the Iwasawa Theory for Unitary Groups

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Abstract

In this thesis we generalize earlier work of Skinner and Urban to construct (p-adic families of) nearly ordinary Klingen Eisensten series for the unitary groups $U(r,s) \hookrightarrow U(r+1,s+1)$ and do some preliminary computations of their Fourier Jacobi coefficients. As an application, using the case of the embedding $U(1,1) \hookrightarrow U(2,2)$ over totally real fields in which the odd prime p splits completely, we prove that for a Hilbert modular form f of parallel weight 2, trivial character, and good ordinary reduction at all places dividing p, if the central critical L-value of f is 0 then the associated Bloch Kato Selmer group has infinite order. We also state a consequence for the Tate module of elliptic curves over totally real fields that are known to be modular.

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To my wife.

Contents

Chapter 1

Introduction

1.1 Conjectures for Motives

1.1.1 Characteristic 0 conjectures

Let M be a motive over a number field F. Suppose p is an odd prime that splits completely in F. (We are mainly interested in the p-adic realization $H_p(M)$ of M, i.e. a Galois representation of F with coefficients a finite extension L of $\mathbb{Q}_p$ and which is unramified outside a finite set of primes and potentially semi-stable at all places dividing p.) Let V be $H_p(M)$. Suppose that for each $v|p$ we have defined a subspace $V_v^+ \subset V$ which is invariant under the local Galois group $G_{F,v}$. Then the Selmer group $H^1_f(\mathcal{K}, V)$ of V relative to the V_v^+'s is defined to be the kernel of the restriction map

$$H^1(F,V) \to \prod_{v\nmid p} H^1(I_v, V) \times \prod_{v|p} H^1(I_v, V/V_v^+),$$

where $I_v \subset G_{F,v}$ is the inertial group.

Greenberg gave a recipe for choosing such V_v^+'s under certain standard conditions. For each v a prime of F dividing p, suppose $H_p(M)$ is Hodge-Tate at v with $H_p(M) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \otimes_i \mathbb{C}_p(i)^{h_i}$ where the h_i are integers and $\mathbb{C}_p(i)$ is the ith Tate twist of $\mathbb{C}_p$. If $d = \dim_{\mathbb{Q}_p}(H_p(M))$ and $d^{\pm}$ is the dimension of the subspace of $H_p(M)$ on which complex conjugation acts by ± 1, then $d^+ + d^- = d$. We assume that $\sum_{i\geq 1} h_i = d^+$ and that:
Panchishkin Condition:

$H_p(M)$ contains a subspace $F^+H_p(M)$ invariant under $G_{F,v}$ with the property that

$$F^+H_p(M) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \oplus_{i\leq 1}\mathbb{C}_p(i)^{h_i}.$$

Then $V_v^+ := F^+H_p(M)$. Examples of motives for which these conditions hold include:

- all Dirichlet characters and their Tate twists;
- elliptic curves with multiplicative or good ordinary reductions at all places dividing p;
- nearly ordinary modular forms.

One can also define the L-function $L(M,s)$ for M which, conjecturally is absolutely convergent for $\mathrm{Re}(s)$ is sufficiently large and has a meromorphically continuation to the whole complex plane. A general philosophy is that the size of the Selmer group for M is controlled by the special value $L(M^*(1),0)$ (up to certain periods and normalization factors), where $*$ means dual and (1) is the Tate twist. More precisely, the characteristic 0 Bloch-Kato conjecture is:

Conjecture 1.1.1. *Suppose V is an irreducible Galois representation of F, then*

$$ord_{s=0}L(M^*(1),s) = rank_L H^1_f(F,M)$$

1.1.2 Iwasawa Main Conjectures

We can choose the coefficient to be the integer ring $\mathcal{O}_L$ instead of L and defined the corresponding 'integral version' Selmer groups as well. We can also deform everything in p-adic families. More precisely, on the arithmetic side consider the integral Selmer group Sel_M for M but over some $\mathbb{Z}_p^d$ extension F_∞ of F. This Selmer group has an action of the Iwasawa algebra $\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$ and can be viewed as interpolating Selmer groups of $H_p(M)$ twisted by characters of $\mathrm{Gal}(F_\infty/F)$. On the analytic side there is a conjectural p-adic L-function $\mathcal{L}_M \in A[[\mathrm{Gal}(F_\infty/F)]]$ which interpolates special values of L-functions for M twisted by Hecke characters. (here A is some finite extension of $\mathcal{O}_L$.) The Iwasawa main conjecture essentially states that:

Conjecture 1.1.2. Sel_M *is a torsion module over* $\mathbb{Z}_p[[\mathrm{Gal}_{F_\infty/F}]]$ *and*

$$\mathrm{Char}(\mathrm{Sel}_M) = (\mathcal{L}_M)$$

as ideals of the Iwasawa algebra. Here Char *means the characteristic ideal to be defined later (see section 2.8).*

Note that in the special case when F is totally real and M is 1-dimensional, this is the classical Iwasawa main conjecture which was proved by Mazur-Wiles [MW] and Wiles [Wiles90].

A strategy to proving such results is introduced in the papers [MW] and [Wiles], which proved the Iwasawa main conjecture for Hecke characters over totally real fields. There they studied the congruences between GL_2 Eisenstein Series, whose associated Galois representations are reducible, and cusp forms, whose Galois representations are irreducible. Recently, this has been generalized successfully by C.Skinner and E.Urban ([SU], [SU1],[SU2],[SU3]), proving many cases of the rank 1 and 2 characteristic 0 Bloch-Kato conjectures and the Iwasawa main conjectures for GL_2 modular forms as well as some other groups. The method of Skinner and Urban is to study the congruences between cusp forms and Eisenstein series on an even larger group ($GU(2,2)$) to construct the Selmer classes.

1.2 Main Results

This thesis is devoted to generalizing some of the work in [SU] to other unitary groups. More precisely, starting from a cusp form on $U(r,s)$ we hope to: (1) construct a (p-adic family) of nearly ordinary Klingen Eisenstein series on $U(r+1,s+1)$ with the constant terms divisible by the p-adic L-functions we hope to study; (2) study the p-adic properties of the Fourier-Jacobi coefficients of the Klingen Eisenstein families and deduce some congruences between this family and cuspidal families; (3) pass to the Galois side to deduce one divisibility of the Iwasawa main conjecture. The first step is done in the first part of the paper. The second step is the most difficult one and we are only able to achieve this for $U(1,1)\hookrightarrow U(2,2)$ and $U(2,0)\rightarrow U(3,1)$. In general we lack general results about non-vanishing modulo p of special L-values. The last step is essentially an argument appearing in [SU]. As a result we are able to prove one divisibility of the Iwasawa main conjecture for two kinds of Rankin-Selberg L-functions. In the thesis we have only explained the proof of the following theorem due to limited time and leaving the write up of the other results to the future:

Theorem 1.2.1. *Let F be a totally real number field. Let p be an odd rational prime that splits completely in F. Let f be a Hilbert modular form over F of parallel weight 2 and trivial character. Let ρ_f be the p-adic Galois representation associated to f such that $L(\rho_f,s)=L(f,s)$. Suppose:*
(i) f is good ordinary at all primes dividing p;
(ii) ***(irred)*** *and* ***(dist)*** *hold for ρ_f.*
If the central critical value $L(f,1)=0$, then the Selmer group $H^1_f(F,\rho_f^*)$ is infinite.

Here $\boldsymbol{(irred)}$ means the residual Galois representation $\bar{\rho}_f$ of F is irreducible and $\boldsymbol{(dist)}$ means that for $V = \rho_f$ and each prime $v|p$, the $\mathcal{O}_L^\times$-valued characters giving the actions of $G_{F,v}$ on V_v^+ and V/V_v^+ are distinct modulo the maximal ideal of $\mathcal{O}_L$.

Corollary 1.2.1. *Let E be an elliptic curve over F with the p-adic Tate module ρ_E. Suppose E has good ordinary reduction at all primes dividing p. Suppose also that the residual Galois representation $\bar{\rho}_E$ is modular and satisfies* $\boldsymbol{(dist)}$ *above are satisfied. If the central critical value $L(E,1) = 0$, then the Selmer group $H^1_f(F, \rho_E)$ is infinite.*

The corollary follows from the theorem immediately by the modularity lifting results of [SW2]. We assume that $\bar{\rho}_E$ is modular since we do not know the Serre conjecture in the totally real case.

In the special case that $F = \mathbb{Q}$ theorem 1.2.1 is essentially proved in [SU], though our result is slightly more general (in particular we do not need to assume that f is special or even square integrable at any finite place).

In the case when the root number is -1 the theorem 1.2.1 is a result of Zhang and Nekovar. We prove it when the root number is $+1$. In fact, our theorem, combined with the parity result of Nekovar, implies that when the order of vanishing is even and at least 2, then the rank of the Selmer group is also at least 2. Also note that the method of [SU2] does not seem to generalize to the totally real field case.

In order to prove theorem 1.2.1 we need to choose a CM extension $\mathcal{K}$ of F and make use of the unitary group $U(1,1)_{/F}$ which is closely related to GL_2. We embed f into a Hida family $\mathbf{f}$ and use some CM character ψ to construct a family of forms on $U(1,1)$. Then our proof consists of four steps: (1) from this family on $U(1,1)$ we construct a p-adic family of Klingen Eisenstein series on $U(2,2)$ such that the constant term is the divisible by the p-adic L-function of $\mathbf{f}$ over $\mathcal{K}$; (2) prove (the Fourier expansion of) the Klingen Eisenstein family is co-prime to the p-adic L-function by a computation using the doubling methods; (3) use the results about the constant terms in step 1 to construct a cuspidal family which is congruent to the Klingen Eisenstein family modulo the p-adic L-function; (4) pass to the Galois side, using the congruence between the Galois representations for the Klingen Eisenstein family and the cuspidal family to prove the theorem.

We first prove the above theorem assuming that d is even and use a base change trick to remove that condition. A large part of the arguments are straightforward generalizations of [SU]. However we do all the computations in the adelic language instead of the mixture of classical and adelic language of [SU]. This simplifies the computations somewhat since we no longer need to compare the classical and adelic pictures. The required non-vanishing modulo p results of some special L-values are known thanks to the recent work of of Ming-lun Hsieh [Hsi11] and Jeanine Van-Order [VAN]. Also we use Hida's work on the anticyclotomic main conjecture to compare the CM periods and canonical periods. To construct the cuspidal family in step (3) we explicitly write it down instead of using the geometric argument in [SU] Chapter 6. This is a much easier way since we only need to do Hida theory for cuspidal forms (which is already available) if we are only interested in proving the characteristic 0 result. In the future we will generalize the geometric argument in [SU] 6.3 to prove the one divisibility of the Iwasawa-Greenberg main conjecture. (In the case when $F \neq \mathbb{Q}$ we need to restrict to a certain subfamily of the whole weight space to have freeness of the nearly ordinary forms over the (sub) weight space and surjectivity to the boundary).

1.3 Summary of the Thesis

This thesis consists of two parts: part one is the first 5 chapters, which are computations for general unitary groups, and part two consists of chapters 6-14, which specializes to $U(1,1) \hookrightarrow U(2,2)$ and proves the main theorem.

Part one is devoted to constructing the nearly ordinary Klingen Eisenstein series for unitary groups. The motivation for computations in this generality is for possible future generalization of part two to general unitary groups, by studying the congruences between such Eisenstein Series and cusp forms. In chapter 2 we recall various backgrounds and formulate our main conjectures for unitary groups and Hilbert modular forms. In chapter 3 we recall the notion of Klingen and Siegel Eisenstein series, the pull-back formulas relating them and their Fourier-Jacobi coefficients. In chapter 4 and 5 we construct the nearly ordinary Klingen Eisenstein series by the pullbacks of a Siegel Eisenstein series from a larger group. We manage to take the Siegel sections so that when we are moving our Eisentein datum p-adically, these Siegel Eisenstein series also move p-adic analytically. The hard part is to choose the sections at p-adic places. For the ℓ-adic cases we just pick one section and might change this choice whenever doing arithmetic applications. At the Archimedean places we restrict ourselves to the parallel scalar weight case which is enough for doing Hida theory. We plan to generalize this to more general weights in the future, which might enable us to do some finite slope arithmetic applications. Also in the first part of this thesis we content ourselves with only computing a single form (instead of a family) and leave the p-adic interpolation for future work. We also do some preliminary computations for Fourier-Jacobi coefficients for the Siegel Eisenstein series on the big unitary group. The Fourier Jacobi coefficients for Klingen Eisenstein series are realized as the Petersson inner-product of that for Siegel Eisenstein series with the cusp form we start with.

The main use for this computation is to prove that the Klingen Eisenstein series is co-prime to the p-adic L-function and thus giving the congruence relations needed for arithmetic applications.

In part two we apply our computations in part one to the case of $U(1,1) \hookrightarrow U(2,2)$ over totally real fields and deduce our main theorem. For convenience we keep the argument parallel to the [SU] paper. In chapter 6 we recall the notion of Hilbert modular forms and record some results on the Iwasawa theory for their Selmer groups. In chapter 7 we recall some results about p-adic automorphic forms and Hida theory for the group $U(2,2)$. We prove our main theorem in chapter 8 (corresponding to step (4)) assuming some constructions and results in later chapters. Chapters 9-13 (corresponding to step (1) and (2)) are parallel to chapters 9-13 of [SU] and we do the local calculations and deduce the required p-adic properties needed in chapter 8. Chapter 14 is to construct a cuspidal family from the nearly ordinary Klingen Eisenstein family (step (3)). This is also needed in chapter 8.

We remark that the materials in part one (chapters 2-5) for general unitary groups, especially the p-adic computations are new. Part two (chapters 6-14) differs from the paper [SU] only in certain technicalities (the adelic computation, a slightly different choice of the Fourier-Jacobi coefficient, the construction in chapter 14 and the use of different results on non-vanishing modulo p of special L-values and comparing periods).

Chapter 2

Background

In this section we recall notations for holomorphic automorphic forms on unitary groups, Eisenstein series and Fourier Jacobi expansions.

2.1 Notations

Suppose F is a totally real field such that $[F:\mathbb{Q}]=d$ and $\mathcal{K}$ is a totally imaginary quadratic extension of F. For a finite place v of F or $\mathcal{K}$ we usually write ϖ_v for a uniformizer and q_v for $|\varpi_v|$. Let c be the non trivial element of $\mathrm{Gal}(\mathcal{K}/F)$. Let r,s be two integers with $r\geq s\geq 0$. We fix an odd prime p that splits completely in $\mathcal{K}/\mathbb{Q}$. We fix $i_\infty:\bar{\mathbb{Q}}\hookrightarrow\mathbb{C}$ and $\iota:\mathbb{C}\simeq\mathbb{C}_p$ and write i_p for $\iota\circ i_\infty$. Denote Σ_∞ to be the set of Archimedean places of F. We take a CM type $\Sigma\subset\mathrm{Hom}_{\mathbb{C}-alg}(\mathcal{K},\mathbb{C})$ of $\mathcal{K}$ (thus $\Sigma\sqcup\Sigma^c$ are all embeddings $\mathcal{K}\to\mathbb{C}$ where $\Sigma^c=\{\tau\circ c,\tau\in\Sigma\}$). There is a associated CM period $\Omega_\infty=(\Omega_{\infty,\sigma})_{\sigma\in\Sigma}\in\mathbb{C}^\Sigma$ (we refer to [Hida07] for the definition). Define: $\Omega_\infty^\Sigma=\prod_{\sigma\in\Sigma}\Omega_{\infty,\sigma}$. We often write S_m to denote the m by m Hermitian matrices either over F or some localization F_v.

We use ϵ to denote the cyclotomic character and ω the Techimuller character. We will often adopt the following notation: for an idele class character $\chi=\otimes_v\chi_v$ we write $\chi_p(x)=\prod_{v|p}\chi_v(x_v)$. For a character ψ or τ of $\mathcal{K}_v$ or $\mathbb{A}_\mathcal{K}^\times$ we often write ψ' for the restriction to $F_v^\times$ or $\mathbb{A}_F^\times$. For a local or adelic character τ we define τ^c by $\tau^c(x)=\tau(x^c)$ where c standards for the non-trivial element in $\mathrm{Gal}(\mathcal{K}/F)$.

(Gauss sums) If v is a prime of F over ℓ and $\mathfrak{d}_v\mathcal{O}_{F,v}=(d_v)$ is the different of $F/\mathbb{Q}$ at v and if ψ_v is a character of $F_v^\times$ and $(c_{\psi,v}) \subset \mathcal{O}_{F,v}$ is the conductor then we define the local Gauss sums:

$$\mathfrak{g}(\psi_v, c_{\psi,v}d_v) := \sum_{a \in (\mathcal{O}_{F,v}/c_{\psi,v})^\times} \psi_v(a) e(Tr_{F_v/\mathbb{Q}_\ell}(\frac{a}{c_{\psi,v}d_v})).$$

If $\otimes\psi_v$ is an idele class character of $\mathbb{A}_F^\times$ then we set the global Gauss sum:

$$\mathfrak{g}(\otimes\psi_v) := \prod_v \psi_v^{-1}(c_{\psi,v}d_v)\mathfrak{g}(\psi, c_{\psi,v}d_v).$$

This is independent of all the choices. Also if $F_v \simeq \mathbb{Q}_p$ and (p^t) is the conductor for ψ_v, then we write $\mathfrak{g}(\psi_v) := \mathfrak{g}(\psi_v, p^t)$. We define the Gauss sums for $\mathcal{K}$ similarly.

Let $\mathcal{K}_\infty$ be the maximal abelian $\mathbb{Z}_p$ extension of $\mathcal{K}$. Write $\Gamma_\mathcal{K} := \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}$. We define: $\Lambda_\mathcal{K} := \mathbb{Z}_p[[\Gamma_\mathcal{K}]]$. For any A a finite extension of $\mathbb{Z}_p$ define $\Lambda_{\mathcal{K},A} := A[[\Gamma_\mathcal{K}]]$. Let $\boldsymbol{\varepsilon}_\mathcal{K} : G_\mathcal{K} \to \Gamma_\mathcal{K} \hookrightarrow \Lambda_\mathcal{K}^\times$ be the canonical character. We define $\Psi_\mathcal{K}$ to be the composition of $\boldsymbol{\varepsilon}_\mathcal{K}$ by the reciprocity map. We make the corresponding definitions for F as well.

2.2 Unitary Groups

We define:

$$\theta_{r,s} = \begin{pmatrix} & & 1_s \\ & \theta & \\ -1_s & & \end{pmatrix}$$

where $\theta = \zeta 1_{r-s}$ with some totally imaginary element $\zeta \in \mathcal{K}$. Let $V = V(r,s)$ be the hermitian space over $\mathcal{K}$ with respect to this metric, i.e. $\mathcal{K}^{r+s}$ equipped with the metric given by $< u, v >:= u\theta_{r,s}{}^t\bar{v}$. We define algebraic groups $GU(r,s)$ and $U(r,s)$as follows: for any F-algebra R, the R points are:

$$G(R) = GU(r,s)(R) := \{g \in GL_{r+s}(\mathcal{K} \otimes_F R) | g\theta_{r,s}g^* = \mu(g)\theta_{r,s}, \mu(g) \in R^\times\}.$$

($\mu : GU(r,s) \to \mathbb{G}_m$ is called the similitude character.) and

$$U(r,s)(R) := \{g \in GU(r,s)(R) | \mu(g) = 1\}.$$

Some times we write GU_n and U_n for $GU(n,n)$ and $U(n,n)$.

We have the following embedding:

$$GU(r,s)\times \mathrm{Res}_{\mathcal{O}_\mathcal{K}/\mathcal{O}_F}\mathbf{G}_m \to G(r+1,s+1)$$

$$g\times x=\begin{pmatrix} a & b & c\\ d & e & f\\ h & l & k\end{pmatrix}\times x\mapsto \begin{pmatrix} a & & b & c & \\ & \mu(g)\bar{x}^{-1} & & & \\ d & & e & f & \\ h & & l & k & \\ & & & & x\end{pmatrix}$$

We write $m(g,x)$ for the right hand side. The image of the above map is the Levi subgroup of the Klingen parabolic subgroup P of $GU(r,s)$, which we denote by M_P. We also write N_P for the unipotent radical of P.

We write $-V$ for the hermitian space whose metric is $-\theta(r,s)$. We define some embeddings of $GU(r+1,s+1)\times GU(-V(r,s))$ into some larger groups. This will be used in the doubling method. First we define $G(r+s+1,r+s+1)'$ to be the unitary similitude group associated to:

$$\begin{pmatrix} & & & & 1_b & & & \\ & & & & & 1 & & \\ & & \theta & & & & & \\ & & & & & & & -1_b\\ -1_b & & & & & & & \\ & -1 & & & & & & \\ & & & & & & -\theta & \\ & & & 1_b & & & & \end{pmatrix}$$

We define an embedding $\alpha:\{g_1\times g_2\in GU(r+1,s+1)\times GU(-V(r,s)),\mu(g_1)=\mu(g_2)\}\to GU(r+s+1,r+s+1)'$ as follows: we consider g_1 as a block matrix with respect to $s+1+(r-s)+s+1$ and g_2 as a block matrix with respect to $s+(r-s)+s$, then we define α by requiring the $1,2,3,4,5$th (block wise) rows and columns of $GU(r+1,s+1)$ embeds to the $1,2,3,5,6$th (block wise) rows and columns of $GU(r+s+1,r+s+1)'$ and the $1,2,3$th (block wise) rows and columns of $GU(-V(r,s)$ embeds to the $8,7,4$th rows and columns (block-wise) of $GU(r+s+1,r+s+1)'$. We also define an isomorphism:

$$\beta: GU(r+s+1,r+s+1)'\xrightarrow{\sim} GU(r+s+1,r+s+1)$$

by:

$$g \mapsto S^{-1} g S$$

where

$$S = \begin{pmatrix} 1 & & & & & & & -\frac{1}{2} \\ & 1 & & & & & & \\ & & 1 & & & & -\frac{\zeta}{2} & \\ & & & -1 & \frac{1}{2} & & & \\ & & & 1 & \frac{1}{2} & & & \\ & & & & & 1 & & \\ & & -1 & & & & -\frac{\zeta}{2} & \\ -1 & & & & & & & -\frac{1}{2} \end{pmatrix}$$

Remark 2.2.1. *(About Unitary Groups) In order to have Shimura varieties for doing p-adic modular forms and Galois representations, we need to use a unitary group defined over $\mathbb{Q}$. More precisely consider V as a Hermitian space over $\mathbb{Q}$ and still denote $\theta_{r,s}$ to be the metric on it then the correct unitary similitude group should be:*

$$GU(A) := \{g \in GL(V \otimes_{\mathbb{Q}} A) | g \text{ is } \mathcal{K} - \text{linear}, g\theta_{r,s}g^* = \mu(g)\theta_{r,s}, \mu(g) \in A\}$$

This group is smaller than the one we defined before. However this group is not convenient with computations. So what we will do (implicitly) is to construct forms on the larger unitary similitude group defined before and then restrict to the smaller one.

2.3 Hermitian Symmetric Domain

Suppose $r \geq s > 0$ then we put the Hermitian symmetric domain for $GU(r,s)$:

$$X_{r,s} = \{\tau = \begin{pmatrix} x \\ y \end{pmatrix} | x \in M_s(\mathbb{C}^\Sigma), y \in M_{(r-s)\times s}(\mathbb{C}^\Sigma), i(x^* - x) > -iy^*\theta^{-1}y\}$$

For $\alpha \in G(F_\infty))$ we write:

$$\alpha = \begin{pmatrix} a & b & c \\ d & e & f \\ h & l & d \end{pmatrix}$$

according to the standard basis of V together with the block decomposition with respect to $s+(r-s)+s$. There is an action of $\alpha \in G(F_\infty)^+$ (here the superscript $+$ means the component with positive determinant at all Archimedean places) on $X_{r,s}$ is defined by:

$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by+c \\ gx+ey+f \end{pmatrix} (hx+ly+d)^{-1}$$

If $rs=0$, $X_{r,s}$ consists of a single point written $\boldsymbol{x}_0$ with the trivial action of G. For an open compact subgroup U of $G(_{F,f})$ put

$$M_G(X^+, U) := G(F)^+ \backslash X^+ \times G(\mathbb{A}_{F,f})/U$$

where U is an open compact subgroup of $G(\mathbb{A}_{F,f})$.

2.3.1 Automorphic forms

We will mainly follow [Hsieh CM] to define the space of automorphic forms with slight modifications. We define the cocycle: $J : R_{F/\mathbb{Q}}G(\mathbb{R})^+ \times X^+ \to GL_r(\mathbb{C}^\Sigma) \times GL_s(\mathbb{C}^\Sigma) := H(\mathbb{C})$ by: $J(\alpha,\tau) = (\kappa(\alpha,\tau), \mu(\alpha,\tau))$ where for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$ and

$$\alpha = \begin{pmatrix} a & b & c \\ d & e & f \\ h & l & d \end{pmatrix}$$

,

$$\kappa(\alpha,\tau) = \begin{pmatrix} \bar{h}^t x + \bar{d} & \bar{h}^t y + l\bar{\theta} \\ -\bar{\theta}^{-1}(\bar{g}^t x + \bar{f}) & -\bar{\theta}^{-1}\bar{g}^t y + \bar{\theta}^{-1}\bar{e}\bar{\theta} \end{pmatrix}, \mu(\alpha,\tau) = hx+ly+d.$$

Fix a point $\boldsymbol{i} \in X^+$ and let K_∞^0 be the stablizer of $\boldsymbol{i}$ in $R_{F/\mathbb{Q}}G(\mathbb{R})$. Then $J : K_\infty^0 \to H(\mathbb{C}), k_\infty \mapsto J(k_\infty, \boldsymbol{i})$ defines an algebraic representation of K_∞^0.

Definition 2.3.1. *A weight $\underline{k}$ is defined by a set $\{\underline{k}_\sigma\}_{\sigma\in\Sigma_\infty}$ where each $\underline{k}_\sigma = (c_{r+s,\sigma}, ..., c_{s+1,\sigma}; c_{1,\sigma}, ..., c_{s,\sigma})$ with $c_{1,\sigma} > ... > c_{r+s,\sigma}$.*

Remark 2.3.1. *Our convention is different from the literature. For example in [Hsieh CM] the a_{r+1-i} there for $\leq i \leq r$ is our c_{s+i} and b_{s+1-j} there for $1 \leq j \leq s$ is our c_j. Also our c_i is the $-c_{r+s+1-i}$ in [SU2].*

We refer to [Hsieh] for the definition of the definition of the algebraic representation $L_{\underline{k}}(\mathbb{C})$ (note the different index for weight) and define a model $L^{\underline{k}}(\mathbb{C})$ of the representation $H(\mathbb{C})$ with the highest weight $\underline{k}$ as follows. The underlying space of $L^{\underline{k}}(\mathbb{C})$ is $L_{\underline{k}}(\mathbb{C})$ and the group action is defined by

$$\rho^{\underline{k}}(h) = \rho_{\underline{k}}({}^t h^{-1}), h \in H(\mathbb{C}).$$

For a weight $\underline{k}$, define $\|\underline{k}\|$ by:

$$\|\underline{k}\| := -c_{s+1} - ... - c_{s+r} + c_1 + ... + c_s \in \mathbb{Z}[\Sigma]$$

and $|\underline{k}| \in \mathbb{Z}^{\Sigma \sqcup \Sigma^c}$ by:

$$|\underline{k}| = \sum_{\sigma \in \Sigma} (c_{1,\sigma} + ... + c_{s,\sigma}).\sigma - (c_{s+1,\sigma} + ... + c_{s+r,\sigma}).\sigma c.$$

Let χ be a Hecke character of $\mathcal{K}$ with infinite type $|\underline{k}|$, i.e. the Archimedean part of χ is given by:

$$\chi(z_\infty) = (\prod_{\sigma} z_{\sigma}^{(c_{1,\sigma}+...+c_{s,\sigma})}.z_{\sigma^c}^{-(c_{s+1,\sigma}+...+c_{s+r,\sigma})}).$$

Definition 2.3.2. *Let U be an open compact subgroup in $G(\mathbb{A}_{F,f})$. We denote by $M_{\underline{k}}(U, \mathbb{C})$ the space of holomorphic $L_{\underline{k}}(\mathbb{C})$-valued functions f on $X^+ \times G(\mathbb{A}_{F,f})$ such that for $\tau \in X^+$, $\alpha \in G(F)^+$ and $u \in U$ we have:*

$$f(\alpha\tau, \alpha g u) = \mu(\alpha)^{-\|\underline{k}\|}\rho^{\underline{k}}(J(\alpha, \tau))f(\tau, g).$$

Now we consider automorphic forms on unitary groups in the adelic language. Let $\mathcal{A}_{\underline{k}}(G, U, \chi)$ be the space of automorphic forms of weight $\underline{k}$ and level U with central character χ, i.e. smooth and slowly increasing functions $F : G(\mathbb{A}_F) \to L_{\underline{k}}(\mathbb{C})$ such that for every $(\alpha, k_\infty, u, z) \in G(F) \times K_\infty^0 \times U \times Z(\mathbb{A}_F)$,

$$F(z\alpha g k_\infty u) = \rho^{\underline{k}}(J(k_\infty, \boldsymbol{i})^{-1})F(g)\chi^{-1}(z).$$

We can associate a $L_{\underline{k}}(\mathbb{C})$-valued function $\underline{AM}(F)$ on $X^+ \times G(\mathbb{A}_{F,f})$ to $F \in \mathcal{A}_{\underline{k}}(G, U, \chi)$ by

$$\underline{AM}(F)(\tau, g) := \chi_f(\nu(g))\rho^{\underline{k}}(J(g_\infty, \boldsymbol{i}))F((g_\infty, g)),$$

where $g_\infty \in R_{F/\mathbb{Q}}G(\mathbb{R})^+$ such that $g_\infty \boldsymbol{i} = \tau$. We put:

$$\mathcal{A}^{Hol}_{\underline{k}}(G,U,\chi) = \{F \in \mathcal{A}_{\underline{k}}(G,U,\chi) | \underline{AM}(F) \text{ is holomorphic on } X^+ \}.$$

2.4 Galois representations Associated to Cuspidal Representations

In this section we follow [Sk10] to state the result of associating Galois representations to cuspidal automorphic representations on $GU(r,s)(\mathbb{A}_F)$. First of all let us fix the notations. Let $\bar{\mathcal{K}}$ be the algebraic closure of $\mathcal{K}$ and let $G_{\mathcal{K}} := \mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$. For each finite place v of $\mathcal{K}$ let $\bar{\mathcal{K}}_v$ be an algebraic closure of $\mathcal{K}_v$ and fix an embedding $\bar{\mathcal{K}} \hookrightarrow \bar{\mathcal{K}}_v$. The latter identifies $G_{\mathcal{K}_v} := \mathrm{Gal}(\bar{\mathcal{K}}_v/\mathcal{K}_v)$ with a decomposition group for v in $G_{\mathcal{K}}$ and hence the Weil group $W_{\mathcal{K}_v} \subset G_{\mathcal{K}_v}$ with a subgroup of $G_{\mathcal{K}}$. Let π be a holomorphic cuspidal irreducible representation of $U(r,s)(\mathbb{A}_F)$ with weight $\underline{k} = (c_{r+s,\sigma}, ..., c_{s+1,\sigma}; c_{1,\sigma}, ..., c_{s,\sigma})_{\sigma\in\Sigma}$ and central character χ_π; Then for some L finite over $\mathbb{Q}_p$, there is a Galois representation (by [Shin], [Morel] and [Sk10]):

$$R_p(\pi) : G_{\mathcal{K}} \to GL_n(L)$$

such that:

(a)$R_p(\pi)^c \simeq R_p(\pi)^\vee \otimes \rho_{p,\chi_\pi^{1+c}}\epsilon^{1-n}$ where χ_π is the central character of π, $\rho_{p,\chi_\pi^{1+c}}$ denotes the associated Galois character by class field theory and ϵ is the cyclotomic character.

(b)$R_p(\pi)$ is unramified at all finite places not above primes in $\Sigma(\pi) \cup \{$ primes dividing p), and for such a place w:

$$\det(1 - R_p(\pi)(\mathrm{frob}_w q_w^{-s}) = L(BC(\pi)_w \otimes \chi^c_{\pi,w}, s + \frac{1-n}{2})^{-1}$$

Here the frob_w is the geometric Frobenius. We write V for the representation space and it is possible to take a Galois stable $\mathcal{O}_L$ lattice which we denote as T. Suppose π_v is nearly ordinary at all primes v dividing p with respect to $\underline{k}$ (to be defined later). Suppose $v|p$ correspond to $\sigma \in \Sigma$ under $\iota : \mathbb{C} \simeq \mathbb{C}_p$, then if we write $\kappa_{i,\sigma} = s - i + c_{i,\sigma}$ for $\leq i \leq s$ and $\kappa_{i\sigma} = c_{i,\sigma} + s + r + s - i$ for $s+1 \leq i \leq r+s$, then:

$$R_p(\pi)|G_{\mathcal{K},v} \simeq \begin{pmatrix} \xi_{r+s,v}\epsilon^{-\kappa_{r+s,\sigma}} & * & * & * \\ & \xi_{r+s-1,v}\epsilon^{\kappa_{r+s-1,\sigma}} & & * \\ & & \cdots & * \\ & 0 & & \\ & & & \xi_{1,v}\epsilon^{-\kappa_{1,\sigma}} \end{pmatrix}$$

where $\xi_{i,v}$ are unramified characters. Using the fact (a) above we know that $R_p(\pi)_{\bar{v}}$ is equivalent to an upper triangular representation as well.

2.5 Selmer groups

We recall the notion of Σ-primitive Selmer groups, following [SU]3.1 with some modifications. In this section F is a subfield of $\bar{\mathbb{Q}}$. For T a free module over a profinite $\mathbb{Z}_p$-algebra A and assume that T is equipped with a continuous action of the absolute Galois group G_F of F. Assume that for each place $v|p$ of F we are given a G_v-stable free A-direct summand $T_v \subset T$. For any finite set of primes Σ we denote by $\mathrm{Sel}_F^{\Sigma}(T, (T_v)_{v|p})$ the kernel of the restriction map:

$$H^1(F, T \otimes_A A^*) \to \prod_{v \notin \Sigma, v \nmid p} H^1(I_v, T \otimes_A A^*) \times \prod_{v|p} H^1(I_v, T/T_v \otimes_A A^*),$$

We also define:

$$\mathbf{X}_F^{\Sigma}(T, (T_v)_{v|p}) := Hom_A(Sel_F^{\Sigma}(T, (T_v)_{v|p}), A^*).$$

Now let us take T to be the Galois representation stated above. Then for each $v \in \Sigma_p$ suppose $R_p(\pi)_v$ is of the above form with respect to the basis $v_{r+s,v}, ..., v_{1,v}$ then we define T_v to be the $\mathcal{O}_L$ span if $v_{r+s}, ..., v_{s+1,v}$. Also, if $R_p(\pi)_{\bar{v}}$ is upper triangular with respect to the basis $v_{1,\bar{v}}, ..., v_{r+s,\bar{v}}$ then we define $T_{\bar{v}}$ to be the $\mathcal{O}_L$ span of $v_{1,\bar{v}}, ...v_{s,\bar{v}}$.

Remark 2.5.1. *The Selmer group defined here is not quite correct. In fact T does not always satisfy Greenberg's Panchishkin condition. But it is correct in the "Iwasawa theoretic sense". We will explain this in a moment (remark 2.6.1).*

2.6 Iwasawa Theory

Let $\mathcal{K}_\infty$ be the maximal abelian $\mathbb{Z}_p$ extension of $\mathcal{K}$. Write $\Gamma_{\mathcal{K}} := \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}$. We define: $\Lambda_{\mathcal{K}} := \mathbb{Z}_p[[\Gamma_{\mathcal{K}}]]$. For any A a finite extension of $\mathbb{Z}_p$ define $\Lambda_{\mathcal{K},A} := A[[\Gamma_{\mathcal{K}}]]$. Let $\varepsilon_{\mathcal{K}} : G_{\mathcal{K}} \to \Gamma_{\mathcal{K}} \hookrightarrow \Lambda_{\mathcal{K}}^{\times}$ be the canonical character. Then by Shapiro's lemma we have:

$$\mathrm{Sel}^{\Sigma}_{\mathcal{K}_\infty}(T) \equiv \mathrm{Sel}^{\Sigma}_{\mathcal{K}}(T \otimes_A \Lambda_{\mathcal{K},A}(\boldsymbol{\varepsilon}^{-1}_{\mathcal{K}}))$$

So we have a $\Lambda_{\mathcal{K},A}$ module structure for $\mathbf{X}^{\Sigma}_{\mathcal{K}_\infty}(T)$. One can define the Selmer groups for intermediate fields between $\mathcal{K}$ and $\mathcal{K}_\infty$ as well.

Remark 2.6.1. *Later on we will see some control theorems for Selmer groups relating the big Selmer groups for $\mathcal{K}_\infty$ to those of its subfields. However $\mathrm{Sel}^{\Sigma}_{\mathcal{K}}(T)$ itself is not a Selmer group since T does not satisfy Greenberg's Panchinshkin conditions. But by twisting T by some Galois character we can make T satisfy this condition. Also the T_v's we put at $v|p$ are indeed Selmer conditions for such twists in the sense of Greenberg. Therefore our Iwasawa module is indeed interpolating Selmer groups for T twisted by some characters.*

2.7 p-adic L-functions

In a recent work of [EEHLS] they constructed the p-adic L-function $\mathcal{L}^{\Sigma}_{\pi,\mathcal{K},\psi} \in A[[\Gamma_{\mathcal{K}}]]$ (where ψ is some fixed Hecke character for $\mathcal{K}$) interpolating the special values of $L^{\Sigma}(\pi, \psi \otimes \chi_\phi, s)$ up to some periods and normalization factors. Here $\phi \in \mathrm{Spec}\Lambda_{\mathcal{K},A}$ and χ_ϕ corresponds to $\phi \circ \boldsymbol{\varepsilon}_{\mathcal{K}}$ under the reciprocity map.

2.8 Characteristic Ideals and Fitting Ideals

In this subsection we let A be a noetherian ring. We write $\mathrm{Fitt}_A(X)$ for the Fitting ideal in A of a finitely generated A-module X. This is the ideal generated by the determinant of the $r \times r$ monors of the matrix giving the first arrow in a given presentation of X:

$$A^s \to A^r \to X \to 0$$

If X is not a torsion A-module then $\mathrm{Fitt}(X) = 0$.
Fitting ideals behave well with respect to base change. For $I \subset A$ an ideal, then:

$$\mathrm{Fitt}_{A/I}(X/IX) = \mathrm{Fitt}_A(X)\mathrm{mod} I$$

Now suppose A is a Krull domain (a domain which is Noetherian and normal), then the characteristic ideal is defined by:

$$\mathrm{Char}_A(X) := \{x \in A : \mathrm{ord}_Q(x) \geq \ell_Q(X) \text{ for any } Q \text{ a height one prime }\},$$

here $\ell_Q(X)$ is the length of X at Q.

2.9 Main Conjectures

Now we are in a position for formulate the Iwasawa-Greenberg main conjecture, we write $\mathrm{Char}^{\Sigma}_{\pi,\mathcal{K},\psi}$ for the characteristic ideal for $\mathbf{X}^{\Sigma}_{\pi,\mathcal{K},\psi}$, then:

Conjecture 2.9.1. $\mathrm{Char}^{\Sigma}_{\pi,\mathcal{K},\psi}$ *is principal and generated by* $\mathcal{L}^{\Sigma}_{\pi,\mathcal{K},\psi}$.

While trying to prove the main conjecture above we need to embed some nearly ordinary $f \in \pi$ into a Hida family $\mathbf{f}$ of nearly ordinary forms with some coefficient ring $\mathbb{I}$ (taken to be a normal domain). We have a Galois representation $R_p(\mathbf{f})$ on some $\mathbf{T}$ a free module over $\mathbb{I}$ of finite rank. It satisfies local conditions at $v|p$ similar to that for f and we define the corresponding Selmer conditions and thus $\mathrm{Sel}^{\Sigma}_{\mathbf{f},\mathcal{K},\psi}$ and $X^{\Sigma}_{\mathbf{f},\mathcal{K},\psi}$ which is a module over $\mathbb{I}[\Gamma_{\mathcal{K}}]]$. Then we have the main conjecture for Hida families as well:

Conjecture 2.9.2.

$$\mathrm{Char}^{\Sigma}_{f,\mathcal{K},\psi} = (\mathcal{L}^{\Sigma}_{f,\mathcal{K},\psi})$$

as ideals of $\mathbb{I}[[\Gamma_{\mathcal{K}}]]$.

2.10 Hilbert modular forms

As mentioned in the introduction we can use unitary groups to study the Iwasawa theory for Hilbert modular forms. Let f ($\mathbf{f}$) be a nearly ordinary Hilbert modular form (or Hida family). Then the associated galois representations satisfy similar local conditions at $v|p$, namely isomorphic to upper triangular representations and one can define Selmer groups $Sel^{\Sigma}_{f,\mathcal{K},\chi}$, $\mathbf{X}^{\Sigma}_{f,\mathcal{K},\chi}$. (see part two for details). Also the p-adic L-functions $\mathcal{L}^{\Sigma}_{f,\mathcal{K},\chi}$, $\tilde{\mathcal{L}}^{\Sigma}_{f,\mathcal{K},\chi}$ (see chapter 12) are essentially those for $U(1,1)$ with some modifications for interpolation formulas (since now we are using GL_2 L-functions of f instead of that for base change of unitary group automorphic forms). We can formulate the following main conjecture as well.

