-vanishing of the Petersson inner product (ensured by our choices) that for some $u_{aux}\in\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})$, $\theta_{\phi_3}(u_{aux})\neq 0$. When we move χ_θ p-adically our θ_{ϕ_3} is fixed so we know that $\frac{\theta_{\phi_3}(u_{aux})}{\Omega_\infty^2}.\theta_{\phi_2}$ is a Λ_2-adic family of forms on $1\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$. Now we take a representative $(\breve{u}_1,...\breve{u}_{h_\mathcal{K}})$ of $\mathrm{U}(1)(\mathbb{Q})\mathrm{U}_1(\mathbb{R})\backslash\mathrm{U}(1)(\mathbb{A}_\mathbb{Q})/\mathrm{U}(1)(\hat{\mathbb{Z}})$ considered as elements of the center of $\mathrm{U}(2)$.

Definition 8.5. *We define:*

$$\theta := \sum_{i=1}^{h_\mathcal{K}}\chi_\theta(\breve{u}_j)^{-1}\omega_{\lambda^{-1}}(\breve{u}_j)(\theta_{\phi_2}\cdot\bar{\lambda}).$$

We denote $\boldsymbol{\theta}$ for the Λ_2-adic family constructed this way (we omit the corresponding map of the weight ring of $\mathrm{U}(2)$ into $\Lambda_\mathcal{D}$).

The property required by the proposition follows from comparing our choices of theta kernel function with the computations for θ_2 in Section 6. $\square$

We can do the same thing to construct a Λ_D-adic family of forms on $\mathrm{U}(2)\times 1 \hookrightarrow \mathrm{U}(2,2)$. This time we define ϕ such that for $v \neq p$ the local components are as before. If $v = p$ recall that $W_p = X'_p \oplus Y'_p$. If $x'_p = (x'_{p,1}, x'_{p,2})$ and $y'_p = (y'_{p,1}, y'_{p,2})$ we define $\phi_p(x'_p, y'_p) = \chi_{\theta,p}(x'_{1,p})$ for $x'_{1,p} \in \mathbb{Z}_p^\times$ and $x'_{2,p}, y'_{1,p}, y'_{2,p} \in \mathbb{Z}_p$ and $\phi_p(x'_p, y'_p) = 0$ otherwise. Direct computation by plugging in the intertwining operator gives, if we write $x'_{1,p} = (x''_{1,p}, x'''_{1,p})$ and $x'_{2,p} = (x''_{2,p}, x'''_{2,p})$, $\delta'^{,-1}_{\psi,p}(\omega_\lambda(\Upsilon)\phi_p)(x'_{1,p}, x'_{2,p}) = \chi_{\theta,p}(x''_{1,p})$ if $x''_{1,p} \in \mathbb{Z}_p^\times$ and $x''_{1,p}, x''_{2,p}, x'''_{2,p} \in \mathbb{Z}_p$ and equals 0 otherwise.

Then as before we move χ_θ p-adically our θ_{ϕ_2} is fixed and non zero at some point $u'_{aux} \in \mathrm{U}(2)(\mathbb{A}_\mathbb{Q})$ and θ_{ϕ_3} is moving p-adic analytically. Thus $\frac{\theta_{\phi_2}(u'_{aux})}{\Omega_\infty^2}\theta_{\phi_3}$ is a $\Lambda_\mathcal{D}$-adic form on $\mathrm{U}(2)\times 1 \subset \mathrm{U}(2,2)$. As before we define $\tilde{\theta}_3$ (or $\tilde{\boldsymbol{\theta}}_3$ for the family) by

$$\sum_{i=1}^{h_\mathcal{K}} \theta_{\phi_3}(\breve{u}_i)\chi_\theta(\breve{u}_i).$$

Definition 8.6. *We define forms θ^D and $\tilde{\theta}_3^D$ on $D^\times(\mathbb{A}_\mathbb{Q})$ using θ and $\tilde{\theta}_3$ and characters χ_θ and $\bar{\chi}_\theta$, using the procedure at the beginning of subsection 8.1. Sometimes we drop the superscript D when it is clear from the context. The key functorial property of it (and some other forms constructed) is summarized in Remark 8.9.*

As before we let $\mathcal{L}^{Katz}_{\lambda\chi_\theta}$ be the Katz p-adic L-function interpolating the values

$$\frac{L(\lambda^2\lambda_{\theta,\phi}\chi_{\theta,\phi}^{-c}, 1)\Omega_p^2}{(\lambda^2\chi_{\theta,\phi}\chi_{\theta,\phi}^{-c})_{2,p}G(\lambda^2\chi_{\theta,\phi}\chi_{\theta,\phi}^{-c})\Omega_\infty^2}.$$

We are going to compute the Petersson inner product of θ and $\tilde{\theta}_3$ at a generic arithmetic point.

Proposition 8.7. *Up to multiplying by a non-zero element of $\bar{\mathbb{Q}}_p$ which is fixed throughout the family (i.e. independent of the arithmetic point ϕ), we have*

$$p^t \int_{[\mathrm{U}(2)]} \theta(g)\tilde{\theta}_3(g)dg = \phi(\mathcal{L}^{Katz}_{\lambda\chi_\theta}).$$

(Here p^t is the conductor of the point ϕ. Note that the θ is the specialization of the $\boldsymbol{\theta}$ to ϕ and similarly for $\tilde{\theta}_3$.)

Proof. For $v \nmid p$ the corresponding local integrals are non-zero constants which are fixed along the p-adic families. For $v = p$. We construct the Schwartz function in $S(W_p) = S(X'_p \oplus Y'_p)$ first and apply the intertwining operators δ_ψ^{-1}. We write $x'_p = (x'_{p,1}, x'_{p,2}) \in X'_p$ and $y'_p = (y'_{p,1}, y'_{p,2}) \in Y'_p$. We define: $\phi_p(x_p, y_p) = \chi_{\theta,p}(x'_{p,1}, y'_{p,1})$ if $x'_{p,1}, y'_{p,1} \in \mathbb{Z}_p^\times$ and $x'_{p,2}, y'_{p,2} \in \mathbb{Z}_p$ and equals 0 otherwise. Now as before we can compute that $\delta'^{,-1}_\psi(\phi_p) = \phi_3 \times \phi_2 \in S(X'_{1,p}) \times S(X_{2,p'})$ where: $\phi_3(x''_{1,p}, x'''_{1,p}) = \chi_{\theta,p}(x''_{1,p})$ if $x''_{1,p} \in \mathbb{Z}_p$, $x'''_{1,p} \in \mathbb{Z}_p$ and equals 0 otherwise,

$$\phi_2(x''_{2,p}, x'''_{2,p}) = \frac{\mathfrak{g}(\chi_{\theta,p})}{p^t}\chi_{\theta,p}^{-1}(x''_{2,p})$$

if $x''_{2,p} \in p^{-t}\mathbb{Z}_p^\times$ and $x'''_{2,p} \in \mathbb{Z}_p$ and equals 0 otherwise. So these are exactly the theta functions whose inner product we want to compute. Now we compute the Siegel section $f \in I_1(\lambda^2)$ on $\mathrm{U}(1,1)(\mathbb{Q}_p)$. From the form of ϕ it is easy to see that $f_{\phi_p} = f_{1,p}f_{2,p}$ for $f_{2,p} \in I_1(\lambda)$ to be the spherical which takes value 1 on the identity and $f_{1,p} \in I_1(\lambda)$ is the Siegel Weil section on $\mathrm{U}(1,1)$ of $\mathrm{U}(1,1) \times \mathrm{U}(1)$ for the Schwartz function $\phi_p \in S(\mathcal{K}_p)$ which with respect to $\mathcal{K}_v \equiv \mathbb{Q}_v \times \mathbb{Q}_v$ is $\phi(x_1, x_2) = \chi_{\theta,p}(x_1 \cdot x_2)$ for $x_1, x_2 \in \mathbb{Z}_p^\times$ and $\phi(x_1, x_2) = 0$ otherwise. But this section is nothing but the Siegel section $f^\dagger$ we constructed in [48, section 4] for the 1-dimensional unitary group case. So the local integral is easily computed to be:

$$\frac{\mathfrak{g}(\chi_{\theta,p})}{p^t}.\lambda_p^{-2}((p^{-t}, p^t))p^t\lambda_p^2(p^{-t}, 1) = \lambda_p^2((1, p^t))\mathfrak{g}(\chi_{\theta,p}).$$

To sum up, up to multiplying by certain fixed constants in $\bar{\mathbb{Q}}_p$, this Petersson inner product is the value interpolated by the Katz p-adic L-function $\mathcal{L}^{Katz}_{\lambda\chi_\theta}$. $\square$

Definition 8.8. *(Constructing* $\mathbf{h}$*) We will repeat the above process to construct another family* $\mathbf{h}$ *which will be used in computing the Fourier-Jacobi expansion as well. First we construct a family* h' *and* $\tilde{h}_3$ *using theta liftings similar to* θ *and* $\tilde{\theta}_3$*, with* χ_h *in place of* χ_θ *and slightly different theta kernels described as follows. For the theta kernel functions* ϕ_2 *and* ϕ_3 *as above, at each primes* v *in* Σ^2*, note that* $\pi_{h',v}$ *is the dual of* $\pi_{\theta,v}$*, we choose the local theta kernel* $\phi_{2,v}$ *at* v *so that the image in* $\pi_{h',v}$ *is* $\tilde{v}_1$ *(notations as before) and the theta kernel* ϕ_3 *whose image is* v_1*. At each prime* v *in* Σ^1*, we take some theta kernel* $\phi_{2,v}$ *and* $\phi_{3,v}$ *such that the images in the local theta lifting are the new vector for* $\mathrm{U}(2)$ *in our sense (subsection 8.1). We make the choices at split primes including* p *similar to the* θ *case. Again we define* h^D *on* $D^\times(\mathbb{A}_\mathbb{Q})$ *using the character* χ_h *using the procedure at the beginning of Section 8.1.*

Remark 8.9. The automorphic representation for h^D (θ^D) is the Jacquet-Langlands correspondence of the CM form associated to $\lambda\chi_h$ ($\lambda\chi_\theta$ respectively). We define h on $\mathrm{GU}(2)(\mathbb{A}_\mathbb{Q})$ using h^D and the character χ_θ^{-1} (not χ_h!) as the central character. This is just a twist of h'. We write π_h for the corresponding automorphic representation. Also the automorphic representation π_θ^D on $D^\times$ is associated with the CM character $\lambda\theta$.