Conjecture 2.10.1. *As ideals of* $\mathbb{I}[[\Gamma_{\mathcal{K}}]]$,

$$(\mathcal{L}^{\Sigma}_{f,\mathcal{K},\chi}) = \mathrm{Char}^{\Sigma}_{f,\mathcal{K},\chi}.$$

We can construct the non-integral p-adic L-function $\tilde{\mathcal{L}}^{\Sigma}_{\mathbf{f},\mathcal{K},\chi}$ in great generality. This is enough for proving the characteristic 0 results (theorem 1.2.1) . However we use certain Gorenstein properties of some Hecke algebras to construct the integral p-adic L-function tha appears in the conjecture above. Let us briefly discuss this issue. Let $\mathbf{f}$ be a Hida family of nearly ordinary Hilbert modular eigenforms with tame level M. Let $\mathbb{I}$ be some finite extension of Λ_W, let $\mathfrak{m}_{\mathbf{f}}$ be the maximal ideal of the Hecke algebra $\mathbb{T}(M, \mathbb{I})$ with $\mathbb{I}$ coefficients corresponds to $\mathbf{f}$. Let $\mathbb{T}(M, A)_{\mathfrak{m}_{\mathbf{f}}}$ be the localization. Then we say that it is Gorenstein if $\mathrm{Hom}_{\mathbb{I}}(\mathbb{T}_{\mathfrak{m}_{\mathbf{f}}}, \mathbb{I})$ is free of rank 1 over $\mathbb{T}_{\mathfrak{m}_{\mathbf{f}}}$. This is used to guarantee the existence of a generator of the congruence module. In the case when $F = \mathbb{Q}$ Wiles [Wiles95] proved that this is true whenever the (irred) and (dist) in [SU] (see theorem 1.2.1.) are satisfied. In general the situation is complicated. We record here a theorem of Fujiwara which gives sufficient conditions for $\mathbb{T}_{\mathfrak{m}_{\mathbf{f}}}$ to be Gorenstein:

Theorem 2.10.1. *(Fujiwara) Let $\bar{\rho}$ be the modulo p Galois representation associated to* $\mathbf{f}$. *Suppose*

- *$p \geq 3$ and $\bar{\rho}|F(\zeta_p)$ is absolutely irreducible. When $p = 5$ the following case is excluded: the projective image $\bar{G}$ of $\bar{\rho}$ is isomorphic to $PGL_2(\mathbb{F}_p)$ and the* mod*p cyclotomic character $\bar{\chi}_{cycle}$ factors through $G_F \to \bar{G}^{ab} \simeq \mathbb{Z}/2$;*
- *There is a minimal modular lifting of $\bar{\rho}$.*
- *The case 0_E defined in [8] section 3.1 does not occur for any finite place v.*
- *In the case when $d := [F : \mathbb{Q}]$ is odd the Ihara's lemma is true for Shimura curves.*

Then the ring $\mathbb{T}_{\mathfrak{m}_f}$ is Gorenstein.

This is [Fuji] theorem 11.2. The third condition is put to ensure that the quaternion algebra considered by Fujiwara is not ramified at any finite places so that the Hecke algebra is the same as the GL_2 Hecke algebra. Recall that 0_E in called "exceptional" by Fujiwara and means that $\bar{\rho}_{I_{F,v}}$ is absolutely irreducible and $q_v \equiv -1 \mod p$.

Chapter 3

Eisenstein Series and Fourier-Jacobi Coefficients

The materials of this chapter are straightforward generalizations of parts of [SU] chapter 9 and 11 and I use the same notations as *loc.cit*; So everything in this chapter should eventually be the same as [SU] when specializing to the group $GU(2,2)_{/\mathbb{Q}}$.

3.1 Klingen Eisenstein Series

Recall that in chapter we denote GU to be $GU(r,s)$ defined there. Let $\mathfrak{gu}$ be the Lie algebra of $GU(r,s)(\mathbb{R})$.

3.1.1 Archimedean Picture

Let v be an infinite place of F so that $F_v \simeq \mathbb{R}$. Let $\boldsymbol{i}'$ and $\boldsymbol{i}$ be the points on the Hermitian symmetric domain for $GU(r,s)$ and $GU(r+1,s+1)$ which are $\begin{pmatrix} i1_s \\ 0 \end{pmatrix}$ and $\begin{pmatrix} i1_{s+1} \\ 0 \end{pmatrix}$ respectively (here 0 means the $(r-s)\times s$ or $(r-s)\times(s+1)$ matrix 0). Let $GU(r,s)(\mathbb{R})^+$ be the subgroup of $GU(r,s)(\mathbb{R})$ whose similitude factor is positive. Let K_∞^+ and $K_\infty^{+,'}$ be the compact subgroups of $U(r+1,s+1)(\mathbb{R})$ and $U(r,s)(\mathbb{R})$ stabilizing $\boldsymbol{i}$ or $\boldsymbol{i}'$ and let K_∞ (K'_∞) be the groups generated by K_∞^+ ($K_\infty^{+,'}$) and $diag(1_{r+s+1}, -1_{s+1})$ ($diag(1_{r+s}, -1_s)$). Let (π, V) be an irreducible $(\mathfrak{gu}(\mathbb{R}), K'_\infty)$-module and suppose that π is unitary ,tempered representation. There is an irreducible, unitary Hilbert representation (π, H) of $GU(\mathbb{R})$, unique up to isomorphism such that (π, V) can be identified with the $(\mathfrak{gu}(\mathbb{R}), K'_\infty)$-module of it. Let χ be the central character of π. Let ψ and τ be unitary characters of $\mathbb{C}^\times$ such that $\psi|_{\mathbf{R}^\times} = \chi$. Now we define a representation ρ of $P(\mathbb{R})$: for $g = mn, n \in N_P(\mathbb{R}), m = m(g, a) \in M_P(\mathbb{R})$ with $a \in \mathbb{C}^\times, g \in GU(\mathbb{R})$, put

$$\rho(g)v := \tau(a)\pi(g)v, v \in H.$$

For any function $f \in C^\infty(K_\infty, H_\infty)$ such that $f(k'k) = \rho(k')f(k)$ for any $k' \in P(\mathbb{R}) \cap K_\infty$, where H_∞ is the space of smooth vector of H, and each $z \in \mathbb{C}$ we define a function

$$f_z(g) := \delta(m)^{(a+2b+1)/2+z}\rho(m)f(k), g = mk \in P(\mathbb{R})K_\infty,$$

where δ is such that $\delta^{a+2b+1} = \delta_P$ and δ_P is the modulus character for the Klingen parabolic P. and we define an action $\sigma(\rho, z)$ of $GU(r+1, s+1)(\mathbb{R})$ on $I(H_\infty)$:

$$(\sigma(\rho, z)(g)f)(k) := f_z(kg).$$

Let $I(\rho)$ be the subspace of K_∞-finite vectors of $I(H_\infty)$ which has a structure of $\mathfrak{gu}(\mathbb{R}), K_\infty$ module structure.

Let $(\pi^\vee, V)$ be the irreducible $(\mathfrak{gu}(\mathbb{R}), K'_\infty)$-module given by $\pi^\vee(x) = \pi(\eta^{-1}x\eta)$ for $\eta = \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix}$. x in $\mathfrak{gu}(\mathbb{R})$ or K'_∞, and denote $\rho^\vee, I(\rho^\vee), I^\vee(H_\infty)$ and $\sigma(\rho^\vee, z), I(\rho^\vee))$ the representations and spaces defined as above but with π, ψ, τ replaced by $\pi^\vee \otimes (\tau \circ \det), \psi\tau\tau^c, \bar{\tau}^c$. We are going to define an intertwining operator. Let $w = \begin{pmatrix} & & 1_{b+1} \\ & 1_a & \\ -1_{b+1} & & \end{pmatrix}$, for any $z \in \mathbb{C}$, $f \in I(H_\infty)$ and $k \in K_\infty$ consider the integral:

$$A(\rho, z, f)(k) := \int_{N_P(\mathbb{R})} f_z(wnk)dn. \tag{3.1}$$

This is absolutely convergent when $Re(z) > \frac{a+2b+1}{2}$ and $A(\rho, z, -) \in Hom_{\mathbf{C}}(I(H_\infty), I^\vee(H_\infty))$ intertwines the actions of $\sigma(\rho, z)$ and $\sigma(\rho^\vee, -z)$.

Now Suppose π is the holomorphic discrete series representation associated to the (scalar) weight $(0, ..., 0; \kappa, ..., \kappa)$, then it is well known that there is a unique (up to scalar) vector $v \in \pi$ such that $k.v = \det \mu(k, i)^{-\kappa}$ (here μ means the second component of the automorphic factor J instead of the similitude character) for any $k \in K_\infty^{+,'} v$ (notation as in section 3.1). Then by Frobenius reciprocity law there is a unique (up to scalar) vector $\tilde{v} \in I(\rho)$ such that $k.\tilde{v} = \det \mu(k,i)^{-\kappa}\tilde{v}$ for any $k \in K_\infty^+$. We fix v and scale $\tilde{v}$ such that $\tilde{v}(1) = v$. In $\pi^\vee$, $\pi(w)v$ (w is defined in section 3.1) has the action of K_∞^+ given by multiplying by $\det \mu(k,i)^{-\kappa}$. We define $w' \in U(a+b+1, b+1)$ by $w' = \begin{pmatrix} 1_b & & & & \\ & & & & 1 \\ & & 1_a & & \\ & & & 1_b & \\ & -1 & & & \end{pmatrix}$. Then there is a unique vector $\tilde{v}^\vee \in I(\rho^\vee)$ such that the action of K_∞^+ is given by $\det \mu(k,i)^{-\kappa}$ and $\tilde{v}^\vee(w') = \pi(w)v$. Then by uniqueness there is a constant $c(\rho, z)$ such that $A(\rho, z, \tilde{v}) = c(\rho, z)\tilde{v}^\vee$.

Definition 3.1.1. *We define $F_\kappa \in I(\rho)$ to be the $\tilde{v}$ as above.*

3.1.2 ℓ-adic picture

Our discussion here follows from [SU] 9.1.2. Let (π, V) be an irreducible, admissible representation of $GU(F_v)$ and suppose that π is unitary and tempered. Denote by χ the central character of π. Let ψ and τ be unitary characters of $\mathcal{K}_v^\times$ such that $\psi|_{F_v^\times} = \chi$. We extend π to a representation ρ of $P(F_v)$ on V as follows. For $g = mn, n \in N_P(F_v)$, $m = m(g,a) \in M_P(F_v), a \in K_v^\times, g \in GU(F_v)$, put

$$\rho(g)v := \tau(a)\psi(b)\pi(s)v, v \in V.$$

Let $I(\rho)$ be the space of functions $f : \mathcal{K}_v \to V$ such that (i) there exists an open subgroup $U \subseteq K_v$ such that $f(gu) = f(g)$ for all $u \in U$ and (ii) $f(k'k) = \rho(k')f(k)$ for $k' \in P(\mathcal{O}_{F,v})$. For each $f \in I(\rho)$ and each $z \in \mathbf{C}$ we define a function f_z on $GU(F_v)$ by

$$f_v(g) := \delta_P(m)^{3/2+z}\rho(m)f(k), g = mk \in P(F_v)K_v$$

We define a representation $\sigma(\rho, z)$ of $GU(r+1, s+1)(F_v)$ on $I(\rho)$ by

$$(\sigma(\rho, z)(g)f)(k) := f_z(kg).$$

Let $(\pi^\vee, V)$ be given by $\pi^\vee(g) = \pi(\eta^{-1}g\eta)$. This representation is also tempered and unitary. We denote by $\rho_\vee, I(\rho^\vee)$, and $(\sigma(\rho_\vee, z), I(\rho^\vee))$ the representations and spaces defined as above but with π, ψ and τ replaced by $\pi^\vee \otimes (\tau \circ det), \psi\tau\tau^c$, and $\bar{\tau}^c$, respectively.

For $f \in I(\rho), k \in K_v$, and $z \in \mathbb{C}$ consider the integral

$$A(\rho, z, v)(k) := \int_{N_P(F_v)} f_z(wnk)dn. \tag{3.2}$$

As a consequence of our hypotheses on π this integral converges absolutely and uniformly for z and k in compact subsets of $z : Re(z) > (a+2b+1)/2 \times K_v$. Moreover, for such z, $A(\rho, z, f) \in I(\rho^\vee)$ and the operator $A(\rho, z, -) \in Hom_{\mathbf{C}}(I(\rho), I(\rho^\vee))$ intertwines the actions of $\sigma(\rho, z)$ and $\sigma(\rho^\vee, -z)$.

For any open subgroup $U \subseteq K_v$ Let $I(\rho)^U \subseteq I(\rho)$ be the finite-dimensional subspace consisting of functions satisfying $f(ku) = f(k)$ for all $u \in U$. Then the function $z \in \mathbf{C} : Re(z) > (a+2b+1)/2 \to Hom_{\mathbb{C}}(I(\rho)^U, I(\rho^\vee)^U)$, $z \mapsto A(\rho, z, -)$, is holomorphic. This map has a meromorphic continuation to all of $\mathbb{C}$.

3.1.3 global picture

We follow [SU]9.1.4 for this part. Let (π, V) be an irreducible cuspidal tempered automorphic representation of $GU(\mathbb{A}_F)$. It is an admissible $(\mathfrak{gu}(\mathbb{R}), K'_\infty)_{v|\infty} \times GU(\mathbf{A}_f)$-module which is a restricted tensor product of local irreducible admissible representations. Let $\tau : \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$ be a Hecke character and let $\tau = \otimes\tau_w$ and $\psi = \otimes\psi_w$ be their local decompositions, w over places of F. We associate with (π, τ) a representation of $(P(F_\infty) \cap K_\infty) \times P(\mathbb{A}_{F,f})$ and $v = \otimes v_w \in V$ put

$$\rho(m)v := \otimes(\rho_w(m_w)w_m),$$

Let $K_f := \prod_{w \not| \infty} K_w$ and $K_{\mathbb{A}_F} := K_{F_\infty} \times K_f$. Let $I(\rho)$ be the space of functions $f : K_{\mathbb{A}_F} \to V$ such that $f(k'k) = \rho(k')f(k)$ for $k' \in P(\mathbb{A}_F) \cap K_{\mathbb{A}}$, and f factors through $K_{F_\infty} \times K_f/K'$ for some open subgroup $K' \subseteq K_f$ and f is K_{F_∞}-finite and smooth as a function on $K_{F,\infty} \times K_f/K'$. This can be identified with the restricted product $\otimes I(\rho_w)$ with respect to the F_{ρ_w}'s at those w at which τ_w, ψ_w, π_w are unramified.

For each $z \in \mathbb{C}$ and $f \in I(\rho)$ we define a function f_z on $G(\mathbb{A})$ as

$$f_z(g) := \otimes f_{w,z}(g_w)$$

where $f_{w,z}$ are defined as before. Also we define an action $\sigma(\rho, z)$ of $\mathfrak{g}, K_{F_\infty}) \otimes GU(r+1, s+1)(\mathbb{A}_f)$ on $I(\rho)$ by $\sigma(\rho, z) := \otimes\sigma(\rho_w, z)$. Similarly we define $\rho^\vee, I(\rho^\vee)$, and $\sigma(\rho^\vee, z)$ but with the corresponding things replaced by their $\vee$'s. For each $z \in \mathbb{C}$ there are maps

$$I(\rho), I(\rho^\vee) \hookrightarrow \mathcal{A}(M_P(F)N_P(F)\backslash P(\mathbb{A}_F)),$$

given by

$$f \mapsto (g \mapsto f_z(g)(1)).$$

In the following we often write f_z for the automorphic form given by this recipe.

Definition 3.1.2. *Let Σ be a finite set of primes of F containing all the infinite places, primes dividing p and places when π or τ is ramified then we call the triple $\mathcal{D} = (\pi, \tau, \Sigma)$ is an Eisenstein Datum.*

I am sorry to use the same notation as the CM type in section 2.1. The meaning should be clear in the context.

3.1.4 Klingen-type Eisenstein series on G

We follow [SU]9.1.5. Let π, ψ, and τ be as above. For $f \in I(\rho), z \in \mathbb{C}$, and $g \in GU(r+1, s+1)(\mathbb{A})$ the series

$$E(f, z, g) := \sum_{\gamma \in P(F)\backslash G(F)} f_z(\gamma g) \tag{3.3}$$

is known to converge absolutely and uniformly for (z, g) in compact subsets of $\{z \in \mathbb{C} : Re(z) > \frac{a+2b+1}{2}\} \times G(\mathbb{A})$ and to define an automorphic form on G. The may $f \mapsto E(f, z, -)$ intertwines the action of $\sigma(\rho, z)$ and the usual action of $(\mathfrak{g}, K_\infty) \times GU(r+1, s+1)(\mathbb{A}_f)$ on $\mathcal{A}(GU(r+1, s+1))$.

The following lemma is well-known (see [SU] lemma 9.1.6)

Lemma 3.1.1. *Let R be a standard F-parabolic of $GU(r+1, s+1)$ (i.e, $R \supseteq B$). Suppose $Re(z) > \frac{a+2b+1}{2}$.*

(i) If $R \neq P$ then $E(f, z, g)_R = 0$;

(ii) $E(f, z, -)_P = f_z + A(\rho, f, z)_{-z}$.

3.2 Siegel Eisenstein series on G_n

Our discussion in this section follows from [SU] 11.1-11.3. Let $Q = Q_n$ be the Siegel parabolic subgroup of GU_n consisting of matrices $\begin{pmatrix} A_Q & B_q \\ 0 & D_q \end{pmatrix}$. It consists of matrices whose lower-left $n \times n$ block is zero. For a place v of F and a character χ of $\mathcal{K}_v^\times$ we let $I_n(\chi)$ be the space of smooth $K_{n,v}$-finite functions $f : K_{n,v} \to \mathbb{C}$ such that $f(qk) = \chi(detD_q)f(k)$ for all $q \in Q_n(F_v) \cap K_{n,v}$ (we write q as block matrix $q = \begin{pmatrix} A_Q & B_q \\ 0 & D_q \end{pmatrix}$). Given $z \in \mathbb{C}$ and $f \in I(\chi)$ we define a function $f(z, -) : G_n(F_v) \to \mathbb{C}$ by $f(z, qk) := \chi(detD_q))|detA_qD_q^{-1}|_v^{z+n/2}f(k), q \in Q_n(F_v)$ and $k \in K_{n,v}$.

For an idele class character $\chi = \otimes\chi_v$ of $\mathbb{A}_\mathcal{K}^\times$ we similarly define a space $I_n(\chi)$ of smooth $K_{n,\mathbb{A}}$ functions on $K_{n,\mathbb{A}}$. We also similarly define $f(z, -)$ given $f \in I_n(\chi)$ and $z \in \mathbb{C}$. There is an identification $\otimes I_n(\chi_v) = I_n(\chi)$, the former being the restricted tensor product defined using the spherical vectors $f_v^{sph} \in I_n(\chi_v), f_v^{sph}(K_{n,v}) = 1$, at the finite places v where χ_v is unramified:$\otimes f_v$ is identified with $k \mapsto \prod_v f_v(k_v)$. Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set. By a meromorphic section of $I_n(\chi)$ on $\mathcal{U}$ we mean a function $\varphi : \mathcal{U} \mapsto I_n(\chi)$ taking values in a finite dimensional subspace $V \subset I_n(\chi)$ and such that $\varphi : \mathcal{U} \to V$ is meromorphic.

Let $\chi = \otimes\chi_v$ be a unitary idele class character of $\mathbb{A}_\mathcal{K}^\times$. For $f \in I_n(\chi)$ we consider the Eisenstein series

$$E(f; z, g) := \sum_{\gamma \in Q_n(F)\backslash G_n(F)} f(z, \gamma g).$$

This series converges absolutely and uniformly for (z, g) in compact subsets of $\{Re(z) > n/2\} \times G_n(\mathbb{A}_F)$ and defines an automorphic form on G_n and a holomorphic function on $\{Re(z) > n/2\}$. The Eisenstein series $E(f; z, g)$ has a meromorphic continuation in z to all of $\mathbb{C}$. If $\varphi : \mathcal{U} \to I_n(\chi)$ is a meromorphic section, then we put $E(\varphi; z, g) = E(\varphi(z); z, g)$. This is clearly a meromorphic function of $z \in \mathcal{U}$ and an automorphic form on G_n for those z where it is holomorphic.

3.2.1 Intertwining operators and functional equations

Let χ be a unitary character of $\mathcal{K}_v^\times$, v a place of F. For $f \in I_n(\chi), z \in \mathbb{C}$, and $k \in K_{n,v}$, we consider the integral

$$M(z, f)(k) := \bar{\chi}^n(\mu_n(k)) \int_{N_{Q_n}(F_v)} f(z, w_n rk)dr.$$

For z in compact subsets of $\{Re(z) > n/2\}$ this integral converges absolutely and uniformly, with the convergence being uniform in k. $M(z,f) \in I_n(\bar{\chi}^c)$. It thus defines a holomorphic section $z \mapsto M(z,f)$ on $\{Re(z) > 3/2\}$. This has a continuation to a meromorphic section on all of $\mathbb{C}$.

Let $\chi = \otimes \chi_v$ be a unitary idele class character. For $f \in I_n(\chi), z \in \mathbf{C}$, and $k \in K_{n,\mathbb{A}_F}$ we consider the integral $M(z,f)(k)$ as above but with the integration being over $N_{Q_n}(\mathbb{A}_F)$. This again converges absolutely and uniformly for z i compact subsets of $\{Re(z) > n/2\}$, with the convergence being uniform in k. Thus $z \mapsto M(z,f)$ defines a holomorphic section $\{Re(z) > n/2\} \to I_n(\bar{\chi}^c)$. This has a continuation to a meromorphic section on $\mathbb{C}$. For $Re(z) > n/2$ at least, we have

$$M(z,f) = \otimes_v M(z,f_v), f = \otimes f_v.$$

3.2.2 The pull-back formulas

Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$. Given a cuspform ϕ on $G(r,s)$ we consider

$$F_\phi(f;z,g) := \int_{U(r,s)(\mathbb{A}_F)} f(z, S^{-1}\alpha(g,g_1 h)S)\bar{\chi}(\det g_1 g)\phi(g_1 h)dg_1,$$

$$f \in I_{r+s+1}(\chi), g \in G(r+1,s+1)(\mathbb{A}_F), h \in G(r,s)(\mathbb{A}_F), \mu(g) = \mu(h)$$

This is independent of h. The pull-back formulas are the identities in the following proposition.

Proposition 3.2.1. *Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$.*
(i) if $f \in I_{r+s}(\chi)$, then $F_\phi(f;z,g)$ converges absolutely and uniformly for (z,g) in compact sets of $\{Re(z) > r+s\} \times G(r,s)(\mathbb{A}_F)$, and for any $h \in G(r,s)(\mathbb{A}_F)$ such that $\mu(h) = \mu(g)$

$$\int_{U(r,s)(F)\backslash U(r,s)(\mathbb{A}_F)} E(f;z,S'^{-1}\alpha(g,g_1 h)S)\bar{\chi}(\det g_1 h)\phi(g_1 h)dg_1 = F_\phi(f;z,g). \tag{3.4}$$

(ii) If $f \in I_{r+s+1}(\chi)$, then $F_\phi(f;z,g)$ converges absolutely and uniformly for (z,g) in compact sets of $\{Re(z) > r+s+1/2\} \times G(r+1,s+1)(\mathbb{A}_F)$ such that $\mu(h) = \mu(g)$

$$\begin{aligned}\int_{U(r,s)(F)\backslash U(r,s)(\mathbb{A}_F)} & E(f;z,S^{-1}\alpha(g,g'h)S)\bar{\chi}(\det g_1 h)\phi(g_1 h)dg_1 \\ &= \sum_{\gamma \in P(F)\backslash G(r+1,s+1)(F)} F_\phi(f;z,\gamma g),\end{aligned} \tag{3.5}$$

with the series converging absolutely and uniformly for (z,g) in compact subsets of $\{Re(z) > r+s+$

$1/2\} \times G(r+1, s+1)(\mathbb{A}_F)$.

Proof. (i) is proved by Piatetski-Shapiro and Rallis and (ii) is a straight-forward generalization by [Shi97]. See also [SU] Proposition 11.2.3. $\square$

3.3 Fourier-Jacobi Expansion

We will usually use the notation $e_{\mathbb{A}}(x) = e_{\mathbb{A}_{\mathbb{Q}}}(\mathrm{Tr}_{\mathbb{A}_F/\mathbb{A}_{\mathbb{Q}}} x)$ for $x \in \mathbb{A}_F$. For any automorphic form φ on $GU(r,s)(\mathbb{A}_F)$, $\beta \in S_m(F)$ for $m \leq s$. We define the Fourier-Jacobi coefficient at $g \in GU(r,s)(\mathbb{A}_F)$:

$$\varphi_\beta(g) = \int_{S_m(F)\backslash S_m(\mathbb{A}_F)} \varphi(\begin{pmatrix} 1_s & 0 & \begin{matrix} S & 0 \\ 0 & 0 \end{matrix} \\ 0 & 1_{r-s} & 0 \\ 0 & 0 & 1_s \end{pmatrix} g) e_{\mathbb{A}}(-\mathrm{Tr}(\beta S)) dS.$$

In fact we are mainly interested in two cases: $m = s$ or $r = s$ and arbitary $m \leq s$. In particular, $G = G_n = U(n,n)$, $0 \leq m \leq n$ are integers, $\beta \in S_m(F)$. Let φ be a function on $G(F)\backslash G(\mathbb{A})$. The β-th Fourier-Jacobi coefficient φ_β of φ at g is defined by

$$\varphi_\beta(g) := \int \varphi(\begin{pmatrix} 1_n & \begin{matrix} S & 0 \\ 0 & 0 \end{matrix} \\ & 1_n \end{pmatrix} g) e_{\mathbb{A}}(-Tr\beta S) dS.$$

Now we prove a useful formula on the Fourier Jacobi coefficients for Siegel Eisenstein series.

Definition 3.3.1. *let:*

$$Z := \{ \begin{pmatrix} 1_n & \begin{matrix} z & 0 \\ 0 & 0 \end{matrix} \\ 0_n & 1_n \end{pmatrix} | z \in \mathrm{Her}_m(\mathcal{K}) \}$$

$$V := \{ \begin{pmatrix} 1_m & x & z & y \\ & 1_{n-m} & y^* & 0 \\ & & 1_m & \\ & 0_n & -x^* & 1_{n-m} \end{pmatrix} | x, y \in M_{m(n-m)}(\mathcal{K}), z - xy^* \in \mathrm{Her}_m(\mathcal{K}) \}$$

$$X := \{\begin{pmatrix} 1_m & x & & \\ & 1_{n-m} & & 0_n \\ & & 1_m & \\ 0_n & & -x^* & 1_{n-m} \end{pmatrix} | x \in M_{m(n-m)}(\mathcal{K})\}$$

$$Y := \{\begin{pmatrix} 1_n & \begin{matrix} z & y \\ y^* & 0 \end{matrix} \\ 0_n & 1_n \end{pmatrix} | y \in M_{m(n-m)}(\mathcal{K})\}$$

Proposition 3.3.1. *Suppose $f \in I_n(\tau)$ and $\beta \in S_m(F)$, β is totally positive. If $E(f;z,g)$ is the Siegel Eisenstein Series on G defined by f for some $Re(z)$ sufficiently large then the β-th Fourier-Jacobi coefficient $E_\beta(f;z,g)$ satisfies:*

$$E_\beta(f;z,g) = \sum_{\gamma \in Q_{n-m}(F)\backslash G_{n-m}(F)} \sum_{y \in Y} \int_{S_m(\mathbb{A})} f(w_n \begin{pmatrix} 1_n & \begin{matrix} S & y \\ {}^t\bar{y} & 0 \end{matrix} \\ & 1_n \end{pmatrix} \alpha_{n-m}(1,\gamma)g) e_{\mathbb{A}}(-Tr\beta S) dS$$

.

Proof. We follow [IKE] section 3. Let H be the normalizer of V in G. Then

$$G_n(F) = \sqcup_{i=1}^m Q_n(F)\xi_i H(F)$$

for $\xi_i := \begin{pmatrix} 0_{m-i} & 0 & -1_{m-i} & 0 \\ 0 & 1_{n-m+i} & 0 & 0 \\ 1_{m-i} & 0 & 0_{m-i} & 0 \\ 0 & 0_{n-m+i} & 0 & 1_{n-m+i} \end{pmatrix}$. then unfold the Eisenstein series we get:

$$E_\beta(f;z,g) =$$

$$\sum_{i>0} \sum_{\gamma \in Q_n(F)\backslash Q_n(F)\xi_i H(F)} \int f(\gamma \begin{pmatrix} 1_n & \begin{matrix} S & 0 \\ 0 & 0 \end{matrix} \\ & 1_n \end{pmatrix} g) e_{\mathbb{A}}(-Tr(\beta S)) dS$$

$$+ \sum_{\gamma \in Q_n(F)\backslash Q_n(F)\xi_0 H(F)} \int f(\gamma \begin{pmatrix} 1_n & \begin{matrix} S & 0 \\ 0 & 0 \end{matrix} \\ & 1_n \end{pmatrix} g) e_{\mathbb{A}}(-Tr(\beta S)) dS$$

by lemma (3.1) in [IKE] (see *loc.cit* P628), the first term vanishes. Also, we have (*loc.cit*)

$$\begin{aligned} & Q_n(F)\backslash Q_n(F)\xi_0 H(F) \\ = \quad & \xi_0 Z(F)X(F)Q_{n-m}(F)\backslash G_{n-m}(F) \\ = \quad & \xi_0 X(F).Q_{n-m}(F)\backslash G_{n-m}(F).Z(F) \\ = \quad & w_n Y(F)S_m(F)w_{n-m}Q_{n-m}(F)\backslash G_{n-m}(F) \end{aligned}$$

(note that S_m commutes with X and G_{n-m}). So

$$E_\beta(f;z,g) = \sum_{\gamma\in Q_{n-m}(F)\backslash G_{n-m}(F)} y \sum_{y\in Y(F)} \int_{S_m(\mathbb{A})} f(w_n \begin{pmatrix} 1_n & \begin{matrix} S & y \\ {}^t\overline{y} & 0 \end{matrix} \\ & 1_n \end{pmatrix} \alpha_{n-m}(1,\gamma)g)e_{\mathbb{A}}(-Tr(\beta S))$$

Note that the final object is a local one. $\square$

Now we record some useful formulas:

Definition 3.3.2. *If $g_v \in U_{n-m}(F_v), x \in GL_m(\mathcal{K}_v)$, then define:*

$$FJ_\beta(f_v;z,y,g,x) = \int_{S_m(F_v)} f(w_n \begin{pmatrix} 1_n & \begin{matrix} S & y \\ {}^t\overline{y} & 0 \end{matrix} \\ & 1_n \end{pmatrix} \alpha(\mathrm{diag}(x, {}^t\overline{x}^{-1}), g))e_{F_v}(-Tr\beta S)dS$$

where if $g_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, g_2 = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ then:

$$\alpha(g_1,g_2) = \begin{pmatrix} A & & B & \\ & D' & & C' \\ C & & D & \\ & B' & & A' \end{pmatrix}$$

Since

$$\begin{pmatrix} 1_n & & S & X \\ & & {}^t\bar{X} & \\ & & & 1_n \end{pmatrix}\begin{pmatrix} 1_m & & & \\ & \bar{A}^{-1} & & \\ & & 1_m & \\ & B\bar{A}^{-1} & & A \end{pmatrix} = \begin{pmatrix} 1_m & XB\bar{A}^{-1} & & \\ & \bar{A}^{-1} & & \\ & & 1_m & \\ & B\bar{A}^{-1} & & A \end{pmatrix}\begin{pmatrix} 1_n & & S - XB{}^t\bar{X} & XA \\ & & \bar{A}{}^t\bar{X} & \\ & & & \\ & & 1_n & \end{pmatrix}.$$

it follows that:

$$FJ_\beta(f; z, X, \begin{pmatrix} A & B\bar{A}^{-1} \\ & \bar{A}^{-1} \end{pmatrix} g, Y) =$$
$$\tau_v^c(\det A)^{-1}|detA\bar{A}|_v^{z+n/2} e_v(-tr({}^t\bar{X}\beta XB))FJ_\beta(f; z, XA, g, Y)$$

Also we have:

$$FJ_\beta(f; z, y, g, x) = \tau_v(\det x)|\det x\bar{x}|_{\mathbb{A}}^{-(z+\frac{n}{2}-m)} FJ_{{}^t\bar{x}\beta x}(f; z, x^{-1}y, g, 1)$$

3.3.1 Weil Representations

Now we briefly recall some formulas for the Weil representations which will be useful for computing Fourier Jacobi coefficients. Let V be the two-dimensional $\mathcal{K}$-space of column vectors.

The local set-up. Let v be a place of F. Let $h \in S_m(F_v), \det h \neq 0$. Let U_h be the unitary group of this matric and denote V_v to be the corresponding Hermitian space. Let $V_1 := \mathcal{K}^{(n-m)} \oplus \mathcal{K}^{(n-m)} := X_v \oplus Y_v$ be the Hermitian space associated to $U(n-m, n-m)$. Let $W := V_v \otimes_{\mathcal{K}_v} V_{1,v}$, where $V_{1,v} := V_1 \otimes F_v$. Then $(-,-) := Tr_{\mathcal{K}_v/F_v}(<-,->_h \otimes_{\mathcal{K}_v} <-,->_1)$ is a F_v linear pairing on W that makes W into an $4m(n-m)$-dimensional symplectic space over F_v. The canonical embeding of $U_h \times U_1$ into $Sp(W)$ realizes the pair (U_h, U_1) as a dual pair in $Sp(W)$. Let λ_v be a character of $\mathcal{K}_v^\times$ such that $\lambda_v|_{F_v^\times} = \chi_{\mathcal{K}/F,v}^m$. In [Ku94], a splitting pair $U_h(F_v) \times U_1(F_v) \hookrightarrow Mp(W, F_v)$ of the metaplectic cover $Mp(W, F_v) \to Sp(W, F_v)$ is associated with the character λ_v; we use this splitting to identify $U_h(F_v) \times U_1(F_v)$ with a subgroup of $Mp(W, F_v)$.

We let $\omega_{h,v}$ be the corresponding Weil representation of $U_h(F_v) \times U_1(F_v)$ (associated with λ_v and e_v) on the Schwartz space $\mathcal{S}(V_v \otimes_{\mathcal{K}_v} X_v)$: the action of (u, g) on $\Phi \in \mathcal{S}(V_v \otimes_{\mathcal{K}_v} X_v)$ is written $\omega_{h,v}(u, g)\Phi$. If $u = 1$ we often omit u, writing $\omega_{h,v}(g)$ to mean $\omega_{h,v}(1, g)$. Then $\omega_{h,v}$ satisfies: for

$X \in M_{m\times(n-m)}(\mathcal{K}_v)$:

- $\omega_{h,v}(u,g)\Phi(X) = \omega_{h,v}(1,g)\Phi(u^{-1}X)$
- $\omega_{h,v}(diag(A, {}^t\bar{A}^{-1}))\Phi(X) = \lambda(\det A)|\det A|_{\mathcal{K}}\Phi(XA)$,
- $\omega_{h,v}(r(S))\Phi(x) = \Phi(x)e_v(\mathrm{tr} < X, X >_h S)$,
- $\omega_{h,v}(\eta)\Phi(x) = |\det h|_v \int \Phi(Y)e_v(Tr_{\mathcal{K}_v/\mathbb{Q}_v}(\mathrm{tr} < Y, X >_h))dY$.

Chapter 4

Local Computations

In this chapter we do the local computations for Klingen Eisenstein sections realized as the pullbacks of Siegel Eisenstein sections. We will mainly compute the Fourier and Fourier-Jacobi coefficients for the Siegel sections and the pullback Klingen Eisenstein section.

4.1 Archimedean Computations

Let v be an Archimedean place of F.

4.1.1 Fourier Coefficients

Now we recall a lemma from [SU] 11.4.2.

Lemma 4.1.1. *If we define* $f_{\kappa,n}(z,g) = J_n(g, i1_n)^{-\kappa}|J_n(g, i1_n)|^{\kappa-2z-n}$*,suppose* $\beta \in S_n(\mathbb{R})$*. Then the function* $z \to f_{\kappa,\beta}(z,g)$ *has a meromorphic continuation to all of* $\mathbb{C}$*. Furthermor, if* $\kappa \geq n$ *then* $f_{\kappa,n,\beta}(z,g)$ *is holomorphic at* $z_\kappa := (\kappa-n)/2$ *and for* $y \in \mathrm{GL}_n(\mathbb{C})$, $f_{\kappa,n,\beta}(z_\kappa, \mathrm{diag}(y, {}^t\bar{y}^{-1})) = 0$ *if* $\det\beta \leq 0$ *and if* $\det\beta > 0$ *then*

$$f_{\kappa,n,\beta}(z_\kappa, \mathrm{diag}(y, {}^t\bar{y}^{-1})) = \frac{(-2)^{-n}(2\pi i)^{n\kappa}(2/\pi)^{n(n-1)/2}}{\prod_{j=0}^{n-1}(\kappa-j-1)!} e(i\mathrm{Tr}(\beta y {}^t\bar{y})) \det(\beta)^{\kappa-n} \det \bar{y}^{\kappa}.$$

Later on our $f_{\kappa,n}$ will be defined differently, but it is just the one defined above translated by matrices of the form $\mathrm{diag}(y, {}^t\bar{y}^{-1})$. So the Fourier coefficient can be deduced from the above lemma.

4.1.2 Pullback Sections

Now we assume that our π is the holomorphic discrete series representation associated to the (scalar) weight $(0, ..., 0; \kappa, ..., \kappa)$ and let ϕ be the unique (up to scalar) vector such that the action of $K_\infty^{+,\prime}$ (see section 3.1) is given by $\det \mu(k, i)^{-\kappa}$. Recall also that in section 3.1 we have defined the Klingen section $F_\kappa(z, g)$ there. Recall that:

$$S = \begin{pmatrix} 1 & & & & & & & -\frac{1}{2} \\ & 1 & & & & & & \\ & & 1 & & & & -\frac{\zeta}{2} & \\ & & & -1 & \frac{1}{2} & & & \\ & & & 1 & \frac{1}{2} & & & \\ & & & & & 1 & & \\ & & -1 & & & & -\frac{\zeta}{2} & \\ -1 & & & & & & & -\frac{1}{2} \end{pmatrix}$$

and

$$S' = \begin{pmatrix} 1 & & & & & -\frac{1}{2} \\ & 1 & & & -\frac{\zeta}{2} & \\ & & -1 & \frac{1}{2} & & \\ & & 1 & \frac{1}{2} & & \\ & -1 & & & -\frac{\zeta}{2} & \\ -1 & & & & & -\frac{1}{2} \end{pmatrix}.$$

Let $\mathbf{i} := \begin{pmatrix} \frac{i}{2} 1_b & & \\ & i & \\ & & \frac{\zeta}{2} 1_a \end{pmatrix}$ be a point in the symmetric domain for $GU(n, n)$ or $GU(n+1, n+1)$ for $n = a + 2b$, where the block matrices i are of size $b \times b$ or $(b + 1) \times (b + 1)$. We define archimedean section to be:

$$f_\kappa(g) = J_{n+1}(g, \mathbf{i})^{-\kappa} |J_{n+1}(g, \mathbf{i})|^{\kappa - 2z - n - 1}$$

and

$$f'_\kappa(g) = J_n(g, \mathbf{i})^{-\kappa} |J_n(g, \mathbf{i})|^{\kappa - 2z - n}$$

and the pull back sections on $GU(a+b+1,b+1)$ and $GU(a+b,a)$ to be

$$F_\kappa(z,g) := \int_{U_{a+b,b}(\mathbf{R})} f_\kappa(z, S^{-1}\alpha(g,g_1)S)\bar{\tau}(\det g_1)\pi(g_1)\phi dg_1$$

and

$$F'_\kappa(z,g) := \int_{U_{a+b,b}(\mathbf{R})} f'_\kappa(z, S'^{-1}\alpha(g,g_1)S')\bar{\tau}(\det g_1)\pi(g_1)\phi dg_1$$

Lemma 4.1.2. *The integrals are absolutely convergent for $Re(z)$ sufficiently large and for such z, we have:*

(i)

$$F_\kappa(z,g) = c_\kappa(z)F_{\kappa,z}(g);$$

(ii)

$$F'_\kappa(z,g) = c'_\kappa(z)\pi(g)\phi;$$

where

$$c'_\kappa(z,g) = 2^\nu |\det\theta|_v^b \begin{cases} \pi^{(a_v+b_v)b_v}\Gamma_{b_v}(z+\frac{n+\kappa}{2}-a_v-b_v)\Gamma_{b_v}(z+\frac{n+k}{2})^{-1}, & b>0 \\ 1, & \textit{otherwise.} \end{cases}$$

and $c_\kappa(z,g) = c'_\kappa(z+\frac{1}{2},g)$. Here $\Gamma_m := \pi^{\frac{m(m+1)}{2}}\prod_{k=0}^{m-1}\Gamma(s-k)$ and $\nu := (a+2b)db$ (recall that $d=[F:\mathbb{Q}]$).

Proof. See [Shi97] 22.2 and A2.9. Note that the action of $(\beta,\gamma)\in U(r,s)\times U(r,s)$ are given by (β',γ') defined there. Taking this into consideration, our conjugation matrix S are Shimura's S times Σ^{-1}, which is defined in (22.1.2) in [Shi97]. Also our result differ from [SU1] 11.4.4 by some powers of 2 since we are using a different S here. ☐

4.1.3 Fourier-Jacobi Coefficients

Lemma 4.1.3. *Let $z_\kappa = \frac{\kappa-n}{2}$, $\beta\in S_m(\mathbb{R})$, $m<n$, $\det\beta>0$. then:*

(i)$FJ_{\beta,\kappa}(z_\kappa,x,\eta,1) = f_{\kappa,m,\beta}(z_\kappa+\frac{n-m}{2},1)e(iTr({}^t\bar{X}\beta X))$;

(ii)if $g\in U_{n-m}(\mathbb{R})$, then

$$FJ_{\beta,\kappa}(z_\kappa,X,g,1) = e(iTr\beta)c_m(\beta,\kappa)f_{\kappa-m,n-m}(z_\kappa,g')w_\beta(g')\Phi_{\beta,\infty}(x).$$

where $g' = \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix} g \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix}$ *and* $c_t(\beta,\kappa) = \frac{(-2)^{-t}(2\pi i)^{t\kappa}(2/\pi)^{t(t-1)/2}}{\prod_{j=0}^{t-1}(\kappa-j-1)} \det \beta^{\kappa-t}$.

Proof. For (i) we first assume that $m \le n/2$, then there is a matrix $U \in U_{n-m}$ such that $XU = (0, A)$ for A a $(m \times m)$ positive semi-definite Hermitian matrix . It then follows that $FJ_{\beta,\kappa}(z, X, \eta, 1) = FJ_\beta(z, (0, A), \eta, 1)$ and $e(iTr({}^t\bar{X}\beta X)) = e(iTr(U^{-1}{}^t\bar{X}\beta XU))$, so we are reduced to the case when $X = (0, A)$.

Let C be a $(m \times m)$ positive definite Hermitian matrix defined by $C = \sqrt{A^2 + 1}$. (Since A is positive semi-definite Hermitian, this C exists by linear algebra.)

$$\begin{pmatrix} & & & A \\ 1_n & & & \\ & A & & \\ & & 1_n & \end{pmatrix} = \begin{pmatrix} C & & & & & \\ & 1 & & & & \\ & & C & & & \\ & & AC^{-1} & C^{-1} & & \\ & & & & 1 & \\ AC^{-1} & & & & & C^{-1} \end{pmatrix} \begin{pmatrix} C^{-1} & & & & & C^{-1}A \\ & 1 & & & & \\ & & C^{-1} & C^{-1}A & & \\ & & -C^{-1}A & C^{-1} & & \\ & & & & 1 & \\ -C^{-1}A & & & & & C^{-1} \end{pmatrix}$$

write $k(a)$ for the second matrix in the right of above which belongs to $K^+_{n,\infty}$, then as in [SU] lemma 11.4.3,

$$w_n \begin{pmatrix} 1_n & \begin{matrix} S & X \\ {}^t\bar{X} & \end{matrix} \\ & 1_n \end{pmatrix} = \begin{pmatrix} C^{-1} & & & \times & \times & \times \\ & 1 & & \times & \times & \times \\ \times & \times & C^{-1} & \times & \times & \times \\ & & & C & & \\ & & & & 1 & \\ & & & \times & \times & C \end{pmatrix} w_n \begin{pmatrix} 1_n & U^{-1}SU^{-1} \\ & 1_n \end{pmatrix} k(a)$$

thus

$$\begin{aligned} FJ_{\beta,\kappa}(z_\kappa, (0, A), \eta, 1) &= (detC)^{2m-2\kappa} FJ_{\beta',\kappa}(z_\kappa, 0, \eta, 1), \qquad \beta' = C\beta C \\ &= (detC)^{2m-2\kappa} f_{\kappa,m,\beta'}(z_\kappa + \tfrac{n-m}{2}, 1) \\ &= f_{\kappa,m,\beta}(z + \tfrac{n-m}{2}, 1) e(iTr(C\beta C - \beta)) \end{aligned}$$

but

$$e(iTr(C\beta C-\beta))=e(iTr(C^2\beta-\beta))=e(iTr((C^2-1)\beta))=e(iTr(A^2\beta))=e(iTr(A\beta A))$$

this proves part (i).

Part (ii) is proved completely the same as in lemma 11.4.3 of [SU].

In the case when $m>\frac{n}{2}$ we proceed similarly as lemma 11.4.3 of [SU], replacing a and u there by corresponding block matrices just as above. we omit the details.

□

4.2 ℓ-adic computations, unramified case

4.2.1 Fourier-Jacobi Coefficients

Let v be a prime of F not dividing p and τ be a character of $\mathcal{K}_v^\times$, for $f\in I_n(\tau)$ and $\beta\in S_m(F_v), 0\leq m\leq n$, we have defined the local Fourier-Jacobi coefficient to be

$$f_\beta(z;g):=\int_{S_m(\mathbb{A})} f(z,w_n\begin{pmatrix}1_n & & S & 0\\ & & 0 & 0\\ & & 1_n & \end{pmatrix}g)e_v(-Tr\beta S)dS$$

We first record a generalization of lemma 11.4.6 in [SU] to any fields (Proposition 18.14 and 19.2 of [Shi97])

Lemma 4.2.1. *Let $\beta\in S_n(F_v)$ and let $r:=rank(\beta)$. Then for $y\in GL_n(\mathcal{K}_v)$,*

$$\begin{aligned}f_{v,\beta}^{sph}(z,diag(y,{}^t\bar{y}^{-1}))= & \tau(dety)|dety\bar{y}|_v^{-z+n/2}D_v^{-n(n-1)/4}\\ & \times\frac{\prod_{i=r}^{n-1}L(2z+i-n+1,\bar{\tau}'\chi_{\mathcal{K}}^i)}{\prod_{i=0}^{n-1}L(2z+n-i,\bar{\tau}\chi_{\mathcal{K}}^i)}h_{v,{}^t\bar{y}\beta y}(\bar{\tau}'(\varpi)q_v^{-2z-n}).\end{aligned}$$

where $h_{v,{}^t\bar{y}\beta y}\in Z[X]$ is a monic polynomial depending on v and ${}^t\bar{y}\beta y$ but not on τ. If $\beta\in S_n(\mathcal{O}_{F,v})$ and $det\beta\in\mathcal{O}_{F,v}^\times$, then we say that β is v-primitive and in this case $h_{v,\beta}=1$.

Lemma 4.2.2. *Suppose v is unramified in $\mathcal{K}$. Let $\beta\in S_m(F_v)$ such that $det\beta\neq 0$. Let $y\in GL_{n-m}(\mathcal{K}_v)$ such that ${}^t\bar{y}\beta y\in S_m(\mathcal{O}_{F_v})$, let λ be an unramified character of $\mathcal{K}_v^\times$ such that $\lambda|_{F_v\times}=1$.*

(i) If $\beta, y \in GL_m(\mathcal{O}_v)$ then for $u \in U_\beta(F_v)$:

$$FJ_\beta(f_n^{sph}; z, x, g, uy) = \tau(detu)|detu\bar{u}|_v^{-z+1/2} \frac{f_{n-m}^{sph}(z,g)\omega_\beta(u,g)\Phi_{0,y}(x)}{\prod_{i=0}^{m-1} L(2z+n-i, \bar{\tau}'\chi_{\mathcal{K}}^i)}.$$

(ii) If ${}^t\bar{y}\beta y \in GL_m(\mathcal{O}_v)$, then for $u \in U_\beta(F_v)$,

$$FJ_\beta(f_n^{sph}; z, x, g, uy) = \tau(detuy)|detuy\bar{u}|_{\mathcal{K}}^{-z+1/2} \frac{f_{n-m}^{sph}(z,g)\omega_\beta(u,g)\Phi_{0,y}(x)}{\prod_{i=0}^{m-1} L(2z+n-i, \bar{\tau}'\chi_{\mathcal{K}}^i)}.$$

4.2.2 Pull-back integrals

Lemma 4.2.3. *Suppose π, ψ and τ are unramified and ϕ is a newvector. If $Re(z) > (a+b)/2$ then the pull back integral converges and*

$$F_\phi(f_v^{sph}; z, g) = \frac{L(\tilde{\pi}, \bar{\tau}^c, z+1)}{\prod_{i=0}^{a+2b-1} L(2z+a+2b+1-i, \bar{\tau}'\chi_{\mathcal{K}}^i)} F_{\rho,z}(g)$$

where F_ρ is the spherical section.

4.3 ℓ-adic computations, ramified case

4.3.1 Pull Back integrals

Again let v be a prime of F not dividing p. The choices in this section is not quite important. In fact in applications we are going to change it according to the needs. The purpose for this section is only to convince the reader that such kinds of section do exist. We define $f^\dagger$ to be the Siegel section supported on the cell $Q(F_v)w_{a+2b+1}N_Q(\mathcal{O}_{F,v})$ where $w_{a+2b+1} = \begin{pmatrix} & 1_{a+2b+1} \\ -1_{a+2b+1} & \end{pmatrix}$ and the value at $N_Q(\mathcal{O}_{F,v})$ equals 1. We fix some x and y in $\mathcal{K}$ which are divisible by some high power of ϖ_v. (When we are moving things p-adically the x and y are not going to change).

Definition 4.3.1.

$$f_{v,sieg}(g) := f(\begin{pmatrix} 1_{a+2b+1} & \\ & \frac{1}{2}1_{a+2b+1} \end{pmatrix} g \begin{pmatrix} 1_{a+2b+1} & \\ & 21_{a+2b+1} \end{pmatrix} \tilde{\gamma}_v)$$

where $\tilde{\gamma}_v$ *is defined to be:*

$$\begin{pmatrix} 1 & & & & & & & \frac{1}{x} \\ & 1 & & & & & & \\ & & 1 & & & & \frac{1}{y\bar{y}} & \\ & & & 1 & \frac{1}{\bar{x}} & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}$$

Lemma 4.3.1. *Let* $K_v^{(2)}$ *be the subgroup of* $G(F_v)$ *of the form* $\begin{pmatrix} 1 & & & & d \\ a & 1 & f & b & c \\ & & 1 & & g \\ & & & 1 & e \\ & & & & 1 \end{pmatrix}$ *where* $e = -{}^t\bar{a}$, $b = {}^t\bar{d}$, $g = -\theta {}^t\bar{f}$, $c = \bar{c}$, $a \in (x)$, $e \in (\bar{x})$, $f \in (y\bar{y})$, $g \in (2\zeta y\bar{y})$. *Then* $F_\phi(z; g, f)$ *is supported in* $PwK_v^{(2)}$ *and is invariant under the action of* $K_v^{(2)}$*.*

Proof. Let $S_{x,y}$ consists of matrices: $S := \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}$ in the space of Hermitian $(a+2b+1) \times (a+2b+1)$ matrices (the blocks are with respect to the partition $b+1+a+b$ such that the entries of S_{13}, S_{23} are divisible by y, the entries of S_{14}, S_{24} are divisible by x, the entries of S_{31}, S_{32} are divisible by $\bar{y}$, the entries of S_{41}, S_{42} are divisible by $\bar{x}$, the entries of S_{33} are divisible by $y\bar{y}$, the entries of S_{34} are divisible by $x\bar{y}$, the entries of S_{43} are divisible by $\bar{x}y$, and the entries of S_{44} are divisible by $x\bar{x}$. Let $Q_{x,y} := Q(F_v)$. $\begin{pmatrix} 1 & \\ S_{x,y} & 1 \end{pmatrix}$.

For $g = \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 \\ a_4 & a_5 & a_6 & b_3 & b_4 \\ a_7 & a_8 & a_9 & b_5 & b_6 \\ c_1 & c_2 & c_3 & d_1 & d_2 \\ c_4 & c_5 & c_6 & d_3 & d_4 \end{pmatrix}$, we have:

$$\begin{aligned} & \gamma(g,1) \in \operatorname{supp} f_{v,sieg} \\ \Leftrightarrow \quad & \gamma(g,1) w_{a+2b+1} d_{x,y} \tilde{\gamma}^{-1} \in Q_{x,y} \\ \Leftrightarrow \quad & \gamma(gw, \eta \operatorname{diag}(\bar{x}^{-1}, 1, x)) w' d_y \tilde{\delta}^{-1} \in Q_{x,y} \end{aligned}$$

Here $d_{x,y} = \operatorname{diag}(1,1,y,x,1,1,\bar{y}^{-1},\bar{x}^{-1})$ and $d_y = \operatorname{diag}(1,1,y,1,1,1,\bar{y}^{-1},1)$. where x and y here stand for the corresponding block matrices of the corresponding size. Recall that $\gamma((m(g_1,1),g_1) \in Q$, by multiplying this to the left for $g_1 = \operatorname{diag}\bar{x}, 1, x^{-1})\eta^{-1}$, we are reduced to proving that if $\gamma(g,1)w'd_y\tilde{\gamma}^{-1} \in Q_{x,y}$, then $g \in PwK_v^{(2)}w^{-1}$. A computation tells us that: $\gamma(g,1)w'd_y\tilde{\gamma}^{-1}$ equals:

$$\begin{pmatrix} a_1 & a_2 & \zeta a_3 y & -b_1 & b_1 & b_2 & a_3\bar{y}^{-1} & \\ a_4 & a_5 & \zeta a_6 y - a_6\bar{y}^{-1} & -b_3 & b_3 & b_4 & a_6\bar{y}^{-1} & \\ a_7/2 & a_8/2 & \frac{\zeta y(a_9-1)}{2} - \frac{(a_9+1)\bar{y}^{-1}}{2} & -\frac{b_5}{2} & \frac{b_5}{2} & \frac{b_6}{2} & \frac{(a_9+1)\bar{y}^{-1}}{2} & \\ & & & 1 & & & & \\ c_1 & c_2 & \zeta c_3 y - c_3\bar{y}^{-1} & 1-d_1 & d_1 & d_2 & c_3\bar{y}^{-1} & \\ c_4 & c_5 & \zeta c_6 y - c_6\bar{y}^{-1} & -d_3 & d_3 & d_4 & c_6\bar{y}^{-1} & \\ -\frac{\zeta^{-1}}{2}a_7 & -\frac{\zeta^{-1}}{2}a_8 & -\frac{(a_9+1)}{2}y + \frac{\zeta^{-1}}{2}(a_9-1)\bar{y}^{-1} & \frac{\zeta^{-1}}{2}b_5 & -\frac{\zeta^{-1}}{2}b_5 & -\frac{\zeta^{-1}}{2}b_6 & \frac{\zeta^{-1}}{2}(1-a_9)\bar{y}^{-1} & \\ a_1-1 & a_2 & \zeta a_3 y - a_3\bar{y}^{-1} & -b_1 & b_1 & b_2 & a_3\bar{y}^{-1} & 1 \end{pmatrix}$$

One first proves that $d_4 \neq 0$ by looking at the second row of the lower left of the above matrix, so by left multiplying g by some matrix in N_P, we may assume that $d_2 = b_2 = b_4 = b_6 = 0$, then the result follows by an argument similarly as lemma 4.4.11 later on. □

Now recall that $g = \begin{pmatrix} a_5 & a_6 & a_4 \\ a_8 & a_9 & a_7 \\ a_2 & a_3 & a_1 \end{pmatrix}$, let $\mathfrak{Y}$ be the set of g's so that the entries of a_2 are integers, the entries of a_3 are divisible by $y\bar{y}$, the entries of $a_1 - 1$ are divisible by $\bar{x}$, the entries of $1 - a_5$ are divisible by x, the entries of a_6 are divisible by $\bar{x}y$, the entries of a_4 are divisible by $x\bar{x}$, $\frac{1-a_9}{2} = y\bar{y}\zeta(1 + y\bar{y}N)$ for some N with integral entries, the entries of a_8 are divisible by $\bar{y}y\zeta$, and the entries of a_7 are divisible by $\bar{y}yx\zeta$.

Lemma 4.3.2. *Let $\phi_x = \pi(\mathrm{diag}(\bar{x}, 1, x^{-1})\eta^{-1})\phi$ where ϕ is invariant under the action of $\mathfrak{Y}$ defined above, then $F_{\phi_x}(z, w) = \tau(y\bar{y}x)|(y\bar{y})^2 x\bar{x}|_v^{-z-\frac{a+2b+1}{2}}\mathrm{Vol}(\mathfrak{Y}).\phi$.*

Proof.

$$\begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && \frac{1}{2} &&&& \frac{1}{2} & \\ &&& 1 &&&& \\ &&& 1 & 1 &&& \\ &&&&& 1 && \\ && -\frac{\zeta^{-1}}{2} &&&& -\frac{\zeta^{-1}}{2} & \\ 1 &&&&&&& 1 \end{pmatrix} \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& a_1 &&& a_3 & a_2 \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&& a_7 &&& a_9 & a_6 \\ &&& a_4 &&& a_6 & a_5 \end{pmatrix} \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&& -\zeta & \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ && -1 &&&& -\zeta & \\ &&&&&&& 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ &&&&&& 1 & \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ && -1 &&&&& \\ &&&&&&& 1 \end{pmatrix} \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && y &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& \bar{y}^{-1} & \\ &&&&&&& 1 \end{pmatrix} \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&& -1 & 1 &&& \\ &&&&& 1 && \\ && -1 &&&& 1 & \\ -1 &&&&&&& 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ \frac{a_8}{2} & & \frac{\zeta y}{2}(1-a_9) - \frac{\bar{y}^{-1}}{2}(1+a_9) & -\frac{a_7}{2} & & & \frac{\bar{y}^{-1}}{2}(1+a_9) & -\frac{a_8}{2} \\ -a_2 & & a_3\zeta y + a_3\bar{y}^{-1} & a_1 & & & -a_3\bar{y}^{-1} & a_2 \\ -a_2 & & a_3\zeta y + a_3\bar{y}^{-1} & a_1 - 1 & 1 & & -a_3\bar{y}^{-1} & a_2 \\ &&&&& 1 && \\ \frac{\zeta^{-1}}{2}a_8 & & -\frac{y}{2}(1+a_9) + \frac{\zeta^{-1}\bar{y}^{-1}}{2}(1-a_9) & -\frac{\zeta^{-1}}{2}a_7 & & & -\frac{\zeta^{-1}\bar{y}^{-1}}{2}(1-a_9) & -\frac{\zeta^{-1}}{2}a_8 \\ 1-a_5 & & a_6\zeta y + a_6\bar{y}^{-1} & a_4 & & & -a_6\bar{y}^{-1} & a_5 \end{pmatrix}$$

One checks the above matrix belongs to $Q_{x,y}$ if and only if the a_i's satisfy the conditions required by the definition of $\mathfrak{Y}$. The lemma follows by a similar argument as in lemma 4.4.12. $\square$

Definition 4.3.2. *We fix a constant C_v such that $C_v\mathrm{Vol}(\mathfrak{Y})$ is a p-adic integer.*

When we are moving things in p-adic families, this constant is not going to change.

4.3.2 Fourier-Jacobi Coefficient

Here we record a lemma on the Fourier-Jacobi coefficient for $f_v^\dagger \in I_n(\tau_v)$ and $\beta \in S_m(F_v)$.

Lemma 4.3.3. *If $\beta \notin S_m(\mathcal{O}_{F_v})^*$ then $FJ_\beta(f^\dagger; z, u, g, hy) = 0$. If $\beta \in S_n(\mathcal{O}_{F_v})^*$ then*

$$FJ_\beta(f^\dagger; z, u, g, 1) = f^\dagger(z, g'\eta)\omega_\beta(h, g'\eta^{-1})\Phi_{0,y}(u).\mathrm{Vol}(S_m(\mathcal{O}_{F_v})),$$

where $g' = \begin{pmatrix} 1_{n-m} & \\ & -1_{n-m} \end{pmatrix} g \begin{pmatrix} 1_{n-m} & \\ & -1_{n-m} \end{pmatrix}$.

The proof is similar to [SU]11.4.16.

4.4 p-adic computations

In this section we first prove that under some 'generic conditions' the unique up to scalar nearly ordinary vector in $I(\rho)$ is just the unique up to scalar vector with certain prescribed level action. Then we construct a section $F^\dagger$ in $I(\rho^\vee)$ which is the pull back of a Siegel section $f^\dagger$ supported in the big cell. We can understand the level action of this section. Then we define F^0 to be the image of $F^\dagger$ under the intertwining operator. By checking the level action of F^0 we can prove that it is just the nearly ordinary vector.

4.4.1 Nearly Ordinary Sections

Let $\lambda_1, ..., \lambda_n$ be n characters of $\mathbb{Q}_p^\times$, $\pi = Ind_B^{GL_n}(\lambda_1, ..., \lambda_n)$.

Definition 4.4.1. *Let $n = r + s$ and $\underline{k} = (c_{r+s}, ..., c_{s+1}; c_1, ..., c_s)$ is a weight. We say $(\lambda_1, ...\lambda_n)$ is nearly ordinary with respect to $\underline{k}$ if the set:*

$$\begin{aligned} & \{\mathrm{val}_p\lambda_1(p), ..., \mathrm{val}_p\lambda_n(p)\} \\ = \quad & \{c_1 + s - 1 - \tfrac{n}{2} + \tfrac{1}{2}, c_2 + s - 2 - \tfrac{n}{2} + \tfrac{1}{2}, ..., c_s - \tfrac{n}{2} + \tfrac{1}{2}, c_{s+1} + r + s - 1 - \tfrac{n}{2} + \tfrac{1}{2}, ..., c_{r+s} + s - \tfrac{n}{2} + \tfrac{1}{2}\} \end{aligned}$$

We denote the above as $\{\kappa_1, ..., \kappa_{r+s}\}$, thus $\kappa_1 > ... > \kappa_{r+s}$.

Let $\mathcal{A}_p := \mathbb{Z}_p[t_1, t_2, ..., t_n, t_n^{-1}]$ be the Atkin-Lehner ring of $G(\mathbb{Q}_p)$, where t_i is defined by $t_i = N(\mathbb{Z}_p)\alpha_i N(\mathbb{Z}_p)$, $\alpha_i = \begin{pmatrix} 1_{n-i} & \\ & p1_i \end{pmatrix}$, t_i acts on $\pi^{N(\mathbb{Z}_p)}$ by

$$v|t_i = \sum_{x \in N|\alpha_i^{-1} N \alpha_i} x_i \alpha_i^{-1} v.$$

We also define a normalized action with respect to the weight $\underline{k}$ ([Hida04]):

$$v||t_i := \delta(\alpha_i)^{-1/2} p^{\kappa_1 + \ldots + \kappa_i} v|t_i$$

Definition 4.4.2. *A vector $v \in \pi$ is called nearly ordinary if it is an eigenvector for all $||t_i$'s with eigenvalues that are p-adic units.*

We identify π as the set of functions on $GL_n(\mathbb{Q}_p)$:

$$\pi = \{f : GL_n(\mathbb{Q}_p) \to \mathbb{C}, f(bx) = \lambda(b)\delta(b)^{1/2} f(x)\}.$$

Let w_ℓ be the longest Weyl element $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & ... & & \\ 1 & & & \end{pmatrix}$, f^ℓ be the element in π such that f^ℓ is supported in $Bw_\ell N(\mathbb{Z}_p)$ and invariant under $N(\mathbb{Z}_p)$; this is unique up to scalar. We have:

Lemma 4.4.1. *f^ℓ is an eigenvector for all t_i's.*

Proof. Note that for any i, $t_i f^\ell$ is invariant under $N(\mathbb{Z}_p)$. By looking at the defining $v|t_i$ under the above model for π it is not hard to see that the section is supported in $B(\mathbb{Q}_p)w_\ell B(\mathbb{Z}_p)$. So $f^\ell||t_i$ must be a multiple of f^ℓ. □

Lemma 4.4.2. *Suppose that $(\lambda_1, ..., \lambda_n)$ is nearly ordinary with respect to $\underline{k}$ and suppose*

$$\nu_p(\lambda_1(p)) > \nu_p(\lambda_2(p)) > ... > \nu_p(\lambda_n(p))$$

then the eigenvalues of $\|t_i$ acting on f^ℓ are p-adic units. In other words f^ℓ is an ordinary vector.

Proof. A straightforward computation gives that

$$f^\ell||t_i = \lambda_1 ... \lambda_i(p^{-1}) p^{\kappa_1 + \ldots + \kappa_i} f^\ell$$

which is clearly a p-adic unit by the definition of $(\lambda_1, ..., \lambda_n)$ to be nearly ordinary with respect to $\underline{k}$. $\square$

Lemma 4.4.3. *Let $\lambda_1, ..., \lambda_{a+2b}$ be a set characters of $\mathbb{Q}_p^\times$ such that $cond(\lambda_{a+2b}) > ..., > cond(\lambda_{b+1}) > cond(\lambda_1) > ... > cond(\lambda_b)$. In this case we say $\lambda := (\lambda_1, ..., \lambda_{a+2b})$ is generic and we defined a subgroup: K_λ is defined to be the subgroup of $GL_{a+2b}(\mathbb{Z}_p)$ whose below diagonal entries of the ith column are divisible by $cond(\lambda_{a+2b+1-i})$ for $1 \leq i \leq a+b$, and the left to diagonal entries of the jth row are divisble by $cond(\lambda_{a+2b+1-j})$for $a+b+2 \leq j \leq a+2b$ and λ^{op} a character defined by:*

$$\lambda_{a+2b}(g_{11})\lambda_{a+2b-1}(g_{22})...\lambda_1(g_{a+2b\ a+2b})$$

then f^ℓ is the unique (up to scalar) vector in π such that the action of K_λ is given by multiplying λ.

Proof. This can be proven in the same way as [SU]9.2.6. $\square$

We let $w_1 := \begin{pmatrix} 1 & & & & & \\ & ... & & & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & ... & \\ & & & 1 & & \end{pmatrix}$

Now let $\tilde{B} = B^{w_1}$ and $\tilde{K}_\lambda = K_\lambda^{w_1}$.

Corollary 4.4.1. *Denote $a_i := \nu_p(\lambda_i(p))$. Suppose $\lambda_1, ..., \lambda_{a+2b}$ are such that $cond(\lambda_1) > ... > cond(\lambda_{a+2b})$ and $a_1 < ... < a_{a+b} < a_{a+2b} < ... < a_{a+b+1}$, then the unique (up to scalar) ordinary section with respect to $\tilde{B}$ is*

$$f(x)^{ord} = \begin{cases} \lambda_1(g_{11})...\lambda_{a+2b}(g_{a+2b,a+2b}), & g \in \tilde{K}_\lambda. \\ 0 & \textit{otherwise} . \end{cases}$$

Proof. We only need to prove that $\pi(w_1)f^{ord}(x)$ is ordinary with respect to $\tilde{B}^{w_1}$. Let $\lambda'_1 = \lambda_{a+b+1}, ..., \lambda'_b = \lambda_{a+2b}, \lambda'_{b+1} = \lambda_{a+b}, ..., \lambda'_{a+2b} = \lambda_1$, then λ' satisfies lemma 4.4.2 and thus the ordinary section for B (up to scalar) is $f^\ell_{\lambda'}$. λ' also satisfies the assumptions of lemma 4.4.3 so $f^\ell_{\lambda'}$ is the unique section such that the action of K_λ is given by $\lambda'_{a+2b}(g_{11})...\lambda'_1(g_{a+2b,a+2b})$. But λ is clearly regular, so $Ind_B^{GL_{a+2b}}(\lambda) \simeq Ind_B^{GL_{a+2b}}(\lambda')$. So the ordinary section of $Ind_B^{GL_{a+2b}}(\lambda)$ for B also has the action of K_λ given by this character. It is easy to check that $\pi(w_1)f^{ord}$ has this property and the uniqueness (up to scalar) gives the result. $\square$

4.4.2 Pull Back Sections

In this section we construct a Siegel section on $U(a+2b+1,a+2b+1)$ which pulls back to the nearly ordinary Klingen sections on $U(a+b+1,b+1)$. We need to re-index the rows and columns since we are going to study large block matrices and the new index will greatly simplify the explanation. One can check that the Klingen Eisenstein series we construct in this section, when going back to our previous index, is nearly ordinary with respect to the Borel : $\begin{pmatrix} * & * & & * & * \\ & * & & * & * \\ & * & * & * & * \\ & & & * & \\ & & & * & * \end{pmatrix}$, where the diagonal blocks are upper, upper, upper, lower, lower triangular, while the one we need is nearly ordinary with respect to the Borel $\begin{pmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & & * \\ & & & * & * \end{pmatrix}$ (it is for this one we can use the Λ-adic Fourier-Jacobi expansions). (here the blocks are with respect to the partition: $b+1+a+b+1$.) However we will see that the nearly ordinary sections with respect to different Borels only differ by right translation by some Weyl element depending on a and b. We will specify this Weyl element when doing arithmetic applications.

Now we explain the new index. Let $V_{s,b}$ be the hermitian space with metric $\begin{pmatrix} \zeta 1_a & & \\ & & 1_b \\ & -1_b & \end{pmatrix}$, $V_{a,b+1}$ be the hermitian space with metric $\begin{pmatrix} \zeta 1_a & & \\ & & 1_{b+1} \\ & -1_{b+1} & \end{pmatrix}$. The matrix S for the embedding:

$U(V_{a,b}) \times U(V_{a,b+1}) \hookrightarrow U(V_{a+2b+1})$ becomes:

$$\begin{pmatrix}
1 & & & & -\frac{\zeta}{2} & & & \\
 & 1 & & & & & & \frac{1}{2} \\
 & & 1 & & & & & \\
 & & & -1 & & \frac{1}{2} & & \\
-1 & & & & -\frac{\zeta}{2} & & & \\
 & & & 1 & & \frac{1}{2} & & \\
 & & & & & & 1 & \\
 & -1 & & & & & & -\frac{1}{2}
\end{pmatrix}$$

Siegel-Weil section at p

τ: character of $\mathcal{K}_{\wp}^{\times} = \mathcal{K}_{v}^{\times} \times \mathcal{K}_{\bar{v}}^{\times} = Q_p^{\times} \times Q_p^{\times}$

$\tau = (\tau_1, \tau_2^{-1})$, p^{s_i} being the conductor of $\tau_i, i = 1, 2$.

Let $\chi_1, ...\chi_a, \chi_{a+1}, ...\chi_{a+2b}$ be characters of $Q_p^{\times}$ whose conductors are $p^{t_1}, ..., p^{t_{a+2b}}$. Suppose we are in the:

Generic case:

$$t_1 > t_2 > ... > t_{a+b} > s_1 > t_{a+b+1} > ... > t_{a+2b} > s_2$$

Also, let $\xi_i = \chi_i \tau_1^{-1}$ for $1 \leq i \leq a+b$

$\xi_j = \chi_j^{-1} \tau_2$ for $a+b+2 \leq j \leq a+2b+1$. $\xi_{a+2b+1} = 1$.

Let Φ_1 be the following Schwartz functions: let Γ be the subgroup of $GL_{a+2b+1}(\mathbb{Z}_p)$ consists of matrices $\gamma = (\gamma_{ij})$ such that p^{t_k} divides the below diagonal entries of the k^{th} column for $1 \leq k \leq a+b$ and p^{s_1} divides γ_{ij} when $a+b+2 \leq j \leq a+2b+1$, $i \leq a+b+1$ or $i > j$.

Let $\xi'_i = \chi_i \tau_2^{-1}$ $1 \leq i \leq a+b$

$\xi_j = \chi_j^{-1} \tau_1$, $a+b+2 \leq j \leq a+2b+1$

$\xi'_{a+b+1} = \tau_1 \tau_2^{-1}$. (thus $\xi'_k = \xi_k \tau_1 \tau_2^{-1}$ for any k).

Definition 4.4.3.

$$\Phi_1(x) = \begin{cases} 0 & x \notin \Gamma \\ \prod_{k=1}^{a+b+1} \xi'_k(x_{kk}) & x \in \Gamma \end{cases}$$

Now we define another Schwartz function Φ_2.

Let $\mathfrak{X}$ be the following set: if $\mathfrak{X} \ni x = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$ is in the block form with respect to the partition: $a + 2b + 1 = a + b + 1 + b$, then:

- x has entries in $\mathbb{Z}_p$;
- $\begin{pmatrix} A_{11} & A_{14} \\ A_{21} & A_{24} \end{pmatrix}$ has all the ℓ-upper-left minors A_ℓ so that $(detA_\ell) \in \mathbb{Z}_p^\times$ for $\ell = 1, 2, ..., a + b$;
- and A_{42} has all the ℓ-upper-left minors B_ℓ so that $(detB_\ell) \in \mathbb{Z}_p^\times$ for $\ell = 1, 2, ..., b$.

We define:

$$\Phi_\xi(x) = \begin{cases} 0 & x \notin \mathfrak{X} \\ \xi_1/\xi_2(detA_1)...\xi_{a+b-1}/\xi_{a+b}(detA_{a+b}) & \\ \times\xi_{a+b+2}/\xi_{a+b+3}(detB_1)...\xi_{a+2b}/\xi_{a+2b+1}(detB_{b-1})\xi_{a+2b+1}(detB_b). & x \in \mathfrak{X} \end{cases}$$

Let $\Phi_2(x) := \hat{\Phi}_\xi(x) = \int_{M_{a+2b+1}}(Q_p)\Phi_\xi(y)e_p(-tryx^t)dy$.

Let Φ be a Schwartz function on

$M_{a+2b+1,2(a+2b+1)}(Q_p)$ by:

$$\Phi(X, Y) := \Phi_1(X)\Phi_2(Y).$$

and define a Siegel-(Weil) section by:

$$f^\Phi(g) = \tau_2(detg)|detg|_p^{-s+\frac{a+2b+1}{2}} \times \int_{GL_{a+2b+1}(Q_p}\Phi((0, X)g)\tau_1^{-1}\tau_2(detX)|detX|_p^{-2s+a+2b+1}d^\times X.$$

Lemma 4.4.4. *If $\gamma \in \Gamma$, then:*

$$\Phi_\xi({}^t\gamma X) = \prod_{k+1}^{a+2b+1}(\xi_k(\gamma_{kk}))\Phi_\xi(X)$$

Proof. Straightforward. □

<u>Fourier Coefficients</u> (at p)

If $\beta \in \mathrm{Herm}_{a+2b+1}(K)$ the Fourier coefficient is defined by:

$$\begin{aligned}
f^{\Phi}_{\beta}(1,s) &= \int_{M_{a+2b+1}}(Q_p) f^{\Phi}(\begin{pmatrix} & 1_{a+2b+1} \\ -1_{a+2b+1} & \end{pmatrix}\begin{pmatrix} 1 & N \\ & 1 \end{pmatrix}) e_p(-tr\beta N)dN \\
&= \int_{M_{a+2b+1}(Q_p)}\int_{GL_{a+2b+1}(Q_p)} \Phi((0,X)\begin{pmatrix} & 1_{a+2b+1} \\ -1_{a+2b+1} & -N \end{pmatrix})\tau_1^{-1}\tau_2(\det X) \\
&\quad \times |\det X|_p^{-2s+a+2b+1} e_p(-tr\beta N)dNd^{\times}X \\
&= \int_{GL_{a+2b+1}(Q_p)} \Phi_1(-X)\Phi_{\xi}(-{}^tX^{-1}{}^t\beta)\tau_1^{-1}\tau_2(detX)|detX|_p^{-2s}d^{\times}X \\
&= \tau_1^{-1}\tau_2(-1)vol(\Gamma)\Phi_{\xi}({}^t\beta).
\end{aligned}$$

Definition 4.4.4. *Let $\tilde{f}^{\dagger} = \tilde{f}^{\dagger}_{a+2b+1}$ be the Siegel section supported on $Q(\mathbb{Q}_p)w_{a+2b+1}\begin{pmatrix} 1 & M_{a+2b+1}(\mathbb{Z}_p) \\ & 1 \end{pmatrix}$ and $\tilde{f}^{\dagger}(w\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}) = 1$ for $X \in M_{a+2b+1}(\mathbb{Z}_p)$.*

Lemma 4.4.5.

$$\tilde{f}^{\dagger}_{\beta}(1) = \begin{cases} 1 & \beta \in M_{a+2b+1}(\mathbb{Z}_p) \\ 0 & \beta \notin M_{a+2b+1}(\mathbb{Z}_p) \end{cases}$$

(here we used the projection of β into its first component in $\mathcal{K}_v = F_v \times F_v$) where the first component correspond to the element inside our CM-type Σ under $\iota := \mathbb{C} \simeq \mathbb{C}_p$ (see section 2.1).

Definition 4.4.5.

$$f^{\dagger} := \frac{f^{\Phi}}{\tau_1^{-1}\tau_2(-1)\mathrm{Vol}(\Gamma)}$$

Thus $f^{\dagger}_{\beta} = \Phi_{\xi}({}^t\beta)$.

Remark 4.4.1. *This ensures that when we are moving our Eisenstein datum p-adically, the Siegel Eisenstein series also move p-adic analytically.*

Now we recall a lemma from [SU]11.4.12. which will be useful later.

Lemma 4.4.6. *Suppose $v|p$ and $\beta \in S_n(\mathbb{Q}_v), \det\beta \neq 0$.*
(i) If $\beta \notin S_n(\mathbb{Z}_v)$ then $M(z, \tilde{f}^{\dagger}_n)_{\beta}(-z,1) = 0$;
(ii) Suppose $\beta \in S_n(\mathbb{Z}_v)$. Let $c := \mathrm{ord}_v(\mathrm{cond}(\tau'))$. Then:

$$M(z, \tilde{f}^{\dagger}_n)_{\beta}(-z,1) = \tau'(\det\beta)|\det\beta|_v^{-2z}\mathfrak{g}(\bar{\tau}')^n c_n(\tau', z).$$

where

$$c_n(\tau', z) := \begin{cases} \tau'(p^{nc})p^{2ncz-cn(n+1)/2} & c > 0 \\ p^{2nz-n(n+1)/2} & c = 0. \end{cases}$$

Note that our $\tilde{f}^\dagger$ is the $f^\dagger$ in [SU] and our τ is their χ.

Now we want to write down our Siegel-Weil section f^Φ in terms of $\tilde{f}^\dagger$. First we prove the following:

Lemma 4.4.7. *Suppose Φ_ξ is the function on $M_n(Q_p)$ defined as follows: if cond$(\xi_i) = (p^{t_i})$ for $i = 1, 2, ..., n$, then*

$$\tilde{\mathfrak{X}}_\xi := N(\mathbb{Z}_p) \begin{pmatrix} p^{-t_1}\mathbb{Z}_p^\times & & \\ & ... & \\ & & p^{-t_n}\mathbb{Z}_p^\times \end{pmatrix} N^{opp}(\mathbb{Z}_p).$$

then the Fourier transform $\tilde{\Phi}_\xi$ is the following function:

$$\frac{\tilde{\Phi}_\xi(x)}{\prod_{i=1}^n G(\xi_i)} = \begin{cases} 0 & x \notin \tilde{\mathfrak{X}}_\xi \\ \prod_{i=1}^n \bar{\xi}_i(x_i p^{t_i}) & \tilde{\mathfrak{X}}_\xi \ni x = \begin{pmatrix} 1 & & \\ ... & ... & \\ ... & ... & 1 \end{pmatrix} \begin{pmatrix} x_1 & & \\ & ... & \\ & & x_n \end{pmatrix} \begin{pmatrix} 1 & ... & ... \\ & ... & ... \\ & & 1 \end{pmatrix} \end{cases}$$

Proof. First suppose x is supported in the "big cell": $N(\mathbb{Q}_p)T(\mathbb{Q}_p)N^{opp}(\mathbb{Q}_p)$ where the superscript 'opp' means the opposite parabolic. It is easily seen that we can write x in terms of block matrices:

$$x = \begin{pmatrix} 1_{n-1} & u \\ & 1 \end{pmatrix} \begin{pmatrix} z & \\ & w \end{pmatrix} \begin{pmatrix} 1_{n-1} & \\ v & 1 \end{pmatrix}$$

where $z \in M_{n-1}(\mathbb{Q}_p)$ $w \in \mathbb{Q}_p$.

A first observation is that $\tilde{\Phi}_\xi$ is invariant under right multiplication by $N^{opp}(\mathbb{Z}_p)$ and left multiplication by $N(\mathbb{Z}_p)$. We show that $v \in M_{1\times(n-1)}(\mathbb{Z}_p)$ if $\tilde{\Phi}_\xi(x) \neq 0$. By definition:

$$\begin{aligned}
\tilde{\Phi}_\xi(x) &= \int_{M_n(Q_p)} \Phi_\xi(y) e_p(try^t x) dy \\
& (y = \begin{pmatrix} 1_{n-1} & \\ \ell & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} & \\ & b \end{pmatrix} \begin{pmatrix} a_{n-1} & m \\ & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} & a_{n-1}m \\ \ell a_{n-1} & \ell a_{n-1} m + b \end{pmatrix}) \\
&= \int_{a \in \mathfrak{X}_{\xi,n-1}, m \in M(\mathbb{Z}_p), \ell \in M(\mathbb{Z}_p), b \in \mathbb{Z}_p^\times} \Phi_\xi(\begin{pmatrix} 1 & \\ \ell & 1 \end{pmatrix} \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}) \\
&\times \; e_p(tr \begin{pmatrix} 1 & \\ {}^t m & 1 \end{pmatrix} \begin{pmatrix} {}^t a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & {}^t \ell \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} z & \\ & w \end{pmatrix} \begin{pmatrix} 1 & \\ v & 1 \end{pmatrix}) dy \\
&= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(tr \begin{pmatrix} 1 & \\ {}^t m + v & 1 \end{pmatrix} \begin{pmatrix} {}^t a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & {}^t \ell + u \\ & 1 \end{pmatrix} \begin{pmatrix} z & \\ & w \end{pmatrix}) dy \\
&= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(tr \begin{pmatrix} a & a(\ell+u) \\ (m+v)a & (m+v)a(\ell+u)+b \end{pmatrix} \begin{pmatrix} z & \\ & w \end{pmatrix}) dy \\
&= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(az + ((m+v)a(\ell+u)+b)w) dy
\end{aligned}$$

(Note that Φ_ξ is invariant under transpose.)