Convention:
sometimes when we constructed $\theta \in \pi_\theta$ then by $\tilde{\theta} \in \pi_{\tilde{\theta}}$ we mean $\theta \cdot (\chi_\theta^{-1} \circ \det)$. If we have constructed $\tilde{\theta}_3$, then by θ_3 we mean $\tilde{\theta}_3 \cdot (\chi_\theta \circ \det)$. We use the same conventions for h's as well. We write $\mathbf{h}$, $\boldsymbol{\theta}$ for the corresponding p-adic families thus constructed.

We have the following immediate corollary:

Corollary 8.10. *The* θ*,* $\tilde{\theta}_3$*,* θ^D*,* $\tilde{\theta}_3^D$*,* h*,* $\tilde{h}_3$*,* h^D*,* $\tilde{h}_3^D$ *constructed before are pure tensors in the corresponding automorphic representations.*

8.4 Triple product formula

Backgrounds for Ichino's Formula
Let π_1, π_2, π_3 be three irreducible cuspidal automorphic representations for $\mathrm{GL}_2/\mathbb{Q}$ such that the product of their central characters is trivial and the archimedean components are holomorphic discrete series of weight 2. Let π_i^D be the Jacquet-Langlands correspondence of them to $D^\times$ (assume

they do exist). Let $\phi_i \in \pi_i^D$ and $\tilde{\phi} \in \tilde{\pi}_i^D$. Write $\Pi = \prod_{i=1}^3 \pi_i$, $\phi = \prod_i \phi_i \in \Pi$, $\tilde{\phi} = \prod_i \tilde{\phi}_i \in \tilde{\Pi}$, and r the natural 8-dimensional representation of $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$. We write

$$I(\phi \otimes \tilde{\phi}) = (\int_{[D]} \phi_1(g)\phi_2(g)\phi_3(g)dg)(\int_{[D]} \tilde{\phi}_1(g)\tilde{\phi}_2(g)\tilde{\phi}_3(g)dg).$$

Now look at the local picture. Suppose $\phi_i = \otimes_v \phi_{i,v}$ and $\tilde{\phi}_i = \otimes_v \tilde{\phi}_{i,v}$. We fix $\langle , \rangle$ a $D_v^\times$ invariant pairing between π_i^D and $\tilde{\pi}_i^D$. Let ζ be the Riemann zeta-function. Define:

$$I_v(\phi_v \otimes \tilde{\phi}_v) = \zeta_v(2)^{-2} \frac{L_v(1, \Pi_v, Ad)}{L_v(1/2, \Pi_v, r)} \cdot \int_{\mathbb{Q}_v^\times \backslash D^\times(\mathbb{Q}_v)} \prod_i \langle \pi_v^D \phi_v(x_v), \tilde{\phi}_v \rangle d^\times x_v.$$

Note that this depends on the choice of the pairing.

Let Σ be a finite set of primes including all bad primes, then we have the following formula of Ichino [24]:

$$\frac{I(\phi \otimes \tilde{\phi})}{\prod_i \langle \phi_i, \tilde{\phi}_i \rangle} = \frac{C}{8} \zeta^2(2) \frac{L^\Sigma(\frac{1}{2}, \Pi, r)}{L^\Sigma(1, \Pi, Ad)} \prod_{v \in \Sigma} \frac{I_v(\phi_v \otimes \tilde{\phi}_v)}{\langle \phi_v, \tilde{\phi}_v \rangle}$$

where C is the Tamagawa number for $D^\times$. This does not depend on the choice of the pairing.

In application our $\langle \phi, \tilde{\phi} \rangle$ is usually 0, thus we need a slight variant of the above formula. Suppose we have elements $g_i = \prod_v g_{i,v}$ such that $\langle \phi_i, \pi(g_i)\tilde{\phi}_i \rangle \neq 0$ for $i = 1, 2, 3$, where $g_{i,v}$ are elements in the group algebra $\bar{\mathbb{Q}}_p[D^\times(\mathbb{Q}_v)]$. Then:

$$\frac{I(\phi \otimes \tilde{\phi})}{\prod_i \langle \phi_i, \pi(g_i)\tilde{\phi}_i \rangle} = \frac{C}{8} \zeta^2(2) \frac{L^\Sigma(\frac{1}{2}, \Pi, r)}{L^\Sigma(1, \Pi, Ad)} \prod_{v \in \Sigma} \frac{I_v(\phi_v \otimes \tilde{\phi}_v)}{\langle \phi_v, \pi(g_v)\tilde{\phi}_v \rangle}$$

with $g_v = \prod_v g_{i,v}$.

Local Triple Product Computations

Split Case Principal Series

Suppose v is a split prime of $\mathcal{K}/\mathbb{Q}$. We assume $\pi_{1,v}$ and $\pi_{2,v}$ are principal series representation and $\pi_{3,v}$ is either principal series representation or special representation with square-free conductor. For $K = \mathrm{GL}_2(\mathbb{Z}_v)$ the maximal compact subgroup. We use the realizations of induced representations as functions on K:

$$\mathrm{Ind}_{B(\mathbb{Q}_v)}^{\mathrm{GL}_2(\mathbb{Q}_v)}(\chi_{1,v}, \chi_{2,v}) = \{v : K \to \mathbb{C}, v(qk) = \chi(q)v(K), q \in B(\mathbb{Q}_v) \cap K\}$$

where $\chi(q) = \chi_{1,v}(a)\chi_{2,v}(d)\delta_B(q)$ for $q = \begin{pmatrix} a & b \\ & d \end{pmatrix}$. We realize the inner products as

$$\langle v_1, v_2 \rangle = \int_K v_1(k)v_2(k)dk$$

for $v_1 \in \mathrm{Ind}_B^{\mathrm{GL}_2}(\chi_{1,v}, \chi_{2,v})$, $v_2 \in \mathrm{Ind}_B^{\mathrm{GL}_2}(\chi_{1,v}^{-1}, \chi_{2,v}^{-1})$. For a positive integer t let $K_t \subset K$ consist of matrices in $B(\mathbb{Z}_v)$ modulo ϖ_v^t. For $f \in \pi(\chi_1, \chi_2)$, $\tilde{f} \in \pi(\chi_1^{-1}, \chi_2^{-1})$, we define the matrix coefficient $\Phi_{f,\tilde{f}}(g) = \langle \pi(g)f, \tilde{f} \rangle$. Let $\sigma_n = \begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix}$.

Lemma 8.11. *Suppose* $t \geq 1$, $\mathrm{cond}(\chi_1\chi_2^{-1}) = (\varpi_v^t)$*. Let* $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ *in this lemma. If*

$$f_\chi(k_v) = \begin{cases} \chi_1(a)\chi_2(d), & k_v \in K_t \\ 0, & \text{otherwise} \end{cases}$$

and $\tilde{f}_{\chi^{-1}}$ *is defined similar to* f_χ *but with* χ *replaced by* χ^{-1}*. Then* $\Phi_{f_\chi, \tilde{f}_{\chi^{-1}}}(g) = 0$ *on*

$$\cup_n K_1 w\sigma_n K_1 \cup_n K_1\sigma_n w K_1.$$

On $\cup_n K_1\sigma_n K_1 \cup_n K_1 w\sigma_n \omega K_1$*, it is supported in*

$$\cup_n \begin{pmatrix} 1 & \\ & \varpi_v^n \end{pmatrix}\begin{pmatrix} 1 & \\ \varpi_v^{t-n}\mathbb{Z}_v & 1 \end{pmatrix} K_t \cup_n \begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \varpi_v^{-n}\mathbb{Z}_v \\ & 1 \end{pmatrix} K_t.$$

The corresponding values are: $\mathrm{Vol}(K_t)\alpha_2^n q^{-\frac{n}{2}}$ *and* $\mathrm{Vol}(K_t)\alpha_1^n q^{-\frac{n}{2}}$ *where* $\alpha_i = \chi_i(\varpi_v)$ *for* $i = 1, 2$.

Proof. It is easy to check by considering the supports of f_χ and $\tilde{f}_\chi$ that $\Phi_{f_\chi, \tilde{f}_\chi}(g) = 0$ on $\coprod_{n \geq 0} K_1\sigma_n w K_1$. ($K_1\sigma_n w K_1$ does not intersect $\mathrm{supp} f_\chi$.)

Now suppose $g \in K_1 w\sigma_n K_1$ for $n \geq 1$, without loss of generality we assume

$$g = \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} w\sigma_n \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} = g = \begin{pmatrix} 1 & \\ & \varpi_v \end{pmatrix}\begin{pmatrix} 1 & \\ \frac{c}{\varpi_v} & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} w$$

for $\varpi_v | b$, $\varpi_v | c$. Plugging in the formula for matrix coefficients, write $K_t \ni g' = \begin{pmatrix} a' & b' \\ & d' \end{pmatrix}\begin{pmatrix} 1 & \\ c' & 1 \end{pmatrix}$ for $a', d' \in \mathbb{Z}_v^\times$, $b' \in \mathbb{Z}_v$, $c' \in \varpi_v^t \mathbb{Z}_v$.

$$\begin{aligned} & g'g \\ = \quad & \begin{pmatrix} a' & b' \\ & d' \end{pmatrix}\begin{pmatrix} 1 & \\ c + c' & 1 \end{pmatrix} w\sigma_n \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} \\ = \quad & \begin{pmatrix} a' & b' \\ & d' \end{pmatrix}\begin{pmatrix} 1 & \\ & \varpi_v^n \end{pmatrix}\begin{pmatrix} 1 & \\ \frac{c'+c}{\varpi_v^n} & \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} w \\ = \quad & \begin{pmatrix} a' & b' \\ & d' \end{pmatrix}\begin{pmatrix} 1 & \\ & \varpi_v^n \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{c''b+1} \\ & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{c''} & \\ & c'' \end{pmatrix}\begin{pmatrix} 1 & \\ \frac{bc''+1}{c''} & 1 \end{pmatrix} \end{aligned}$$

Here we write $c'' = \frac{c'+c}{\varpi_v^n}$. We need to fix g and do the integration for g'. The first observation is that we only need to consider integration with respect to c'. Next we can integrate for those c' such that $p^t | b + \frac{1}{c''}$. We divide the problem into four cases according to whether $\varpi_v^t | c$ and whether $\varpi_v^t | b$. In any case it is not hard to check that the integration is 0 since $\mathrm{cond}(\chi_1\chi^{-1}) = (\varpi_v^t)$. We leave the verification for $g \in K_1\sigma_n K_1$ and $K_1 w\sigma w K_1$ to the reader. □