If $\tilde{\Phi}_\xi(x) \neq 0$, then it follows from the last expression that: $w \in p^{-t_n}\mathbb{Z}_p^\times$ and suppose $v \notin M_{1\times(n-1)}(\mathbb{Z}_p)$, then $m + v \notin M_{1\times(n-1)}(\mathbb{Z}_p)$. We let a, m, b to be fixed and let ℓ to vary in $M_{1\times(n-1)}(\mathbb{Z}_p)$, we find that this integral must be 0. (Notice that $a \in \mathfrak{X}_{\xi,n-1}$ and $w \in p^{-t_n}\mathbb{Z}_p^\times$, thus $(m+v)aw \notin M_{1\times n-1}(\mathbb{Z}_p)$) Thus a contradiction. Therefore, $v \in M_{1\times n-1}(\mathbb{Z}_p)$, similarly $u \in M_{n-1,1}(\mathbb{Z}_p)$. Thus by the observation at the beginning of the proof we may assume $u = 0$ and $v = 0$ without lose of generality.

Thus if we write $\phi_{\xi,n-1}$ as the restriction of Φ_ξ to the up-left $(n-1)\times(n-1)$ minor,

$$\begin{aligned}
\tilde{\Phi}_\xi(x) &= \int \Phi_\xi(\begin{pmatrix} a & \\ & b \end{pmatrix}) e_p(az + (ma\ell + b)w) dy \\
&= p^{-nt_n} \mathfrak{g}(\xi_n) \bar{\xi}_n(wp^{t_n}) \int_{a \in \mathfrak{X}_{\xi,n-1}} \Phi_{\xi,n-1}(a) e_p(az) dy
\end{aligned}$$

by an induction procedure one gets:

$$\tilde{\Phi}_\xi(x) = \begin{cases} 0 & x \notin \tilde{\mathfrak{X}}_{\xi,n} \\ p^{-\sum_{i=1}^n ic_i} \prod_{i=1}^n \mathfrak{g}(\xi_i) \prod_{i=1}^n \bar{\xi}_i(x_i p^{t_i}) & x \in \tilde{\mathfrak{X}}_\xi. \end{cases}$$

Since $\tilde{\mathfrak{X}}_{\xi,n}$ is compact, now that we have proved that $\tilde{\Phi}_{\xi,n}$ when restricting to the "big cell" has support in $\tilde{\mathfrak{X}}_{\xi,n}$, therefore $\tilde{\Phi}_{\xi,n}$ itself must be supported in $\tilde{\mathfrak{X}}_{\xi,n}$. ☐

Lemma 4.4.8. *Let $\tilde{\mathfrak{X}}_\xi$ be the support of $\Phi_2 = \hat{\Phi}_\xi$, then a complete representative of $\tilde{\mathfrak{X}}_\xi \mathrm{mod} M_{a+2b+1}(\mathbb{Z}_p)$ is given by:*

$$\begin{pmatrix} A & & B \\ C & & D \\ & & \\ & E & \end{pmatrix}$$

where the blocks are with respect to the partition $a+b+1+b$ where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ runs over the following set:

$$\begin{pmatrix} 1 & m_{12} & \dots & m_{1,a+b} \\ & \dots & \dots & \dots \\ & & \dots & m_{a+b-1,a+b} \\ & & & 1 \end{pmatrix} \begin{pmatrix} x_1 & & \\ & \dots & & \\ & & \dots & \\ & & & x_{a+b} \end{pmatrix} \begin{pmatrix} 1 & & & \\ n_{21} & \dots & & \\ \dots & \dots & \dots & \\ n_{a+b,1} & \dots & n_{a+b,a+b-1} & 1 \end{pmatrix}$$

where x_i runs over $p^{-t_i}\mathbb{Z}_p^\times$ mod $\mathbb{Z}_p$, m_{ij} runs over $\mathbb{Z}_p$ modp^{t_j} and n_{ij} runs over $\mathbb{Z}_p$ mod p^{t_i}, and E runs over the following set:

$$\begin{pmatrix} 1 & k_{12} & \dots & k_{1,b} \\ & \dots & \dots & \dots \\ & & \dots & k_{b-1,b} \\ & & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & \\ & \dots & & \\ & & \dots & \\ & & & y_b \end{pmatrix} \begin{pmatrix} 1 & & & \\ \ell_{21} & \dots & & \\ \dots & \dots & \dots & \\ \ell_{b,1} & \dots & \ell_{b,b-1} & 1 \end{pmatrix}$$

where y_i runs over $p^{-t_{i+a+b}}\mathbb{Z}_p$ mod$\mathbb{Z}_p$; k_{ij} runs over $\mathbb{Z}_p$ mod$p^{t_{a+b+j}}$; ℓ_{ij} runs over $\mathbb{Z}_p$ mod$p^{t_{a+b+i}}$.

Proof. This is elementary and we omit it here. ☐

Now we define several sets: Let $\mathfrak{B}'$ be the set of $(a+b) \times (a+b)$ upper triangular matrices of the form

$$\begin{pmatrix} 1 & m_{12} & ... & m_{1,a+b} \\ & ... & ... & ... \\ & & ... & m_{a+b-1,a+b} \\ & & & 1 \end{pmatrix} \begin{pmatrix} x_1 & & \\ & ... & & \\ & & ... & \\ & & & x_{a+b} \end{pmatrix}$$

where x_i runs over $p^{-t_i}\mathbb{Z}_p^\times \ mod\ \mathbb{Z}_p$, m_{ij} runs over $\mathbb{Z}_p\ modp^{t_j}$.

Let $\mathfrak{C}'$ be the set of $b \times b$ lower triangular matrices of the form

$$\begin{pmatrix} 1 & & & \\ n_{21} & ... & & \\ ... & ... & ... & \\ n_{a+b,1} & ... & n_{a+b,a+b-1} & 1 \end{pmatrix}$$

where n_{ij} runs over $\mathbb{Z}_p\ modp^{t_i}$

Let $\mathfrak{E}'$ be the set of $b \times b$ upper triangular matrices of the form

$$\begin{pmatrix} 1 & k_{12} & ... & k_{1,b} \\ & ... & ... & ... \\ & & ... & k_{b-1,b} \\ & & & 1 \end{pmatrix}$$

where k_{ij} runs over $\mathbb{Z}_p\ modp^{t_{a+b+j}}$.

Let $\mathfrak{D}'$ be the set of $(a+b) \times (a+b)$ lower triangular matrices of the form

$$\begin{pmatrix} y_1 & & & \\ & ... & & \\ & & ... & \\ & & & y_b \end{pmatrix} \begin{pmatrix} 1 & & & \\ \ell_{21} & ... & & \\ ... & ... & ... & \\ \ell_{b,1} & ... & \ell_{b,b-1} & 1 \end{pmatrix}$$

where y_i runs over $p^{-t_{i+a+b}}\mathbb{Z}_p\ mod\mathbb{Z}_p$; ℓ_{ij} runs over $\mathbb{Z}_p\ modp^{t_{a+b+i}}$. Also we define for $g \in GL_{a+2b}(\mathbb{Q}_p)$, $g^\iota = \begin{pmatrix} & 1_{a\times a} & \\ & & 1_{b\times b} \\ 1_{b\times b} & & \end{pmatrix} g \begin{pmatrix} & & 1_{b\times b} \\ 1_{a\times a} & & \\ & 1_{b\times b} & \end{pmatrix}$.

Corollary 4.4.2.

$$f^\dagger(z,g) = p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+1+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)$$

$$\times \sum_{A,B,C,D,E} \prod_{i=1}^{a} \bar{\xi}_i(A_{ii}) \prod_{i=1}^{b} \bar{\xi}_{a+i,a+i}(D_{ii}) \times \prod_{i=1}^{b} \bar{\xi}_{a+b+1+i}(E_{ii}) \tilde{f}^\dagger(z, g \begin{pmatrix} & A & & B \\ & C & & D \\ 1_{a+2b+1} & & & \\ & & & \\ & & E & \\ & & 1_{a+2b+1} & \end{pmatrix})$$

Proof. using the lemma above, we see that both hand sides have the same β'th fourier coefficients for all $\beta \in S_{a+2b+1}(Q_p)$ thus they must be the same. $\square$

thus if B', C', D', E' runs over the set $\mathfrak{B}', \mathfrak{C}', \mathfrak{D}', \mathfrak{E}'$, then

$$\begin{aligned}
f^\dagger(z,g) &= p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+1+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1) \\
&\quad \times \sum_{B',C',D',E'} \prod_{i=1}^{a+b} \bar{\xi}_i(B'_{ii}) \prod_{i=1}^{b} \xi_{a+b+i}(D'_{ii}) \\
&\quad \times \tilde{f}^\dagger(z, g\alpha(\begin{pmatrix} B' & & & \\ & 1 & & \\ & & C' & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} E' & \\ & D' \end{pmatrix}^{\iota}) \begin{pmatrix} 1 & A' \\ & 1 \end{pmatrix} \alpha(\begin{pmatrix} B' & & & \\ & 1 & & \\ & & C' & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} E' & \\ & D' \end{pmatrix}^{\iota})^{-1}) \\
&= p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+1+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1) \\
&\quad \times \sum_{B',C',D',E'} \prod_{i=1}^{a+b} \bar{\xi}_i(B'_{ii}) \prod_{i=1}^{b} \xi_{a+b+i}(D'_{ii}) \prod_{i=1}^{B} \bar{\tau}_1(B'_{ii}) \prod_{i=1}^{b} \bar{\tau}_2(D'_{ii}) \\
&\quad \times \tilde{f}^\dagger(z, g\alpha(\begin{pmatrix} B' & & & \\ & 1 & & \\ & & C' & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} E' & \\ & D' \end{pmatrix}^{\iota}) \begin{pmatrix} 1 & A' \\ & 1 \end{pmatrix})
\end{aligned}$$

where $A' = \begin{pmatrix} p^{-t_1} & & & & & & & & \\ & \cdots & & & & & & & \\ & & p^{-t_2} & & & & & & \\ & & & & & & p^{-t_{a+1}} & & \\ & & & & & & & \cdots & \\ & & & & & & & & p^{-t_{a+b}} \\ & & & p^{-t_{a+b+1}} & & & & & \\ & & & & \cdots & & & & \\ & & & & & p^{-t_{a+2b}} & & & \end{pmatrix}$

Definition 4.4.6. *(pull back section) If f is a Siegel section and $\phi \in \pi_p$, then*

$$F_\phi(z, f, g) := \int_{GL_{a+2b}(\mathbf{Q}_p)} f(z, \gamma\alpha(g, g_1)\gamma^{-1})\bar{\tau}(\det g_1)\rho(g_1)\phi dg_1$$

Now we define a subset K of $GL_{a+2b+2}(\mathbb{Z}_p)$ to be so that $k \in K$ if and only if: p^{t_i} divides the below diagonal entries of the ith column for $1 \leq i \leq a+b$, p^{s_1} divides the below diagonal entries of the $(a+b+1)$th column, and $p^{t_{a+b+j}}$ divides the right to diagonal entries of the $(a+b+1+j)$ th ROW for $1 \leq j \leq b-1$.

We also define ν, a character of K by:

$$\nu(k) = \tau_1(k_{a+b+1,a+b+1})\tau_2(k_{a+2b+2,a+2b+2}) \prod_{i=1}^{a+b} \chi_i(k_{ii}) \prod_{i=1}^{b} \chi_{a+b+i}(k_{a+b+i+1,a+b+i+1})$$

for any $k \in K$, we also define $\tilde{\nu}$ a character of $\tilde{K}$ by:

$$\tilde{\nu}(k) = \prod_{i=1}^{b} \chi_{a+i}(k_{i,i}) \prod_{i=1}^{a} \chi_i(k_{b+i,b+i}) \prod_{i=1}^{b} \chi_{a+b+i}(k_{a+b+i,a+b+i})$$

Lemma 4.4.9. *Let $K' \subset K$ be the compact subset defined by:*

$K' \ni k = \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 \\ a_4 & a_5 & a_6 & b_3 & b_4 \\ a_7 & a_8 & a_9 & b_5 & b_6 \\ c_1 & c_2 & c_3 & d_1 & d_2 \\ c_4 & c_5 & c_6 & d_3 & d_4 \end{pmatrix}$ *(here the blocks are with respect to $a+b+1+b+1$) if and only if: $p^{t_{a+b+i}+t_j}$ divides the (i,j)th entry of c_1 for $1 \leq i \leq b$, $1 \leq j \leq a$ and $p^{t_{a+b+i}+t_{a+j}}$ divides the*

(i,j)th entry of c_2 for $1 \leq i \leq b$, $1 \leq j \leq b$.

(it is not hard to check that this is a group).

then: $F_\phi(z, f^\dagger, gk) = \nu(k) F_\phi(a, f^\dagger, g)$ for any $\phi \in \pi$ and $k \in K'$

Proof. this follows directly from the action of K' on the Siegel Weil section $f^\dagger$. $\square$

We define K'' to be the subset of K consists of matrices

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ c_1 & c_2 & & 1 & \\ & & & & 1 \end{pmatrix}$$

such that p^{t_j} divides the (i,j)th entry of c_1 for $1 \leq i \leq b$, $1 \leq j \leq a$ and $p^{t_{a+j}}$ divides the (i,j)th entry of c_2 for $1 \leq i \leq b$, $1 \leq j \leq b$.

Definition 4.4.7. *$\tilde{K} \subset GL_{a+2b}(\mathbb{Z}_p)$: $\begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}$ (blocks are with respect to $(b+a+b)$). the column's of a_3, a_6 are divisible by $p^{t_1}, \ldots, p^{t_a}$, the column's of a_4 are divisible by $p^{t_{a+1}}, \ldots, p^{t_{a+b}}$, $p^{t_{a+j}}$ divides the below diagonal entries of the i'th column of a_1, $(1 \leq i \leq b)$, p^{t_j} divides the below diagonal entries of the j's column of a_9 $(1 \leq j \leq a)$, $p^{t_{a+b+k}}$ divides the above diagonal entries of the k'th ROW of a_5.*

$\tilde{K}' \subset \tilde{K}$ is the set of $p^{t_{a+b+i}+t_{a+j}}$ divides the (i,j)th entry of a_4 for $1 \leq i \leq b$, $1 \leq j \leq b$ and $p^{t_{a+b+i}+t_j}$ divides the (i,j)th entry of a_6 for $1 \leq i \leq b$, $1 \leq j \leq a$. We also define $\tilde{K}''$ to be the subset of $\tilde{K}$ consists of matrices:

$$\begin{pmatrix} 1 & & \\ & 1 & \\ a_4 & a_6 & 1 \end{pmatrix}$$

such that $p^{t_{a+j}}$ divides the (i,j)th entry of a_4 for $1 \leq i \leq b$, $1 \leq j \leq b$ and p^{t_j} divides the (i,j)th entry of a_6 for $1 \leq i \leq b$, $1 \leq j \leq a$.

The following lemma would be useful in identifying our pull back section:

Lemma 4.4.10. *Suppose $F_\phi(z, f^\dagger, g)$ as a function of g is supported in PwK and $F_\phi(z, f^\dagger, gk) = \nu(k) F_\phi(z, f^\dagger, g)$ for $k \in K'$, and $F_\phi(z, f^\dagger, w)$ is invariant under the action of $(\tilde{K}'')^\iota$. then $F_\phi(a, f^\dagger, g)$ is the unique section (up to scalar) whose action by $k \in K$ is given by multiplying $\nu(k)$.*

Proof. This is easy from the fact that $K = K'K'' = K''K'$. The uniqueness follows from lemma 4.4.3. $\square$

We define a matrix w to be $\begin{pmatrix} 1_a & & \\ & & 1_{b+1} \\ & -1_{b+1} & \end{pmatrix}$. We also define Υ to be the element in $U(n,n)(F_v)(= U(n,n)(\mathbb{Q}_p))$ such that the projection to the first component of $\mathcal{K}_v = F_v \times F_v$ equals that of γ. (note that $\gamma \notin U(n,n)$).

Lemma 4.4.11. *If $\gamma\alpha(g,1)\gamma^{-1} \in supp\rho(\Upsilon)f^\dagger$ then $g \in PwK$.*

Proof. since $f^\dagger$ is of the form $\sum_{A\in\mathfrak{X}} \tilde{f}^\dagger(-\begin{pmatrix} 1 & A \\ & 1 \end{pmatrix})$, where $\mathfrak{X}$ is some set, we only have to check the lemma for each term.

First recall we defined: $$A' = \begin{pmatrix} -p^{-t_1} & & & & & & & & \\ & \cdots & & & & & & & \\ & & p^{-t_a} & & & & & & \\ & & & & & & -p^{t_{a+1}} & & \\ & & & & & & & \cdots & \\ & & & & & & & & -p^{t_{a+b}} \\ & & & & & & & & \\ & & & -p^{-t_{a+b+1}} & & & & & \\ & & & & \cdots & & & & \\ & & & & & -p^{-t_{a+2b}} & & & \end{pmatrix}$$

where the blocks are with respect to $a+b+1+b$. Let ζ_v and γ_v be the projection of ζ and γ_v to the first component of $\mathcal{K}_v \equiv F_v \times F_v$, then:

$$\gamma_v = \begin{pmatrix} \zeta_v^{-1} & & & & -\zeta_v^{-1} & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ \frac{1}{2} & & & & \frac{1}{2} & & & \\ & & & 1 & & 1 & & \\ & & & & & & 1 & \\ & 1 & & & & & & 1 \end{pmatrix} = \begin{pmatrix} 2\zeta_v^{-1} & & & & -\zeta_v^{-1} & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \frac{1}{2} & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ 1 & & & & 1 & & & \\ & & & 1 & & 1 & & \\ & & & & & & 1 & \\ & 1 & & & & & & 1 \end{pmatrix}$$

we denote the last term $\tilde{\gamma}_v$. Some times we omit the subscript v if no confusions arise.

Using the expression for $f^\dagger$ involving the B', C', D', E''s as above and the fact that $\gamma(m(g,1),g) \in Q$ and that K is invariant under the right multiplication of B's and C's, we only need to check that if $\tilde{\gamma}_v \alpha(g,1)\tilde{\gamma}_v^{-1} \in supp\rho(\Upsilon)\rho(\begin{pmatrix} 1 & A \\ & 1 \end{pmatrix})\tilde{f}^\dagger$, then $g \in PwK$. if $gw = \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 \\ a_4 & a_5 & a_6 & b_3 & b_4 \\ a_7 & a_8 & a_9 & b_5 & b_6 \\ c_1 & c_2 & c_3 & d_1 & d_2 \\ c_4 & c_5 & c_6 & d_3 & d_4 \end{pmatrix}$ then this is equivalent to

$$\begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ 1 & & & & 1 & & & \\ & & & 1 & & 1 & & \\ & & & & & & 1 & \\ & 1 & & & & & & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & & & b_1 & b_2 & \\ a_4 & a_5 & a_6 & & & b_3 & b_4 & \\ a_7 & a_8 & a_9 & & & b_5 & b_6 & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ c_1 & c_2 & c_3 & & & d_1 & d_2 & \\ c_4 & c_5 & c_6 & & & d_3 & d_4 & \\ & & & & & & & 1 \end{pmatrix} w_{a+2b+1} \times$$

$$
\begin{pmatrix}
p^{t_1} &&&&&&&&&&&&&& \\
& \dots &&&&&&&&&&&&& \\
&& p^{t_a} &&&&&&&&&&&& \\
&&& 1 &&&&&&&&&&& \\
&&&& 1 &&&&&&&&&& \\
&&&&& p^{t_{a+1}} &&&&&&&&& \\
&&&&&& \dots &&&&&&&& \\
&&&&&&& p^{t_{a+b}} &&&&&&& \\
&&&&&&&& 1 &&&&&& \\
&&&&&&&&& 1 &&&&& \\
&&&&&&&&&& 1 &&&& \\
&&&&&&&&&&& p^{-t_{a+b+1}} &&& \\
&&&&&&&&&&&& \dots && \\
&&&&&&&&&&&&& p^{-t_{a+2b}} &
\end{pmatrix}
$$

$$
\times \begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 1 &&&& \\
-1 &&&& 1 &&& \\
&&& -1 && 1 && \\
&&&&&& 1 & \\
& -1 &&&&&& 1
\end{pmatrix}
$$

$$
\times \begin{pmatrix}
p^{-t_1} &&&&&&&&&&&&& \\
& \dots &&&&&&&&&&&& \\
&& p^{-t_a} &&&&&&&&&&& \\
&&& 1 &&&&&&&&&& \\
&&&& 1 &&&&&&&&& \\
&&&&& p^{-t_{a+1}} &&&&&&&& \\
&&&&&& \dots &&&&&&& \\
&&&&&&& p^{-t_{a+b}} &&&&&& \\
&&&&&&&& 1 &&&&& \\
&&&&&&&&& 1 &&&& \\
&&&&&&&&&& 1 &&& \\
&&&&&&&&&&& p^{t_{a+b+1}} && \\
&&&&&&&&&&&& \dots & \\
&&&&&&&&&&&&& p^{t_{a+2b}}
\end{pmatrix} w_{a+2b+1}^{-1}
$$

is in $supp\tilde{f}^\dagger$. which is equivalent to

$$\tilde{\gamma}\alpha(g, \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix} \begin{pmatrix} p^{t_{a+b+1}} & & & & & \\ & \cdots & & & & \\ & & p^{-t_1} & & & \\ & & & \cdots & & \\ & & & & p^{-t_{a+1}} & \\ & & & & & \cdots \end{pmatrix}) w_{a+2b+1}\tilde{\gamma}^{-1}$$

belongs, and thus also, $\tilde{\gamma}\alpha(g,1)w_{a+2b+1}\tilde{\gamma}^{-1}$ belongs to:

$$supp\rho(\begin{pmatrix} p^{-t_1} &&&&&&&&&&&&& \\ & \cdots &&&&&&&&&&&& \\ && p^{-t_a} &&&&&&&&&&& \\ &&& 1 &&&&&&&&&& \\ &&&& 1 &&&&&&&&& \\ &&&&& p^{-t_{a+1}} &&&&&&&& \\ &&&&&& \cdots &&&&&&& \\ &&&&&&& p^{-t_{a+b}} &&&&&& \\ &&&&&&&& 1 &&&&& \\ &&&&&&&&& 1 &&&& \\ &&&&&&&&&& 1 &&& \\ &&&&&&&&&&& p^{t_{a+b+1}} && \\ &&&&&&&&&&&& \cdots & \\ &&&&&&&&&&&&& p^{t_{a+2b}} \end{pmatrix} w_{a+2b+1})\tilde{f}^\dagger.$$

the right hand side is contained in: $Q_t := Q.\{\begin{pmatrix} 1 & \\ S & 1 \end{pmatrix} : S \in S_t = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}\}$ where the blocks for S_t is with respect to $a+b+1+b$ and it consists of matrices such that $S_{ij} \in M(\mathbb{Z}_p)$, p^{t_i} divides the ith column for $1 \le i \le a$, $p^{t_{a+i}}$ divides the $(a+b+1+i)$th column for $1 \le i \le b$, $p^{t_{a+b+i}}$ divides the $(a+b+1+i)$th row for $1 \le i \le b$, and the (i,j)-th entry of S_{41} and S_{44} are divisible by $p^{t_{a+b+i}+t_j}$ and $p^{t_{a+b+i}+t_{a+j}}$ respectively. Observe that we have only to show that if

$$\tilde{\gamma}\alpha(gw,1)\begin{pmatrix} &&&&1&&& \\ &1&&&&&& \\ &&1&&&&& \\ &&&1&&&& \\ 1&&&&&&& \\ &&&&&1&& \\ &&&&&&1& \\ &&&&&&&1 \end{pmatrix}\tilde{\gamma}^{-1} \in Q_t \text{ then } g \in PwK, \text{ i.e. } gw \in PK^w \text{ for } K^w := wKw.$$

$$\tilde{\gamma}\alpha(g,1)w\tilde{\gamma}^{-1} = \begin{pmatrix} -a_1 & a_2 & a_3 & -b_1 & a_1 & b_1 & b_2 & \\ -a_4 & a_5 & a_6 & -b_3 & a_4 & b_3 & b_4 & \\ -a_7 & a_8 & a_9 & -b_5 & a_7 & b_5 & b_6 & \\ &&& 1 &&&& \\ 1-a_1 & a_2 & a_3 & -b_1 & a_1 & b_1 & b_2 & \\ -c_1 & c_2 & c_3 & -d_1 & c_1 & d_1 & d_2 & \\ -c_4 & c_5 & c_6 & -d_3 & c_4 & d_3 & d_4 & \\ -a_4 & a_5-1 & a_6 & -b_3 & a_4 & b_3 & b_4 & 1 \end{pmatrix} := H$$

thus if $H \in Q_t$, then $\exists S \in S_t$ such that:

$$\begin{pmatrix} 1-a_1 & a_2 & a_3 & -b_1 \\ -c_1 & c_2 & c_3 & 1-d_1 \\ -c_4 & c_5 & c_6 & -d_3 \\ -a_4 & a_5-1 & a_6 & -b_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & b_2 & \\ c_! & d_1 & d_2 & \\ c_4 & d_3 & d_4 & \\ a_4 & b_3 & b_4 & 1 \end{pmatrix} S$$

By looking at the 3rd row (block-wise), one finds $d_4 \neq 0$, by multiplying g by a matrix $\begin{pmatrix} 1 &&&& \times \\ & 1 &&& \times \\ && 1 & \times & \times \\ &&& 1 & \times \\ &&&& d_4^{-1} \end{pmatrix}$ (which does not change the assumption and conclusion) we may assume that

$d_4 = 1$ and $d_2 = 0, b_2 = 0, b_4 = 0, b_6 = 0, b_5 = 0$. So we assume that gw is of the form:

$$\begin{pmatrix} a_1 & a_2 & a_3 & b_1 & \\ a_4 & a_5 & a_6 & b_3 & \\ a_7 & a_8 & a_9 & & \\ c_1 & c_2 & c_3 & d_1 & \\ c_4 & c_5 & c_6 & d_3 & 1 \end{pmatrix}$$

Next by looking at the 2nd row (block-wise) and note that $d_2 = 0$ we find that d_1 is of the form

$$\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & ... & ... & \mathbb{Z}_p \\ p^{t_{a+1}}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... & \mathbb{Z}_p \\ ... & p^{t_{a+2}}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... \\ ... & ... & ... & ... & ... \\ p^{t_{a+1}}\mathbb{Z}_p & ... & ... & ... & \mathbb{Z}_p^\times \end{pmatrix}$$

and by looking at the 3rd row again we see $c_4 = (p^{t_1}\mathbb{Z}_p, ..., p^{t_a}\mathbb{Z}_p)$, $d_3 \in (p^{t_{a+1}}, ..., p^{t_{a+b}})$., $c_1 \in M_{b\times 1}(p^{t_1}\mathbb{Z}_p), M_{b\times 1}(p^{t_2}\mathbb{Z}_p), ..., M_{b\times 1}(p^{t_a}\mathbb{Z}_p))$, $c_2 \in M_{b\times b}(\mathbb{Z}_p)$, $c_3 \in M_{b\times 1}(\mathbb{Z}_p)$.

By looking at the 1st row and note that $b_2 = 0$ we know $a_1 \in \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & ... & ... & \mathbb{Z}_p \\ p^{t_1}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... & \mathbb{Z}_p \\ ... & p^{t_2}\mathbb{Z}_p & \mathbb{Z}_p^\times & ... & ... \\ ... & ... & ... & ... & ... \\ p^{t_1}\mathbb{Z}_p & ... & ... & ... & \mathbb{Z}_p^\times \end{pmatrix}$,
$b_1 \in (M_{a\times 1}(p^{t_{a+1}}\mathbb{Z}_p), M_{a\times 1}(p^{t_{a+2}}\mathbb{Z}_p), ..., M_{a\times 1}(p^{t_{a+b}}\mathbb{Z}_p))$. Finally look at the 4th row (block-wise), note that $b_4 = 0$, similarly, $a_4 \in (M_{b\times 1}(p^{t_1}\mathbb{Z}_p), M_{b\times 1}(p^{t_2}\mathbb{Z}_p), ..., M_{b\times 1}(p^{t_a}\mathbb{Z}_p))$,
$b_3 \in (M_{b\times 1}(p^{t_{a+1}}\mathbb{Z}_p), M_{b\times 1}(p^{t_{a+2}}\mathbb{Z}_p), ..., M_{b\times 1}(p^{t_{a+b}}\mathbb{Z}_p))$.
$a_5 - 1 \in \begin{pmatrix} M_{1\times b}(p^{t_{a+b+1}}\mathbb{Z}_p) \\ M_{1\times b}(p^{t_{a+b+2}}\mathbb{Z}_p) \\ ... \\ M_{1\times b}(p^{t_{a+2b}}\mathbb{Z}_p) \end{pmatrix}$, $a_6 \in \begin{pmatrix} p^{t_{a+b+1}}\mathbb{Z}_p \\ p^{t_{a+b+2}}\mathbb{Z}_p \\ ... \\ p^{t_{a+2b}}\mathbb{Z}_p \end{pmatrix}$, $a_2 \in M_{a\times b}(\mathbb{Z}_p)$, $a_3 \in M_{a\times 1}(\mathbb{Z}_p)$.
Now we prove that $gw \in PK^w$ using the properties proven above. First we multiply gw by

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ -d_1^{-1}c_1 & -d_1^{-1}c_2 & -d_1^{-1}c_3 & d_1^{-1} & \\ -c_4 & -c_5 & -c_6 & -d_3 & 1 \end{pmatrix}$$ $\in K^w$, which does not change the above properties and what needs to be proven, so without loss of generality we assume that $c_4 = 0, c_5 = 0, c_6 = 0, d_3 = 0, c_1 = 0, c_2 = 0, c_3 = 0, d_1 = 0$. Moreover we set $\begin{pmatrix} a_1 & a_2 \\ a_4 & a_5 \end{pmatrix}^{-1} \begin{pmatrix} a_3 \\ a_6 \end{pmatrix} := T$, then

$$\begin{pmatrix} 1 & & T_1 & & \\ & 1 & T_2 & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$ $\in K^w$, now it is clear that $gw \in PK^w$. □

Now suppose that π is nearly ordinary with respect to $\underline{k}$. We denote ϕ to be the unique (up to scalar) nearly ordinary vector in π. Let $\phi_w = \pi(w)\phi$, $\phi_{aux} = \sum_{x \in J} \pi \begin{pmatrix} 1 & & \\ & 1 & \\ & & x \end{pmatrix} \phi_w$ where x runs through the representatives of $\begin{pmatrix} 1 & x_{12} & ... & x_{1b} \\ & ... & ... & ... \\ & & ... & ... \\ & & & 1 \end{pmatrix}$ so that x_{ij} runs through representatives of $[\mathbb{Z}_p : p^{t_{a+b+i} - t_{a+b+j}} \mathbb{Z}_p]$. ϕ_{aux} apparently depends on the choices of the representatives.

Now write

$$\phi' = \rho(\begin{pmatrix} p^{-t_{a+b+1}} & & & & & \\ & ... & & & & \\ & & p^{t_1} & & & \\ & & & ... & & \\ & & & & p_{t_{a+1}} & \\ & & & & & ... \end{pmatrix}^{\iota} \begin{pmatrix} & & -1_b \\ & 1_a & \\ 1_b & & \end{pmatrix}^{\iota})\phi_{aux}$$

we want to compute the value $F_{\phi'}(z, f^\dagger, w)$. In fact it is equal to:

$$\sum_{B,C,D,E}\int_{GL_{a+2b}(Q_p)} \tilde{f}^\dagger(\tilde{\gamma}\alpha(w\begin{pmatrix} B & & & \\ & 1 & & \\ & & C & \\ & & & 1\end{pmatrix}w,(g_1\begin{pmatrix} E & \\ & D\end{pmatrix}\begin{pmatrix} p^{-t_{a+b+1}} & & & & & \\ & ... & & & & \\ & & p^{t_1} & & & \\ & & & ... & & \\ & & & & p^{t_{a+1}} & \\ & & & & & ...\end{pmatrix}$$

$$\times\begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix})^\iota)w'\tilde{\gamma}^{-1}\begin{pmatrix} p^{-t_1} & & & & & & & & & & & & & \\ & ... & & & & & & & & & & & & \\ & & p^{-t_a} & & & & & & & & & & & \\ & & & 1 & & & & & & & & & & \\ & & & & 1 & & & & & & & & & \\ & & & & & p^{-t_{a+1}} & & & & & & & & \\ & & & & & & ... & & & & & & & \\ & & & & & & & p^{-t_{a+b}} & & & & & & \\ & & & & & & & & 1 & & & & & \\ & & & & & & & & & 1 & & & & \\ & & & & & & & & & & 1 & & & \\ & & & & & & & & & & & p^{t_{a+b+1}} & & \\ & & & & & & & & & & & & ... & \\ & & & & & & & & & & & & & p^{t_{a+2b}}\end{pmatrix}$$

$$\times w^{-1}_{a+2b+1})\bar{\tau}(\det g_1)\rho(g_1^\iota)\phi' dg_1$$

where $w' = \begin{pmatrix} & & & & 1 & & & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ -1 & & & & & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1\end{pmatrix}$ and the sum is over $B \in \mathfrak{B}, C \in \mathfrak{C}, D \in \mathfrak{D}, E \in \mathfrak{E}$. A direct computation gives: $\tilde{\gamma}\alpha(1, \begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}^{\iota})w'\tilde{\gamma}^{-1}$ equals

$$\begin{pmatrix} -1 & & & & 1 & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ -a_3 & -a_2 & & a_1 & & & & a_2 \\ -a_9-1 & -a_8 & & a_7 & 1 & & & a_8 \\ -a_3 & -a_2 & & a_1-1 & & 1 & & a_2 \\ & & & & & & 1 & \\ -a_6 & 1-a_5 & & a_4 & & & & a_5 \end{pmatrix}$$

Now we define $\mathfrak{Y}$ to be the subset of $GL_{a+2b}(\mathbb{Z}_p)$ to be the set of block matrices $\begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}$ such that $\tilde{\gamma}\alpha(1, \begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}^{\iota})w'\tilde{\gamma}^{-1}$ is in the Q_t defined in the proof of the above lemma. It is not hard to prove that it can be described as: the i-th column of $a_9 - 1, a_3$ are divisible by p^{t_i} for $1 \leq i \leq a$, the i-th column of $a_7, a_1 - 1$ are divisible by $p^{t_{a+i}}$, the (i,j)-th entry of a_6 is divisible by $p^{t_{a+b+i}+t_j}$, the (i,j)-th entry of a_4 is divisible by $p^{t_{a+b+i}+t_{a+j}}$, the i-th row of $1 - a_5$ is divisible by

$p^{t_{a+b+i}}$. The entries in a_2 and a_8 are in $\mathbb{Z}_p$. then the pull back section is equal to

$$\sum_{B,C,D,E} \int \tilde{f}^{\dagger}(\tilde{\gamma}\alpha(1,g_1)w'\tilde{\gamma}^{-1} \begin{pmatrix} p^{-t_1} &&&&&&&&&&&& \\ & ... &&&&&&&&&&& \\ && p^{-t_a} &&&&&&&&&& \\ &&& 1 &&&&&&&&& \\ &&&& 1 &&&&&&&& \\ &&&&& p^{-t_{a+1}} &&&&&&& \\ &&&&&& ... &&&&&& \\ &&&&&&& p^{-t_{a+b}} &&&&& \\ &&&&&&&& 1 &&&& \\ &&&&&&&&& 1 &&& \\ &&&&&&&&&& 1 && \\ &&&&&&&&&&& p^{t_{a+b+1}} & \\ &&&&&&&&&&&& ... \\ &&&&&&&&&&&&& p^{t_{a+2b}} \end{pmatrix}$$

$$\times w_{a+2b+1}^{-1})\bar{\tau}(\det g_1)\rho(g_1^{\iota})\phi dg_1$$

where the integration is over the set:

$$g_1 \in \begin{pmatrix} B & \\ & C \end{pmatrix} \mathfrak{Y} \begin{pmatrix} E & \\ & D \end{pmatrix}_{conj} \begin{pmatrix} && 1_b \\ & 1_a & \\ -1_b && \end{pmatrix} \begin{pmatrix} p^{t_{a+b+1}} &&&&& \\ & ... &&&& \\ && p^{-t_1} &&& \\ &&& ... && \\ &&&& p^{-t_{a+b}} & \\ &&&&& ... \end{pmatrix}$$

for:

$$\begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix} \begin{pmatrix} p^{t_{a+b+1}} & & & & & \\ & ... & & & & \\ & & p^{-t_1} & & & \\ & & & ... & & \\ & & & & p^{-t_{a+1}} & \\ & & & & & ... \end{pmatrix} \begin{pmatrix} E & \\ & D \end{pmatrix}$$

$$\times \begin{pmatrix} p^{-t_{a+b+1}} & & & & & \\ & ... & & & & \\ & & p^{t_1} & & & \\ & & & ... & & \\ & & & & p^{t_{a+1}} & \\ & & & & & ... \end{pmatrix} \begin{pmatrix} & & -1_b \\ & 1_a & \\ 1_b & & \end{pmatrix} := \begin{pmatrix} E & \\ & D \end{pmatrix}_{conj}$$

and the value of $\tilde{f}^\dagger$ when $g_1 = \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix} \begin{pmatrix} p^{t_{a+b+1}} & & & & & \\ & ... & & & & \\ & & p^{-t_1} & & & \\ & & & ... & & \\ & & & & p^{-t_{a+b}} & \\ & & & & & ... \end{pmatrix}$ is

$$\tau((p^{t_{a+b+1}+...+t_{a+2b}}, p^{t_1+...+t_{a+b}}))|p^{t_1+...+t_{a+2b}}|^{-z-\frac{a+2b+1}{2}}$$

thus straightforward computation tells us the following:

Lemma 4.4.12. *If ϕ and ϕ' be defined right after the proof of lemma 4.4.11 then:*

$$\begin{aligned} F_{\phi'}(z, f^\dagger, w) &= \tau((p^{t_1+...+t_{a+b}}, p^{t_{a+b+1}+...+t_{a+2b}}))|p^{t_1+...+t_{a+2b}}|^{-z-\frac{a+2b+1}{2}}\mathrm{Vol}(\tilde{K}') \\ &\times p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+1+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1) \prod_{i=1}^{b} \mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)\phi_w \end{aligned}$$

Combining the 3 lemmas above, we get the following:

Proposition 4.4.1. *Assumptions are as in the above lemma. $F_{\phi'}(z, f^\dagger, g)$ is the unique section supported in PwK such that the right action of K is given by multiplying the character ν and its* *value at w is:*

$$F_{\phi'}(z, f^\dagger, w) \qquad = \tau((p^{t_1+\ldots+t_{a+b}}, p^{t_{a+b+1}+\ldots+t_{a+2b}}))|p^{t_1+\ldots+t_{a+2b}}|^{-z-\frac{a+2b+1}{2}}\mathrm{Vol}(\tilde{K}')$$
$$\times p^{-\sum_{i=1}^{a+b} it_i - \sum_{i=1}^{b} it_{a+b+1+i}} \prod_{i=1}^{a+b} \mathfrak{g}(\xi_i)\xi_i(-1)\prod_{i=1}^{b}\mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)\phi_w$$

Proof. ϕ_w is clearly invariant under $(\tilde{K}'')^\iota$. □

This $F_{\phi'}(z, f^\dagger, g)$ we constructed is not going to be the nearly ordinary vector unless we apply the intertwining operator to it. So now we start with a $\rho = (\pi, \tau)$, we require that $\rho^\vee = (\pi^\vee, \bar{\tau}^c)$ satisfies the conditions at the beginning of this section about the conductors. We define our Siegel section $f^0 \in I_{a+2b+1}(\tau)$ to be:

$$f^0(z; g) := M(-z, f^\dagger)_z(g)$$

where $f^\dagger \in I_{a+2b+1}(\bar{\tau}^c)$. We recall the following proposition from [SU] (in a generalized form)

Proposition 4.4.2. *There is a meromorphic function $\gamma^{(2)}$ such that*

$$F_{\phi^\vee}(M(z, f); -z, g) = \gamma^{(2)}(\rho, z)A(\rho, z, F_\phi(f; z, -))_{-z}(g)$$

moreover if $\pi_v \simeq \pi(\chi_1, ..., \chi_{a+2b})$ then if we write $\gamma^{(1)}(\rho, z) = \gamma^{(2)}(\rho, z - \frac{1}{2})$ then

$$\gamma^{(1)}(\rho, z) = \psi(-1)c\epsilon(\tilde{\pi}, \bar{\tau}^c, z + \frac{1}{2})\frac{L(\pi, \tau^c, 1/2 - z)}{L(\tilde{\pi}, \bar{\tau}^c, z + 1/2)}$$

where c is the constant appearing in lemma 4.4.6

Proof. The same as [SU]11.4.13. □

Remark 4.4.2. *Note that here we are using the L-factors for the base change from the unitary groups while [SU] uses the GL_2 L-factor for π so our formula is slightly different.*

Now we are going to show that:

$$F_v^0(z; g) := F_{\phi'}(f^0, z; g)$$

is a constant multiple of the nearly ordinary vector if our ρ comes from the local component of the global Eisenstein data (see section 3.1). Return to the situation of our Eisenstein Data. Suppose that at the archimedean places our representation is a holomorphic discrete series associated to the (scalar) weight: $\underline{k} = (0, ...0; \kappa, ...\kappa)$ with r 0's and s κ's. Here $r = a + b, s = b$. Suppose $\pi \simeq \mathrm{Ind}(\chi_1, ..., \chi_{a+2b})$ is nearly ordinary with respect to the weight $\underline{k}$. We suppose $\nu_p(\chi_1(p)) =$

$s-\frac{n}{2}+\frac{1}{2},...,\nu_p(\chi_r(p))=r+s-1-\frac{n}{2}+\frac{1}{2},\nu_p(\chi_{r+1}(p))=\kappa-\frac{n}{2}+\frac{1}{2},...,\nu_p(\chi_{r+s}(p))=\kappa+s-1-\frac{n}{2}+\frac{1}{2}$, and $\nu_p(\tau_1(p))=\frac{\kappa}{2},\nu_p(\tau_2(p))=\frac{\kappa}{2}$, so

$$\nu_p(\chi_1(p))<...<\nu_p(\chi_{a+b}(p))<\nu_p(\tau_1(p)p^{-z_\kappa})<\nu_p(\tau_2(p)p^{z_\kappa})<\nu_p(\chi_{a+v+1}(p))<...<\nu_p(\chi_{a+2b}(p))$$

where $z_\kappa=\frac{\kappa-r-s-1}{2}$. It is easy to see that $I(\rho_v,z_\kappa)\equiv\mathrm{Ind}(\chi_1,...\chi_{r+s},\tau_1|.|^{z_\kappa},\tau_2|.|^{-z_\kappa})$. By definition $I(\rho_v,z_\kappa)$ is nearly ordinary with respect to the weight $(0,...,0;\kappa,...,\kappa)$ with $(r+1)$ 0's and s κ's.

First of all from the form of $F_{\phi'}(z,f^\dagger;g)$ and the above proposition we have a description for $F_v^0(z,g)$: it is supported in $P(\mathbb{Q}_p)K_v$,

$$\begin{aligned}F_v^0(z,1)= \quad & \gamma^{(2)}(\rho,-z)\bar{\tau}^c((p^{t_1+...+t_{a+b}},p^{t_{a+b+1}+...+t_{a+2b}}))|p^{t_1+...+t_{a+2b}}|^{z-\frac{a+2b+1}{2}}\mathrm{Vol}(\tilde{K}')\\ & \times p^{-\sum_{i=1}^{a+b}(i+1)t_i-\sum_{i=1}^{b}(i+1)t_{a+b+1+i}}\prod_{i=1}^{a+b}\mathfrak{g}(\xi_i)\xi_i(-1)\prod_{i=1}^{b}\mathfrak{g}(\xi_{a+b+1+i})\xi_{a+b+1+i}(-1)\phi\end{aligned}$$

and the right action of K_v is given by the character

$$\chi_1(g_{11})...\chi_{a+b}(g_{a+b\ a+b})\tau_1(g_{a+b+1\ a+b+1})\chi_{a+b+1}(g_{a+b+2\ a+b+2})...\chi_{a+2b}(g_{a+2b+1\ a+2b+1})\tau_2(g_{a+2b+2\ a+2b+2}).$$

(It is easy to compute $A(\rho,z,F_{\phi'}(f;z,-))_{-z}(1)$ and we use the uniqueness of the vector with the required K_v action. Here on the second row of the above formula the power for p is slightly different from that for the section $F(z,f^\dagger,w)$. This comes from the computations for the intertwining operators for Klingen Eisenstein sections.)

Thus Corollary 4.4.1 tells us that $F_v^0(z,g)$ is a nearly ordinary vector in $I(\rho)$.

Now we describe f^0:

Definition 4.4.8. *Suppose* $(p^t)=\mathrm{cond}(\tau')$ *for* $t\geq 1$ *then define* f_t *to be the section supported in* $Q(\mathbb{Q}_p)K_Q(p^t)$ *and* $f_t(k)=\tau(d_k)$ *on* $K_Q(p^t)$.

Lemma 4.4.13.

$$f_x^0:=M(-z,\tilde{f}^\dagger)_z=f_{t,z}.$$

Proof. This is just [SU] 11.4.10. □

4.4.3 Fourier Coefficients for f^0

We record a formula here for the Fourier Coefficients for f^0 which will be used in p-adic interpolation.

Lemma 4.4.14. *Suppose* $|\det\beta| \neq 0$ *then:*

(i) If $\beta \notin S_{a+2b+1}(\mathbb{Z}_p)$ *then* $f^0_\beta(z,1) = 0$*;*

(ii) Let $t := \mathrm{ord}_p(\mathrm{cond}(\tau'))$*. If* $\beta \in S_{a+2b+1}(\mathbb{Z}_p)$*, then:*

$$f^0_\beta(z,1) = \bar{\tau}'(\det\beta)|\det\beta|_p^{2z}\mathfrak{g}(\tau')^{a+2b+1}c_{a+2b+1}(\bar{\tau}', z)\Phi_\xi({}^t\beta).$$

where c_{a+2b+1} *is defined in lemma 4.4.6 and* Φ_ξ *is defined at the beginning of this section.*

Proof. This follows from [SU]11.4.12. and the argument of corollary 4.4.2 where we deduce the form of $f^\dagger$ from the section $\tilde{f}^\dagger$. □

4.4.4 Fourier-Jacobi Coefficients

Now let $m = b+1$. For $\beta \in S_m(F_v) \cap GL_m(\mathcal{O}_v)$ we are going to compute the Fourier Jacobi coefficient for f_t at β

Lemma 4.4.15. *Let* $x := \begin{pmatrix} 1 & \\ D & 1 \end{pmatrix}$ *(this is a block matrix with respect to* $(a+b)+(a+b)$*).*

(a) $FJ_\beta(f_t; -z, v, x\eta^{-1}, 1) = 0$ *if* $D \in p^t M_{a+b}(\mathbb{Z}_p)$*;*

(b) if $D \in p^t M_n(\mathbb{Z}_p)$ *then* $FJ_\beta(f_t; -z, v, x\eta^{-1}, 1) = c(\beta, \tau, z)\Phi_0(v)$*; where*

$$c(\beta,\tau,z) := \bar{\tau}(-\det\beta)|\det\beta|_v^{2z+n-m}\mathfrak{g}(\tau')^m c_m(\tau', -z - \frac{n-m}{2})$$

where c_m *is defined in lemma 4.4.6*

Proof. We only give the detailed proof for the case when $a = 0$. The case when $a > 0$ is even easier to treat.

Assume $a = 0$, we temporarily write n for b and save the letter b for other use, we have:

$$w_{2n+1}\begin{pmatrix} 1_{2n+1} & \begin{matrix} S & v \\ {}^t\bar{v} & D \end{matrix} \\ & 1_{2n+1} \end{pmatrix}\alpha(1,\eta^{-1}) = \begin{pmatrix} & & 1_{n+1} & \\ & -1_n & & \\ -1_{n+1} & v & -S & \\ & D & -{}^t\bar{v} & -1_n \end{pmatrix}.$$

This belongs to $\mathbf{Q}_{2n+1}(\mathbb{Q}_p)K_{Q_{2n+1}}(p^t)$ if and only if S is invertible, $S^{-1} \in p^t M_{n+1}(\mathcal{O}_\wp)$, $S^{-1}v \in p^t M_{(n+1)\times n}(\mathcal{O}_\wp)$ and ${}^t\bar{v}S^{-1}v - D \in p^t M_n(\mathbb{Z}_p)$.

Since $v = \gamma{}^t(b,0)$ for some $\gamma \in SL_{n+1}(\mathcal{O}_\wp)$ and $b \in M_n(\mathcal{K}_\wp)$ we are reduced to the case $v = {}^t(b,0)$. Writing $b = (b_1, b_2)$ with $b_i \in M_n(\mathbb{Q}_p)$ and $S = (T, {}^t\bar{T})$ with $T \in M_{n+1}(\mathbb{Q}_p)$ and $T^{-1} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ where $a_1 \in M_n(\mathbb{Q}_p)$, $a_2 \in M_{n\times 1}(\mathbb{Q}_p)$,$a_3 \in M_{1\times n}(\mathbb{Q}_p)$, $a_4 \in M_1(\mathbb{Q}_p)$, the conditions on S and v can be rewritten as:

(*) $detT \neq 0$, $a_i \in p^t M_n(\mathbb{Z}_p)$, $a_1 b_1 \in p^t M_n(\mathbb{Z}_p)$, $a_3 b_1 \in p^t M_{1\times n}(\mathbb{Z}_p)$, ${}^ta_1 b_2 \in p^t M_n(\mathbb{Z}_p)$, ${}^ta_2 b_2 \in p^t \mathbb{Z}_p$, ${}^t\bar{b}_2 a_1 b_1 - D \in p^t M_n(\mathbb{Z}_p)$

Now we prove that: if the integral for FJ_β i non zero then $b_1, b_2 \in M_n(\mathbb{Z}_p)$.

Suppose otherwise, then without lose of generality we assume b_1 has an entry which has the maximal p-adic absolute value among all entries of b_1 and b_2, Suppose it is p^w for $w > 0$ (throughout the paper w means this only inside this lemma). Also, for any matrix A of given size we say $A \in {}^t\bar{b}_2^\vee$ if and only ${}^t\bar{b}_2 A$ has all entries in $\mathbb{Z}_p$ (of course we assume the sizes of the matrices are correct so that the product makes sense).

Now,let

$$\Gamma := \left\{ \begin{array}{ll} \gamma \begin{pmatrix} h & j \\ k & l \end{pmatrix} \in GL_{n+1}(\mathbb{Z}_p): & h \in GL_{n+1}(\mathbb{Z}_p), l \in \mathbb{Z}_P^\times, \\ h - 1 \in {}^t\bar{b}_2^\vee \cap p^t M_n(\mathbb{Z}_p), & j \in \mathbb{Z}_p^n \cap {}^t\bar{b}_2^\vee, k \in p^t M_{1\times n}(\mathbb{Z}_p) \end{array} \right\}$$

Suppose that our b_1, b_2, D are such that there exist a_i's satisfying (*), then one can check that Γ is a subgroup, and if T satisfies (*), so does $T\gamma$ for any $\gamma \in \Gamma$. Let $\mathcal{T}$ denote the set of $T \in M_{n+1}(\mathbb{Q}_p)$ satisfying (*). then $FJ_\beta(f_t; z, v, \begin{pmatrix} 1 & \\ D & 1 \end{pmatrix} \eta^{-1}, 1)$ equals

$$\sum_{T \in \mathcal{T}/\Gamma} |\det T|_p^{3n+2-2z} \int_\Gamma \tau'(-\det T\gamma) e_p(-tr\beta T\gamma) d\gamma.$$

Let $T' := \beta T = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$ (blocks with respect to $n+1$), then the above integral is zero unless $c_1 \in p^{-t}M_n(\mathbb{Z}_p) + [{}^t\bar{b}_2]_{n\times n}, c_4 \in p^{-t}\mathbb{Z}, c_2 \in p^{-t}M_{n+1}(\mathbb{Z}_p), c_3 \in [{}^t\bar{b}_2]_{1\times n} + M_{1\times n}(\mathbb{Z}_p)$, here $[{}^t\bar{b}_2]_{i\times n}$ means the set of $i \times n$ matrices such that each row is a $\mathbb{Z}_p$-linear combination of the rows of ${}^t\bar{b}_2$.

But then

$$\beta \begin{pmatrix} b_1 \\ 0 \end{pmatrix} = T'T^{-1} \begin{pmatrix} b_1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 a_1 b_1 + c_2 a_2 b_1 \\ c_3 a_1 b_1 + c_4 a_3 b_1 \end{pmatrix}$$

since $\beta \in GL_{n+1}(\mathbb{Z}_p)$, the left must contain some entry with p-adic absolute value p^w. But it is not hard to see that all entries on the right hand side have p-adic values strictly less than p^w, a contradiction, thus we conclude that $b_1 \in M_n(\mathbb{Z}_p)$ and $b_2 \in M_n(\mathbb{Z}_p)$
By (*): ${b_2}^t a_1 b_1 - D \in p^t M_n(\mathbb{Z}_p), a_1 \in p^t M_n(\mathbb{Z}_p)$ so $D \in p^t M_n(\mathbb{Z}_p)$.

The value claimed in part (ii) can be deduced similarly as in [SU]11.4.22. $\square$

Chapter 5

Global Computations

5.1 Klingen Eisenstein Series

Now we are going to construct the nearly ordinary Klingen Eisenstein series (and will p-adically interpolate in families). First of all, recall that for a Hecke character τ which is of infinite type $(\frac{\kappa}{2}, -\frac{\kappa}{2})$ at all infinite places (here the convention is that the first infinite place of $\mathcal{K}$ is inside our CM type) we construct a Siegel Eisenstein series E associated to the Siegel section:

$$f = \prod_{v|\infty} f_\kappa \prod_{v|p} \rho(\Upsilon) f_v^0 \prod_{v\in\Sigma, v\nmid p} f_{v,sieg} \prod_v f_v^{sph} \in I_{a+2b+1}(\tau, z).$$

Recall that we write $\mathcal{D} := \{\pi, \tau, \Sigma\}$ for the Eisenstein datum where Σ is a finite set containing all the infinite places, primes dividing p and the places where π or τ is ramified.then define the normalization factor:

$$\begin{aligned} B_{\mathcal{D}} : \quad &= \tfrac{1}{\Omega_\infty^{2\kappa\Sigma}} \Big(\tfrac{(-2)^{-d(a+2b+1)}(2\pi i)^{d(a+2b+1)\kappa}(2/\pi)^{d(a+2b+1)(a+2b)/2}}{\prod_{j=0}^{a+2b}(\kappa-j-1)^d}\Big)^{-1} \prod_{i=0}^{(a+2b)} L^\Sigma(2z_\kappa + a + 2b + 1 - i, \bar{\tau}'\chi_{\mathcal{K}}^i) \\ &\prod_{v|p} (\mathfrak{g}(\bar{\tau}'_v)^{a+2b+1} c_{a+2b+1}(\tau'_v, z_\kappa))^{-1} \prod_{v\nmid p, v\in\Sigma} \tau^{-1}(y_v\bar{y}_v x_v) |(y_v\bar{y}_v)^2 x_v \bar{x}_v|_v^{z_\kappa + \frac{a+2b+1}{2}} C_v \end{aligned}$$

Here Ω_∞ is the CM period in section 2.1. First note that since π is nearly ordinary with respect to the scalar weight κ. Then its contragradient is also nearly ordinary. (But the nearly ordinary vector is not the one whose neben-type is the inverse of ϕ^{ord}). We denote this representation as π^c. We choose a nearly ordinary vector of this representation which we choose to be "p-adically primitive", i.e. integral but not divisible by p in terms of Fourier Jacobi expansion. In general we will need some Gorenstein properties of certain Hecke algebras to make primitive forms in Hida families. But we do not touch this at the moment. We consider $E(\gamma(g,-))$ as an automorphic form on $U(a+b,b)$. For each $v \not| p$ there is a level group $\tilde{K}_{v,s} \subset U(a+b,b)_v$ such that

$$\prod_{v\nmid p} \rho(\gamma(1, \eta\mathrm{diag}(\bar{x}_v^{-1}, 1, x_v)))(E(\gamma(g,-)) \otimes \bar{\tau}(\det -))$$

is invariant under its action. Suppose $\phi^{c,ord}$ is a "new form" with level group $\tilde{K}_v$ so that $\pi_v^{\tilde{K}_v}$ is 1-dimensional for each v. (In fact it is better to use K-types. But here we content ourselves with new forms for simplicity.) Assume also that there is a Hecke action 1_{π^c} with respect to the level group $\prod_v \tilde{K}_v$ which takes any nearly ordinary automorphic form for this level to its π^c component (which is a multiple of $\phi^{c,ord}$).

Remark 5.1.1. *In a future work we will see that when deforming everything in families, the (Fourier coefficients of the) Siegel Eisenstein series $B_{\mathcal{D}}E$ moves p-adic analytically. This enables us to construct the p-adic analytic family E_{Kling}. This is the reason for introducing $B_{\mathcal{D}}$.*

We define E_{Kling} by:

$$B_{\mathcal{D}} 1_{\pi^c}^{low} e^{low} \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_{v,s}} \rho(\gamma(1, \eta\mathrm{diag}(\bar{x}_v^{-1}, 1, x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)) = E_{Kling}(g) \boxtimes \phi^{c,ord}$$

Here we used the superscript *low* to mean that under $U(a+b+1,b+1)\times U(a+b,b) \hookrightarrow U(a+2b+1, a+2b+1)$ the action is for the group $U(a+b,b)$.

Definition 5.1.1. *Let ϕ be an automorphic form on $GU(\mathbb{A}_F)$ or $U(\mathbb{A}_F)$ we define:*

$$\phi^c(g) := \overline{\phi(\begin{pmatrix} 1_{a+b} & \\ & -1_b \end{pmatrix} g \begin{pmatrix} 1_{a+b} & \\ & -1_b \end{pmatrix})}.$$

Here the overline means complex conjugation.

Recall that we have defined $\phi_{aux} = \sum_{x\in J} \pi \begin{pmatrix} 1 & & \\ & 1 & \\ & & x \end{pmatrix} \phi_w$ where x runs through $\begin{pmatrix} 1 & x_{12} & ... & x_{1b} \\ & ... & ... & ... \\ & & ... & ... \\ & & & 1 \end{pmatrix}$ with x_{ij} running through representatives of $[\mathbb{Z}_p : p^{t_{a+b+i}-t_{a+b+j}}\mathbb{Z}_p]$. ϕ_{aux} apparently depends on the choices of the representatives.

$$< \prod_{v \nmid p} tr_{\tilde{K}_v/\tilde{K}_{v,s}} \rho(\gamma(1, \eta \mathrm{diag}(\bar{x}_v^{-1}, 1, x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)),$$

$$\rho(\begin{pmatrix} p^{-t_{a+b+1}} & & & & & \\ & ... & & & & \\ & & p^{t_1} & & & \\ & & & ... & & \\ & & & & p_{t_{a+1}} & \\ & & & & & ... \end{pmatrix}^{\iota} \begin{pmatrix} & & -1_b \\ & 1_a & \\ 1_b & & \end{pmatrix}^{\iota})\phi_{aux} >$$

$$= \quad < \rho^{low}(\begin{pmatrix} p^{t_{a+b+1}} & & & & \\ & ... & & & \\ & & p^{t_{a+2b}} & & \\ & & & 1_a & \\ & & & & 1_b \end{pmatrix}^{\iota}) \prod_{v \nmid p} tr_{\tilde{K}_v/\tilde{K}_{v,s}} \rho(\gamma(1, \eta \mathrm{diag}(\bar{x}_v^{-1}, 1, x_v)))(E(\gamma(g,-))\bar{\tau}(\det -),$$

$$\rho(\begin{pmatrix} 1_b & & & & \\ & p^{t_1} & & & \\ & & ... & & \\ & & & p_{t_{a+1}} & \\ & & & & ... \end{pmatrix}^{\iota} \begin{pmatrix} & & -1_b \\ & 1_a & \\ 1_b & & \end{pmatrix}^{\iota})\phi_{aux} >$$

Since $E(\gamma(g,-))$ satisfies the property that if $\tilde{K}'''$ is the subgroup of $\tilde{K}$ (defined in the last chapter) consisting of matrices $\begin{pmatrix} a_1 & a_3 & a_2 \\ a_7 & a_9 & a_8 \\ a_4 & a_6 & a_5 \end{pmatrix}^{\iota}$ such that the (i,j)-th entry of a_7 is divisible by $p_{t_i + t_{a+b+j}}$ and the (i,j)-th entry of a_4 is divisible by $p^{t_{a+i}+t_{a+b+j}}$, the i-th row of a_8 and the right to diagonal entries of a_9 are divisible by p^{t_i} for $i = 1, ..., a$, the i-th column of the below diagonal entries of a_1 are divisible by $p^{t_{a+b+i}}$, the i-th row of the up to diagonal entries of a_5 are divisible by $p^{t_{a+i}}$. Then the right action of $\tilde{K}'''$ on $E(\gamma(g,-))$ is given by the character $\lambda(g\iota) = \bar{\chi}_{a+b+1}(g_{11})...\bar{\chi}_{a+2b}(g_{bb})\bar{\chi}_1(g_{b+1,b+1})..\bar{\chi}(g_{a+b,a+b})\bar{\chi}_{a+1}(g_{a+b+1,a+b+1})...\bar{\chi}_{a+b}(g_{a+2b,a+2b})$ so the above expression equals:

$$\frac{1}{\prod_{i=1}^{t} p^{t_{a+b+i(a+b)}}} < \sum_y \rho^{low}(g)\rho^{low}(\begin{pmatrix} p^{a+b+1} & & & \\ & \dots & & \\ & & 1_a & \\ & & & 1_b \end{pmatrix}^{\iota} \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))$$

$$(E(\gamma(g,-))\bar{\tau}(\det -)), \rho(\begin{pmatrix} 1_b & & & & \\ & p^{t_1} & & & \\ & & \dots & & \\ & & & p^{t_{a+1}} & \\ & & & & \dots \end{pmatrix}^{\iota} \begin{pmatrix} & & -1_b \\ & 1_a & \\ 1_b & & \end{pmatrix}^{\iota})\phi_w >$$

Let $\beta = \begin{pmatrix} p^{t_{a+b+1}} & & & \\ & \dots & & \\ & & 1_a & \\ & & & 1_b \end{pmatrix}$ and define T_β^{low} to be the Hecke action corresponding to β just in terms of double cosets. (no normalization factors involved). By checking the level actions we can see that the π^c component of the left part when viewed as an automorphic form on $U(a+b,a)$ is a multiple of $\phi^{c,ord}$ defined right before remark 5.1.1. (Note that this is not the same as ϕ^c). Suppose its eigenvalue for the Hecke operator T_β^{low} is λ_β^c,

Proposition 5.1.1. *With these notations we have:*

$$E_{Kling}(g) = B_{\mathcal{D}} \frac{\prod_{i=0}^{t} p^{t_{a+b+i(a+b)}}}{\lambda_\beta^c} . \frac{< \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))E(\gamma(g,-)),\phi' >}{< \phi^{c,ord},\phi'' >}$$

Proof. The π^c-component of the left part of the inner product above is:

$$\begin{aligned}
& T_\beta^{low}.\pi^c - \text{component of } \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)) \\
= \; & e_{low}.T_\beta^{low}.\pi^c - \text{component of } \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)) \\
= \; & T_\beta^{low}.e^{low}.\pi^c - \text{component of } \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)) \\
= \; & T_\beta^{low}.\pi^c - \text{component of } e^{low} \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)) \\
= \; & \lambda_\beta^c.\pi^c - \text{component of } e^{low} \prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -))
\end{aligned}$$

Thus

$$\begin{aligned}
&\pi^c-\text{component of } e^{low}\prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -))\\
= &\frac{<\prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)),\phi''>}{<\phi^{c,ord},\phi''>}\phi^{c,ord}\\
= &\frac{\prod_{i=0}^{t} p^{t_{a+b+i}(a+b)}}{\lambda_\beta^c}\cdot\frac{<\prod_{v\nmid p} tr_{\tilde{K}_v/\tilde{K}_v,s}\rho(\gamma(1,\eta\mathrm{diag}(\bar{x}_v^{-1},1,x_v)))(E(\gamma(g,-))\bar{\tau}(\det -)),\phi'>}{<\phi^{c,ord},\phi''>}\phi^{c,ord}
\end{aligned}$$

where

$$\phi'=\prod_{v\notin\Sigma}\phi^{sph}\prod_{v\in\Sigma,v\nmid p}\phi_v\prod_{v|p}\rho(\begin{pmatrix} p^{-t_{a+b+1}} &&&&& \\ & \cdots &&&& \\ && p^{t_1} &&& \\ &&& \cdots && \\ &&&& p_{t_{a+1}} & \\ &&&&& \cdots \end{pmatrix}^{\iota}\begin{pmatrix} && -1_b \\ & 1_a & \\ 1_b && \end{pmatrix}^{\iota})\phi_{aux}$$

and

$$\phi''=\prod_{v\notin\Sigma}\phi^{sph}\prod_{v\in\Sigma,v\nmid p}\phi_v\prod_{v|p}\rho(\begin{pmatrix} p^{-t_{a+b+1}} &&&&& \\ & \cdots &&&& \\ && p^{t_1} &&& \\ &&& \cdots && \\ &&&& p_{t_{a+1}} & \\ &&&&& \cdots \end{pmatrix}^{\iota}\begin{pmatrix} && -1_b \\ & 1_a & \\ 1_b && \end{pmatrix}^{\iota})\phi_{w}$$

Thus we get the proposition. $\square$

5.2 Constant Terms

5.2.1 Archimedean Computation

Suppose π is associated to the weight $(0,...,0;\kappa,...,\kappa)$, then it is well known that there is a unique (up to scalar) vector $v\in\pi$ such that $k.v=\det\mu(k,i)^{-\kappa}$ for any $k\in K_\infty^{+,'}v$ (notation as in section 3.1). Then by Frobenius reciprocity law there is a unique (up to scalar) vector $\tilde{v}\in I(\rho)$ such that $k.\tilde{v}=\det\mu(k,i)^{-\kappa}\tilde{v}$ for any $k\in K_\infty^+$. We fix v and scale $\tilde{v}$ such that $\tilde{v}(1)=v$. In $\pi^\vee$, $\pi(w)v$ (w is defined in section 3.1) has the action of K_∞^+ given by multiplying by $\det\mu(k,i)^{-\kappa}$. We define

$w' \in U(a+b+1,b+1)$ by $w' = \begin{pmatrix} 1_b & & & & \\ & & & & 1 \\ & & 1_a & & \\ & & & 1_b & \\ & -1 & & & \end{pmatrix}$. Then there is a unique vector $\tilde{v}^\vee \in I(\rho^\vee)$ such that the action of K_∞^+ is given by $\det \mu(k,i)^{-\kappa}$ and $\tilde{v}^\vee(w') = \pi(w)v$. Then by uniqueness there is a constant $c(\rho,z)$ such that $A(\rho,z,\tilde{v}) = c(\rho,z)\tilde{v}^\vee$.

Lemma 5.2.1. *Assumptions are as above, then:*

$$\begin{aligned} c(\rho,z) &= \pi^{a+2b+1} \textstyle\prod_{i=0}^{b-1} (\frac{1}{z+\frac{\kappa}{2}-\frac{1}{2}-i-a})(\frac{1}{z-\frac{\kappa}{2}+\frac{1}{2}-i}) \prod_{i=0}^{a-1}(\frac{1}{-1+i-2z+2b}) \\ &\quad \times \frac{\Gamma(2z+a)2^{-1-2z+2b}}{\Gamma(\frac{a+1}{2}+z+\frac{\kappa}{2})\Gamma(\frac{a+1}{2}+z-\frac{\kappa}{2})} \det(i\theta/2)^{-2}. \end{aligned}$$

Proof. It follows the same way as [SU]9.2.2. □

Corollary 5.2.1. *In case when $\kappa > \frac{3}{2}a + 2b$ or $\kappa \geq 2b$ and $a = 0$, we have $c(\rho,z) = 0$ at the point $z = \frac{\kappa-a-2b-1}{2}$.*

Let F be the Klingen section which is the tensor product of the local Klingen sections defined in the last Chapter by pulling back of the corresponding Siegel sections. In the case when κ is sufficiently large the intertwining operator:

$$A(\rho, z_\kappa, F) = A(\rho_\infty, z_\kappa, F_\kappa) \otimes A(\rho_f, z_\kappa, F_f)$$

and all terms are absolutely convergent. Thus as a consequence of the above corollary we have $A(\rho, z_\kappa, F) = 0$. Therefore the constant term of E_{Kling} is just $B_{\mathcal{D}} F_{z_\kappa}$. It is essentially

$$\frac{L^\Sigma(\tilde{\pi}, \bar{\tau}^c, z_\kappa+1)\prod_{v|p}\gamma^{(2)}(\rho_v, z_\kappa)}{\Omega_\infty^{2\kappa\Sigma}\prod_{v|p} c_{a+2b}(\tau_v', -z_\kappa - \frac{1}{2})} \cdot \frac{L^\Sigma(2z_\kappa+1, \bar{\tau}'\chi_{\mathcal{K}}^{a+2b})\prod_{v|p} c_{a+2b}(\tau_v', -z_\kappa-\frac{1}{2})}{\prod_{v|p} c_{a+2b+1}(\tau_v', -z_\kappa)} \phi.$$

up to normalization factors at ∞ and each term in the above coefficient can be interpolated p-adic analytically. Here the c_m are defined in lemma 4.4.6.

Chapter 6

Hilbert modular forms and Selmer groups

From now on we are in part two where we specialize to $U(1,1) \hookrightarrow U(2,2)_{/F}$ and prove our main theorem.

6.1 More Notations

We define μ_{p^∞} as the set of roots of unity with order powers of p. Let $\delta_{\mathcal{K}}, \mathfrak{d} = \mathfrak{d}_F, D_{\mathcal{K}}, D_F$ be the different and discriminant of $\mathcal{K}$ and F. We denote N to be the level of f and M the prime to p part of it. Here $N, M, \delta_{\mathcal{K}}, \mathfrak{d}, D_{\mathcal{K}}, D_F$ are all elements in the ideles of $F, \mathcal{K}$ or $\mathbb{Q}$ supported at the finite primes (also the $M_{\mathcal{D}}$ defined later)! This is much more convenient when working in the adelic language. For each $v|p$ we suppose $p^{r^v}\|N_v$ (we save the notation r_v for other use). We assume that $\mathcal{K}$ is split over all primes dividing the $\mathfrak{d}_F$. This assumption makes the computation for Fourier-Jacobi coefficients easier. Let $h = h_F$ be the narrow ideal class number of F, we divide the ideal classes $Cl(\mathcal{K})$ into $I_1 \sqcup ... \sqcup I_h$ corresponding to the image of the norm map to $Cl_n(F)$ and suppose I_1 are those mapping to the trivial class. (Here n stands for narrow). We assume that $\mathcal{K}$ is disjoint from the narrow Hilbert class field of F and thus it is easy to see that the norm map above is surjective. Also we write $<,>$ (integration over $U(1,1)(F)\backslash U(1,1)(\mathbb{A}_F)$) to be the inner product on the unitary group. For f and g Hilbert modular forms such that the product of the central characters of f and $\bar{g}$ are trivial then we denote $<,>_{GL_2}$ to be the inner product on GL_2 (integration over $GL_2(F)\mathbb{A}_F\backslash GL_2(\mathbb{A}_F)$, note that we need to modulo the center here). We also write, for example $<,>_{U_{\mathcal{D}}}$, $<,>_{GL_2,\Gamma_0(N)}$ the inner product with respect to the indicated level group.

For v a finite prime of F, as in [SU] chapter 8 we define the level group $K_{r,t} \subset GU(F_v)$ for $r, t > 0$ as follows: for Q and P being the Siegel and Klingen parabolic, $K_{r,t} = K_{Q,v}(\varpi_v^r) \cap w_2' K_P(\varpi_v^t) w_2'$ where ϖ_v is a uniformizer for v, $K_{Q,v}(\varpi_v^r)$ means the matrices which are in $Q(\mathcal{O}_{F,v})$ modulo ϖ_v^r and $K_{P,v}(\varpi_v^t)$ means matrices which are in $P(\mathcal{O}_{F,v})$ modulo ϖ_v^t and $w_2' = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$.

6.2 Hilbert modular forms

6.2.1 Hilbert modular forms

We set up the basic notions of Hilbert modular forms, following [Hida91] with minor modifications. Let I be the set of all field embedding of F into $\bar{\mathbb{Q}}$. We may regard I as the set of infinite places of F via ι_∞ and the weight of modular forms is a pair of elements (κ, w) in the free module $Z[I]$ generated by embeddings in I. We identify $F_\infty = F \otimes_Q \mathbb{R}$ with $\mathbf{R}^I$ and embed F into $\mathbb{R}^I$ via the diagonal map $a \mapsto (a^\sigma)_{\sigma \in I}$. Then the identity component G_∞^+ of $GL_2(F_\infty)$ naturally acts on $\mathscr{L} = \mathscr{H}^I$ for the Poincare half plane $\mathscr{H}$. We write C_∞^+ for the stabilizer in G_∞^+ of the center point $z_0 = (\sqrt{-1}, \sqrt{-1}, \cdots, \sqrt{-1})$ in $\mathscr{L}$. Then for each open compact subgroup U of $GL_2(F_{\mathbb{A}_f})$, we denote by $\mathbf{M}_{\kappa,w}(U; \mathbb{C})$ the space of holomorphic modular forms of weight (κ, w) with respect to S. Namely $\mathbf{M}_{\kappa,w}(U; \mathbf{C})$ is the space of functions $f : GL_2(\mathbb{A}_F \to \mathbf{C})$ satisfying the automorphic condition:

$$f(\alpha x u) = f(x) j_{\kappa,w}(u_\infty, z_0)^{-1} \text{ for } \alpha \in GL_2(F) \text{ and } u \in UC_{\infty+},$$

where $j_{\kappa,w}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) = (ad - bc)^{-w}(cz + d)^\kappa$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F_\infty)$ and $z \in \mathscr{L}$ and such that for any $g_f \in GL_2(\mathbb{A}_f)$ the associated classical form defined by $f_{cl}(z, g_f) := f(g).j_{\kappa,w}(g_\infty, z_0)$ g such that $g_\infty.z_0 = z$ and finite type g_f is holomorphic on the symmetric domain together with all cusps. We write $\mathbf{S}_{\kappa,w}(U; \mathbf{C})$ for the subspace of $\mathbf{M}_{\kappa,w}(U; \mathbf{C})$ consisting of cusp forms. Here we used the convention that $c^s = \prod_{\sigma \in I} c_\sigma^{s_\sigma}$, the correspondence is given by: for $c = (c_\sigma)_{\sigma \in I} \in \mathbf{C}^I$ and $s = \sum_{\sigma \in I} s_\sigma \sigma \in \mathbf{C}[I]$. Setting $t = \sum_\sigma \sigma$, we sometimes use another pair (n, v) to denote the weight, for $n = \kappa - 2t$ and $v = t - w$. Each automorphic representation π spanned by forms in $\mathbf{S}_{\kappa,w}(U; \mathbb{C})$

has the central character $|\cdot|_{\mathbb{A}}^{-m}$ up to finite order characters for the adelic absolute value $|\cdot|_{\mathbb{A}}$. The twist $\pi^u = \pi \otimes |\cdot|_{\mathbb{A}}^{m/2}$ is called the unitarization of π.

Let h be the narrow class number of F and decompose

$$\mathbb{A}_F^{\times} = \sqcup_{i=1}^{h} F^{\times} a_i (\hat{\mathcal{O}}_F)^{\times} F_{\infty+}^{\times} \text{ with } a_i \in \mathbb{A}_{F,f}^{\times}.$$

Then by strong approximation

$$G(\mathbb{A}_F) = \cup_{i=1}^{h} GL_2(F) t_i U_0(N) G_{\infty+} \text{ for } t_i = \begin{pmatrix} a_i^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

For any ideal N of $\mathcal{O}_F$ let $U_0(N)$ be the open compact subgroup of $GL_2(\hat{\mathcal{O}}_F)$ whose image modulo N is inside $B(\hat{\mathcal{O}}_F)$. Any automorphic form in the space $M_{\kappa,w}(U_0(Mp^{\alpha}), \varepsilon; A)$ is determined by its restriction to the connected component of t_i in $GL_2(F) \setminus GL_2(\mathbf{A}_F)/U_0(Mp^{\alpha})G_{\infty+}$. So we identify the above space with the space of h-tuples: $\{f_i\}$ where f_i are forms in $M_{\kappa,w}(\Gamma_i, A)$ for $\Gamma_i := t_i U_0(Mp^{\alpha}) t_i^{-1}$ with $f_i(g_{\infty}) := f(g_{\infty} t_i)$. Each f_i has a q-expansion:

$$f_i(z) = a(0, f_i) + \sum_{0<<\xi \in F^{\times}} a(\xi, f_i) e_F(\xi z).$$

Denote $\mathbb{A}_{F+}^{\times}$ be the set of ideles whose Archimedean parts are totally positive, we have the following theorem in [Hida91] about the q-expansion for Hilbert modular forms:

Theorem 6.2.1. *Each $f \in \mathbf{M}_{\kappa,w}(U;\mathbb{C})$ has the Fourier expansion of the following type:*

$$f(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) = |y|_{\mathbb{A}} \{a_0(yd, f)|y|_{\mathbb{A}}^{-[v]} \sum_{0<<\xi \in F^{\times}} a(\xi yd, f)\{(\xi yd)^v\}(\xi y_{\infty})^{-v} e_F(i\xi y_{\infty}) e_F(\xi x)\},$$

where $\mathbb{A}_{F+}^{\times} \ni y \mapsto a_0(y, f)$ is a function invariant under $F_+^{\times} U_F(N) F_{\infty+}^{\times}$ and vanishes identically unless $w \in \mathbb{Z} \cdot t$, and $\mathbb{A}_{F+}^{\times} \ni y \mapsto a(y, f)$ is a function vanishing outside $\hat{\mathcal{O}}_F F_{\infty+}^{\times}$ and depending only on the coset of $y_f U_F(N)$.

This adelic q-expansion is deduced from the usual q-expansions. We omit the details and refer to [Hida91].

6.2.2 Hida families

First of all let us define the weight space for Hilbert modular Hida families. For A some finite extension of $\mathbb{Q}_p$ let $\Lambda'_W := A[[\{W_{1,v}, W_{2,v}\}_{v|p}]]$. A point $\phi \in \mathrm{Spec}(\Lambda'_W)$ is called arithmetic if $(1+W_{1,v}) \to \zeta_{1,,v,\phi} \in \mu_{p^\infty})$ and $(1+W_{2,v}) \to (1+p)^{\kappa_{\phi,v}-2}\zeta_{2,v,\phi}$ where $\zeta_{2,v,\phi} \in \mu_{p^\infty}$ and $\kappa_\phi \geq 2$ is some integer. We also require that $\kappa_{\phi,v}$ to be the same for all v. (this means we only consider Hilbert modular forms of parallel weight, which is already enough for constructing the whole Hida family.)

Define Λ_W such that $\mathrm{Spec}\Lambda_W$ is the closed subspace of $\mathrm{Spec}\Lambda'_W$ defined as the Zariski closure of the arithmetic points satisfying: $\phi((1+W_{1,v})(1+W_{2,v}))$ to be equal for all $v|p$. for any $a \in \mathcal{O}_F^\times$. It is naturally a power series ring with $d+1$ variables. We only consider this weight space for simplicity. In fact if the Leopoldt conjecture is true, then this is the whole weight space for the Hida families of Hilbert modular forms.

Now we define the neben-typus associated to ϕ:

$$\varepsilon'_{1,\phi,v}(1+p) = \zeta_{1,\phi_v}, \varepsilon'_{2,\phi,v}(1+p) = \zeta_{2,\phi,v}$$

we extend these to be characters on $\mathcal{O}_v^\times$ as follows: for a such that $a \equiv 1 \mathrm{mod} p$ it is obvious how to extend and then we require them to be trivial on the torsion part of $\mathcal{O}_v^\times$. Define:

$$\varepsilon_{\phi,v}(\begin{pmatrix} a & \\ & b \end{pmatrix}) = \varepsilon'_{1,v,\phi}(a)\varepsilon'_{2,v,\phi}(b)\omega^{\kappa_\phi-2}(b)$$

for $a, b \in \mathcal{O}_v^\times$ (Recall that ω is the Techimuller character).

Remark 6.2.1. *Suppose f is a nearly ordinary eigenform with neben typus ε_ϕ and whose v component at $v|p$ is $\pi(\mu_{1,v}, \mu_{2,v})$ where $\mathrm{val}\mu_{1,v}(0) = p^{-\frac{\kappa_\phi-1}{2}}, \mathrm{val}_p\mu_{2,v}(p) = p^{\frac{\kappa_\phi-1}{2}}$. Then $\mu_{1,v}$, $\mu_{2,v}$ have the same restriction to $\mathcal{O}_{F,v}^\times$ with $\varepsilon'_{1,\phi,v}$ and $\varepsilon'_{2,\phi,v}\omega^{\kappa_\phi-2}$.*

Let $\mathbb{I}$ be a finite integral extension of Λ_W.

Definition 6.2.1. *By an $\mathbb{I}$-adic ordinary cusp form $\mathbf{f}$ of level $V_1(N)$ is a set of elements of $\mathbb{I}$ given by the data:*

$$\{c_i(\xi, \mathbb{I}) \text{ for } \xi \in F^\times, c_i(0, \mathbb{I}) \text{ for } i = 1, \cdots, h\}$$

with the property that for a Zariski densely populated set of primes ϕ of $\mathbb{I}$ which maps to an arithmetic point in $\mathrm{Spec}(\Lambda_W)$, the specialization of $\mathbf{f}_\phi$ is the q-expansion of some form in $S^{ord}_{\kappa_\phi, \frac{\kappa_\phi}{2}}(U_0(Np^\alpha), \varepsilon_\phi, \psi; A)$

where A is some finite extension of $\mathbb{Q}_p$.