Lemma 8.12. *Suppose* $\mathrm{cond}(\chi_1\chi_2^{-1}) = (\varpi_v^t)$*,* $t > 0$ *and* ϑ *is a character with conductor* (ϖ_v^s)*. Define*

$$f_{\chi,\vartheta}(k_v) = \begin{cases} \chi_1(a)\chi_2(d)\vartheta(\frac{c}{\varpi_v^t}), & a, d \in \mathcal{O}_v^\times, c \in \varpi_v^t\mathbb{Z}_v^\times \\ 0, & \text{otherwise} \end{cases}$$

for $k_v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*. Then on* $\cup_n K_1\sigma_n K_1 \cup_n K_1 w\sigma_n w K_1$ *,* $\Phi_{f,\tilde{f}}(g)$ *is supported in* K_{t+s-1}*. More over if* $g = \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$ *with* $\varpi_v^{t+s-1}\|c$ *then* $\Phi(g) = -q^{-t-1}$*. If* $\varpi_v^{t+s}|c$ *then* $\Phi(g) = (q-1)q^{-t-1}$*.*

Proof. Suppose $\Phi_{f_{\chi,\vartheta},\tilde{f}_{\chi^{-1},\vartheta^{-1}}}(g) \neq 0$. If $g \in K_1 \begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix} K_1$ for some $n \geq 0$, then as before we have $g \in \begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \varpi_v^{-n}\mathbb{Z}_v \\ & 1 \end{pmatrix} K_t$. For $g' \in K_t$, $g' = \begin{pmatrix} a' & b' \\ & d' \end{pmatrix} \begin{pmatrix} 1 & \\ c' & 1 \end{pmatrix}$. Without loss of generality assume $g = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$ for $b \in \mathbb{Z}_v$, $\varpi_v^t|c$. Then

$$g'g = \begin{pmatrix} a' & b' \\ & d' \end{pmatrix} \begin{pmatrix} 1 & \\ c' & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}.$$

Plugging in the formula for matrix coefficients we need to integrate for a', b', c', d'. Again we only need to consider integration with respect to $c' \in \varpi_v^t\mathbb{Z}_v$. But

$$\begin{pmatrix} 1 & \\ c' & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{c'b+1} \\ & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{bc'+1} & \\ & bc'+1 \end{pmatrix} \begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ \frac{c'\varpi_v^n}{c'b+1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}.$$

If $n \geq 1$ then the integration is 0. If $n = 0$ and $\Phi(g) \neq 0$ then $g \in K_t$. Suppose $g = \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$ and $g' = \begin{pmatrix} 1 & \\ c' & 1 \end{pmatrix}$ we have: if $p^{t+s-1}\|c$ then $\Phi(g) = -q^{-t-1}$; if $p^{t+s}|c$ then $\Phi(g) = q^{-t-1}(q-1)$. In other cases $\Phi(g) = 0$. If $g \in K_1 w \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} wK_1$ again if $\Phi(g) \neq 0$ then $g \in \begin{pmatrix} 1 & \\ & \varpi_v^n \end{pmatrix} \begin{pmatrix} 1 & \\ \varpi_v^{t-n}\mathbb{Z}_v & 1 \end{pmatrix} K_t$. Without loss of generality write $g \in \begin{pmatrix} 1 & \\ & \varpi_v^n \end{pmatrix} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$ for $c \in \varpi_v^{t-n}\mathbb{Z}_v$, $K_t \ni g' = \begin{pmatrix} a' & b' \\ & d' \end{pmatrix} \begin{pmatrix} 1 & \\ c' & 1 \end{pmatrix}$, $g'g = \begin{pmatrix} a' & b' \\ & d' \end{pmatrix} \begin{pmatrix} 1 & \\ & \varpi_v^n \end{pmatrix} \begin{pmatrix} 1 & \\ \frac{c'}{\varpi_v^n} + c & 1 \end{pmatrix}$. If $n \geq 1$, then one can check that the integration is again 0. $\square$

Suppose $\chi^{-1}\chi_2$ is unramified and ϑ has conductor (ϖ_v^s) then we define $f_{\chi,\vartheta} \in \pi$ by:

$$f_{\chi,\vartheta}(k_v) = \begin{cases} \chi_1(a)\chi_2(d)\vartheta(\frac{c}{\varpi_v}), & g = \begin{pmatrix} a & b \\ & d \end{pmatrix} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ & d \end{pmatrix} \in B(\mathbb{Z}_v), c \in \varpi_v\mathbb{Z}_v^\times \\ 0 & \text{otherwise} \end{cases}$$

We define similarly $\tilde{f}_{\tilde{\chi},\tilde{\vartheta}} \in \tilde{\pi}$ by replacing χ, ϑ by $\chi^{-1}, \vartheta^{-1}$.

Lemma 8.13. *Write* $f = f_{\chi,\vartheta}$*,* $\tilde{f} = \tilde{f}_{\tilde{\chi},\tilde{\vartheta}} \in \tilde{\pi}$ *then on* $\sqcup_n K_0\sigma_n K_0 \sqcup_n K_0 w\sigma_n w K_0$ *it is supported in* K_s *and:*

$$\Phi_{f,\tilde{f}}(g) = \begin{cases} -q_v^{-1}\mathrm{Vol}(K_0(\varpi_v)) & \textit{if } g = \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}, \varpi_v^s\|c \\ \frac{q_v-1}{q_v}\mathrm{Vol}(K_0(\varpi_v)) & \textit{if } g = \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} \varpi_v^{s+1}|c \end{cases}$$

Proof. Similar to the above lemma. ☐

Lemma 8.14.

$$\sum_{a\in\frac{\varpi_v\mathbb{Z}_v}{\varpi_v^{1+s}\mathbb{Z}_v}} \vartheta_v(-\frac{a}{\varpi_v})\pi(\begin{pmatrix}1 & \\ a & 1\end{pmatrix}\begin{pmatrix}\varpi_v^{-s} & \\ & 1\end{pmatrix})(1-q_v\chi_1/\chi_2(\varpi_v))^{-1}(1-q_v^{-\frac{1}{2}}\chi_2^{-1}(\varpi_v)\pi_v(\begin{pmatrix}\varpi_v & \\ & 1\end{pmatrix}))f^{sph}$$

equals $\chi_1(\varpi_v^{-s})q_v^{-\frac{s}{2}}f_\vartheta$.

Proof. Straightforward computations. ☐

Now we are going to evaluate the local triple product integral for certain sections. The following lemma follows from the lemmas above.

Lemma 8.15. *Let* $\chi_{f,1},\chi_{f,2},\chi_{\theta,1},\chi_{\theta,2},\chi_{h,1},\chi_{\theta,2},\vartheta$ *are characters of* $\mathbb{Q}_v^\times$ *and* $t_1<s<t_2$ *are non-negative integers such that if* $t_1\neq 0$ *then* $t_1+s=t_2$, *if* $t_1=0$ *then* $s+1=t_2$. *Suppose* $\mathrm{cond}(\chi_{f,1}\chi_{f,2}^{-1})=(\varpi_v^{t_1})$ *and* $\mathrm{cond}(\vartheta)=(\varpi_v)^s$ *and* $\mathrm{cond}(\chi_{\theta,1}\chi_{\theta,2}^{-1})=\mathrm{cond}(\chi_{h,1}\chi_{h,2}^{-1})=(\varpi_v^{t_2})$. *Assume:* $\chi_{f,1}\cdot\chi_{\theta,1}\cdot\chi_{h,1}.\vartheta=1$ *and* $\chi_{f,2}\cdot\chi_{\theta,2}\cdot\chi_{h,2}.\vartheta^{-1}=1$. *We also define* $f_{\chi_f,\vartheta}\in\pi(\chi_{f,1},\chi_{f,2}), f_{\chi_\theta}\in\pi(\chi_{\theta,1},\chi_{\theta,2}), f_{\chi_h}\in\pi(\chi_{h,1},\chi_{h,2})$ *as above. Similarly for* $\tilde{f}_{\tilde{\chi}_f,\tilde{\vartheta}},\tilde{f}_{\tilde{\chi}_\theta},\tilde{f}_{\tilde{\chi}_h}$. *Then Ichino's local triple product:*

$$I_v(f_{\chi_f,\vartheta}\otimes f_{\chi_\theta}\otimes f_{\chi_h},\tilde{f}_{\tilde{\chi}_f,\tilde{\vartheta}}\otimes\tilde{f}_{\tilde{\chi}_\theta}\otimes\tilde{f}_{\tilde{\chi}_h})=\frac{q_v}{q_v-1}\mathrm{Vol}(K_{t_2}).$$

(In this lemma the χ_h,χ_θ *are defined using* $\chi_{h,1},\chi_{h,2},\chi_{\theta,1},\chi_{\theta,2}$ *similarly as in lemma 8.11.)*

Special Representations

We consider the induced representation $\pi(\chi_1,\chi_2)=\{f:K_v\to K, f(qk)=\chi_1(a)\chi_2(d)\delta_B(q), q=\begin{pmatrix}a & b\\ & d\end{pmatrix}\in B(\mathbb{Z}_v)\}$ where $\chi_2=\chi_1|.|$. The special representation $\sigma(\chi_1,\chi_2)\subset\pi(\chi_1,\chi_2)$ consisting of functions f such that $\int_K f(k)dk=0$. We consider the case when π_3 is the special representation $\sigma(\chi_{v,1},\chi_{v,2})\subset\mathrm{Ind}_{B(\mathbb{Q}_v)}^{\mathrm{GL}_2(\mathbb{Q}_v)}(\chi_{v,1},\chi_{v,2})$ at v with square free conductor. Here $\chi_{v,i}$ are unramified characters. Similar to the unramified principal series case, we use the model of induced representations. It is easy to see that the $f_{\chi,\vartheta}$ defined above is inside $\sigma(\chi_{1,v},\chi_{2,v})$. The inner product of $\sigma(\chi_{1,v},\chi_{2,v})$ is still given by $\langle v_1,v_2\rangle=\int_K v_1(k)v_2(k)dk$. The formula for the triple product integral is the same as the one in the case of principal series representations.
Let $f_{new}\in\sigma(\chi_1,\chi_2)$ be such that $f(k)=q_v$ for $k\in K_1$ and $f(k)=-1$ otherwise. Then we have the following lemma