6.2.3 Galois representations of Hilbert modular forms

Let A be a finite extension of $\mathbb{Q}_p$. For $f \in S_{\kappa,w}(U_0(Np^\alpha), \varepsilon, \chi; A)$, $\kappa \geq 2$ be a normalized eigenform, we fix $L \subset \bar{\mathbb{Q}}_p$ a finite extension of $\mathbb{Q}_p$ containing $\mathbb{Q}(f)$. Let $\mathcal{O}_L$ be the integer ring of L and $\mathbb{F}$ its residue field. Then we have a continuous semi-simple Galois representation (ρ_f, V_f): $\rho_f : G_Q \to GL_L(V_f)$, characterized by being unramified at primes $v \nmid p$ such that π_v is unramified and satisfying:

$$\mathrm{tr}\rho_f(frob_v) = a(v, f)$$

where $a(v, f)$ is the Hecke eigenvalue of f under the Hecke operator T_v (Recall that this is associated to $\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix}$ where ϖ_v is a uniformizer at v). Further more, if f is nearly ordinary at all primes dividing p, then we have the following description of ρ_f restricting to the decomposition groups for all primes v dividing p:

$$\rho|_{G_{F_v}} = \begin{pmatrix} \sigma_{\mu_{1,v}} & * \\ & \sigma_{\mu_{2,v}} \end{pmatrix}.$$

Here σ is the local reciprocity map and $\pi_v \simeq \pi(\mu_{1,v}, \mu_{2,v})$ where $\mu_{1,v}(p)$ has smaller p-valuation than $\mu_{2,v}(p)$.

Therefore for each $v|p$ we have a one-dimensional subspace $V_f^+ \subset V_f$ such that the action of G_v on V_f^+ is given by the character $\sigma_{\mu_{1,v}}$ and G_v acts on the quotient $V_f^- := V_f/V_f^+$ by $\sigma_{\mu_{2,v}}$. Recall that as in [SU] we have distinguished the following situation:

(dist): ψ_v^+ and ψ_v^- are distinct modulo the maximal ideal of $\mathcal{O}_L$ for each $v|p$.

6.3 Selmer groups

We recall the notion of Σ-primitive Selmer groups, following [SU]3.1 with some modifications. $\mathcal{F}$ is a subfield of $\bar{\mathbf{Q}}$. For T a free module of finie rank over a profinite $\mathbb{Z}_p$-algebra A and assume that T is equipped with a continuous action of G_F. Denote also A^* as the Pontryagin dual of A. Assume further more that for each place $v|p$ of $\mathcal{F}$ we are given a G_v-stable free A-direct summand $T_v \subset T$. For any finite set of primes Σ we denote by $Sel_F^\Sigma(T, (T_v)_{v|p})$ the kernel of the restriction map:

$$H^1(\mathcal{F}, T \otimes_A A^*) \to \prod_{v \notin \Sigma, v \nmid p} H^1(I_v, T \otimes_A A^*) \times \prod_{v|p} H^1(I_v, T/T_v \otimes_A A^*),$$

We denote

$$\mathcal{L}_v(T^*) := ker\{H^1(G_v, T\otimes_A A^*) \to H^1(I_v, T\otimes_A A^*)\}$$

$$\mathcal{L}_v(V) := ker\{H^1(G_v, V) \to H^1(I_v, V)\}.$$

Using the inflation-restriction sequence these can be identified with $H^1(G_v/I_v, (T\otimes_A A^*)^{I_v})$ and $H^1(G_v/I_v, V^{I_v})$, respectively. A useful fact is that under Tate local duality:

$$H^1(F_v, V)\times H^1(F_v, V^*(1)) \to L$$

the orthogonal complement $\mathcal{L}_v(V)^\perp$ is precisely $\mathcal{L}_v(V^*(1))$.

We always assume that Σ contains all primes at which T is ramified. We put

$$\mathbf{X}_F^\Sigma(T, (T_v)_{v|p}) := Hom_A(Sel_F^\Sigma(T, (T_v)_{v|p}), A^*).$$

If $E/\mathcal{F}$ is an extension, we put $Sel_E^\Sigma(T) := Sel_E^{\Sigma_E}(T, (T_w)_{w|p})$ and $\mathbf{X}_E^\Sigma(T) := \mathbf{X}_E^{\Sigma_E}(T, (T_w)_{w|p})$, where Σ_E is the set of places of E over those in Σ and if $w|v|p$ then $T_w = g_w T_v$ for $g_w \in G_F$ such that $g_w^{-1}G_{E,w}g_w \subseteq G_{F.v}$. If E/F is infinite we set: $Sel_E^\Sigma(T) = \varinjlim_{E\subseteq F'\subseteq E} Sel_{F'}^\Sigma(T)$ and $\mathbf{X}_E^\Sigma(T) = \varprojlim_{F\subseteq F'\subseteq E}\mathbf{X}_{F'}^\Sigma(T)$, where F' suns over the finite extensions of F contained in E.

Suppose F/F^+ is a CM number field over its maximal totally real subfield, c being the nontrivial element of G_{F^+}/G_F. Then we have an action of c on the Selmer groups of F. We have the following lemma as in [SU]3.1.5. (Recall that we have assumed $p\neq 2$.)

Lemma 6.3.1. *There is a decomposition*

$$Sel_F^\Sigma(T) = Sel_F^\Sigma(T)^+ \oplus Sel_F^\Sigma(T)^-,$$

according to the ± 1 *eigenspaces of the action by* c*. Also, restriction induces isomorphisms*

$$Sel_{F^+}^{\Sigma^+}(T)\to Sel_F^\Sigma(T)^+ \qquad Sel_{F^+}^{\Sigma^+}(T\otimes\chi_F)\to Sel_F^\Sigma(T)^-.$$

6.4 Iwasawa theory of Selmer groups

We let F_∞ be the cyclotomic $\mathbb{Z}_p$ extension of F. The Galois group, which we denote as Γ_F, is isomorphic to $\mathbb{Z}_p$. Let $\mathcal{K}_\infty^-$ be the maximal anticyclotomic (the complex conjugation acting by -1 on the Galois group) unramified outside p abelian $\mathbb{Z}_p$ extension of $\mathcal{K}$ with Galois group denoted as $\Gamma_{\mathcal{K}}^-$. This is isomorphic to $\mathbb{Z}_p^d$. Write $\mathcal{K}_\infty^+$ for $F_\infty\mathcal{K}$ with $\Gamma_{\mathcal{K}}^+$ to be the Galois group (identified with Γ_F). Let $\mathcal{K}_\infty := \mathcal{K}_\infty^- F_\infty$. This is a galois extension with Galois group $\mathbb{Z}_p^{d+1}$. Conjecturally (Leopoldt) this is the maximal unramified outside p abelian $\mathbb{Z}_p$ extension of $\mathcal{K}$. Recall that in chapter 2 we have defined the Iwasawa algebras $\Lambda_{\mathcal{K}}$ and $\Lambda_{\mathcal{K},A}$. We define more Iwasawa algebras $\Lambda_{\mathcal{K}}^-$, $\Lambda_{\mathcal{K}}^+$, $\Lambda_{\mathcal{K},A}^-$, $\Lambda_{\mathcal{K},A}^+$ in an obvious way.

We fix topological generators for each group above: $\gamma := \mathrm{rec}_F(\prod_{v|p}(1+p)_v)$, $\gamma^+ := \mathrm{rec}_{\mathcal{K}}(\prod_{v|p}(1+p, 1+p)_v^{\frac{1}{2}})$ and $\gamma_v^- := \mathrm{rec}_{\mathcal{K}}((1+p,(1+p)^{-1})_v^{\frac{1}{2}})$. Here *rec* means the reciprocity map of class field theory.

6.4.1 control of Selmer groups

We recall some results in [SU] 3.2. with minor modifications to the totally real situation. These would be useful in deducing various main conjectures from our main theorem. In this subsection (only in this subsection) we denote A as any profinite $\mathbb{Z}_p$ algebra and $\mathfrak{a}$ an ideal of it. Let T be a free A module equipped with a G_F action and $T^* := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^*$. It is noted in [SU] 3.2.7 that there is a canonical map:

$$Sel_F^\Sigma(T/\mathfrak{a}T) \to Sel_F^\Sigma(T)[\mathfrak{a}].$$

Here $[\mathfrak{a}]$ on the right hand side means the $\mathfrak{a}$ torsion part.

Proposition 6.4.1. *Suppose there is no nontrivial A-subquotient of T^* on which $G_{\mathcal{K}_\infty^+}$ acts trivially. Suppose also that for any prime $\mathfrak{p}|p$ of F the action of $I_{\mathfrak{p}}$ on $T/T_{\mathfrak{p}}$ factors through the image of $I_{\mathfrak{p}}$ in Γ_F and that $\Sigma \cup \{p\}$ contains all primes at which T is ramified. Let $\tilde{F} = F_\infty, \mathcal{K}_\infty^+$. Then the above map induces isomorphisms:*

$$Sel_{\tilde{F}}^\Sigma(T/\mathfrak{a}T) \simeq Sel_{\tilde{F}}^\Sigma(T)[\mathfrak{a}]$$

and

$$X_{\tilde{F}}^\Sigma(T) \simeq X_{\tilde{F}}^\Sigma(T)/\mathfrak{a}X_{\tilde{F}}^\Sigma(T).$$

Descent from $\mathcal{K}_\infty$ to $\mathcal{K}_\infty^+$. We have the following corollary of the above proposition:

Corollary 6.4.1. *Under the hypotheses of the above proposition. If $\tilde{F}$ is $F_\infty, \mathcal{K}_\infty^+$ then: $Ft_{\tilde{F},A/\mathfrak{a}}^\Sigma(T/\mathfrak{a}T) = Ft_{\tilde{F},A}^\Sigma(T) mod \mathfrak{a}$;*

This will be used in proving the main theorem in chapter 8.

Corollary 6.4.2. *Let I^- be the kernel of the natural map $\Lambda_{\mathcal{K}} \to \Lambda_{\mathcal{K}}^+$. Then under the hypotheses* *of the above proposition, we have an isomorphism:*

$$X^{\Sigma}_{\mathcal{K}_\infty}(T)/I^- X^{\Sigma}_{\mathcal{K}_\infty}(T) \to X^{\Sigma}_{\mathcal{K}^+_\infty}(T)$$

of $\Lambda^+_{\mathcal{K},A}$*-modules.*

specializing the cyclotomic variable.

Let $(T, T_v^+(v|p))$ be as above. Let ϕ be a algebra homomorphism $\Lambda_F \to \mathbb{C}_p$ and I_ϕ be its kernel.

Proposition 6.4.2. *Let* (T', T'_p) *be* (T, T_p) *twisted by* $\boldsymbol{\varepsilon}_\phi$*. Suppose there is no nontrivial* A*-subquotient of* $T^{'*}$ *on which* G_F *acts trivially. Assume:*

(i) $\Sigma \cup \{$ *primes above* $p\}$ *contains all primes at which* T *is ramified;*

(ii)for any $v|p$, $(H^0(I_v, T/T_v \otimes_A \Lambda^*_{F,A}(\boldsymbol{\varepsilon}^{-1})) \otimes_{\Lambda_F} \Lambda_F/I_\phi)^{D_v} = 0$.

Then restriction yields isomorphisms:

$$Sel^{\Sigma}_F(T') \to Sel^{\Sigma}_{F_\infty}(T)[I_\phi] \quad and \quad Sel^{\Sigma}_{\mathcal{K}}(T') \to Sel^{\Sigma}_{\mathcal{K}_{\infty^+}}(T)[I_\phi]$$

This is only a slight generalization of [SU] Proposition 3.2.13 and the proofs are identical.

Chapter 7

Hida Theory for Unitary Hilbert modular forms

In this chapter we recall basic results about ordinary Hida families for Unitary groups over totally real fields. We also recall generalizations of certain results in [SU] chapter 6 which are mostly due to Hida. Some results are only stated for cuspidal forms since this is enough for our use. However as a trade off we make the ad hoc construction in chapter 14 in which we explicitly write down a cuspidal family given the Klingen Eisenstein family.

7.1 Iwasawa Algebras

We let $\mathbb{I}_{\mathcal{K}} := \mathbb{I}[[\Gamma_{\mathcal{K}}]]$ and $\Lambda_{\mathcal{D}} := \mathbb{I}[[\Gamma_{\mathcal{K}}^- \times \Gamma_{\mathcal{K}}]] = \mathbb{I}_{\mathcal{K}}[[\Gamma_{\mathcal{K}}^-]]$. Here we used the notation $\mathcal{D}$ which stands for the Eisenstein datum to be defined in the beginning of chapter 12. Let

$$\alpha : A[[\Gamma_{\mathcal{K}}]] \to \mathbb{I}_{\mathcal{K}}^-, \alpha(\gamma^+) = (1+W_{1,v})^{\frac{1}{2}}(1+W_{2,v})^{\frac{1}{2}}(1+p)^1, \gamma_v^- \to \gamma_v^-(1+p)^{\frac{\kappa}{2}}$$

$$\beta : \mathbb{Z}_p[[\Gamma_{\mathcal{K}}]] \to \mathbb{Z}_p[[\Gamma_{\mathcal{K}}]], \beta(\gamma^+) = \gamma^+, \beta(\gamma_v^-) = \gamma_v^-$$

for each v. We also let $\Lambda := \Lambda_W[[\Gamma_{\mathcal{K}}^-, \Gamma_{\mathcal{K}}]]$. Thus $\Lambda_{\mathcal{D}}$ is finite over Λ.

Definition 7.1.1. *A $\bar{\mathbb{Q}}_p$ point $\phi \in \mathrm{Spec}\mathbb{I}[[\Gamma_{\mathcal{K}}]]$ is called arithmetic if $\phi|_{\mathbb{I}}$ is arithmetic and $\phi(\gamma^+) = (1+p)^{\frac{\kappa}{2}}\zeta^+$ for $\zeta^+ \in \mu_{p^\infty}$ and $\phi(\gamma_v^-) = \zeta_v^-$ for $\zeta_v^- \in \mu_{p^\infty}$. Here $\kappa = \kappa_{\phi|\mathbb{I}}$.*

We write $\mathcal{X}_{\mathbb{I}_{\mathcal{K}}}^a$ for the set of arithmetic points. Next let $W_2 := \prod_{v|p}(1+p\mathbb{Z}_p^\times)_v^4$ and Λ_2 be the completed group algebra of it. We give a Λ_2-algebra structure for $\Lambda_{\mathcal{D}}$ by: for each $v|p$,

$$(t_1,t_2,t_3,t_4)\to(\alpha\otimes\beta)(\mathrm{rec}_{\mathcal{K}_v}(t_3t_4,t_1^{-1}t_2^{-1})\times\mathrm{rec}_{\mathcal{K}_v}(t_4^{-1},t_2)(1+W_{1,v})^{\log_{1+p}(t_1t_3^{-1})}.$$

This way Λ becomes a quotient of Λ_2.

Remark 7.1.1. *When $F=\mathbb{Q}$ then $\Lambda_2=\Lambda$. In general Λ is of lower dimension. In other words we are only considering a subfamily of the whole weight space.*

7.2 Igusa tower and p-adic automorphic forms

We refer the definition of Shimura varieties $S(K)$ for the unitary similitude group and open compact K and the automorphic sheaves $\omega_{\underline{k}}$ to [Lan], [Hida04] and [Hsieh CM] respectively. Recall that a weight $\underline{k}=\{\underline{k}_\sigma\}_{\sigma\in\Sigma}$ where $\underline{k}_\sigma=(c_{s+1,\sigma},...,c_{r+s,\sigma};c_{1,\sigma},...,c_{s,\sigma})$. We write $M_{\underline{k}}(K,R)$ for the space of automorphic forms with weight $\underline{k}$, level K and coefficient R. We write $M_{\underline{k}}(K,R)$ for the cuspidal part.

For any $v|p$, $U(2,2)\simeq\mathrm{GL}_4(\mathbb{Z}_p)$ under projection to the first factor of $\mathcal{K}_v=F_v\times F_v$. (Recall that our convention is the first factor correspond to the Archimedean place inside the CM type under $\iota:\mathbb{C}\simeq\mathbb{C}_p$.) Define B to be the standard Borel $\begin{pmatrix}\times&\times&\times&\times\\ &\times&\times&\times\\ &&\times&\\ &&\times&\times\end{pmatrix}$ and B^u the unipotent radical. Let $I_{0,s}$ ($I_{1,s}$) consists of elements in $U(n,n)(\mathbb{Z}_p)$ which are in $B(\mathbb{Z}_p/p^s\mathbb{Z}_p)$ ($B^u(\mathbb{Z}_p/p^s)$) modulo p^s. (see [SU]5.3.6.)

Let L be a finite extension of $\mathbb{Q}_p$. Recall that as in [SU]6.1, if K is neat and maximal at p, we have $\mathcal{S}=\mathcal{S}_K$ a fixed toroidal compactification of $S_G(K)$ over $\mathcal{O}_L$. Let $\mathcal{I}_{\mathcal{S}}$ be the ideal of the boundary of $\mathcal{S}$. There is a section H of $det(\omega)$, called the Hasse invariant. Since $\det(\omega)$ is ample on the minimal compactification $\mathcal{S}^*$, one finds E, a lifting of H^m over $\mathcal{O}_L$ for sufficiently large m. Then $\mathcal{S}^*[1/E]$ is affine. For any positive integer m, set $S_m:=\mathcal{S}[1/E]\times_{\mathcal{O}_L}/p^m$. Let $H=\mathrm{GL}_2\times\mathrm{GL}_2$. For any integers $s\geq m$, we have the Igusa variety $T_{s,m}$ which is an etale Galois covering of S_m with Galois group canonically isomorphic to $\prod_{v|p}GL_2(\mathcal{O}_{F,v}/p^s)^+\times GL_2(\mathcal{O}_{F,v}/p^s)^-=H(\prod_{v|p}\mathcal{O}_{F,v}/p^s)$. We put $V_{s,m}:=\Gamma(T_{s,m},\mathcal{O}_{T_{s,m}}\otimes_{\mathcal{O}_{\mathcal{S}}}\mathcal{I}_{\mathcal{S}})$. For $j=0,1$ let $I^H_{j,s}:=I_{j,s}\cap H(\prod_{v|p}\mathcal{O}_v/p^s)$, define

$$W_{s,m}:=H^0(I^H_{1,s},V_{s,m})$$

and

$$\mathcal{W} := \varinjlim_m (\varprojlim_s W_{s,m}).$$

We also write $V^0_{s,m}$, $W^0_{s,m}$, $\mathcal{W}^0$ to be the cuspidal part of the corresponding spaces.

For $q = 0$ or ϕ we also defines the space of p-adic automorphic forms on G of weight $\underline{k}$ and level $K = K^0_p K^p$ with p divisible coefficients:

$$V^q_{\underline{k}}(K, L/\mathcal{O}_L) := \varinjlim_m \Gamma(S_m, \omega_{\underline{k}} \otimes_{\mathcal{O}_S} \mathcal{I}_S).$$

and similarly, if A is an $\mathcal{O}_L$-algebra the the space of p-adic automorphic forms with coefficients in A are defined as the inverse limits:

$$V_{\underline{k}}(K, A) := \varprojlim_m \Gamma(S_m, (\omega_{\underline{k}} \otimes_{\mathcal{O}_S} \mathcal{I}_S) \otimes_{\mathcal{O}_L} A).$$

Finally for any $\underline{a} = \{a_v\}_{v|p}$ where each $a_v \in (\mathbb{F}_p^\times)^4$ we define the modules: $V^q_{\underline{a},\underline{k}}(K, L/\mathcal{O}_L)$, etc, in the same way as [SU] 6.2.

7.3 Ordinary automorphic forms

Hida defined an idempotent e_{ord} on the space of p-adic automorphic forms (see [SU] chapter 6) and we define W_{ord}, $\mathcal{W}_{ord}$ $V_{\underline{k},ord}(K, A)$ to be the image of e_{ord} acting on the corresponding spaces. Now we recall the following important theorem of Hida (see [SU]6.2.10):

Theorem 7.3.1. *For any sufficiently regular weight $\underline{k}$ there is a constant $C(\underline{k}) > 0$ depending on $\underline{k}$ such that for any integer $l > C(\underline{k})$, the canonical map:*

$$e_{ord} M^0_{\underline{k}+l(p-1)\underline{t}}(K, L/\mathcal{O}_L) \hookrightarrow V^0_{\underline{k}+l(p-1)\underline{t},ord}(K, L/\mathcal{O}_L)$$

with $\underline{t} = (0, 0; 1, 1)$ at all infinite places σ.

From this theorem we know that there are enough classical forms in our family and thus can construct families of (pseudo)-Galois representations from the classical ones. This is also used in the proof of theorem 8.2.1 where we used Harris' result that there are no (CAP) form with sufficiently regular weight.

Lemma 7.3.1. *For any weight $\underline{k}$, we have canonical isomorphisms;*

$$V^q_{\underline{k},ord}(K,\mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathcal{W}^q_{ord}[\underline{k}] := \{w \in \mathcal{W}^q :: t.w = t^{\underline{k}} w \forall t \in T_H(\mathbb{Z}_p)\}$$

and

$$V^q_{\underline{k},ord}(K^p I_s, \psi, \mathbb{Q}_p/\mathbb{Z}_p) \simeq (\mathcal{W}^q \otimes_{\mathbb{Z}} A)[\psi_{\underline{k}}] := \{w \in \mathcal{W}^q \otimes_{\mathbb{Z}} A : t.w = \psi_{\underline{k}}(t) w \forall t \in T_H(\mathbb{Z}_p)\}$$

for any $\mathbb{Z}_p(\psi)$-algebra A.

Proof. the same as [SU] 6.2.3 □

Proposition 7.3.1. *For $q = 0$ we have for any sufficiently regular weight $\underline{k} \geq 0$, the canonical base-change morphism*

$$e_{ord}.\Gamma(\mathcal{S}^*[1/E], \pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}}} \pi^*\mathcal{I}^q) \otimes \mathbb{Z}/p^m\mathbb{Z}) \to e_{ord}.\Gamma(\mathcal{S}^*[1/E], \pi_*(\omega_{\underline{k}} \otimes_{\mathcal{O}_{\mathcal{S}}} \pi^*\mathcal{I}^q \otimes \mathbb{Z}/p^m\mathbb{Z}))$$

is an isomorphism.

This proposition fails for $q \neq 0$, thus we can't get a good control theorem for non-cuspidal Hida families.

The following corollary is immediate from the above proposition.

Corollary 7.3.1. *For $q = 0$ and any sufficiently regular weight $\underline{k}$ the module $V^q_{\underline{k},ord}(K,\mathbb{Q}_p/\mathbb{Z}_p)$ is divisible.*

7.4 Λ-adic ordinary automorphic forms

Recall that we have defined the Iwasawa algebra Λ_n. There is an action of it on the space of p-adic automorphic forms given by neben characters. (see [SU]) We define $\mathbf{V}_{ord}$ ($\mathbf{V}^0_{ord}$) to be the Pontrjagin dual of $\mathcal{W}_{ord}$ ($\mathcal{W}^0_{ord}$). As in [SU] we have the following theorem by the above corollary:

Theorem 7.4.1. *$\boldsymbol{V}_{ord}$ is finite over Λ_n and $\boldsymbol{V}^0_{ord}$ is free of finite rank over Λ_2.*

Proof. This is proved by Hida. See [SU] theorem 6.3.3. Note that the freeness is no longer true if the base field is not $\mathbb{Q}$. □

Now we define the space of ordinary cuspidal Λ_n-adic forms to be

$$\mathcal{M}^0_{ord}(K^p,\Lambda_2)=\mathrm{Hom}_{\Lambda_n}(\mathbf{V}^0_{ord},\Lambda_2).$$

Recall that in 7.1 we have defined a quotient Λ of Λ_2. Then $\boldsymbol{V}^0_{ord}\otimes_{\Lambda_2}\Lambda$ is also free over Λ. So we define the space of Λ-adic forms to be:

$$\mathcal{M}^0_{ord}(K^p,\Lambda)=\mathrm{Hom}_{\Lambda}(\mathbf{V}^0_{ord}\otimes_{\Lambda_2}\Lambda,\Lambda_n).$$

This is a closed subfamily of Λ_2-adic forms.

7.5 q-expansions

The q-expansion principle will be crucial for our later argument. Similar as in [SU], for x running through a (finite) set of representatives of $G(F)\backslash G(\mathbb{A}_{F,f})/K$ with $x_p\in Q(\mathcal{O}_{F,p})$, we have that the Λ_n-adic q-expansion map

$$\mathcal{M}^0_{ord}(K^p,\Lambda)\hookrightarrow\oplus_x\Lambda[[q^{S^+_x}]]$$

is injective. Here S^+_x is the set of Hermitian matrices h in $M_2(\mathcal{K})$ such that $\mathrm{Tr}_{F/\mathbb{Q}}\mathrm{Tr}hh'\in\mathbb{Z}$ for all Hermitian matrices h' such that $\begin{pmatrix}1 & h'\\ & 1\end{pmatrix}\in N_Q(F)\cap xKx^{-1}$ and K is the open compact of $G(\hat{\mathcal{O}}_F)$ maximal at primes dividing p which we fix from the very beginning. This follows from the irreducibility of the Igusa Tower. Let A be a finite torsion-free Λ-free algebra finite over Λ and let Σ be a Zariski dense subset of primes of A such that $Q\cap\Lambda=P_{\psi_{\underline{k}}}$ for some pair $(\underline{k},\psi)$ (we refer the definitions to [SU] chapter 6). Let $\mathcal{N}^0_{\Sigma,ord}(A)$ be the set of elements $(F_x)_x\in\oplus_x A[[q^{S^+_x}]]$ such that for each $Q\in\Sigma$ above $P_{\psi_{\underline{k}}}$ the reduction of $(F_x)_x$ is the q-expansion of some element $f\in V^0_{\underline{k},ord}(K^pI_s,\psi,A/Q)$. Then we have:

Lemma 7.5.1. *the inclusion:*

$$\mathcal{M}^0_{\underline{a},ord}(K^p,A)\hookrightarrow\mathcal{N}^0_{\underline{a},\Sigma,ord}(A)$$

is an equality.

Proof. See [SU] 6.3.7. $\square$

We will use this lemma to see that the family constructed in the last chapter by formal q-

expansions comes from some Λ-adic form.

Chapter 8

Proof of the Main results

We prove the main results in this chapter, assuming certain constructions and results of later chapters. In this chapter we will use f and $\mathbf{f}$ to define a nearly ordinary Hilbert modular form or Hida family with some coefficient ring $\mathbb{I}$. Let ψ, τ be Hecke characters of $\mathbb{A}_{\mathcal{K}}^{\times}$ and $\boldsymbol{\psi}$, $\boldsymbol{\tau}$ be p-adic families of Hecke characters of $\mathcal{A}_{\mathcal{K}}^{\times}$. We require that the restrictions of ψ and $\boldsymbol{\psi}$ to $\mathbb{A}_F^{\times}$ to be the same as the central character of the f or $\mathbf{f}$. Let ξ and $\boldsymbol{\xi}$ be $\frac{\psi}{\tau}$ and $\frac{\boldsymbol{\psi}}{\boldsymbol{\tau}}$. These are part of the Eisenstein datum $\mathcal{D}$ which we are going to define at the beginning of in chapter 12. (We are going to use this notation in this chapter though.)

8.1 The Eisenstein ideal

8.1.1 Hecke operators

Let $K' = K'_{\Sigma}K^{\Sigma} \subset G(\mathbb{A}_f^p)$ be an open compact subgroup with $K^{\Sigma} = G(\hat{\mathcal{O}}_F^{\Sigma})$ and such that $K := K'K_p^0$ is neat. The Hecke operators we are going to consider are at the unramified places and at primes dividing p. We follow closely to [SU]9.5 and 9.6.

<u>Unramified Inert Case</u>

Let v be a prime of F inert in $\mathcal{K}$. Recall that as in [SU] 9.5.2 that $Z_{v,0}$ is the Hecke operator associated to the matrix $z_0 := \mathrm{diag}(\varpi_v, \varpi_v, \varpi_v, \varpi_v$ by the double coset Kz_0K where K is the maximal compact subgroup of $G(\mathcal{O}_{F,v})$. Let $t_0 := \mathrm{diag}(\varpi_v, \varpi_v, 1, 1)$, $t_1 := \mathrm{diag}(1, \varpi_v, 1, \varpi_v^{-1})$ and

$t_2 := \mathrm{diag}(\varpi_v, 1, \varpi_v^{-1}, 1)$. We define the Hecke operators T_i for $i = 1, 2, 3, 4$ by requiring that

$$1 + \sum_{i=1}^{4} T_i X^i = \prod_{i=1}^{2} (1 - q_v^{\frac{3}{2}} [t_i] X)(1 - q_v^{\frac{3}{2}} [t_i]^{-1} X)$$

is an equality of polynomials of the variable X. Here $[t_i]$ means the Hecke operator defined by the double coset $K t_i K$. We also define:

$$Q_v(X) := 1 + \sum_{i=1}^{4} T_i (Z_0 X)^i.$$

Unramified Split Case

Suppose v is a prime of F split in $\mathcal{K}$. In this case we define $z_0^{(1)}$ and $z_0^{(2)}$ to be $(\mathrm{diag}(\varpi_v, \varpi_v, \varpi_v, \varpi_v), 1)$ and $(1, \mathrm{diag}(\varpi_v, \varpi_v, \varpi_v, \varpi_v)$ and define the Hecke operators $Z_0^{(1)}$ and $Z_0^{(2)}$ as above but replacing z_0 by $z_0^{(1)}$ and $z_0^{(2)}$. Let $t_1^{(1)} := \mathrm{diag}(1, (\varpi_v, 1), 1, (1, \varpi_v^{-1}))$, $t_2^{(1)} := \mathrm{diag}((\varpi_v, 1), 1, (1, \varpi_v^{-1}), 1)$. Define $t_i^{(2)} := \bar{t}_i^{(1)}$ and $t_i = t_i^{(1)} t_i^{(2)}$ for $i = 1, 2$. Then we define Hecke operators $T_i^{(j)}$ for $i = 1, 2, 3, 4$ and $j = 1, 2$ by requiring that the following

$$1 + \sum_{i=1}^{4} T_i^{(j)}) X^i = \prod_{i=1}^{2} (1 - q_v^{\frac{3}{2}} [t_i^{(j)}] X)(1 - q_v^{\frac{3}{2}} [t_i^{(j')}]^{-1} X)$$

to be equalities of polynomials of the variable X. Here $j' = 3 - j$ and $[t_i^{(j)}]$'s are defined similarly to the inert case. Now let $v = w\bar{w}$ for w a place of $\mathcal{K}$. Define $i_w = 1$ or 2 depending on whether the valuation associated to 2 comes from the projection onto the first or second factor of $\mathcal{K}_v = F_v \times F_v$. Then we define:

$$Q_w(X) := 1 + \sum_{i=1}^{4} T_i^{(i_w)} (Z_0^{(3-i_w)} X)^i.$$

p-adic Case

Let $t = \mathrm{diag}(p^{a_1}, p^{a_2}, p^{a_4}, p^{a_3})$, u_t is the Hida's normalized u_t operator defined in [SU] (6.2.2.a).

Let $\mathbf{h}_{\mathcal{D}} = \mathbf{h}_{\mathcal{D}}(K')$ be the reduced quotient of the universal ordinary cuspidal Hecke algebra which is defined by the ring of elements in $\mathrm{End}_{\Lambda_{\mathcal{D}}}(S^{ord}(K', \Lambda_{\mathcal{D}}))$ generated by the Hecke operators $Z_{v,0}$, $Z_{v,0}^{(i)}$, $T_{i,v}$, $T_{i,v}^{(j)}$,$u_{t,v}$ defined above. This is a finite reduced $\Lambda_{\mathcal{D}}$-algebra. Now we define for each prime w of $\mathcal{K}$ a polynomial $Q_{w,\mathcal{D}}(X)$ to be $\det(1 - \rho_{\mathcal{D}}(\mathrm{frob}_w) X)$ where $\mathcal{D}$ is the Eisenstein datum mentioned at the beginning of this chapter and $\boldsymbol{\rho}_{\mathcal{D}}$ is the Galois representation defined in subsection 8.2.2.

We define the Eisenstein ideal (which is actually the kernel of homomorphism from the abstract hecke algebra to $\Lambda_{\mathcal{D}}$ determined by the Eisenstein family we will construct later) generated by:

- the coefficients of $Q_w(X) - Q_{w,\mathcal{D}}(X)$ for all finite places v of $\mathcal{K}$ and not dividing a prime in Σ.
- $Z_{v,0} - \boldsymbol{\sigma}_\psi \boldsymbol{\sigma}_\xi^{-1}(\mathrm{frob}_v)$ for v a finite place outside Σ.
- $Z_{v,0}^{(i)} - \boldsymbol{\sigma}_\psi \boldsymbol{\sigma}_\xi^{-1}(\mathrm{frob}_{w_i})$ for all v outside Σ such that $v = w_1 w_2$ being the factorization of $\mathcal{K}_v = F_v \times F_v$.
- For all $v|p$, $u_{t,v} - \lambda_{\mathbf{E}_{\mathcal{D}}}(u_{t,v})$ with $a_1 \leq \cdots \leq a_4$

Here $\boldsymbol{\sigma}$ is the reciprocity map of class field theory, $\lambda_{\mathbf{E}_{\mathcal{D}}}$ is the Hecke eigenvalue for $u_{v,t}$ acting on $\mathbf{E}_{\mathcal{D}}$. It follows from the computations in part one that these are elements in $\mathbb{I}[[\Gamma_{\mathcal{K}}]]$. We omit the precise formulas.

aaaaaa

The structure map $\Lambda_{\mathcal{D}} \to \mathbf{h}_{\mathcal{D}}/I_{\mathcal{D}}$ is surjective and we denote $\mathcal{E}_{\mathcal{D}} \subset \Lambda_{\mathcal{D}}$ to be kernel of this map so that:

$$\Lambda_{\mathcal{D}}/\mathcal{E}_{\mathcal{D}} \to \mathbf{h}_{\mathcal{D}}/I_{\mathcal{D}}.$$

We define ϕ_0 to be the point on the weight space corresponding to the special L-value $L(f_2, 1)$ where f_2 is the nearly ordinary form in our Hida family of parallel weight 2 and trivial neben typus at p. (In fact this notion is a little bit ambiguous since we might have several f_2's inside the Hida family and what we are going to prove is true for any of such point ϕ_0). We have the following theorem which is [SU] 6.5.4 in our situation:

Theorem 8.1.1. *Assumptions are as above. Then there is a finite normal extension $\mathbb{J}$ of $\mathbb{I}$ such that if $P \subset \Lambda_{\mathcal{D}}$ is a height one prime of $\Lambda_{\mathcal{D},\mathbb{J}}$ passing through ϕ_0 such that $\boldsymbol{E}_{\mathcal{D}}$ is non-zero modulo P (i.e. if the ideal generated by the Fourier coefficient of $\boldsymbol{E}_{\mathcal{D}}$ is not contained in P), and that P is not a pull back of a height one prime from $\mathbb{J}[[\Gamma_{\mathcal{K}}^+]]$ then:*

$$ord_P(\mathcal{E}_{\mathcal{D}}) \geq ord_P(\mathcal{L}_{\mathcal{D}}).$$

Proof. The proof is completely the same as [SU]6.5.4. except that we use the construction of Chapter 8, which explains why we need to take the extension $\mathbb{J}$. $\square$

8.2 Galois Representations

8.2.1 Galois theoretic argument

In this section we summarize the main results in [SU] chapter 4, which would be used to construct elements in the Selmer group.

Let G be a group and C a ring. $r :\to Aut_C(V)$ a representation of G with $V \simeq C^n$. This can be extended to $r : C[G] \to End_C(V)$. For any $x \in C[G]$, define: $Ch(r, x, T) := det(id - Tr(x)) \in C[T]$.

Let (V_1, σ_1) and (V_2, σ_2) be two C representations of G. Assume both are defined over a local henselian subring $B \subseteq C$, we say σ_1 and σ_2 are residually disjoint modulo the maximal ideal $\mathfrak{m}_B$ if there exists $x \in B[G]$ such that $Ch(\sigma_1, x, T)$ mod $\mathfrak{M}_B$ and $Ch(\sigma_2, x, , T)$ mod $\mathfrak{m}_B$ are relatively prime in $\kappa_B[T]$, where $\kappa_B := B/\mathfrak{m}_B$.

Let H be a group with a decomposition $H = G \rtimes \{1, c\}$ with $c \in H$ an element of order two normalizing G. For any C representations (V, r) of G we write r^c for the representation defined by $r^c(g) = r(cgc)$ for all $g \in G$.

<u>Polarizations</u>:

Let $\theta : G \to GL_L(V)$ be a representation of G on a vector space V over field L and let $\psi : H \to L^\times$ be a character. We assume that θ satisfies the ψ-polarization condition:

$$\theta^c \simeq \psi \otimes \theta^\vee.$$

By a ψ-polarization of θ we mean an L-bilinear pairing $\Phi_\theta : V \times V \to L$ such that

$$\Phi_\theta(\theta(g)v, v') = \psi(g)\Phi_\theta(v, \theta^c(g)^{-1}v').$$

Let $\Phi_\theta^t(v, v') := \Phi_\theta(v', v)$, which is another ψ-polarization. We say that ψ is compatible with the polarization Φ_θ if

$$\Phi_\theta^t = -\psi(c)\Phi_\theta.$$

Suppose that:

(1)A_0 is a pro-finite $\mathbf{Z}_p$ algebra and a Krull domain;

(2)$P \subset A_0$ is a height one prime and $A = \hat{A_{0,P}}$ is the completion of the localization of A_0 at P. This is a DVR.

(3) R_0 is local reduced finite A_0-algebra;

(4) $Q \subset R_0$ is prime such that $Q \cap A_0 = P$ and $R = \hat{R_{0,Q}}$;

(5) there exist ideals $J_0 \subset A_0$ and $I_0 \subset R_0$ such that $I_0 \cap A_0 = J_0, A_0/J_0 = R_0/I_0, J = J_0 A, I = I_0 R, J_0 = J \cap A_0$ and $I_0 = I \cap R_0$;

(6) G and H are pro-finite groups; we have subgroups $D_i \subset G$ for $i = 1, \cdots, d$.

the set up: suppose we have the following data:

(1) a continuous character $\nu : H \to A_0^\times$;

(2) a continuous character $\xi : G \to A_0^\times$ such that $\bar{\chi} \neq \bar{\nu}\bar{\chi}^{-c}$; Let $\chi' := \nu\chi^{-c}$;

(3) a representation $\rho : G \to Aut_A(V), V \simeq A^n$, which is a base change from a representation over A_0, such that:

$a.\rho^c \simeq \rho^\vee \otimes \nu,$

$\bar{\rho}$ is absolutely irreducible ,

ρ is residually disjoint from χ and χ';

(4) a representation $\sigma : G \to Aut_{R\otimes_A F}(M), M \simeq (R \otimes_A F)^m$ with $m = n + 2$, which is defined over the image of R_0 in R, such that:

$a.\sigma^c \simeq \sigma^\vee \otimes \nu,$

$b.tr\sigma(g) \in R$ for all $g \in G$,

$c.$ for any $v \in M$,$\sigma(R[G])v$ is a finitely-generated R-module

(5) a proper ideal $I \subset R$ such that $J := A \cap I \neq 0$, the natural map $A/J \to R/I$ is an isomorphism, and

$$tr\sigma(g) \equiv \chi'(g) + tr\rho(g) + \chi(g) \ mod\ I$$

for all $g \in G$.

(6) ρ is irreducible and ν is compatible with ρ.

(7) (local conditions for ρ) For each $i = 1, \cdots, d$ there is a D_i stable sub A_0 module $V_{0,i}^+ \subset V_0$ such that $V_{0,i}^+$ and $V_{0,i}^- := V_{0,i}/V_{0,i}^+$ are free A_0 modules.

(8) (local conditions for σ). For each $i = 1, \cdots, d$ there is a D_i-stable sub-$R \otimes_A F$-module $M_i^+ \subseteq M$ such that M_i^+ and $M_i^- := M/M_i^+$ are free $R \otimes_A F$ modules.

(9) (compatibility with the congruence condition) Assume that for all $x \in R[D_i]$, we have congruence relation:

$$Ch(M_i^+, x, T) \equiv Ch(V_i^+, x, T)(1 - T\chi(x)) \ mod \ I$$

(then we automatically have:

$$Ch(M_i^-, x, T) \equiv Ch(V_i^-, x, T)(1 - T\chi'(x)) \ mod \ I$$

(10) For each F-algebra homomorphism $\lambda : R \otimes_A F \to K$, K a finite field extension of F, the representation $\sigma_\lambda : G \to GL_m(M \otimes_{R \otimes F} K)$ obtained from σ via λ is either absolutely irreducible or contains an absolutely irreducible two-dimensional sub K-representation σ'_λ such that $tr\sigma'_\lambda(g) \equiv \chi(g) + \chi'(g) mod I$.

One defines the Selmer groups $X_H(\chi'/\chi) := ker\{H^1(G, A_0^*(\chi'/\chi)) \to H^1(D, A_0^*(\chi'\chi))\}^*$. and $X_G(\rho_0 \otimes \chi^{-1}) := ker\{H^1(G, V_0 \otimes_{A_0} A_0^*(\chi^{-1})) \to H^1(D, V_0^- \otimes_{A_0} A_0^*(\chi^{-1}))\}^*$

Our result is:

Proposition 8.2.1. *under the above assumptions, if $ord_P(Ch_H(\chi'/\chi)) = 0$ then $ord_P(Ch_G(\rho_0 \otimes \chi^{-1}) \geq ord_P(J)$.*

We record here an easy lemma about Fitting ideals and characteristic ideals which will be used later.