Lemma 8.16.

$$\sum_{a\in\frac{\varpi_v\mathbb{Z}_v}{\varpi_v^{1+s}\mathbb{Z}_v}} \vartheta_v(-\frac{a}{\varpi_v})\pi(\begin{pmatrix}1 & \\ a & 1\end{pmatrix}\begin{pmatrix}\varpi_v^{-s} & \\ & 1\end{pmatrix})f_{new}$$

is $\chi_1(\varpi_v^{-s})(q_v^{1-\frac{s}{2}}+q_v^{-\frac{s}{2}})f_{\chi,\vartheta}$ *where* $f_{\chi,\vartheta}$ *is defined above.*

Proof. Let f_0 and f_1 be he characteristic functions on K_1 and K_1wK_1. Then $f_{new}=q_vf_0-f_1$. A computation shows that

$$\sum_{a\in\frac{\varpi_v\mathbb{Z}_v}{\varpi_v^{1+s}\mathbb{Z}_v}} \vartheta_v(-\frac{a}{\varpi_v})\pi(\begin{pmatrix}1 & \\ a & 1\end{pmatrix}\begin{pmatrix}\varpi_v^{-s} & \\ & 1\end{pmatrix})f_0=\chi_1(\varpi_v^{-s})q_v^{-\frac{s}{2}}f_\vartheta,$$

$$\sum_{a\in\frac{\varpi_v\mathbb{Z}_v}{\varpi_v^{1+s}\mathbb{Z}_v}}\vartheta_v(-\frac{a}{\varpi_v})\pi(\begin{pmatrix}1 & \\ a & 1\end{pmatrix}\begin{pmatrix}\varpi_v^{-s} & \\ & 1\end{pmatrix})f_1=-\chi_2(\varpi_v^{-s})q_v^{-\frac{s}{2}}f_\vartheta.$$

The lemma follows. □

Remark 8.17. The reason why the local integrals showing up in the triple product formula later on are the ones considered in this subsection is a consequence of the computations in subsection 6.7 and the remarks at the end of subsection 4.3.

Non-split Case 1

If $\mathrm{U}(2)(\mathbb{Q}_v)$ is compact. This case is easier since we are in the representation theory for finite groups. By our assumptions on $\chi_{\theta,v},\chi_{h,v}$ and $\pi_{f,v}$ and our chosen vectors $h,\theta,\tilde{\theta}_3,\tilde{h}_3$ and that $\pi_{f,v}^D$ is 1-dimensional, we conclude that: by the inner product formula of matrix coefficients of representations of finite groups the local triple product integral is $\frac{1}{d_{\pi_{h,v}}}$ where $d_{\pi_{h,v}}$ is the dimension of the representation $\pi_{h,v}$. (see e.g. [5]). When we are moving our datum p-adic analytically, this integration is not going to change.

Non-split Case 2

Recall that we fix a generic arithmetic point. By [37] we know that there are

$$g_{1,v},g_{2,v},g_{3,v},g_{4,v}\in D^\times(\mathbb{Q}_v)\subset\mathrm{GU}(2)(\mathbb{Q}_v)$$

such that

$$I_v(\pi_{h,v}(g_{2,v})h_v^D\otimes v_{\phi_v}^D\otimes\pi_{f,v}(g_{1,v})f_{\vartheta,v}^D,\tilde{\pi}_{h,v}(g_{4,v})\tilde{h}_v^D\otimes\tilde{v}_{\phi_v}^D\otimes\tilde{\pi}_{f,v}(g_{3,v})\tilde{f}_{\tilde{\vartheta},v}^D)\neq 0.$$

(We write $v_{\phi_v}^D$ for the image of v_{ϕ_v} in the corresponding $D^\times(\mathbb{Q}_v)$ representation and similarly for $\tilde{v}_{\phi_v}^D$. Note that the local signs are correct by our choices.)

Definition 8.18. *We define*

$$g_i=\prod_v g_{i,v}$$

for $i=1,2,3,4$. (We take $g_{i,v}=1$ if v is non-split in $\mathcal{K}/\mathbb{Q}$.

The local triple product integrals are also non-zero by our computations. By Ichino's formula and that the special L-values are non-vanishing (by our choices of characters, note that the product of the central characters for f,θ,h is obviously trivial) the global trilinear form is also non-zero. So by our definitions for h^D,θ^D, etc, we know that $\prod_v g_{2,v}$ is in $\check{D}^\times(\mathbb{A}_\mathbb{Q})$. We can find

$$\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})\ni g_2'=a_2g_2$$

for $a_2\in\mathbb{A}_\mathbb{Q}^\times$. So

$$\int_{[D^\times]}(\pi(g_2)h^D)(g)\theta^D(g)(\pi(g_1)f_\vartheta)(g)dg=\frac{1}{C_U^D}\chi_h(a)\int_{[U(2)]}(\pi(g_2')h)(g)\theta(g)(\pi(g_1)f_\vartheta)(g)dg.$$

Similarly we have $\prod_v g_{4,v}\in\check{D}^\times(\mathbb{A}_\mathbb{Q})$ and can find $\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})\ni g_4'=a_4g_4$ for $a_4\in\mathbb{A}_\mathbb{Q}^\times$.

Remark 8.19. These $g_{i,v}$ and $\tilde{g}_{i,v}$ are chosen only at the arithmetic point ϕ. We are going to fix them when moving the Eisenstein datum in p-adic families. Then for the arithmetic points ϕ with fixed $\phi|_{\mathbb{I}}$, the $\pi^D_{h,v}$ and $\pi^D_{\theta,v}$ only move by twisting by unramified characters which are converse to each other. (In fact the Weil representations on unitary groups are unchanged since $\chi_\theta|_{\mathrm{U}(1)}$ are. The difference only comes from extending a form on U(2) to GU(2)). Thus if the theta kernel we used to define h and θ are fixed then the local triple product integrals $\frac{I_v(\pi_{h,v}(g_{2,v})h^D_v\otimes v^D_{\phi_v}\otimes\pi_{f,v}(g_{1,v})f^D_{\vartheta,v},\tilde{\pi}_{h,v}(g_{4,v})\tilde{h}^D_v\otimes\tilde{v}^D_{\phi_v}\otimes\tilde{\pi}_{f,v}(g_{3,v})\tilde{f}^D_{\tilde{\vartheta},v})}{\langle h^D_v,\tilde{h}^D_{3,v}\rangle\langle v^D_1,\tilde{v}^D_1\rangle\langle f^D_{\vartheta,v},\tilde{f}^D_{\tilde{\vartheta},v}\rangle}$ does not change (note that since π has square-free conductor, the $\langle f^D_{\vartheta,v},\tilde{f}^D_{\tilde{\vartheta},v}\rangle$ has non-zero inner product). Later on we will see that certain product of such local factors move p-adic analytically. Thus this is an analytic function depending only on the Hida family variable of f. This observation is crucial in studying the p-adic properties of the Fourier-Jacobi coefficients.

Interpolation the f_ϑ's

We will interpolate f_ϑ's as $\mathbb{I}$-adic families from the family $\mathbf{f}$ using Lemma 8.14 and 8.16. However in order to do so we make have to replace $\mathbb{I}$ by a larger normal domain finite over $\mathbb{I}$, so that the $\chi_{1,v}(\varpi_v)$ and $\chi_{2,v}(\varpi_v)$'s at primes in Σ where π_f is unramified will be elements of this newly defined $\mathbb{I}$. We choose this family as the $\boldsymbol{\varphi}$ in Proposition 7.7.

8.5 Evaluating the Integral

Now we construct: for any $\Lambda_{\mathcal{D}}$-adic family F,

$$l_{\theta_1}(F) := l'_{\theta_1}FJ_1(\prod_{v\in\Sigma^1\cup\Sigma^2}(\sum_i C_{v,i}\rho(\begin{pmatrix} u_{v,i} & & \\ & I_2 & \\ & & u_{v,i}\end{pmatrix})(F)))$$

whose value is a $\Lambda_{\mathcal{D}}$-adic family on U(2). Here $C_{v,i}$ are defined in lemma 6.21.

Lemma 8.20. *The $l_{\theta_1}(F)$ is a $\Lambda_{\mathcal{D}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$-adic form on* U(2)*.*

Proof. The proof is simple: as before we just take a basis of $\mathcal{O}_L$-dual space $(\theta'_1,\cdots,\theta'_m)$ of the finite dimensional space $H^1(\mathcal{B},\mathcal{L}(\beta))$. Pairing the $\Lambda_{\mathcal{D}}$-adic Fourier-Jacobi coefficient of F with these θ'_i we get a $\Lambda_{\mathcal{D}}$-adic family in Λ. But our θ_1 is in the L-linear combination of the θ'_i's. Thus we get a $\Lambda_{\mathcal{D}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$-adic family on U(2). □

We also define

$$B_1 = \int l_{\theta_1}(\mathbf{E}_{\mathcal{D},Kling})d\mu_{\pi(g'_2)\mathbf{h}}.$$

We can evaluate this expression by Ichino's formula for triple products.

Definition 8.21. *For any nearly ordinary form f or family $\mathbf{f}$ (we use the same notations for $\mathbf{h}$, $\boldsymbol{\theta}$, etc) we define $f_{low}(g) := f(g\begin{pmatrix} & 1\\ 1 & \end{pmatrix}_p)$(under the identification $D_p^\times\simeq\mathrm{GL}_2(\mathbb{Q}_p)$ given by the v_0 projection). Also, if χ is the (family of) central characters of $\mathbf{f}$ we define $\tilde{\mathbf{f}}=\mathbf{f}.(\chi^{-1}\circ\mathrm{Nm})$. We also define $f^{ss}(g)=f(g\begin{pmatrix} & 1\\ p^t & \end{pmatrix}_p)$.*

We have the following

Proposition 8.22. *Up to multiplying by a constant in $\bar{\mathbb{Q}}_p^\times$ which is fixed along the family and a unit in $\hat{\mathbb{I}}^{ur}$, we have for any generic arithmetic point ϕ,*

$$\phi(B_1) = p^t C^D_{\mathrm{U}(2)}\mathcal{L}_5\mathcal{L}_6 \times (\int_{\mathbb{A}_\mathbb{Q}^\times D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_\mathbb{Q})} \pi(g_2)h^D_\phi(g)\pi(g_1)f^D_{\phi,\vartheta}(g)\theta^D_{\phi,low}(g)dg)$$

where $\mathcal{L}_5$ and $\mathcal{L}_6$ are certain p-adic L-functions interpolating the the L-values

$$\mathrm{Eul}_p \cdot \frac{\Gamma(\kappa-1)\Omega_p^\kappa}{\Omega_\infty^\kappa} L(\chi_{aux}\tau^{-c}, \frac{\kappa-2}{2})$$

and

$$\mathrm{Eul}_p \cdot \frac{\Gamma(\kappa-2)\Omega_p^{\kappa-2}}{\Omega_\infty^{\kappa-2}} L(\lambda^2\chi_\theta^{-c}\chi_\theta\chi_{aux}\tau^{-1}, \frac{\kappa-2}{2}).$$

(We refer the corresponding normalization factors Eul_p *at p to [20, (4.16)]) are the Euler factors defined).*

Proof. This is a consequence of the pullback formula. We first apply (5) and then further factorize the $E_{sieg,2}$ via the embedding $\mathrm{U}(2)\times\mathrm{U}(2) \hookrightarrow \mathrm{U}(2,2)$ of $E_{sieg,2}$ and pair with h. The $\mathcal{L}_5$ and $\mathcal{L}_6$ come from the factors show up here. Also note that the θ part appearing in the first integral above should be $\theta_{\phi_2}\otimes\bar{\lambda}$ constructed before (not an eigenform). However in view of the central character of h and f, only the component θ of θ_2 with the correct central character matters. Also by the construction this θ part is a multiple of θ^{ss} which, after applying the operator $\sum_{n\in\mathbb{Z}_p/p^t\mathbb{Z}_p}\pi_\theta(\begin{pmatrix} 1 & \\ n & 1\end{pmatrix})$ is θ_{low} from the definition of $\phi_{2,p}$. Comparing $\mathcal{L}_5\cdot\mathcal{L}_6$ with the factor coming from the doubling method for h under $\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$, it is easy to see that the ratio comes from the local Euler factors at p computed in Subsection 6.9, Euler factor for $\mathcal{L}_5$ at primes in Σ^2 and Euler factors for $\mathcal{L}_6$ at p, which are units by our choice of characters (See [20, (4.16)] for the Euler factor at p). Then the lemma follows. □

By our choices for characters the corresponding non-Σ-primitive p-adic L-functions $\mathcal{L}_5$ and $\mathcal{L}_6$ are units in $\Lambda_\mathcal{D}^\times$.

In the following we often omit the superscript D for simplicity. We construct a form $\tilde{f}_{\tilde{\vartheta}}\in\tilde{\pi}$ in a similar way to f_ϑ at primes outside p and is the nearly ordinary vector (with respect to the upper triangular Borel) at p. Up to a constant in $\bar{\mathbb{Q}}_p^\times$ (which does not change along the family) we have

$$\begin{aligned}
& p^t B_1\cdot(\int(\pi(g_4')\tilde{h}_3)(g)(\pi_f(g_3)\tilde{f}_{\tilde{\vartheta}})(g)\tilde{\theta}_{3,low}(g)dg) \\
= \quad & (\lambda_{p,2}\chi_{\theta,2})^{-t}(p)(\chi_{\theta,1}\lambda_{p,1})^t(p)p^{3t}(\int(\pi(g_2)h)(g)(\pi_f(g_1)f_\vartheta)(g)\theta^{ss}(g)dg) \\
& \times(\int(\pi_{\tilde{h}}(g_4')\tilde{h}_3)(g)(\pi_{\tilde{f}}(g_3)\tilde{f}_{\tilde{\vartheta}})(g)\tilde{\theta}_3^{ss}(g)dg) \\
= \quad & \lambda_{p,2}^{-2t}(p)\chi_{\theta,2}^{-2t}(p)p^{3t}(\int(\pi(g_2)h)(g)(\pi_f(g_1)f_\vartheta)(g)\theta^{ss}(g)dg) \\
& \times(\int(\pi(g_4')\tilde{h_3})^{ss}(g)(\pi_{\tilde{f}}(g_3)\tilde{f}_{\tilde{\vartheta}})^{ss}(g)\tilde{\theta}_3(g)dg)
\end{aligned}$$

(Note that by our discussion before Proposition 7.4, we know up to multiplying by an element in $\bar{\mathbb{Q}}_p^\times$, the

$$(p^t \int (\pi(g_4)\tilde{h}_3)(g)(\pi_f(g_3)\tilde{f}_{\tilde{\vartheta}})(g)\tilde{\theta}_{3,low}(g)dg)$$

is interpolated by an element in $\hat{\mathbb{I}}^{ur}[[\Gamma_\mathcal{K}]]$.)

Definition 8.23. *Suppose $\pi_{f,p}$ is the principal series representation $\pi(\chi_{f,s},\chi_{f,o})$ where $\mathrm{val}_p(\chi_{f,s}) = \frac{1}{2}$ and $\mathrm{val}_p(\chi_{f,o}) = -\frac{1}{2}$.*

We write the p-component of $\lambda^2\chi_\theta\chi_h$ as $((\lambda^2\chi_\theta\chi_h)_1, (\lambda^2\chi_\theta\chi_h)_2)$ and f the normalized GL_2 ordinary form new outside p. We also define the following (to make notations less cumbersome in the next few formulas we omit the subscript ϕ, e.g. $\chi_{\theta,2}$ stands for $\chi_{\theta,2,\phi}$ for an arithmetic point ϕ): the p-adic L-function $\mathcal{L}_1$ interpolates values

$$\frac{\mathfrak{g}(\chi_{\theta,2})\mathfrak{g}(\chi_{h,1}^{-1})L(f,\lambda^2\chi_\theta\chi_h,\frac{1}{2})(\chi_{f,s}(p)\chi_{f,o}(p))^{-t}((\lambda^2\chi_\theta\chi_h)_2(p).p)^{-2t}p^t}{\pi\Omega_\infty^2};$$

the p-adic L-function $\mathcal{L}_2$ interpolates values

$$\frac{(1-a_p(f)^{-1}\chi_{\theta,p,1}\chi_{h,p,2}(p^{-1}))(1-a_p(f)\chi_{\theta,p,2}\chi_{h,p,1}(p^{-1})p^{-1})}{(1-a_p(f)\chi_{\theta,p,1}\chi_{h,p,2}(p)p^{-1})\cdot(1-a_p(f)^{-1}\chi_{\theta,p,2}\chi_{h,p,1}(p))}$$
$$\times\frac{\mathfrak{g}(\chi_f^{-1})L(f,\chi_\theta^c\chi_h,\frac{1}{2})(\chi_{f,o}(p)p^{\frac{1}{2}}(\chi_h\chi_\theta^c)_1(p))^{-t}}{\pi^2\langle f, f^c|\begin{pmatrix} & -1\\ N & \end{pmatrix}\rangle_{\Gamma_0(N)}};$$

the p-adic L-function $\mathcal{L}_3$ interpolates values

$$\frac{\zeta_{\chi_\mathcal{K}}(1)}{\pi}\frac{\mathfrak{g}(\chi_\theta)L(\lambda^2\chi_\theta\chi_\theta^{-c},1)((\lambda^2\chi_\theta\chi_\theta^{-c})_2(p)p)^{-t}p^t}{\pi\Omega_\infty};$$

the p-adic L-function $\mathcal{L}_4$ interpolates values

$$\frac{\zeta_{\chi_\mathcal{K}}(1)}{\pi}.\frac{\mathfrak{g}(\chi_{h,1}^{-1})L(\lambda^2\chi_h\chi_h^{-c},1)((\lambda^2\chi_h\chi_h^{-c})_2(p).p)^{-t}.p^t}{\pi\Omega_\infty}$$

We refer to [18], [20] for the justification of their interpolation formulas. These values are interpolated by some p-adic L-functions in $\hat{\mathbb{I}}_\mathcal{K}^{ur}$. Note that

$$\frac{\mathfrak{g}(\chi_f^{-1})L(\mathrm{ad},f,1)M\varphi(M)(p-1)(\chi_{f,o}\chi_{f,s}^{-1}(p).p)^{-t}}{2^4\pi^3\langle f, f|\begin{pmatrix} & -1\\ N & \end{pmatrix}\rangle_{\Gamma_0(N)}} = 1$$

Note also that $\mathcal{L}_2$ is in fact a non-zero element in $\mathrm{Frac}(\mathbb{I})$ (i.e. does not depend on the variable $\Gamma_\mathcal{K}$) by checking the p-components of $\chi_\theta^c\chi_h$ and nonzero by our choice of characters in subsection 8.2. It can actually be written as the ratio of two elements whose specializations to all but finitely many generic arithmetic points are non-zero. By our choices for χ_θ and χ_h we know that $\mathcal{L}_1$ is in $\hat{\mathbb{I}}^{ur}[[\Gamma_\mathcal{K}]]^\times$. We consider the expression:

$$C^D_{\mathrm{U}(2)}\frac{\langle f,\tilde{f}^{ss}\rangle\langle h,\tilde{h}^{ss}\rangle\langle\tilde{\theta},\theta^{ss}\rangle\chi^{-1}_{\theta,1}(p^t)\mathcal{L}_1\mathcal{L}_2\mathcal{L}_5\mathcal{L}_6}{\mathcal{L}_3\mathcal{L}_4}(\chi_{f,s}(p)\lambda^2_{p,1}(p)\chi_{\theta,1}(p)\chi_{h,1}(p)p^{\frac{3}{2}})^t p^{3t} \tag{9}$$

$$= \quad C^D_{\mathrm{U}(2)}\frac{\langle f,\tilde{f}_{low}\rangle\langle h,\tilde{h}_{low}\rangle\langle\theta,\tilde{\theta}_{low}\rangle\mathcal{L}_1\mathcal{L}_2\mathcal{L}_5\mathcal{L}_6}{\mathcal{L}_3\mathcal{L}_4}p^{3t}. \tag{10}$$

We first give the following lemma for the local triple product integral at p.