Lemma 8.2.1. *Let A be a Krull domain and T is a A-module. Suppose $f \in A$ is such that for any height one primes P of A, $\mathrm{ord}_P(\mathrm{Fitt}_A T) \geq \mathrm{ord}_P(f)$ then $\mathrm{char}_A(T) \leq (f)$.*

Proof. for any $g \in \mathrm{char}_A(T)$ the assumption and the definition for characteristic ideals ensures that for any height 1 prime P, $\mathrm{ord}_P(\frac{g}{f}) \geq 0$. Since A is normal this implies $\frac{g}{f} \in A$. Thus $g \in (f)$. □

8.2.2 Galois representations

We define a semi-simple representation

$$\boldsymbol{\rho}_{\mathcal{D}} := \boldsymbol{\sigma}_{\boldsymbol{\psi}}^{c}\epsilon^{-3} \oplus (\rho_{\boldsymbol{f}} \otimes \boldsymbol{\xi}^{-c}\boldsymbol{\psi}^{c}\epsilon^{-2}) \oplus \epsilon^{-1}det\rho_{\boldsymbol{f}}\boldsymbol{\sigma}_{\boldsymbol{\xi}'}^{-1}\boldsymbol{\sigma}_{\boldsymbol{\psi}}^{c}.$$

Recall that here $\boldsymbol{\sigma}$ means the reciprocity map. We will see that this is the Galois representation associated to the Eisenstein family constructed in the next chapter.

Let $\mathcal{E}_{\mathcal{D}} \subseteq \Lambda_{\mathcal{D}}$ be the Eisenstein ideal associated with $\mathcal{D}$. For any prime $v|p$ of F we let $T_{\boldsymbol{f},v}^{+} \subseteq T_{\boldsymbol{f}}$ be the rank one $\mathbb{I}$-summand of $T_{\boldsymbol{f}}$ that is G_v-stable. Given a height one prime P of $\Lambda_{\mathcal{D}}$ containing $\mathcal{E}_{\mathcal{D}}$ let:

- $H := G_{F,\Sigma}, G := G_{\mathcal{K},\Sigma}, c =$ the usual complex conjugation;
- $A_0 := \Lambda_{\mathcal{D}}, A := \hat{\Lambda}_{\mathcal{D},P}$;
- $J_0 := \mathcal{E}, J := \mathcal{E}_{\boldsymbol{D}}A$;
- $R_0 : \boldsymbol{T}_{\mathcal{D}}, I_0 := I_{\mathcal{D}}$;
- $Q \subset R_0$ is the inverse image of $Pmod\mathcal{E}_{\mathcal{D}}$ under $\boldsymbol{T}_{\mathcal{D}} \to \boldsymbol{T}_{\mathcal{D}}/I_{\mathcal{D}} = \Lambda_{\mathcal{D}}/\mathcal{E}_{\mathcal{D}}$;
- $R := \hat{\boldsymbol{T}}_{\boldsymbol{D},Q}, I := I_{\mathcal{D}}R$;
- $V_0; = T_{\boldsymbol{f}} \otimes_{\mathbb{I}} \Lambda_{\mathcal{D}}, \rho := \rho_{\boldsymbol{f}} \otimes \boldsymbol{\sigma}_{\xi}^{-c}\boldsymbol{\sigma}_{\psi}^{c}\epsilon^{-2}$;
- $V_0^{+} := T_{\boldsymbol{f}}^{+} \otimes_{\mathbb{I}} A_0, V_0^{-} := T_{\boldsymbol{f}}/T_{\boldsymbol{f}}^{+}) \otimes_{A_0} A$;
- $V = V_0 \otimes_{A_0} A, \rho = \rho_0 \otimes_{A_0} A, V^{\pm} := V_0^{\pm} \otimes_{A_0} A$;
- $\chi := \epsilon^{-1}det\rho_{\boldsymbol{f}}\boldsymbol{\sigma}_{\xi'}^{-1}\boldsymbol{\sigma}_{\psi}^{c}, \nu := \boldsymbol{\sigma}_{\psi'}\boldsymbol{\sigma}_{\xi'}^{-1}\epsilon^{-4}$;
- $\chi' := \boldsymbol{\sigma}_{\psi}^{c}\epsilon^{-3}$ so $\chi' = \nu\chi^{-c}$;
- $M := (R \otimes_A F_A)^4$, F_A is the field of fractions of A and the Galois action is given by the Galois representation associated to the cuspidal Hida families on $U(2,2)(\mathbb{A}_F)$;
- σ is the representation on M obtained from R_D .

Let $\mathcal{T} := (T_{\boldsymbol{f}} \otimes_{\mathbb{I}} \mathbb{I}[[\Gamma_{\mathcal{K}}]])(\boldsymbol{\varepsilon}_{\mathcal{K}})$ (see section 2.1) and $\mathcal{T}^{+} := (T_{\boldsymbol{f}}^{+} \otimes_{\mathbb{I}} \mathbb{I}[[\Gamma_{\mathcal{K}}]])(\boldsymbol{\varepsilon}_{\mathcal{K}})$. Let $Ch_{\mathcal{K}}^{\Sigma}(\rho_{\boldsymbol{f}}) \subset \mathbb{I}[[\Gamma_{\mathcal{K}}]]$ be the characteristic ideal of the dual Selmer group $X_{\mathcal{K}}^{\Sigma}(\mathcal{T}, \mathcal{T}^{+})$.

Theorem 8.2.1. *Suppose $\mathbb{I}$ is an integrally closed domain. Let $P_0 \subset \mathbb{I}[[\Gamma_{\mathcal{K}}]]$ be a height one prime and let $P = P_0\Lambda_{\mathcal{D}}$ that is not a pullback of one of $\mathbb{I}[[\Gamma_{\mathcal{K}}^+]]$ be the height one prime of $\Lambda_{\mathcal{D}}$ it generates. Suppose also that:*

$$V^+ \oplus A(\chi) \text{ and } V^- \oplus A(\chi') \text{ modulo } P \text{ do not have common irreducible pieces .} \tag{8.1}$$

Then

$$ord_{P_0}(Ch^{\Sigma}_{\mathcal{K}}(\rho_{\boldsymbol{f}} \otimes \boldsymbol{\varepsilon}_{\mathcal{K}})) \geq ord_P(\mathcal{E}_{\mathcal{D}}).$$

Proof. One just apply proposition 8.2.1. The condition (10) there is guaranteed by an argument similar to [SU] theorem 7.3.1. We use the modularity lifting results in [SW] for ordinary Galois representations satisfying (irred) and (dist) and Harris's result that there is no (CAP) forms when the weight $\underline{k}$ is sufficiently regular. We also use the main conjecture for totally real field F proven in [Wiles90] to conclude that $\mathrm{ord}_P(Ch_H(\chi'/\chi)) = 0$ since it is non zero by [Wiles90] and only involves the cyclotomic variable. □

We are going to define two conditions (NV1) and (NV2) in Chapter 12 and give sufficient conditions for them.

Now for f a Hilbert modular form with trivial characters and neben typus, then we write Sel_f briefly for the Selmer group for the motive $\rho_f \otimes \det \rho_f^{-1}$. Then:

Theorem 8.2.2. *Let p be a rational odd prime that splits completely in F. Let f be a Hilbert modular form over F of parallel weight 2 and trivial character. Suppose:*
(i) f is ordinary at all primes of F dividing p;
(ii) ***(irred)*** *and* ***(dist)*** *in [SU1] hold for ρ_f.*
If the central critical value $L(f,1) = 0$, then the Selmer group $H^1_f(F, \rho_f)$ is infinite.

Proof. We only need to prove the theorem in the case when the root number for f is $+1$ since otherwise it is a well known result of Nekovar [Nek]. First suppose that $d = [F : \mathbb{Q}]$ is even then we choose an imaginary quadratic extension $\mathcal{K}$ of F so that $\mathcal{K}/F$ is split at all primes at which f is ramified and $L(f, \chi_{\mathcal{K}/F}, 1) \neq 0$ where $\chi_{\mathcal{K}/F}$ is the quadratic character of $\mathbb{A}_F^\times$ associated to $\mathcal{K}/F$. This is possible by Waldspurger. Then the $S(1)$ defined in [Vatsal07] p123 consists of exactly all the infinite places and since d is even we are in the definite case there.

We put f in a Hida family $\mathbf{f}$. Now we do not have the Gorenstein properties. We have to replace $\ell_{\boldsymbol{f}}$ by $1_{\mathbf{f}}$ everywhere. Our p-adic L-function is not integral. (in $F_{\mathbb{I}} \otimes_{\mathbb{I}} \mathbb{I}[[\Gamma_{\mathcal{K}}]]$ actually). Suppose

$\tilde{\mathcal{L}} = \frac{h}{g}$, it follows from the definition for $1_{\mathbf{f}}$ in chapter 12 and the congruence number for f_2 is finite that we may choose $g \in \mathbb{J}$ so that $g(\phi_0) \neq 0$. Suppose $L(f,1) = 0$. Start with the 1-dimensional family of cylotomic twists of f. Since $L(f,1) = 0$ then there is a height 1 prime of the 1-dimensional weight space passing through ϕ_0 and containing the image of h. Here we must notice that by our construction we did not include ϕ_0 as an interpolation point. But by the interpolation property we know that our $\tilde{\mathcal{L}}^{\Sigma}$ is the same as Hida's ([Hida91]) up to Euler factors at Σ at a sub family containing the above 1-dimensional family.

Now we consider the specialization step by step. At each step the Iwasawa algebra is a Krull domain. Suppose $\mathrm{Spec}\Lambda_1 \hookrightarrow \mathrm{Spec}\Lambda_2$ where Λ_2 has one more variable than Λ_1, i.e. $\Lambda_1 = \Lambda_2/x\Lambda_2$ for some variable x. If P_1 is a height 1 prime of Λ_1 passing through ϕ_0 and containing the image of h in Λ_1 then we can find P_2 a height one prime of Λ_2 also passing through ϕ_0 and contains the image of h in Λ_2 such that $\mathrm{Supp}P_1 \subset \mathrm{Supp}P_2$ under $\mathrm{Spec}\Lambda_1 \hookrightarrow \mathrm{Spec}\Lambda_2$. Finally we found some P a height one prime of the full dimensional Iwasawa algebra passing through ϕ_0 and containing h. Note also that P does not contain g since $g(\phi_0) \neq 0$. In chapter 12 we will see that (NV1) is satisfied in our situation and thus h is not contained in any height one prime of $\mathbb{I}[[\Gamma_{\mathcal{K}}^{+}]]$ passing through ϕ_0. In the construction of chapter 14 after replacing $\ell_{\mathbf{f}}$ by $1_{\mathbf{f}}$ the cuspidal family we construct still have P-adically integral coefficients and has some coefficient prime to P. (although things there are in $F_{\mathbb{J}} \otimes_{\mathbb{J}} \Lambda_{\mathcal{D},\mathbb{J}}$, however as above we can make sure that things showing up in the denominator are non zero at ϕ_0 thus outside P). The argument of theorem 8.1.1 still gives:

$$1 \leq \mathrm{ord}_P \tilde{\mathcal{L}} \leq \mathrm{ord}_P(\mathcal{E}_{\mathcal{D},\mathbb{J}})$$

theorem 8.2.1 and proposition 13.3.2 (should be the $\tilde{E}$ version which we did not state there) gives that:

$$\mathrm{ord}_P Ft^{\Sigma}_{\mathbf{f},\mathcal{K},1} \geq 1.$$

Then we need to specialize the variables back step by step to prove the theorem. Using the control theorem for the Selmer groups we have at each step: we get $\mathrm{ord}_{P_i} Ft^{\Sigma}_{\mathcal{K}} \geq 1$ here the P_i and the Selmer modules are interpreted in the context of each step. Finally we specialize to the point ϕ_0 to get that the Σ-primitive Selmer group over $\mathcal{K}$ is infinity. But this implies that Sel_f is itself infinity since L_{Σ} is non zero. However this Selmer group is the product of Selmer groups for f and $f \otimes \chi_{\mathcal{K}}$. By [YZZ] and our choice of $\mathcal{K}$, we know that the Selmer group for $f \otimes \chi_{\mathcal{K}}$ is finite. So our theorem is true. Next we assume d is odd. Then again by Waldspurger we can find a real quadratic character $\chi_{F,/F}$ such that F' is split at all primes at which f is ramified and $L(f,\chi_{F'/F},1) \neq 0$. We consider $f_{F'}$ the base change of f to F'. Then $[F':\mathbb{Q}]$ is even and we deduce that at least one of Sel_f and $Sel_{f\otimes\chi_{F'/F}}$ is infinite. But by [YZZ] we know that $Sel_{f\otimes\chi_{\mathcal{K}}}$ is finite. So Sel_f must be infinite. $\square$

Chapter 9

Klingen Eisenstein Series

Now we recall some notions for Klingen Eisenstein Series in this setting. We need to use a character ψ to pass form the GL_2 picture to the unitary group $U(1,1)$ similar to [SU] chapter 9. Then the p-adic constructions are just as in part one. Note that this is slightly different from [SU] since there the ordinary vector is the new vector. However in the Hilbert modular case we do not assume this in order to get the whole Hida family. For the ℓ-adic construction the one used by [SU] is much better than the one used in part one so we just follow [SU].

9.1 Induced representations

9.1.1 archimedean picture

Let (π, V) be an irreducible $(\mathfrak{gl}_2, K'_\infty)$-module and suppose that π is unitary ,tempered representation. There is an irreducible, unitary Hilbert representation (π_1, H) of $GL_2(\mathbb{R})$, unique up to isomorphism such that π, V can be identified with the $\mathfrak{g}, \mathfrak{l}, K'_\infty)$-module of it. Let χ be the central character of π_1. Let ψ and τ be unitary characters of $\mathbf{C}^\times$ such that $\psi|_{\mathbb{R}^\times} = \chi$. Now we define a representation ρ of $P(\mathbf{R})$: for $g = mn, n \in N_P(\mathbb{R}), m = m(bx, a) \in M_P(\mathbf{R})$ with $a, b \in \mathbb{C}^\times, x \in GL_2(\mathbb{R})$, put

$$\rho(g)v := \tau(a)\psi(b)\pi(x)v, v \in H.$$

For any function $f \in C^\infty(K_\infty, H_\infty)$ such that $f(k'k) = \rho(k')f(k)$ for any $k' \in P(\mathbb{R}) \cap K_\infty$, where H_∞ is the space of smooth vector of H, any each $z \in \mathbb{C}$ we define a function

$$f_z(g) := \delta_P(m)^{3/2+z}\rho(m)f(k), g = mk \in P(\mathbb{R})K_\infty,$$

and we define an action $\sigma(\rho, z)$ of $G(\mathbb{R})$ on $I(H_\infty)$:

$$(\sigma(\rho, z)(g)f) := f_z(kg).$$

Let $(\pi^\vee, V)$ be the irreducible $(\mathfrak{gl}_2, K'_\infty)$-module given by $\pi^\vee(x) = \pi(\eta^{-1}x\eta)$ for x in $\mathfrak{gl}_2$ or K'_∞, denote $\rho^\vee, I(\rho^\vee), I^\vee(H_\infty)$ and $\sigma(\rho^\vee, z), I(\rho^\vee))$ the representations and spaces defined as above but with π, ψ, τ replaced by $\pi^\vee \otimes (\tau \circ det), \psi\tau\tau^c, \bar{\tau}^c$. Let $\tilde{\pi} := \pi^\vee \otimes \chi^{-1}$. Also, for any $z \in \mathbb{C}$, $f \in I(H_\infty)$ and $k \in K_\infty$ consider the integral:

$$A(\rho, z, f)(k) := \int_{N_P(\mathbb{R})} f_z(wnk)dn. \tag{9.1}$$

$A(\rho, z, -) \in Hom_{\mathbb{C}}(I(H_\infty), I^\vee(H_\infty))$ intertwines the actions of $\sigma(\rho, z)$ and $\sigma(\rho^\vee, -z)$.

9.1.2 ℓ-adic picture

Let v be a prime of F and (π, V) be an irreducible, admissible representation of $GL_2(F_v)$ and suppose that π is unitary and tempered. Denote by χ the central character of π. Let ψ and τ be unitary characters of $\mathcal{K}_v^\times$ such that $\psi|_{F_V^\times} = \chi$. We extend π to a representation ρ of $P(F_v)$ on V as follows. For $g = mn, n \in N_P(F_v)$, $m = m(bx, a) \in M_P(F_v), a, b \in K_v^\times, x \in GL_2(F_v)$, put

$$\rho(g)v := \tau(a)\psi(b)\pi(s)v, v \in V.$$

Let $I(\rho)$ be the space of functions $f : \mathcal{K}_v \to V$ such that (i) there exists an open subgroup $U \subseteq K_v$ such that $f(gu) = f(g)$ for all $u \in U$ and (ii) $f(k'k) = \rho(k')f(k)$ for $k' \in P(\mathcal{O}_{F,v})$. For each $f \in I(\rho)$ and each $z \in \mathbf{C}$ we define a function f_z on $G(F_v)$ by

$$f_v(g) := \delta_P(m)^{3/2+z}\rho(m)f(k), g = mk \in P(F_v)K_v$$

We define a representation $\sigma(rho, z)$ of $G(F_v)$ on $I(\rho)$ by

$$(\sigma(\rho, z)(g)f)(k) := f_z(kg).$$

If π, ψ, τ are unramified the

$$dim_{\mathbf{C}} I(\rho)^{K_v} = 1.$$

In particular if $\phi \in V$ is a newvector for π and F_ρ is defined by $F_\rho(mk)\rho(m)\phi, mk \in P(\mathcal{O}_{F,v})K_v$, $I(\rho)^{K_v}$ is spanned by F_ρ.

Let $(\pi^\vee, V)$ be given by $\pi^\vee(g) = \pi(\eta^{-1}g\eta)$. This representation is also tempered and unitary. We denote by $\rho^\vee, I(\rho^\vee)$, and $(\sigma(\rho_\vee, z), I(\rho^\vee))$ the representations and spaces defined as above but with π, ψ and τ replaced by $\pi^\vee \otimes (\tau \circ det), \psi\tau\tau^c$, and $\bar{\tau}^c$, respectively. Let $\tilde{\pi} := \pi^\vee \otimes \chi^{-1}$.

For $f \in I(\rho), k \in K_v$, and $z \in \mathbf{C}$ consider the integral

$$A(\rho, z, v)(k) := \int_{N_P(F_v)} f_z(wnk)dn. \tag{9.2}$$

As a consequence of our hypotheses on π this integral converges absolutely and uniformly for z and k in compact subsets of $z : Re(z) > 3/2 \times K_v$. Moreover, for such z, $A(\rho, z, f) \in I(\rho^\vee)$ and the operator $A(\rho, z, -) \in Hom_{\mathbf{C}}(I(\rho), I(\rho^\vee))$ intertwines the actions of $\sigma(\rho, z)$ and $\sigma(\rho^\vee, -z)$.

As in [SU]9.1.3, this has a meromorphic continuation (in the sense defined there) to $\mathbb{C}$ and the poles can only occur when $Rez = 0, \pm\frac{1}{2}$.

9.1.3 p-adic picture

Now assume $v|p$. We need to study the relations between the GL_2 picture and the computations in part one for $U(1,1)$. Suppose $\pi_v \equiv \pi(\mu_1, \mu_2)$ where $\mathrm{val}_p(\mu_1(p) = -\frac{1}{2}$ and $\mathrm{val}(\mu_2(p) = \frac{1}{2}$. From now on we write $\xi = \frac{\psi}{\tau}$ and $\xi = (\xi_1, \xi_2)$ with respect to $\mathcal{K}_v \equiv F_v \times F_v$. Similarly for $\tau_1, \tau_2, \psi_1, \psi_2$. Note that our ξ here is different from part one. In fact the $\xi_1\xi_2, \chi_1\chi_2$ there are $\mu_1\bar{\xi_2}, \mu_1\bar{\xi_1}, \mu_1\psi_2^{-1}, \mu_2\psi_2^{-1}$. Note that $\psi_1\psi_2 = \mu_1\mu_2$.

<u>Generic Case</u> The generic case mentioned in part one correspond to:cond$(\chi_1) >$ cond$(\tau_2) >$ cond$(\tau_1) >$ cond(χ_2). (note that the τ in $\rho^\vee$ is $\bar{\tau}^c$. We assume cond$(\psi_2) >$ cond$(\tau_2) >$ cond$(\tau_1) >$ cond$(\psi_1) >$ cond(μ_1). Then the datum is generic in the sense of part one.

9.1.4 global picture

Let (π, V) be an automorphic representation of GL_2/F. It is an admissible $(\mathfrak{gl}_2^d, K'_\infty) \times GL_2(\mathbb{A}_f)$-module which is a restricted tensor product of local irreducible admissible representations. Let $\tau, \psi : \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$ be Hecke characters such that $\psi|_{\mathbb{A}_F^\times} = \chi_\pi$ and he let $\tau_\otimes\tau_w$ and $\psi = \otimes\psi_w$ be their local decompositions, w over places of F. We associat with triple (τ, ψ, τ) a representation of

$(P(F_\infty) \cap K_\infty) \times P(\mathbb{A}_{F,f})$ and $v = \otimes v_w \in V$ put

$$\rho(m)v := \otimes(\rho_w(m_w)w_m),$$

Let $K_f := \prod_{w \not| \infty} K_w$ and $K_{\mathbb{A}_F} := K_{F_\infty} \times K_f$. Let $I(\rho)$ be the space of functions $f : K_{\mathbb{A}_F} \to V$ such that $f(k'k) = \rho(k')f(k)$ for $k' \in P(\mathbb{A}_F) \cap K_{\mathbb{A}}$, and f factors through $K_{F_\infty} \times K_f/K'$ for some open subgroup $K' \subseteq K_f$ and f is K_{F_∞}-finite and smooth as a function on $K_{F,\infty} \times K_f/K'$. This can be identified with the restricted product $\otimes I(\rho_w)$ with respect to the F_{ρ_w}'s at those w at which τ_w, ψ_w, π_w are unramified.

For each $z \in \mathbb{C}$ and $f \in I(\rho)$ we define a function f_z on $G(\mathbb{A}_F)$ as

$$f_z(g) := \otimes f_{w,z}(g_w)$$

where $f_{w,z}$ are defined as before. Also we define an action $\sigma(\rho, z0$ of $\mathfrak{g}, K_{F_\infty}) \otimes G(\mathbb{A}_f)$ on $I(\rho)$ by $\sigma(\rho, z) := \otimes\sigma(\rho_w, z)$. Similarly we define $\rho^\vee, I(\rho^\vee)$, and $\sigma(\rho^\vee, z)$ but with the corresponding things replaced by their $\vee$'s. For each $z \in \mathbb{C}$ there are maps

$$I(\rho), I(\rho^\vee) \hookrightarrow \mathcal{A}(M_P(F)N_P(F)\backslash P(\mathbb{A}_F)),$$

given by

$$f \mapsto (g \mapsto f_z(g)(1)).$$

In the following we often write f_z for the automorphic form given by this recipe.

9.1.5 Klingen-type Eisenstein series on G

Let π, ψ, and τ be as above. For $f \in I(\rho), z \in \mathbf{C}$, and $g \in G(\mathbb{A}$ the series

$$E(f, z, g) := \sum_{\gamma \in P(F)\backslash G(F)} f_z(\gamma g) \tag{9.3}$$

is known to converge absolutely and uniformly for (z, g) in compact subsets of $\{z \in \mathbb{C} : Re(z) > 3/2\} \times G(\mathbb{A})$ and to define an automorphic form on G.The may $f \mapsto E(f, z, -)$ intertwines the action of $\sigma(\rho, z)$ and the usual action of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ on $\mathcal{A}(G)$.

We state a well known lemma of [SU] here for the field F.

Lemma 9.1.1. *Let R be a standard F-parabolic of G (i.e, $R \supseteq B$). Suppose $Re(z) > \frac{3}{2}$.*

(i) If $R \neq P$ *then* $E(f,z,g)_R = 0$*;*

(ii) $E(f,z,-)_P = f_z + A(\rho,f,z)_{-z}$.

9.2 Induced representations:good sections

9.2.1 Archimedean sections

The choices made here are completely the same as [SU] chapter 9 for all infinite places (see also part one). So we omit here and denote the Klingen section as F_κ.

9.2.2 ℓ-adic sections

Let v be a prime of F not dividing p. The sections chosen here are the same as in [SU] chapter 9. We define a character ν' of $K_{r_{\psi'},s'}$ by

$$\nu'(\begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & * \\ * & * \end{pmatrix}) := \psi'(ad-bc)\bar{\xi}'(d).$$

For $K \subseteq K_{r'_\psi,s'}$ let

$$I(\rho',K) := \{f \in I(\rho') : \rho'(k)f = \nu'(k)f, k \in K\}.$$

Let $\phi \in V$ be any vector having a conductor with respect to $\pi^\vee$ and let $(\lambda^{t_\phi}) := cond_{\tilde{\pi}}(\phi)$. For any $K_{r,t}$ with $r \geq max(r_\psi, r_\phi)$ and $t > s$ we define $f_{\phi,r,t} \in I(\rho, K_{r,t})$ by

$$F_{\phi,r,t}(g) := \begin{cases} \nu(k)\rho(p)\phi & g = pmk \in P(\mathcal{O}_{F,v}wK_{r,t} \\ 0 & otherwise. \end{cases}$$

Since $P(\mathcal{O}_{F,v})wK_Q(\lambda) = P(\mathcal{O}_{F,v})wQ(\mathcal{O}_{F,v})$, if $r,r' \geq \{r_\phi, 1\}$ then $F_{\phi,r,t} = F_{\phi,r',t}$..

9.2.3 p-adic sections

We define our p-adic section to be the F_v^0 defined in part one. This is nearly ordinary as proved there.

9.3 Good Eisenstein series

9.3.1 Eisenstein data

(π, V) is an irreducible $(\mathfrak{gl}_2, K'_\infty)^d \times GL_2(\mathbf{A}_{F,f})$-sub representation of $\mathcal{A}^0(GL_2/F)$ and let $V = \otimes V_\pi$ and $\pi = \otimes \pi_w$. By an Eisenstein datum for π we will mean a 4-tuple $\mathcal{D} = \{\Sigma, \varphi, \psi, \tau\}$ consisting of a finite set of primes Σ, a cuspform $\varphi \in V$ that is completely reducible, and unitary Hecke characters $\psi = \otimes \psi_w$ and $\tau = \otimes \tau_w$ of $\mathbf{A}_\mathcal{K}^\times/\mathcal{K}^\times$, all satisfying:

- Σ contains all primes dividing p, primes ramified in $\mathcal{K}/\mathbf{Q}$, and all primes v such that π_v, ψ_v or τ_v is ramified
- for all $k \in K'_+, \pi_\infty(k)\phi_\infty = j(k,i)^{-\kappa}\phi_\infty$;
- if $v \notin \Sigma$ then ϕ_v is the newvector;
- if $v \in \Sigma, v \nmid p$, then ϕ_v has a conductor with respect to $\pi_v^\vee$
- if $v|p$, then ϕ_v is the one choosen in part one.
- $\psi|_{\mathbf{A}_F^\times} = \chi$;
- $\tau_\infty(x) = (x/|x|)^{-\kappa} = \psi_\infty(x)$for any infinit place ∞ of F.

Let $\xi = \otimes \xi_w = \psi/\tau$ and define $F_f := \prod_{v\notin\Sigma} F^{sph} \prod_{v\in\Sigma, v\nmid p} F_{\phi,r,t} \prod_{v|p} F_v^0$. Then as in [SU] 9.3.1 we define

Lemma 9.3.1. *Suppose $\kappa > 6$ and let $z_\kappa := (\kappa - 3)/2$. Let $F = F_\kappa \otimes F_f \in I(\rho) = I(\rho_\infty) \otimes I(\rho_f)$.*

(1)$A(\rho, z_\kappa, F) = 0$.
(2)$E(F, z_\kappa, g)_P = F_{z_\kappa}(g)$.

Thus $A_\mathcal{D}(z_\kappa, g) = 0$ and $E_\mathcal{D}(z_\kappa, -)_P = \varphi_\mathcal{D}(z_\kappa)$.

For $\kappa > 6$, then for any $F = F_\kappa \otimes F_f \in I(\rho)$ we define a function of $(Z, x) \in \mathbf{H} \otimes G(\mathbf{A}_{F,f})$:

$$E(Z, x; F) := J(g, \boldsymbol{i})^\kappa \mu(g)^\kappa E(F, z_\kappa, gx), g \in G^+(\mathbf{R}), g(\boldsymbol{i}) = Z.$$

we write $E_\mathcal{D}(Z, x)$ for $E(Z, x; \varphi_\mathcal{D}(z_\kappa))$. The following proposition is essentially [SU]9.3.3.

Proposition 9.3.1. *Suppose $\kappa > 6$ and $F = F_\kappa \otimes F_f$. Then $E(Z, x; F)$ is a hermitian modular* *form of weight κ. In particular, $E_{\mathcal{D}} \in M_\kappa(K_{\mathcal{D}}, \nu_{\mathcal{D}})$.*

9.4 Hecke operators

We refer to [SU]9.5 for the definitions of the Hecke operators at unramified primes. The local situations are the same when the base field is F instead of $\mathbb{Q}$. We only record the following proposition ([SU]9.6.1):

Proposition 9.4.1. *Suppose $\kappa > 6$. Then the prime to Σ part of the L-function $L^\Sigma(E_{\mathcal{D}}, s)$ is given by:*

$$L^\Sigma_{\mathcal{K}}(f, \bar{\xi}^c\psi^c, s-2)L^\Sigma(\psi^c, s-3)L^\Sigma(\chi\bar{\xi}'\psi^c, s-\kappa).$$

This explains the reason why the Galois representation associated to the Klingen Eisenstein series is the one given in the last chapter.

Chapter 10

Hermitian Theta Functions

As in [SU] chapter 10, we recall the Weil representations and theta functions associated with certain definite hermitian matrices in and define some specific Schwartz functions and inter into our later expressions for fourier coefficients of the Eisenstein series $E_{\mathcal{D}}$.

10.1 Generalities.

Let V be the two-dimensional $\mathcal{K}$-space of column vectors.

The local set-up

Let v be a place of F. Let $h \in S_2(F_v), deth \neq 0$. Then $< x, y >_h := \bar{x}^t hy$ defines a non-degenerate hermitian pairing on $V_v := V \otimes F_v$. Let U_h be the unitary group of this pairing and let GU_h be its similitude group with similitude character $\mu_h : GU_h \to \mathbf{G}_m$. Let $V_1 := \mathcal{K}^2$ and $< -, - >_1$ be the pairing on V_1 defined by $< x, y >_1 = xw_1\bar{y}^t$. Let $W := V_v \otimes_{\mathcal{K}_v} V_{1,v}$, where $V_{1,v} := V_1 \otimes F_V$. Then $(-, -) := Tr_{\mathcal{K}_v/F_v}(< -, - >_h \otimes_{\mathcal{K}_v} < -, - >_1)$ is a F_v linear pairing on W that makes W into an 8-dimensional symplectic space over F_v. The canonical embeding of $U_h \times U_1$ into $Sp(W)$ realizes the pair (U_h, U_1) as a dual pair in $Sp(W)$. Let λ_v be a character of $\mathcal{K}_v^\times$ such that $\lambda_v|_{F_v^\times} = 1$. In [Ku94], a splitting pair $U_h(F_v) \times U_1(F_v) \hookrightarrow Mp(W, F_v)$ of the metaplectic cover $Mp(W, F_v) \to Sp(W, F_v)$ is associated with the character λ_v; we use this splitting to identify $U_h(F_v) \times U_1(F_v)$ with a subgroup of $Mp(W, F_v)$.

We let $\omega_{h,v}$ be the corresponding Weil representation of $U_h(F_v) \times U_1(F_v)$ on the Schwartz space $\mathcal{S}(V_v)$: the action of (u, g) on $\Phi \in \mathcal{S}(V_v)$ is written $\omega_{h,v}(u, g)\Phi$. If $u = 1$ we often omit u, writing

$\omega_{h,v}(g)$ to mean $\omega_{h,v}(1,g)$. Then $\omega_{h,v}$ satisfies:

- $\omega_{h,v}(u,g)\Phi(x) = \omega_{h,v}(1,g)\Phi(u^{-1}x)$
- $\omega_{h,v}(diag(a,\bar{a}^t))\Phi(x) = \lambda(a)|a|_{\mathcal{K}}\Phi(xa), a \in \mathcal{K}^\times$.
- $\omega_{h,v}(r(S))\Phi(x) = \Phi(x)e_v(< x,x >_h S), S \in F_v$;
- $\omega_{h,v}(\eta)\Phi(x) = |deth|_v \int_{V_v} \Phi(y)e_v(Tr_{\mathcal{K}/Q} < y,x >_h)dy$.

The global set-up

Let $h \in S_2(F), h > 0$. We can define global versions of U_h, GU_h, W, and $(-,-)$, analogously to the above. Fixing an idele class character $\lambda = \otimes\lambda_v$ of $\mathbb{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ such that $\lambda|_{F^\times} = 1$, the associated local splitting described above then determine a global splitting $U_h(\mathbb{A}_F) \times U_1(\mathbb{A}_F) \hookrightarrow Mp(W,\mathbb{A}_F)$ and hence an action $\omega_h := \otimes\omega_{h,v}$ of $U_h(\mathbb{A}_F) \times U_1(\mathbb{A}_F)$ on the Schwartz space $\mathcal{S}(V \otimes \mathbb{A}_F)$.

10.1.1 Theta Functions

Given $\Phi \in \mathcal{S}(V \otimes \mathbb{A}_F)$ we let

$$\Theta_h(u,g;\Phi) := \sum_{x \in V} \omega_h(u,g)\Phi(x).$$

This is an automorphic form on $U_h(\mathbb{A}_F) \times U_1(\mathbb{A}_F)$.

10.2 Some useful Schwartz functions.

We now define various Schwartz function that show up in later formulas.

10.2.1 Archimedean Schwartz functions

Let $\Phi_{h,\infty} \in \mathcal{S}(V \otimes \mathbb{R})$ be

$$\Phi_{h,\infty}(x) = e^{-2\pi < x,x >_h}.$$

Henceforth we assume that

$$\lambda_\infty(z) = (z/|z|)^{-2}.$$

Lemma 10.2.1. *Given $z \in \mathfrak{h}$, let $\Phi_{h,z}(x) := e(< x,x >_h z)$ (so $\Phi_{h,i} = \Phi_{h,\infty}$). For any $g \in U_1(\mathbb{R})$,*

$$\omega_h(g)\Phi_{h,z} = J_1(g,z)^{-2}\Phi_{h,g(z)}.$$

In particular, if $k \in K_{\infty,1}^+$ then $\omega(k)\Phi_{h,\infty} = J_1(k,i)^{-2}\Phi_{h,\infty}$.

Proof. this is just [SU]10.2.2. ☐

10.2.2 ℓ-adic Schwartz functions.

For a finite place v of F dividing a rational prime ℓ, let $\Phi_0 \in \mathcal{S}(V_v)$ be the characteristic function of the set of column vectors with entries in $\mathcal{O}_{\mathcal{K},v}$. For $y \in GL_2(\mathcal{K}_v)$ we let $\Phi_{0,y}(x) := \Phi_0(y^{-1}x)$.

Lemma 10.2.2. *Let $h \in S_2(F_v)$, $det\ h \neq 0$. Let $y \in GL_2(\mathcal{K}_v)$. Suppose $\bar{y}^t hy \in S_2(\mathscr{O}_{F,v})^\times$.*
(i) if λ is unramified, v is unramified in $\mathcal{K}$, and $h, y \in GL_2(\mathcal{O}_{F,v})$. Then

$$\omega_h(U_1(\mathcal{O}_{F,v}))\Phi_{0,y} = \Phi_{0,y}$$

(ii) if $D_v det\bar{y}^t hy | \varpi_v^r$, $r > 0$. Then

$$\omega_h(k)\Phi_{0,y} = \lambda(a_k)\Phi_{0,y},\ k \in \{k \in U_1(\mathcal{O}_{F,v}) : \varpi_v^r | c_k\}.$$

.

Proof. See [SU]10.2.4. ☐

Let θ be a character of $\mathcal{K}_v^\times$ and let $0 \neq x \in cond(\theta)$. Let

$$\Phi_{\theta,x}(u) := \sum_{a \in (\mathcal{O}_{\mathcal{K},v}/x)^\times} \theta(a)\Phi_0((u_1 + a/x, u_2)^t), u = (u_1, u_2)^t.$$

For $y \in GL_2(\mathcal{K}_v)$ we let $\Phi_{\theta,x,y}(u) := \Phi_{\theta,x}(y^{-1}u)$. We let $\Phi_{h,\theta,x} := \omega_h(\eta^{-1})\Phi_{\theta,x}$ and $\Phi_{h,\theta,x,y} := \omega_h(\eta^{-1})\Phi_{\theta,x,y}$.

Lemma 10.2.3. *Let $h \in S_2(F_v), det\ h \neq 0$. Let $y \in GL_2(\mathcal{K}_v)$. Suppose $\bar{y}^t hy \in S_2(\mathcal{O}_{F,v})^\times$. Let θ be a character of $\mathcal{K}_v^\times$ and Let $0 \neq x \in cond(\theta)$ be such that $\varpi_v | x$. Let $(c) := cond(\theta) \cap (\tilde{\varpi}_v)$ where $\tilde{\varpi}_v = \varpi_v$ if v splits in $\mathcal{K}$ and otherwise a uniformizer of $\mathcal{K}_v$ at v.*
(i) if $cD_v det\bar{y}^t hy \| x$ and $y^{-1}hy \in GL_2(\mathcal{O}_{F,v})$ and $D_v = 1$ or $y^{-1}h^{-1}y^{-1t} = \begin{pmatrix} * & * \\ * & d \end{pmatrix}$ with $d \in \mathcal{O}_{F,v}$, then

$$\omega_h(k)\Phi_{\theta,x,y} = \lambda\theta(a_k)\Phi_{\theta,x,y},\ k \in U_1(\mathcal{O}_{F,v}), \mathfrak{d}^{-1}D_v | c_k, x\bar{x} | b_k$$

(ii) if $h = diag(\alpha, \beta)$, then $\Phi_{h,\theta,x,y}$ is supported on the lattice $h^{-1}y^{-1^t}L^*_{\theta,x}$ where if v is non split in $\mathcal{K}$ then

$$L^*_{\theta,x} = \{(u_1, u_2)^t : u_2 \in \delta_{\mathcal{K}}{}^{-1}\mathcal{O}_{\mathcal{K},v}, \bar{u}_1 \in \frac{x}{c\delta_{\mathcal{K}}}\begin{cases}\mathcal{O}_{\mathcal{K},v}, & cond(\theta) = \mathcal{O}_{\mathcal{K},v}, \\ \mathcal{O}^{\times}_{\mathcal{K},v}, & cond(\theta) \neq \mathcal{O}_{\mathcal{K},v}.\end{cases}\} \tag{10.1}$$

and if v splits in $\mathcal{K}$, then

$$L^*_{\theta,x} := \{(u_1, u_2)^t : u_2 \in \delta^{-1}_{\mathcal{K}}\mathcal{O}_{\mathcal{K},v}, \bar{u}_{1,i} \in \frac{x_i}{c_i\delta_{\mathcal{K}}}\begin{cases}\mathcal{O}_{F,v}, & cond(\theta) = \mathcal{O}_{F,v}, \\ \mathcal{O}^{\times}_{F,v}, & cond(\theta) \neq \mathcal{O}_{F,v}.\end{cases}\} \tag{10.2}$$

with $\bar{u}_1 = (\bar{u}_{1,1}, \bar{u}_{1,2}), x = (x_1, x_2), c = (c_1, c_2) \in \mathcal{K}_v = F_v \times F_v$ and $\theta = (\theta_1, \theta_2)$. Further more for $v = h^{-1}y^{-1^t}u$ with $u \in L^*_{\theta,x}$.

$$\Phi_{h,\theta,x,y}(v) = |det\ hy\bar{y}|_v D_v^{-1}\lambda(-1) \sum_{a \in (\mathcal{O}_{\mathcal{K},v}/x)^{\times}} \theta(s)e_{\ell}(Tr_{\mathcal{K}/\mathbf{Q}}a\bar{u}_1/x)$$

.

Proof. See [SU]10.2.5. □

Lemma 10.2.4. *suppose $v|p$ splits completely in $\mathcal{K}$. Let $(c) := cond(\theta)$ and suppose $c = (p^r, p^s)$ with $r, s > 0$. Let $\gamma = (\eta, 1) \in SL_2(\mathcal{O}_{\mathcal{K},v}) = SL_2(\mathcal{O}_{F,v}) \times SL_2(\mathcal{O}_{F,v})$. Suppose $h = diag(\alpha, \beta)$ with $\alpha, \beta \in F_v^{\times}$. Then*

(i)$\Phi_{h,\theta,c,\gamma}$ is supported on

$$L' := \{u = (a, b)^t : a \in \mathcal{O}^{\times}_{F,v} \times \mathcal{O}_{F,v}, b \in \mathcal{O}_{F,v} \times \mathcal{O}^{\times}_{F,v}\}$$

and for $u \in L'$

$$\Phi_{h,\theta,c,\gamma}(u) = \bar{\theta}_1(\beta b_2)g(\theta_1)\bar{\theta}_2(\alpha a_1)g(\theta_2)$$

where $a = (a_1, a_2), b = (b_!, b_2) \in \mathcal{O}_{F,v} \times \mathcal{O}_{F,v}$, and $\theta = (\theta_1, \theta_2)$.