Lemma 8.24. *At a generic point the local triple product integral at p is given by:*

$$\frac{p^{-t}(1-p)}{1+p}\frac{1}{1-a_p(f)\chi_{\theta,p,1}\chi_{h,p,2}(p)p^{-1}}\cdot\frac{1}{1-a_p(f)^{-1}\chi_{\theta,p,2}\chi_{h,p,1}(p)}.$$

Proof. This follows from Lemma 8.11. □

We examine the ratio between (9) and the expression for

$$p^tB_1(\int(\pi(g_4)\tilde{h}_3)(g)(\pi_f(g_3)\tilde{f})(g)\tilde{\theta}_{3,low}(g)dg) \tag{11}$$

using Ichino's formula. By our calculations for the local triple product integrals, and the Petersson inner product of $<\theta,\tilde{\theta}_3>$ and $<h,\tilde{h}_3>$, the ratio is a product of:

- Euler factors for $\zeta_{\chi_\mathcal{K}}(1)$ at Σ;
- the local Euler factors for $L(ad,f,1)$ at Σ;
- p^t times the local triple product integral for $v=p$, which are units times a constant in $\bar{\mathbb{Q}}_p$ by our choices;
- $\langle f^D_\vartheta,\tilde{f}^D_{\vartheta,low}\rangle/\langle f^D,\tilde{f}^D_{low}\rangle$ which is interpolated by a non-zero element in $\mathbb{I}$;
- The local Euler factors of $\mathcal{L}_5$ and $\mathcal{L}_6$ at Σ^2 and p which are units by our choices;
- The local integrals showing up in the Rallis inner product formula at Σ, which is fixed along the family;
- The local Euler factors at Σ^2 of $L(f,\chi^c_\theta\chi_h,\frac{1}{2})$ which are non-zero elements in $\mathbb{I}$.
- The local triple product integrals for $v\nmid p$;

Thus we have the following proposition:

Proposition 8.25. *Any height 1 prime of $\hat{\mathbb{I}}^{ur}[[\Gamma_\mathcal{K}]]$ containing $\int l_{\theta_1}(\mathbf{E}_{Kling,\mathcal{D}})d\mu_{\pi(g_2')\mathbf{h}}$ must be the pullback of a height 1 prime of $\hat{\mathbb{I}}^{ur}$.*

Proof. The last item we listed above has two parts: at split primes and non-split primes. The integrals at split primes are non-zero numbers in $\bar{\mathbb{Q}}_p^\times$ which are fixed throughout the family. At non-split primes we do not know much about it. We only know that at the generic arithmetic point ϕ at which we choose our g_1, g_2, g_3, g_4, this integral is not zero. We may assume that at this ϕ the expression (9) is non-zero (in fact just need $\mathcal{L}_2$ to be non-zero finite number here). Thus the expression (11) is not identically zero. So the ratio of (11) over (9) is a non-zero element of $\mathrm{Frac}(\hat{\mathbb{I}}^{ur}[[\Gamma_\mathcal{K}]])$. On the other hand the integral in the last item above only depends on $\phi|_\mathbb{I}$ as observed in Remark 8.19. So if we evaluate this ratio at the generic arithmetic points where (9) is not zero, it depends only on $\phi|_\mathbb{I}$. From this it is not hard to prove that (say using the following lemma) the ratio is a non-zero element of $\mathrm{Frac}(\hat{\mathbb{I}}^{ur})$ and the proposition is true. $\square$

Lemma 8.26. *Suppose A is an element in $\hat{\mathbb{I}}[[\Gamma_\mathcal{K}]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If for any generic arithmetic points $\phi, \phi' \in \hat{\mathbb{I}}[[\Gamma_\mathcal{K}]]$ such that $\phi|_{\hat{\mathbb{I}}} = \phi'|_{\hat{\mathbb{I}}}$, we have $\phi(A) = \phi'(A)$. Then $A \in \hat{\mathbb{I}}$.*

Proof. This lemma is easily proved by observing that if ζ_1, ζ_2 are p^t-roots of unity and ϕ is a generic arithmetic point with conductors being $p^{t'}$ such that $t' > t$, then the composition ϕ' of ϕ with the ring automorphism $\iota_{\zeta_1,\zeta_2} : \hat{\mathbb{I}}[[\Gamma_\mathcal{K}]] \to \hat{\mathbb{I}}[[\Gamma_\mathcal{K}]]$ given by identity on $\hat{\mathbb{I}}$ and $\gamma^+ \mapsto \gamma^+\zeta_1$, $\gamma^- \mapsto \gamma^-\zeta_2$ is still a generic arithmetic point. Let F be the element considered in the lemma. Then $F - F \circ \iota_{\zeta_1,\zeta_2}$ is 0 at a Zariski dense set of points, and is thus identically zero. The arbitrariness of ζ_1, ζ_2 implies the lemma. In the proof of the above proposition we apply this lemma to $B_2 \cdot \mathcal{L}_2$ times (11) over (9), where B_2 interpolates $p^t\langle f, \tilde{f}_{low}\rangle$ (which are indeed interpolated by an Iwasawa algebra element by our previous discussion). $\square$

9 Proof of the Theorems

9.1 Hecke Operators

Let $K' = K'_{\Sigma\backslash\{p\}}K^\Sigma \subset G(\mathbb{A}_f^p)$ be an open compact subgroup with $K^\Sigma = G(\hat{\mathbb{Z}}^\Sigma)$ and such that $K := K'K_p^0$ is neat. For each v outside Σ we have $\mathrm{GU}(3,1)(\mathbb{Q}_v) \simeq \mathrm{GU}(2,2)(\mathbb{Q}_v)$ with the isomorphism given by conjugation by some elements in $\mathrm{GL}_4(\mathcal{O}_{\mathcal{K},v})$. So we only need to study the unramified Hecke operators for $\mathrm{GU}(2,2)$ with respect to $\mathrm{GU}(2,2)(\mathbb{Z}_v)$. We follow closely to [45, 9.5,9.6].

Unramified Inert Case

Let v be a prime of $\mathbb{Q}$ inert in $\mathcal{K}$. Recall that as in [45, 9.5.2] that $Z_{v,0}$ is the Hecke operator associated to the matrix $z_0 := \mathrm{diag}(\varpi_v, \varpi_v, \varpi_v, \varpi_v)$ by the double coset Kz_0K where K is the maximal compact subgroup of $G(\mathbb{Z}_v)$. Let $t_0 := \mathrm{diag}(\varpi_v, \varpi_v, 1, 1)$, $t_1 := \mathrm{diag}(1, \varpi_v, 1, \varpi_v^{-1})$ and $t_2 := \mathrm{diag}(\varpi_v, 1, \varpi_v^{-1}, 1)$. As in [45, 9.5.2] we define

$$\mathcal{R}_v := \mathbb{Z}[X_v, q^{1/2}, q^{-1/2}]$$

where X_v is $T(\mathbb{Q}_v)/T(\mathbb{Z}_v)$ and write $[t]$ for the image of t in X_v. Let $\mathcal{H}_K$ be the abstract Hecke ring with respect to the level group K. There is a Satake map: $\mathcal{S}_K : \mathcal{H}_K \to \mathcal{R}_v$ given by

$$\mathcal{S}_K(KgK) = \sum \delta_B^{1/2}(t_i)[t_i]$$

if $KgK = \sqcup t_i n_i K$ for $t_i \in T(\mathbb{Q}_v), n_i \in N_B(\mathbb{Q}_v)$ and extend linearly. We define the Hecke operators T_i for $i = 1, 2, 3, 4$ by requiring that

$$1 + \sum_{i=1}^{4} \mathcal{S}(T_i)X^i = \prod_{i=1}^{2}(1 - q_v^{\frac{3}{2}}[t_i]X)(1 - q_v^{\frac{3}{2}}[t_i]^{-1}X)$$

is an equality of polynomials of the variable X. We also define:

$$Q_v(X) := 1 + \sum_{i=1}^{4} T_i(Z_0 X)^i.$$

Unramified Split Case

Suppose v is a prime of F split in $\mathcal{K}$. In this case we define $z_0^{(1)}$ and $z_0^{(2)}$ to be $(\mathrm{diag}(\varpi_v, \varpi_v, \varpi_v, \varpi_v), 1)$ and $(1, \mathrm{diag}(\varpi_v, \varpi_v, \varpi_v, \varpi_v))$ and define the Hecke operators $Z_0^{(1)}$ and $Z_0^{(2)}$ as above but replacing z_0 by $z_0^{(1)}$ and $z_0^{(2)}$. Let $t_1^{(1)} := \mathrm{diag}(1, (\varpi_v, 1), 1, (1, \varpi_v^{-1}))$, $t_2^{(1)} := \mathrm{diag}((\varpi_v, 1), 1, (1, \varpi_v^{-1}), 1)$. Define $t_i^{(2)} := \bar{t}_i^{(1)}$ and $t_i = t_i^{(1)} t_i^{(2)}$ for $i = 1, 2$. We define R_v and $\mathcal{S}_K$ in the same way as the inert case. Then we define Hecke operators $T_i^{(j)}$ for $i = 1, 2, 3, 4$ and $j = 1, 2$ by requiring that the following

$$1 + \sum_{i=1}^{4} \mathcal{S}_K(T_i^{(j)})X^i = \prod_{i=1}^{2}(1 - q_v^{\frac{3}{2}}[t_i^{(j)}]X)(1 - q_v^{\frac{3}{2}}[t_i^{(j')}]^{-1}X)$$

to be equalities of polynomials of the variable X. Here $j' = 3 - j$ and $[t_i^{(j)}]$'s are defined similarly to the inert case. Now let $v = w\bar{w}$ for w a place of $\mathcal{K}$. Define $i_w = 1$ and $i_{\bar{w}} = 2$. Then we define:

$$Q_w(X) := 1 + \sum_{i=1}^{4} T_i^{(i_w)}(Z_0^{(3-i_w)}X)^i,$$

$$Q_{\bar{w}}(X) := 1 + \sum_{i=1}^{4} T_i^{(i_{\bar{w}})}(Z_0^{(3-i_{\bar{w}})}X)^i.$$

9.2 Eisenstein Ideals

Let $K_{\mathcal{D}}$ be an open compact subgroup of $\mathrm{GU}(3,1)(\mathbb{A}_{\mathbb{Q}})$ maximal at p and all primes outside Σ such that the Klingen Eisenstein series we construct is invariant under $K_{\mathcal{D}}$. We consider the ring $\mathbb{T}_{\mathcal{D}}$ of reduced Hecke algebras acting on the space of $\hat{\Lambda}_{\mathcal{D}}^{ur}$-adic nearly ordinary cuspidal forms with level group $K_{\mathcal{D}}$. It is generated by the Hecke operators $Z_{v,0}$, $Z_{v,0}^{(i)}$, $T_{i,v}$, $T_{i,v}^{(j)}$ defined above and then taking the maximal reduced quotient. It is well known that one can interpolate the Galois representations attached to nearly ordinary cusp forms to get a pseudo-character $R_{\mathcal{D}}$ of $G_{\mathcal{K}}$ with values in $\mathbb{T}_{\mathcal{D}}$. We define the ideal $I_{\mathcal{D}}$ of $\mathbb{T}_{\mathcal{D}}$ to be generated by $\{t - \lambda(t)\}_t$ for t's in the abstract Hecke algebra and $\lambda(t)$ is the Hecke eigenvalue of t on $\mathbf{E}_{\mathcal{D},Kling}$. Then it is easy to see that the structure map $\hat{\Lambda}_{\mathcal{D}}^{ur} \to \mathbb{T}_{\mathcal{D}}/I_{\mathcal{D}}$ is surjective. Suppose the inverse image of $I_{\mathcal{D}}$ in $\hat{\Lambda}_{\mathcal{D}}^{ur}$ is $\mathcal{E}_{\mathcal{D}}$. We call it the Eisenstein ideal. It measures the congruences between the Hecke eigenvalues of cusp forms and Klingen Eisenstein series. We have:

$$R_{\mathcal{D}}(\mathrm{mod} I_{\mathcal{D}}) \equiv \mathrm{tr}\rho_{\mathbf{E}_{\mathcal{D},Kling}}(\mathrm{mod}\mathcal{E}_{\mathcal{D}}).$$

Now we prove the following lemma:

Lemma 9.1. *Let P be a height 1 prime of $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$ which is not the pullback of a height 1 prime of $\hat{\mathbb{I}}^{ur}$. Then*

$$\mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathbf{f},\xi,\mathcal{K}}) \le \mathrm{ord}_P(\mathcal{E}_{\mathcal{D}}).$$

Proof. Suppose $t := \mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}) > 0$. By the fundamental exact sequence Theorem 3.6 there is an $\mathbf{H} = \mathbf{E}_{\mathcal{D},Kling} - \mathcal{L}^{\Sigma}_{\mathbf{f},\xi,\mathcal{K}}F$ for some $\Lambda_{\mathcal{D}}$-adic form F such that $\mathbf{H}$ is a cuspidal family. We write ℓ for the functional $\ell(G) = \int l_{\theta_1}(G)d\mu_{\pi(g_2')\mathrm{h}}$ constructed in subsection 8.5 on the space of $\Lambda_{\mathcal{D}}$-adic forms. By our assumption on P we have proved that $\ell(\mathbf{H}) \not\equiv 0(\mathrm{mod}P)$. Consider the $\Lambda_{\mathcal{D}}$-linear map:

$$\mu : \mathbb{T}_{\mathcal{D}} \to \Lambda_{\mathcal{D},P}/P^r\Lambda_{\mathcal{D},P}$$

given by: $\mu(t) = \ell(t.\mathbf{H})/\ell(\mathbf{H})$ for t in the Hecke algebra. Then:

$$\ell(t.\mathbf{H}) \equiv \ell(t\mathbf{E}_{\mathcal{D}}) \equiv \lambda(t)\ell(\mathbf{E}_{\mathcal{D}}) \equiv \lambda(t)\ell(\mathbf{H})(\mathrm{mod}P^t)$$

so $I_{\mathcal{D}}$ is contained in the kernel of μ. Thus it induces: $\Lambda_{\mathcal{D},P}/\mathcal{E}_{\mathcal{D}}\Lambda_{\mathcal{D},P} \twoheadrightarrow \Lambda_{\mathcal{D},P}/P^t\Lambda_{\mathcal{D},P}$ which proves the lemma. □

9.3 Galois Theoretic Argument

In this section, for ease of reference we repeat the set-up and certain results from [45, Chapter 4] with some modifications, which are used to construct elements in the Selmer group.

Let G be a group and C a ring. $r :\to Aut_C(V)$ a representation of G with $V \simeq C^n$. This can be extended to $r : C[G] \to \mathrm{End}_C(V)$. For any $x \in C[G]$, define: $\mathrm{Ch}(r, x, T) := \det(id - r(x)T) \in C[T]$.

Let (V_1, σ_1) and (V_2, σ_2) be two C representations of G. Assume both are defined over a local henselian subring $B \subseteq C$, we say σ_1 and σ_2 are residually disjoint modulo the maximal ideal $\mathfrak{m}_B$ if there exists $x \in B[G]$ such that $\mathrm{Ch}(\sigma_1, x, T)$ mod $\mathfrak{m}_B$ and $\mathrm{Ch}(\sigma_2, x, , T)$ mod $\mathfrak{m}_B$ are relatively prime in $\kappa_B[T]$, where $\kappa_B := B/\mathfrak{m}_B$.

Let H be a group with a decomposition $H = G \rtimes \{1, c\}$ with $c \in H$ an element of order two normalizing G. For any C representations (V, r) of G we write r^c for the representation defined by $r^c(g) = r(cgc)$ for all $g \in G$.

<u>Polarizations</u>:
Let $\theta : G \to \mathrm{GL}_L(V)$ be a representation of G on a vector space V over field L and let $\psi : H \to L^\times$ be a character. We assume that θ satisfies the ψ-polarization condition:

$$\theta^c \simeq \psi \otimes \theta^\vee.$$

By a ψ-polarization of θ we mean an L-bilinear pairing $\Phi_\theta : V \times V \to L$ such that

$$\Phi_\theta(\theta(g)v, v') = \psi(g)\Phi_\theta(v, \theta^c(g)^{-1}v').$$

Let $\Phi^t_\theta(v, v') := \Phi_\theta(v', v)$, which is another ψ-polarization. We say that ψ is compatible with the polarization Φ_θ if

$$\Phi^t_\theta = -\psi(c)\Phi_\theta.$$

Suppose that:
(1) A_0 is a pro-finite $\mathbb{Z}_p$ algebra and a Krull domain;
(2) $P \subset A_0$ is a height one prime and $A = \hat{A}_{0,P}$ is the completion of the localization of A_0 at P. This is a discrete valuation ring.
(3) R_0 is local reduced finite A_0-algebra;
(4) $Q \subset R_0$ is prime such that $Q \cap A_0 = P$ and $R = \hat{R}_{0,Q}$;
(5) there exist ideals $J_0 \subset A_0$ and $I_0 \subset R_0$ such that $I_0 \cap A_0 = J_0, A_0/J_0 = R_0/I_0, J = J_0 A, I = I_0 R, J_0 = J \cap A_0$ and $I_0 = I \cap R_0$;
(6) G and H are pro-finite groups; we have subgroups $D_i \subset G$ for $i = 1, \cdots, d$.

Set Up:
suppose we have the following data:
(1) a continuous character $\nu : H \to A_0^\times$;
(2) a continuous character $\xi : G \to A_0^\times$ such that $\bar{\chi} \neq \bar{\nu}\bar{\chi}^{-c}$; Let $\chi' := \nu\chi^{-c}$;
(3) a representation $\rho : G \to Aut_A(V), V \simeq A^n$, which is a base change from a representation over A_0, such that:

$$a. \rho^c \simeq \rho^\vee \otimes \nu,$$
$$\bar{\rho} \text{ is absolutely irreducible },$$
$$\rho \text{ is residually disjoint from } \chi \text{ and } \chi';$$

(4) a representation $\sigma : G \to \mathrm{Aut}_{R\otimes_A F}(M), M \simeq (R \otimes_A F)^m$ with $m = n+2$, which is defined over the image of R_0 in R, such that:

$$a.\ \sigma^c \simeq \sigma^\vee \otimes \nu,$$
$$b.\ \mathrm{tr}\sigma(g) \in R \text{ for all } g \in G\ ,$$
$$c.\ \text{ for any } v \in M, \sigma(R[G])v \text{ is a finitely-generated } R\text{-module}$$

(5) a proper ideal $I \subset R$ such that $J := A \cap I \neq 0$, the natural map $A/J \to R/I$ is an isomorphism, and

$$\mathrm{tr}\sigma(g) \equiv \chi'(g) + \mathrm{tr}\rho(g) + \chi(g)\ mod\ I$$

for all $g \in G$.

(6) ρ is irreducible and ν is compatible with ρ.

(7) (local conditions for σ) For $p = v\bar{v}$ there is a $G_{\bar{v}}$-stable sub-$R \otimes_A F$-module $M_{\bar{v}}^+ \subseteq M$ such that $M_{\bar{v}}^+$ and $M_{\bar{v}}^- := M/M_{\bar{v}}^+$ are free $R \otimes_A F$ modules.

(8) (compatibility with the congruence condition) For $p = v\bar{v}$ Assume that for all $x \in R[G_{\bar{v}}]$, we have congruence relation:

$$\mathrm{Ch}(M_{\bar{v}}^+, x, T) \equiv (1 - T\chi(x))\ mod\ I$$

(then we automatically have:

$$\mathrm{Ch}(M_{\bar{v}}^-, x, T) \equiv \mathrm{Ch}(V_{\bar{v}}, x, T)(1 - T\chi'(x))\ mod\ I)$$

(9) For each F-algebra homomorphism $\lambda : R \otimes_A F \to K$, K a finite field extension of F, the representation $\sigma_\lambda : G \to \mathrm{GL}_m(M \otimes_{R\otimes F} K)$ obtained from σ via λ is either absolutely irreducible or contains an absolutely irreducible two-dimensional sub K-representation σ'_λ such that $\mathrm{tr}\sigma'_\lambda(g) \equiv \chi(g) + \chi'(g)$ *mod* I.

One defines the Selmer groups $\mathbf{X}_H(\chi'/\chi) := \ker\{H^1(G, A_0^*(\chi'/\chi)) \to H^1(D, A_0^*(\chi'\chi))\}^*$. and $\mathbf{X}_G(\rho_0 \otimes \chi^{-1}) := \ker\{H^1(G, V_0 \otimes_{A_0} A_0^*(\chi^{-1})) \to H^1(D, V_0^- \otimes_{A_0} A_0^*(\chi^{-1}))\}^*$. Let $\mathrm{Ch}_H(\chi'/\chi)$ and $\mathrm{Ch}_G(\rho_0 \otimes \chi^{-1})$ be their characteristic ideals as A_0-modules.

Proposition 9.2. *Under the above assumptions, if* $ord_P(\mathrm{Ch}_H(\chi'/\chi)) = 0$ *then*

$$ord_P(\mathrm{Ch}_G(\rho_0 \otimes \chi^{-1})) \geq ord_P(J).$$

Proof. This can be proved in the same way as [45, Corollary 4.16]. The only difference is the Selmer condition at p where we use the description of Section 3.3 to take care of. Note that the part corresponding to ρ_0 corresponds to the upper-left two by two block here while in [45] the ρ_f contains the highest and the lowest Hodge-Tate weights. □

Before proving the main theorem we first prove a useful lemma, which appears in an earlier version of [45] .

Lemma 9.3. *Let* $Q \subset \mathbb{I}[[\Gamma_\mathcal{K}]]$ *be a height one prime such that* $\mathrm{ord}_Q(\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi}) \geq 1$ *and* $\mathrm{ord}_Q(\mathcal{L}_{\chi_\mathbf{f}\bar{\xi}'}) = 0$*, then* $\mathrm{ord}_Q(\mathcal{L}^\Sigma_{\chi_\mathbf{f}\bar{\xi}'}) = 0$.

Proof. Let $\theta = \chi_{f_0}\bar{\xi}'$. If $\mathrm{ord}_Q(\mathcal{L}^\Sigma_{\chi_\mathbf{f}\bar{\xi}'}) \geq 1$. Then for some $\ell \in \Sigma\backslash\{p\}$,

$$\prod_{\ell \in \Sigma\backslash\{p\}} (1 - \theta^{-1}(\gamma_+^{-1}(1+W))^e \ell^{2-\kappa}) \in Q.$$

where $e \in \mathbb{Z}_p$ be such that $\ell = \omega^{-1}(\ell)(1+p)^e$.

$$\theta(\ell) \equiv \gamma_+^{-e}\omega(\ell)^{\kappa-2}(1+p)^{e(2-\kappa)}(\mathrm{mod}Q).$$

Thus there is some integer f such that:

$$1 \equiv (\gamma_+(1+W)^{-1}(1+p)^{\kappa-2})^{-fe}(\mathrm{mod}Q)$$

which implies that for some p-power root of unity ζ_+, Q is contained in the kernel of any ϕ' such that $\phi'(\gamma_+(1+W)^{-1}) = \zeta_+(1+p)^{2-\kappa}$. This implies, by the work of [11] for the interpolation formula, that at the central critical point $L_\mathcal{K}(f_\phi, \theta_1, 1) = 0$ where θ_1 is some fixed CM character of infinite type $(\frac{\kappa}{2}, -\frac{\kappa}{2})$ and ϕ any arithmetic point. But then we can specialize $\mathbf{f}$ to some point ϕ'' of weight 4 (this is not an arithmetic point in our definition, but is an interpolation point, by *loc.cit*). By temperedness the specialization is not 0. This is a contradiction. □

Now we apply the above result to prove the theorem.

- $H := G_{\mathbb{Q},\Sigma}$, $G = G_{\mathcal{K},\Sigma}$, c is complex conjugation.
- $A_0 = \hat{\Lambda}^{ur}_\mathcal{D}, A := \hat{\Lambda}_{\mathcal{D},P}$.

- $J_0 := \mathcal{E}_{\mathcal{D}}, J := \mathcal{E}_{\mathcal{D}} A$.
- $R_0 := \mathbb{T}_{\mathcal{D}}, I_0 := I_{\mathcal{D}}$.
- $Q \subset R_0$ is the inverse image of P modulo $\mathcal{E}_{\mathcal{D}}$ under $\mathbb{T}_{\mathcal{D}} \to \mathbb{T}_{\mathcal{D}}/I_{\mathcal{D}} = \Lambda_{\mathcal{D}}/\mathcal{E}_{\mathcal{D}}$.
- $R := T_{\mathbf{f}} \otimes_{\mathbb{I}} \Lambda_{\mathcal{D}}$, $\rho_0 := \rho_{\mathbf{f}} \sigma_{(\boldsymbol{\psi}/\boldsymbol{\xi})^c} \epsilon^{-\frac{\kappa+3}{2}}$.
- $V = V_0 \otimes_{A_0} A$, $\rho = \rho_0 \otimes_{A_0} A$.
- $\chi = \sigma_{\boldsymbol{\psi}^c}$, $\chi' = \sigma_{\boldsymbol{\psi}^c} \sigma_{(\boldsymbol{\psi}/\boldsymbol{\xi})'} \epsilon^{-\kappa}$. $\nu = \chi^c \chi'$.
- $M := (R \otimes_A F_A)^4$, F_A is the fraction field of A.
- σ is the representation on M obtained as the pseudo-representation associated to $\mathbb{T}_{\mathcal{D}}$, as in [45, Proposition 7.2.1].

Now we are ready to prove the main theorem in the introduction.

Proof. We first remark that we need only to prove the corresponding inclusion for the Σ-primitive Selmer groups and L-functions since locally the size of the unramified extensions at primes outside p are controlled by the local Euler factors of the p-adic L-functions since $\mathbb{Q}_\infty \subseteq \mathcal{K}_\infty$. (See [8, Proposition 2.4]).

Recall that we have enlarged our $\mathbb{I}$ at the end of Subsection 8.4 which we denote as $\mathbb{J}$ in this proof. We first prove the main theorem with $\hat{\mathbb{I}}^{ur}$ replaced by $\hat{\mathbb{J}}^{ur}$. Under the assumption of Theorem 1.1, as in [45] we know that by the discussion for the anticyclotomic μ-invariant, $\mathcal{L}_{\mathbf{f},\mathcal{K},\xi}$ is not contained in any height 1 prime which is the pullback of a prime in $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}^+]]$, and so is $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}$. Thus by lemma 9.1 for any such height one prime P of $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$,

$$\mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}) = \mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}) \le \mathrm{ord}_P(\mathcal{E}_{\mathcal{D}}).$$

Applying proposition 9.2, we prove the first part of the theorem for $\hat{\mathbb{J}}^{ur}$ in place of $\mathbb{I}$.

We replace $\hat{\mathbb{J}}^{ur}$ by $\hat{\mathbb{I}}^{ur}$. We write $\mathcal{L}$ for $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}$. Thus $\mathrm{Fitt}(X \otimes_{\hat{\mathbb{I}}^{ur}} \hat{\mathbb{J}}^{ur}) = \mathrm{Fitt}(X) \otimes_{\hat{\mathbb{I}}^{ur}} \hat{\mathbb{J}}^{ur}$. We claim that for any $x \in \mathrm{Fitt}(X), x\mathcal{L}^{-1} \in \hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$. In fact from what we proved for $\hat{\mathbb{J}}^{ur}$ we have $\mathrm{Fitt}(V \otimes_{\hat{\mathbb{I}}^{ur}} \hat{\mathbb{J}}^{ur}) \subseteq (\mathcal{L})$ as ideals of $\hat{\mathbb{J}}^{ur}[[\Gamma_{\mathcal{K}}]]$. Note $x\mathcal{L}^{-1} \in \hat{\mathbb{J}}^{ur}[[\Gamma_{\mathcal{K}}]] \cap F_{\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]}$ where $F_{\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]}$ is the fraction field of $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$. Since $\hat{\mathbb{I}}^{ur}$ is normal and $\hat{\mathbb{J}}^{ur}$ is finite over $\hat{\mathbb{I}}^{ur}$. We have $x\mathcal{L}^{-1} \in \hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$. Thus $\mathrm{Fitt}(X) \subseteq (\mathcal{L})$, this in turn implies that $\mathrm{char}(X) \subseteq (\mathcal{L})$. This proves Theorem 1.1.

Now assume we are under the assumption of Theorem 1.2. Note that in this case $\mathcal{L}_{\chi\bar{\xi}'} = 1$. Thus by the last lemma $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}$ is coprime to $\mathcal{L}^{\Sigma}_{\chi\bar{\xi}'}$. Suppose $P_1, ..., P_t$ are the height 1 primes of $\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}$ that are pullbacks of height 1 primes in $\hat{\mathbb{I}}^{ur}$. Note that none of the primes passes through ϕ_0 since the 2 variable p-adic L-function is not identically 0. We consider the ring $\hat{\mathbb{I}}^{ur}_{p,P_1,...,P_t}[[\Gamma_{\mathcal{K}}]]$ where the subscripts denote localizations. Then the argument as in (1) proves that $(\mathcal{L}^{\Sigma}_{\mathbf{f},\mathcal{K},\xi}) \supseteq \mathrm{Fitt}_{\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]_{p,P_1,...,P_t}}(X^{\Sigma}_{p,P_1,...,P_t})$ as ideals of $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]_{p,P_1...,P_t}$. Specialize with respect to ϕ_0' we find:

$$(\mathcal{L}^{\Sigma}_{f_0,\mathcal{K},\xi}) \supseteq \mathrm{Fitt}_{\hat{\mathcal{O}}_L[[\Gamma_{\mathcal{K}}]] \otimes L}(X_{f_0}).$$

Thus

$$\mathrm{Fitt}_{\hat{\mathcal{O}}_L^{ur}[[\Gamma_{\mathcal{K}}]]\otimes\mathbb{Q}_p}(X_{\mathbf{f}}^{\Sigma}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/(\ker\phi_0')X_{\mathbf{f}}^{\Sigma}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\subseteq(\mathcal{L}^{\Sigma}_{f_0,\mathcal{K},\xi}).$$

Thus

$$\mathrm{char}_{\hat{\mathcal{O}}_L^{ur}[[\Gamma_{\mathcal{K}}]]\otimes\mathbb{Q}_p}(X_{\mathbf{f}}^{\Sigma}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/(\ker\phi_0')X_{\mathbf{f}}^{\Sigma}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\subseteq(\mathcal{L}^{\Sigma}_{f_0,\mathcal{K},\xi}).$$

By Proposition 2.4 (which is also true if we replace the ring $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]$ by $\hat{\mathbb{I}}^{ur}[[\Gamma_{\mathcal{K}}]]_{p,P_1,\ldots,P_t}$),

$$\mathrm{char}_{\hat{\mathcal{O}}_L^{ur}[[\Gamma_{\mathcal{K}}]]\otimes\mathbb{Q}_p}(X_{\mathbf{f}}^{\Sigma}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/(\ker\phi_0')X_{\mathbf{f}}^{\Sigma}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)=\mathrm{char}_{\hat{\mathcal{O}}_L^{ur}[[\Gamma_{\mathcal{K}}]]\otimes\mathbb{Q}_p}(X_{f_0}^{\Sigma}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)$$

and we conclude Theorem 1.2. □

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