(ii)$\omega_h(u, k)\Phi_{h,\theta,c} = \theta^{-1}(d_g)\theta_2(\det g)\lambda\theta(d_k)\Phi_{h,\theta,c}$ for $u = (g, g') \in U_h(\mathbb{Z}_p)$ with $p^{max(r,s)}|c_g$ and for $k \in U_1(\mathbb{Z}_p)$ such that $p^{max(r,s)}|c_k$.

Proof. See [SU]10.2.6. □

Chapter 11

Siegel Eisenstein Series and Their Pull-backs

11.1 Siegel Eisenstein series on G_n; the general set up

For a place v of F and a character χ of $\mathcal{K}_v^\times$ we let $I_n(\chi)$ be the space of smooth $K_{n,v}$-finite functions $f : K_{n,v} \to \mathbb{C}$ such that $f(qk) = \chi(detD_q)f(k)$ for all $q \in Q_n(F_v) \cap K_{n,v}$. Given $z \in \mathbb{C}$ and $f \in I(\chi)$ we define a function $f(z,-) : G_n(F_v) \to \mathbb{C}$ by $f(z,qk) := \chi(\det D_q))|\det A_q D_q^{-1}|_v^{z+n/2} f(k), q \in Q_n(F_v)$ and $k \in K_{n,v}$.

For an idele class character $\chi = \otimes\chi_v$ of $\mathbb{A}_\mathcal{K}^\times$ we similarly define a space $I_n(\chi)$ of smooth $K_{n,\mathbb{A}}$ functions on $K_{n,\mathbb{A}}$. We also similarly define $f(z,-)$ given $f \in I_n(\chi)$ and $z \in \mathbf{C}$. There is an identification $\otimes I_n(\chi_v) = I_n(\chi)$, the former being the restricted tensor product defined using the spherical vectors $f_v^{sph} \in I_n(\chi_v), f_v^{sph}(K_{n,v}) = 1$, at the finite places v where χ_v is unramified:$\otimes f_v$ is identified with $k \mapsto \prod_v f_v(k_v)$.. Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set. By a meromorphic section of $I_n(\chi)$ on $\mathcal{U}$ we mean a function $\varphi : \mathcal{U} \mapsto I_n(\chi)$ taking values in a finite dimensional subspace $V \subset I_(\chi)$ and such that $\varphi : \mathcal{U} \to V$ is meromorphic.

Let $\chi = \otimes\chi_v$ be a unitary idele class character of $\mathbb{A}_\mathcal{K}^\times$. for $f \in I_n(\chi)$ we consider the Eisenstein series

$$E(f;z,g) := \sum_{\gamma \in Q_n(F)\backslash G_n(F)} f(z,\gamma g).$$

This series converges absolutely and uniformly for (z,g) in compact subsets of $\{Re(z) > n/2\} \times G_n(\mathbb{A}_F)$ and defines an automorphic form on G_n and a holomorphic function on $\{Re(z) > n/2\}$. The Eisenstein series $E(f;z,g)$ has a meromorphic continuation in z to all of $\mathbb{C}$. If $\varphi : \mathcal{U} \to I_n(\chi)$ is a meromorphic section, then we put $E(\varphi;z,g) = E(\varphi(z);z,g)$. This is clearly a meromorphic function of $z \in \mathcal{U}$ and an automorphic form on G_n for those z where it is holomorphic.

11.1.1 Intertwining operators and functional equations

Let χ be a unitary character of $\mathcal{K}_v^\times$, v a place of F. For $f \in I_n(\chi), z \in \mathbb{C}$, and $k \in K_{n,v}$, we consider the integral

$$M(z,f)(k) := \bar{\chi}^n(\mu_n(k)) \int_{N_{Q_n}(F_v)} f(z, w_n r k) dr.$$

For z in compact subsets of $\{Re(z) > n/2\}$ this integral converges absolutely and uniformly, with the convergence being uniform in k. $M(z,f) \in I_n(\bar{\chi}^c)$. It thus defines a holomorphic section $z \mapsto M(z,f)$ on $\{Re(z) > 3/2\}$. This has a continuation to a meromorphic section on all of $\mathbb{C}$.

Let $\chi = \otimes \chi_v$ be a unitary idele class character. For $f \in I_n(\chi), z \in \mathbb{C}$, and $k \in K_{n,\mathbb{A}_F}$ we consider the integral $M(z,f)(k)$ as above but with the integration being over $N_{Q_n}(\mathbb{A}_F)$. This again converges absolutely and uniformly for z is compact subsets of $\{Re(z) > n/2\}$, with the convergence being uniform in k. Thus $z \mapsto M(z,f)$ defines a holomorphic section $\{Re(z) > n/2\} \to I_n(\bar{\chi}^c)$. This has a continuation to a meromorphic section on $\mathbb{C}$. For $Re(z) > n/2$ at least, we have

$$M(z,f) = \otimes_v M(z,f_v), f = \otimes f_v.$$

11.2 Pull-backs of Siegel Eisenstein series.

As in [SU]11.2,we recall the pull-back formulas of Garrett and Shimura which expresses Klingen-type Eisenstein series in terms of restrictions (pull-backs) of Siegel Eisenstein series to subgroups. But first we define various maps between groups that intervene in the statement of the general formula as well as in the particular instance used in subsequent sections.

11.2.1 Some isomorphisms and embeddings.

Let $V_n := \mathcal{K}^{2n}$. Then w_n defines a skew-hermitian pairing $< -,- >_n$ on $V_n :< x,y >_n := x w_n \bar{y}^t$. The group G_n/F is the unitary similitude group $GU(V_n)$ of the hermitian space $(V_n, < -,- >_n)$.

Let $W_n := V_{n+1} \oplus V_n$ and $W'_n : V_n \oplus V_n$. The matrices $w_{n+1} \oplus -w_n$ and $w_n \oplus w_n$ define define hermitian pairings on W_n and W'_n, respectively.

One can define isomorphisms:, $\alpha_n : GU(W_n) \simeq G_{2n+1}$, $\alpha'_n : GU(W'_n) \simeq G_{2n}$, $\gamma : GU(W_n) \simeq G_{2n+1}$ and $\gamma' : GU(W'_n) \simeq G_{2n}$ we omit the details and referring to [SU] 11.2.1. Also as in [SU] we use S and S' to denote the matrix

$$\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & -1 & 1 & & \\ & & & & 1 & \\ -1 & & & & & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ -1 & & & 1 \end{pmatrix}$$

This is different from the convention of part one of this thesis.

11.2.2 The pull-back formulas

Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$. Given a cuspform ϕ on G_n we consider

$$F_\phi(f; z, g) := \int_{U_n(\mathbb{A}_F)} f(z, \gamma(g, g_1 h))\bar{\chi}(\det g_1 g)\phi(g_1 h) dg_1,$$

$$f \in I_{m+n}(\chi), g \in G_m(\mathbb{A}_F), h \in G_n(\mathbb{A}_F), \mu_m(g) = \mu_n(h), m = n+1 \text{ or } n,$$

with $\gamma = \gamma_n$ or γ' depending on whether $m = n+1$ or $m = n$. This is independent of h. The pull-back formulas are the identities in the following proposition.

Proposition 11.2.1. *Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$.*
(i) if $f \in I_{2n}(\chi)$, then $F_\phi(f; z, g)$ converges absolutely and uniformly for (z, g) in compact sets of $\{Re(z) > n\} \times G_n(\mathbb{A}_F)$, and for any $h \in G_n(\mathbb{A}_F)$ such that $\mu_n(h) = \mu(g)$

$$\int_{U_n(F)\backslash U_n(\mathbb{A}_F)} E(f; z, \gamma'_n(g, g_1 h))\bar{\chi}(detg_1 h)\phi(g_1 h) dg_1 = F_\phi(f; z, g). \tag{11.1}$$

(ii) If $f \in I_{2n+1}(\chi)$, then $F_\phi(f;z,g)$ converges absolutely and uniformly for (z,g) in compact sets of $\{Re(z) > n+1/2\} \times G_{n+1}(\mathbb{A}_F)$ such that $\mu_n(h) = \mu_{n+1}(g)$

$$\int_{U_n(F)\backslash U_n(\mathbb{A}_F)} E(f;z,\gamma_n(g,g'h))\bar{\chi}(\det g_1 h)\phi(g_1 h)dg_1 = \sum_{\gamma \in P_{n+1}(F)\backslash G_{n+1}(F)} F_\phi(f;z,\gamma g), \tag{11.2}$$

with the series converging absolutely and uniformly for (z,g) in compact subsets of $\{Re(z) > n+1/2\} \times G_{n+1}(\mathbf{A}_F)$.

Proof. See [SU] 11.2.3. □

11.3 fourier-jacobi expansions: generalities.

Let $0 < r < n$ be an integer. Each Eisenstein series $E(f;z,g)$ has a fourier-jacobi expansion

$$E(f;z,g) = \sum_{\beta \in S_{n-r}(F)} E_\beta(f;z,g). \tag{11.3}$$

where

$$E_\beta(f;z,g) := \int_{S_{n-r}(F)\backslash S_{n-r}(\mathbb{A}_F)} E(f;z,\begin{pmatrix} 1_n & \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \\ & 1_n \end{pmatrix} g)e_{\mathbb{A}}(-Tr_{\mathcal{K}/\mathbb{Q}}(\beta S))dS. \tag{11.4}$$

Lemma 11.3.1. *Let $f = \otimes_v f_v \in I_n(\chi)$ be such that for some prime v the support of f_v is in $Q_n(F_v)w_n Q_n(F_v)$. Let $\beta \in S_n(F)$ and $q \in Q_n(\mathbb{A}_F)$. If $Re(z) > n/2$ then*

$$E_\beta(f;z,g) = \prod_v \int_{S_n(F_v)} f_v(z, w_n r(S_v)q_v)e_v(-Tr\beta S_v)dS_v. \tag{11.5}$$

In particular, the integrals on the right-hand side converge absolutely for $Re(z) > n/2$.

Proof. see [SU] 11.3.1. □

Lemma 11.3.2. *Suppose $f \in I_3(\chi)$ and $\beta \in S_2(F), \beta > 0$. Let V be the two-dimensional $\mathcal{K}$-vector space of column vectors. If $Re(z) > 3/2$ then*

$$E_\beta(f;z,g) = \sum_{\gamma\in Q_1(F)\backslash G_1(F),\gamma\in U_1(F)} \sum_{x\in V} \int_{S_2(\mathbb{A}_F)} f(w_3 \begin{pmatrix} 1_3 & \begin{pmatrix} S & x \\ \bar{x}^t & 0 \end{pmatrix} \\ & 1_3 \end{pmatrix} \alpha_1(1,\gamma)g)$$
$$\times e_{\mathbb{A}}(-Tr_{\mathcal{K}/\mathbb{Q}}\beta S)dS.$$

Proof. See [SU] 11.3.2. □

We also recall a few identities which would be useful later on. Letting:

$$FJ_\beta(f;z,x,g,y) := \int_{S_2(F_v)} f(z, w_3 \begin{pmatrix} 1_n & \begin{pmatrix} S & x \\ \bar{x}^t & 0 \end{pmatrix} \\ & 1_n \end{pmatrix} \alpha_1(diag(y, {y^t}^{-1}), g))e_v(-Tr\beta S)dS.$$

Then:

$$FJ_\beta(f;z,x, \begin{pmatrix} a & \bar{a}^{-1}b \\ & \bar{a}^{-1} \end{pmatrix} g, y) = \chi_v^c(a)^{-1}|a\bar{a}|_v^{z+3/2} e_v(\bar{x}^t\beta x b) FJ_\beta(f;z,xa,g,y). \tag{11.6}$$

For $u \in U_\beta(\mathbf{A}_F)$, U_β being the unitary group associated to β.

$$FJ_\beta(f;z,x,g,uy) = \chi(detu)|detu\bar{u}|_{\mathbb{A}_F}^{-z+1/2} FJ - \beta(f;z,u^{-1}x,g,y). \tag{11.7}$$

If as a function of x, $FJ_\beta(f;z,x,g,y) \in \mathcal{S}(V\otimes F_v)$ then:

$$\begin{aligned} &FJ_\beta(f;z,x, \begin{pmatrix} a & \bar{a}^{-1}b \\ & \bar{a}^{-1} \end{pmatrix} g, y) \\ &\qquad = (\lambda_v/\chi_v^c)(a)|a\bar{a}|_v^{z+1/2} \omega_\beta(\begin{pmatrix} a & \bar{a}^{-1}b \\ & \bar{a}^{-1} \end{pmatrix}) FJ_\beta(f:z,x,g,y). \end{aligned} \tag{11.8}$$

11.4 Some good Siegel sections

11.4.1 Archimedean Siegel sections

We summarize the results of [SU] 11.4.1. Let $k \geq 0$ be an integer. Then $\chi(x) = (x/|x|)^{-k}$ is a character of $\mathbb{C}^{\times}$.

The sections. We let $f_{\kappa,n} \in I_n(\chi)$ be defined by $f_{\kappa,n}(k) := J_n(k,i)^{-\kappa}$. Then

$$f_{\kappa,n}(z,qk) = J_n(k,i)^{-\kappa}\chi(detD_q)|detA_qD_q^{-1}|^{z+1/2}, q \in Q_n(\mathbb{R}), k \in K_{n,\infty}. \qquad (11.9)$$

If $g \in U_n(\mathbb{R})$ then $f_{\kappa,n}(z,g) = J_n(g,i)^{-\kappa}|J_n(g,i)|^{\kappa-2z-n}$.

Fourier-jacobi coefficients. Given a matrix $\beta \in S_2(\mathbb{R})$ we consider the local fourier coefficient:

$$f_{\kappa,n,\beta}(z,g) := \int_{S_n(\mathbb{R})} f_\kappa(z, w_n \begin{pmatrix} 1_n & S \\ & 1_n \end{pmatrix} g) e_\infty(-Tr\beta S) dS.$$

This converges absolutely and uniformly for z in compact sets of $\{Re(z) > n/2\}$.

Lemma 11.4.1. *Suppose $\beta \in S_n(\mathbb{R})$. The function $z \mapsto f_{\kappa,\beta}(z,g)$ has a meromorphic continuation to all of $\mathbb{C}$. Furthermore, if $\kappa \geq n$, then $f_{\kappa,n,\beta}(z,g)$ is holomorphic at $z_\kappa := (\kappa-n)/2$ and for $y \in GL_n(\mathbb{C})$, $f_{\kappa,n,\beta}(z_\kappa, diag(y, \bar{y^t}^{-1})) = 0$ if $det\beta \leq 0$, and if $det\beta > 0$ then*

$$f_{\kappa,n,\beta}(z_\kappa, diag(y, \bar{y^t}^{-1})) = \frac{(-2)^{-n}(2\pi i)^{n\kappa}(2/\pi)^{n(n-1)/2}}{\prod_{j=0}^{n-1}(\kappa-j-1)!} e(iTr(\beta y\bar{y}^t))(det\beta)^{\kappa-n}det\bar{y}^\kappa.$$

Proof. See [SU]11.4.2. $\square$

Suppose now that $n = 3$. For $\beta \in S_2(\mathbb{R})$ let $FJ_{\beta,\kappa}(z,x,g,y) := FJ_\beta(f_\kappa; z,x,g,y)$.

Lemma 11.4.2. *Let $z_\kappa := (\kappa-3)/2$. Let $\beta \in S_2(\mathbb{R})$, $det\beta > 0$.*
(i)$FJ_{\beta,\kappa}(z_\kappa, x, \eta, 1) = f_{\kappa,2,\beta}(z_\kappa + 1/2, 1)e(i < x, x >_\beta)$. (ii) For $g \in U_1(\mathbb{R})$

$$FJ_{\beta,\kappa}(z_\kappa,x,g,y) = e(i\mathrm{Tr}\beta y\bar{y}^t)\det\bar{y}^\kappa c(\beta,k) f_{\kappa-2,1}(z_\kappa, g')\omega_\beta(g')\Phi_{\beta,\infty}(x),$$

where $g' = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ *and*

$$c(\beta, \kappa) = \frac{(2\pi i)^{2\kappa}(2/\pi)}{4(k-1)!(k-2)!} \det \beta^{\kappa-2}$$

and the Weil representation ω_β *is defined using the character* $\lambda_\infty(z) = (z/|z|)^{-2}$.

Pull-back integrals

The Archimedean situation is completely the same as the [SU] situation. Let $f_\kappa \in I_3(\tau)$ be as before and let

$$\begin{gathered} F_\kappa(z, g) := \int_{U_1(\mathbb{R})} f_\kappa(z, S^{-1}\alpha_1(g, g_1 h))\bar{\tau}(detg_1 h)\pi_1(g_1 h)\phi dg_1, \\ g \in G_2(\mathbb{R}), h \in G_1(\mathbb{R}), \mu_1(h) = \mu_2(g). \end{gathered} \tag{11.10}$$

Similarly, for $f_\kappa \in I_2(\tau)$ and $g \in G_1(\mathbf{R})$ we let

$$\begin{gathered} F_\kappa(z, g) := \int_{U_1(\mathbb{R})} f_\kappa(z, S^{-1}\alpha_1'(g, g_1 h))\bar{\tau}(\det g_1 h)\pi_1(g_1 h)\phi dg_1, \\ g, h \in G_1(\mathbb{R}), \mu_1(h) = \mu_1(g). \end{gathered} \tag{11.11}$$

Lemma 11.4.3. *([SU]11.4.4.) The integrals converge if* $Re(z) \geq (\kappa - m - 1)/2$ *and* $Re(z) > (m - 1 - \kappa)/2$, $m = 2$ *and* 1, *respectively, and for such* z *we have:*
(i)$F_\kappa(z, g) = \pi 2^{-2z-1} \frac{\Gamma(z+(1+\kappa)/2)}{\Gamma(z+(3+\kappa)/2)} F_{\kappa,z}(g)$;
(ii)$F'_\kappa(z, g) = \pi 2^{-2z} \frac{\Gamma(z+\kappa/2)}{\Gamma(z+1+\kappa/2)} \pi_\psi(g)\phi$.

11.4.2 ℓ-adic Siegel sections: the unramified case

Lemma 11.4.4. *Let* $\beta \in S_n(F_v)$ *and let* $r := rank(\beta)$. *Then for* $y \in GL_n(\mathcal{K}_v)$.

$$f^{sph}_{v,\beta}(z, diag(y, {y^t}^{-1})) = \chi(det\ y)|det\ y\bar{y}|_v^{-z+n/2} Vol(S_n(\mathcal{O}_{F,v}))$$

$$\times \frac{\prod_{i=r}^{n-1} L(2z+i-n+1, \bar{\chi}'\chi^i_{\mathcal{K}})}{\prod_{i=0}^{n-1} L(2z+n-i, \bar{\chi}'\chi^i_{\mathcal{K}})} h_{v,\bar{y}^t\beta y}(\bar{\chi}'(\varpi_v) q_v^{-2z-n})$$

where $h_{v,\bar{y}^t\beta y}$ *is a monic polynomial depending on* v *and* $\bar{y}^t\beta y$ *but not on* χ.

Proof. See [SU]11.4.6. $\square$

Lemma 11.4.5. *Suppose v is unramified in $\mathcal{K}$, let $\beta \in S_2(F_v)$ such that det $\beta \neq 0$. Let $y \in GL_2(\mathcal{K}_v)$ such that $\bar{y}^t\beta y \in S_2(\mathcal{O}_{F,v})$. Let λ be an unramified character of $\mathcal{K}_v^\times$ such that $\lambda|_{F_v^\times} = 1$.*
(i) if $\beta, y \in GL_2(\mathcal{O}_{\mathcal{K},v})$ then for $u \in U_\beta(F_v)$.

$$FJ_\beta(f_3^{sph}; z, x, g, uy) = \chi(det\ u)|det\ u\bar{u}|_v^{-z+1/2} \frac{f_1^{sph}(z,g)\omega_\beta(u,g)\Phi_{0,y}(x)}{\prod_{i=o}^{1} L(2z+3-i, \bar{\chi}'\chi_{\mathcal{K}}^i)}$$

(ii)if $\bar{y}^t\beta y \in GL_2(\mathcal{O}_{\mathcal{K},v})$. Then for $u \in U_\beta(F_v)$.

$$FJ_\beta(f_3^{sph}; z, x, g, uy) = \chi(det\ uy)|det\ uy|_{\mathcal{K}}^{-z+1/2} \frac{f_1^{sph}(z,g)\omega_\beta(u,g)\Phi_{0,y}(s)}{\prod_{i=0}^{1} L(2z+3-i, \bar{\chi}'\chi_{\mathcal{K}}^i)}$$

.

Proof. (i) is the same as [SU]11.4.7. Note that in (ii) we have removed the assumption in *loc.cit* that g is of the form $\begin{pmatrix} 1 & \\ n & 1 \end{pmatrix}$. In fact since

$$FJ_\beta(f_3^{sph}; z, x, g, uy) = \chi(\det uy)|\det uy|_{\mathcal{K}}^{-z-\frac{1}{2}} FJ_{{}^t\bar{y}\beta y}(f_3^{sph}; z, y^{-1}u^{-1}x, g, 1)$$

by (i) we have only to prove that

$$\omega_{{}^t\bar{y}\beta y}(1,g)\Phi_0(y^{-1}u^{-1}x) = \omega_\beta(u,g)\Phi_{0,y}(x) = (\omega_\beta(1,g)\Phi_{0,y})(u^{-1}x)$$

i.e.

$$(\omega_{{}^t\bar{y}\beta y}(1,g)\Phi_0)_{,y}(x) = (\omega_\beta(1,g)\Phi_{0,y})(x)$$

Here we write $\Phi_{,y}$ to be the function defined by: $\Phi_{,y}(x) = \Phi(y^{-1}x)$. By definition one checks that for any ϕ

$$\omega_\beta(g)\Phi_{,y} = (\omega_{{}^t\bar{y}\beta y}(1,g)\Phi)_{,y}(x)$$

for $g \in \begin{pmatrix} a & \\ & \bar{a}^{-1} \end{pmatrix}, \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}, \eta$, thus for all $g \in U_1(F_v)$. In particular, for $\Phi = \Phi_0$ $\square$

<u>Pull-back integrals</u> We let $(\pi, V), \psi, \tau, \rho, \xi := \psi/\tau$ be as before. Then the pair (π, ψ) determines a representation of $G_1(F_v)$ on V, which we denote as π_ψ. Let $\phi \in V$. Let $m = 1$ or 2. Given $f \in I_{m+1}(\tau)$ we consider the integral:

$$F_\phi(f;z,g) := \int_{U_1(F_v)} f(z,\gamma(g,g_1h))\bar{\tau}(detg_1h)\pi_\psi(g_1h)\phi dg_1, \tag{11.12}$$

where $\gamma = \gamma_1$ or γ_1' depending on whether $m=2$ or $m=1$. (similar to [SU] 11.4.)

Lemma 11.4.6. *Suppose π, ψ and τ are unramified and ϕ is a newvector. If $Re(z) > (m+1)/2$ then the above integral converges and*

$$F_\phi(f_v^{sph};z,g) = \begin{cases} \frac{L(\tilde{\pi},\xi,z+1/2)}{\prod_{i=0}^{1} L(2z+2-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}\pi_\psi(g)\phi & m=1 \\ \frac{L(\tilde{\pi},\xi,z+1)}{\prod_{i=0}^{1} L(2z+3-i,\bar{\tau}'\chi_{\mathcal{K}}^i)}F_{\rho,z}(g) & m=2. \end{cases}$$

Here, F_ρ is the spherical section as defined in [SU] 9.1.2.

Proof. See [SU]11.4.8. ☐

11.4.3 ℓ-adic Siegel sections: ramified cases

The sections. We let $f_n^\dagger \in I_n(\chi)$ be the function supported on $Q_n(\mathcal{O}_{F,v})w_nN_{Q_n}(\mathcal{O}_{F,v})(= Q_n(\mathcal{O}_{F,v})w_nK_{Q_n}(\lambda^t)$ for any $t>0$) such that $f_n^\dagger(w_nr) = 1, r \in N_{Q_n}(\mathcal{O}_{F,v})$. Given $(\lambda^u) \subseteq \mathcal{O}_{\mathcal{K},v}$ contained in the conductor of χ, we let $f_{u,n} \in I_n(\chi)$ be the function such that $f_{u,n}(k) = \chi(detD_k)$ if $k \in K_{Q_n}(\lambda^u)$ and $f_{u,n}(k) = 0$ otherwise.

Lemma 11.4.7. *Suppose v is not ramified in $\mathcal{K}$ and suppose χ is such that $\mathcal{O}_{\mathcal{K},v} \neq cond(\chi) \supseteq cond(\chi\chi^c)$. Let $(\lambda^u) := cond(\chi)$. Then*

$$M(z,f_n^\dagger) = f_{u,n}\cdot Vol(S_n(\mathcal{O}_{F,v})) \in I_n(\bar{\chi}^c)$$

for all $z \in \mathbb{C}$.

Proof. See [SU]11.4.10. ☐

Lemma 11.4.8. *Let $A \in GL_n(\mathcal{K}_v)$. If* det $\beta \neq 0$*, then*

$$f_{n,\beta}^\dagger(z, diag(A,{\bar{A}^t}^{-1})) = \begin{cases} \chi(det\ A)|det\ A|_v^{-z+n/2}Vol(S_n(\mathcal{O}_{F,v})) & \bar{A}^t\beta A \in S_n(\mathcal{O}_{F,v})^* \\ 0, & otherwise. \end{cases} \tag{11.13}$$

Proof. See [SU]11.4.11. ☐

Lemma 11.4.9. *Suppose $\beta \in S_n(F_v)$, det $\beta \neq 0$, char$(v) = \ell$ and ℓ splits completely in $\mathcal{K}$.*

(i) if $\beta \notin S_n(\mathcal{O}_{F,v})$, then $M(z, f_n^{\dagger})_\beta(-z, 1) = 0$.

(ii) Suppose $\beta \in S_n(\mathcal{O}_{F,v})$. Let $c := ord_v(cond(\chi'))$. If $c > 0$, then

$$M(z, f_n^{\dagger})_\beta(-z, 1) = \chi'(det\ \beta)|det\ \beta|_v^{-2z} g(\bar{\chi}')^n c_n(\chi', z).$$

where

$$c_n(\chi', z) = \begin{cases} \chi'(\ell^{nc})\ell^{2ncz - cn(n+1)/2} & c > 0 \\ \ell^{2nz - n(n+1)/2} & c = 0 \end{cases} \tag{11.14}$$

Proof. See [SU]11.4.12. □

Now We use the convention for $m = 1$ or 2 as in the last subsection.

Proposition 11.4.1. *Let $m = 1$ or 2. There exists a meromorphic function $\gamma^{(m)}(\rho, z)$ on $\mathbb{C}$ such that:*

(i) If $m = 1$. Then $F_{\phi^\vee}(M(z, f); -z.g) = \gamma^{(1)}(\rho, z)\tau(\mu_1(g))F_\phi(f; z, \eta g)$

Moreover, if $\pi \simeq \pi(\chi_1, \chi_2)$ and v splits in $\mathcal{K}$. Then

$$\gamma^{(1)}(\rho, z) = \Psi(-1)g(\bar{\tau}', \varpi_v^e)^2 \cdot \tau'(\varpi_v^{nc})|\varpi_v|_v^{-2ncz+n(n+1)c/2} \cdot \epsilon(\tilde{\pi} \otimes \xi^c, z + 1/2)\frac{L(\pi \otimes \bar{\xi}^c, 1/2 - z)}{L(\tilde{\pi} \otimes \xi^c, z + 1/2)}$$

.

(ii) If $m = 2$ and π, Ψ, τ are the v constituents of a global triple. Then

$$F_{\phi^\vee}(M(z, f); -z, g) = \gamma^{(2)}(\rho, z)A(\rho, z, F_\phi(f; z, -))_{-z}(g)$$

each of these equalities is an identity of meromorphic functions of z.

(iii) Suppose moreover that $\mathcal{O}_v \neq \mathrm{cond}(\tau) \supset \mathrm{cond}(\tau\tau^c)$ then:

$$\gamma^{(2)}(\rho, z) = \gamma^{(1)}(\rho, z - \frac{1}{2}).$$

Proof. See [SU]11.4.13. □

11.4.4 ℓ-adic sections: ramified cases again

The sections. As in [SU]11.4.14, we define modified version of the sections $f^\dagger$.

Let $m = 1$ or 2. For $x \in \mathcal{O}_{\mathcal{K}.v} \cap \mathcal{K}_v^\times$ let

$$f_x^{\dagger,(m)}(z,g) = f_{m+1}^\dagger(z, g \begin{pmatrix} 1 & & & & & 1/x \\ & 1_{m-1} & & & 0_{m-1} & \\ & & 1 & 1/\bar{x} & & \\ & & & 1_{m-1} & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}).$$

Lemma 11.4.10. *Let $\beta = (b_{i,j}) \in S_{m+1}(F_v)$. Then for all $z \in \mathbb{C}, f_{x,\beta}^{\dagger,(m)}(z,1) = 0$ if $\beta \notin S_{m+1}(\mathcal{O}_{F,v})^*$. If $\beta \in S_{m+1}(\mathcal{O}_{F,v})^*$, then*

$$f_{x,\beta}^{\dagger,(m)}(z,1) = Vol(S_{m+1}(\mathcal{O}_{F,v}))e_\ell(Tr_{\mathcal{K}_v\mathbf{Q}_v}(b_{m+1,1}/x))$$

.

Proof. See [SU]11.4.15. □

Lemma 11.4.11. *Let $\beta \in S_2(F_v)$, $\det\ \beta \neq 0$. Let $y \in GL_2(\mathcal{K}_v)$ and suppose $\bar{y}^t\beta y \in S_2(\mathcal{O}_{F,v})^*$. Let λ, θ be characters of $\mathcal{K}_v^\times$ and suppose $\lambda|_{F_v^\times} = 1$. Let $(c) := cond(\lambda) \bigcap cond(\theta) \bigcap (\varpi_v)$. Let $x \in \mathcal{K}_v^\times$ be such that $D_v|x$, $cond(\chi^c)|x$, and $cD_v \det\ \bar{y}^t\beta y|x$, where $D_v := N_{\mathcal{K}/F}(\mathfrak{D}_{\mathcal{K}/\mathbf{Q}})$. Suppose $y^{-1}\beta^{-1}\bar{y^t}^{-1} = \begin{pmatrix} * & * \\ * & d \end{pmatrix}$ with $d \in F_v$. Denote $\tilde{D}_v := N_{\mathcal{K}/F}(\mathfrak{D}_{\mathcal{K}/F})$ then for $h \in U_\beta(F_v)$,*

$$\sum_{a \in (\mathcal{O}_v/x)^\times} \theta\bar{\chi}^c(a) FJ_\beta(f_x^{\dagger,(2)}; z, u, g \begin{pmatrix} a^{-1} & \\ & \bar{a} \end{pmatrix}, hy)$$

$$= \chi(\det\ hy)|\det\ hy|_{\mathcal{K}}^{-z+1/2} Vol S_2(\mathcal{O}_{F,v}) \cdot \sum_{b \in (\mathcal{O}_v/\tilde{D}_v\mathfrak{d}\mathcal{O}_v)} f_{-b}(z, g'\eta)\omega_\beta(h, g' \begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix})\Phi_{\theta,x,y}(u)$$

. *Where recall that* $g' = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ *and*

$$f_b(g) = \begin{cases} \frac{\chi}{\lambda}(d_p), & g = p\eta \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}, p \in B_1(\mathcal{O}_{F,v}), m - b \in \tilde{D}_v \mathfrak{d} \mathcal{O}_{F,v} \\ 0 & \text{otherwise} . \end{cases}$$

Proof. See [SU]11.4.16. □

Pull-back integrals

Let $\mathcal{T}$ denote a triple (ϕ, ψ, τ) with $\phi \in V$ having a conductor with respect to $\tilde{\pi}$. Let $\phi_x := \pi_\psi(\eta diag(\bar{x}^{-1}, x))\psi$ and let

$$F^{(m)}_{\mathcal{T},x}(z, g) := \int_{U_1(F_v)} f_x^{\dagger,(m)}(z, S^{-1}\alpha(g, g'h))\bar{\tau}(detg'h)\pi_\psi(g'h)\phi_x dg',$$

where $\alpha = \alpha_1$ or α_1' depending on whether $m = 2$ or 1, again using the convention of subsection 11.4.2. If $f(z, g) = f_x^{\dagger,(m)}(z, gS^{-1})$ then $F - \mathcal{T}, x(z, g) = F_{\phi_x}(f; z, g)$.

Proposition 11.4.2. *Suppose* $x = \lambda^t, t > 0$ *is contained in the conductors of* τ *and* ψ *and* $x\bar{x} \in (\lambda^{r_\phi}) = cond_{\tilde{\pi}}(\phi)$. *Then* $F^{(m)}_{\mathcal{T},x}(z, g)$ *converges for all* z *and* g *and*

$$F^{(1)}_{\mathcal{T},x}(z, \eta) = [U_1(\mathcal{O}_{F,v}) : K_x]^{-1}\tau(x)|x\bar{x}|_v^{-z-1}\phi$$

and

$$F^{(2)}_{\mathcal{T},x} = [U_1(\mathcal{O}_{F,v}) : K_x]^{-1}\tau(x)_x\bar{x}|_v^{-z-3/2}F_{\phi,r,t}.$$

for any $r \geq max\{r_\phi, t\}$. *Here* K_x *is the subgroup defined as:*

$$K_x := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_1(\mathcal{O}_{F,v}) : a - 1 \in (\bar{x}), b \in (x\bar{x}), c \in \mathcal{O}_v, d - 1 \in (x)\}.$$

Proof. See [SU]11.4.17. □

Proposition 11.4.3. *For* $m = 1$ *or* 2, *Let* $\gamma^{(m)}(\rho, z)$ *be as above. Assuming* $char(v) = \ell$ *which is unramified in* $\mathcal{K}$. *If* $\mathcal{O}_v \neq cond(\tau) \supseteq cond(\tau\tau^c)$ *then* $\gamma^{(2)}(\rho, z) = \gamma^{(1)}(\rho, z - 1/2)$.

Proof. See [SU]11.4.18. □

11.4.5 p-adic sections

Since we have done a large part of this in part one we only record the formulas for Fourier-Jacobi coefficients and pull-back sections below. These are only slightly different from [SU] 11.4.

Lemma 11.4.12. *Suppose ψ and τ are as in the Generic case and let $(p^m) := cond(\tau')$. Let $\beta \in S_2(F_v)$,$\det\beta \neq 0$, and suppose $\beta \in GL_2(\mathcal{O}_v)$. Let $y \in GL_2(\mathcal{O}_v)$. Let λ be an unramified character of $\mathcal{K}_v^\times$ such that $\lambda|_{F_v^\times} = 1$. Then for $h \in U_\beta(F)v)$*

$$\sum_{a\in(\mathcal{O}_v/x)^\times} \mu_{1,v}^{-1}\xi^c\tau(a)FJ_\beta(f_z^{0,(2)}; -z, u, gdiag(a_{-1}, \bar{a}), hy)$$

$$\xi(-1)c(\beta,\tau,z)\tau(dethy)|deth\bar{h}|_v^{-z+1/2}f_{m,1}(z, g\eta)\omega_\beta(h,g)\Phi_{\mu_{1,v}^{-1}\xi^c,x,y}(u), \tag{11.15}$$

where ω is defined using λ,and

$$c(\beta,\tau,z) := \bar{\tau}'(-det\beta)|det\beta|_v^{2z+1}\mathfrak{g}(\tau')^2\bar{\tau}'(p^2m)p^{-4mz-5m}.$$

Proof. See [SU]11.4.22. □

Now we use the convention for $m = 1$ or 2 as in subsection 11.4.2.

Proposition 11.4.4. *Let $\phi \in V$ be an eigenvector for π such that $v|cond_\pi(\phi)$. Let $(x) := cond(\xi_1) = (\varpi_v^t) = (\varpi_v^{t_1}, \varpi_v^{t_2})$. Suppose $t > 0$ and that x is contained in $cond(\tau_1)$ and $cond(\psi_1)$ and that $x\bar{x} \in cond_\pi(\phi)$. Let $\phi_x^\vee := \Psi_v(-1)\pi(diag(x, \bar{x}^{-1}))\phi$. Then*

$$F_{\phi_x^\vee}(\tilde{f}_{-z}^{0,(m)}; z, g) = \gamma^{(m)}(\rho_1, -z)[U_1(\mathcal{O}_{F,v}) : K_x]^{-1}\bar{\tau}^c(x)|x\bar{x}|_v^{z-\frac{m+1}{2}}\begin{cases} F_{\phi,z}^0(g), & m = 2 \\ \pi_\psi(g)\phi, & m = 1. \end{cases} \tag{11.16}$$

where $\tilde{f}_{-z}^{0,(m)}(z,g) = f_{-z}^{0,m}(z, gS^{-1})$.

Proof. See part one. □

11.5 Good Siegel Eisenstein Series

From now on we assume that the characters ψ and τ are unramified outside p. Let $(\pi, V) = (\otimes\pi_v, \otimes V_v)$ be as before and let $\mathcal{D} = (\sum, \varphi, \psi, \tau)$ be an Eisenstein datam for ψ). We augment this datum with a choice of an integer $M_{\mathcal{D}}$ satisfying

- $M_{\mathcal{D}}$ is divisible only by primes in $\Sigma \setminus \{v|p\}$;
- for $v \in \Sigma \setminus \{v|p\}, M_{\mathcal{D}}$ is contained in $\delta_{\mathcal{K}}, \mathrm{cond}(\xi_v), \mathrm{cond}(\psi_v), \mathrm{cond}(\tau_v)$, and $\mathrm{cond}_{\tilde{\pi}_v}(\phi_v)$.

All constructions to follow and subsequent formulas depend on this choice. In our applications we are free to choose a suitable $M_{\mathcal{D}}$.

Let $x_v := p^{t_v} \in \mathcal{O}_{\mathcal{K},v}$ be such that $(x_v) = cond(\xi_v^c)$. Let

$$U_{\mathcal{D}} := \prod_{v|p} K_{x_v,v} \prod_{v|\Sigma\setminus\{p\}} K_{M_{\mathcal{D}},v} \prod_{v\nmid\Sigma} U_1(\mathcal{O}_{F,v}),$$

with $K_{x,v}$ as in the last section.

Remark 11.5.1. *Later we will use $U_{\mathcal{D}}$ to denote the corresponding groups with same level in GL_2 as well.*

For $m = 1$ or 2 we define a meromorphic section $f_{\mathcal{D}}^{(m)} : \mathbb{C} \to I_{m+1}(\tau)$ as follows: $f_{\mathcal{D}}^{(m)}(z) = \otimes f_{\mathcal{D},w}^{(m)}(z)$ where

- $f_\infty^{(m)}(z) := f_\kappa \in I_{m+1}(\tau_\infty)$ for any infinite place;
- $v \nmid \Sigma$ then $f_{\mathcal{D},v}^{(m)}(z) := f_v^{sph} \in I_{m+1}(\tau_v)$.
- if $v|\Sigma, v \nmid p,$ then $f_{\mathcal{D},v}^{(m)}(z) := f_{M_{\mathcal{D}},v}^{(m)} \in I_{m+1}(\tau_v)$;
- for $v|p, f_{\mathcal{D},v}^{(m)}(z) := f_{-z}^{0,(m)} \in I_{m+1}(\tau_v),$ where x_v is used to define $f_{-z}^{0,(m)}$.

We let $H_{\mathcal{D}}^{(m)}(z,g) := E(f_{\mathcal{D}}^{(m)}; z, g)$.

Let

$$K_{\mathcal{D}}^{(m)} := \{k \in G_{m+1}(\hat{\mathcal{O}}_F) : 1 - k \in M_{\mathcal{D}}^2 \prod_{v|p} (x_v \bar{x}_v) M_{2(m+1)}(\mathcal{O})\}.$$

Then it easily follows from the definition of the $f_{\mathcal{D},v}^{(m)}(z)$'s that

$$H_{\mathcal{D}}^{(m)}(z, gk) = H_{\mathcal{D}}^{(m)}(z, g), k \in K_{\mathcal{D}}^{(m)}, \tag{11.17}$$

and that if for any $v|p$, $t_v > 0, x_v \in cond(\psi)$, and $x_v\bar{x}_v \in cond_\pi(\phi_v)$ then

$$H_{\mathcal{D}}^{(m)}(z, g\alpha(1,k)) = \tau(a_{k_p})H_{\mathcal{D}}^{(m)}(z,g), k \in U_{\mathcal{D}}.$$

For $u \in GL_{m+1}(\mathbb{A}_{\mathcal{K},f})$ let

$$L_v^{(m)} := \{\beta \in S_{m+1}(F) : \beta \geq 0, Tr\beta\gamma \in \hat{\mathcal{O}}_F, \gamma \in uS_{m+1}(\hat{\mathcal{O}}_F){}^t\bar{u}\}.$$

Lemma 11.5.1. *(i) if $k \geq m+1$, then $H_{\mathcal{D}}^{(m)}$ is holomorphic at $z_k := (k-m-1)/2$;*
(ii) if $k \geq m+1$ and if $g \in Q_{m+1}(\mathbb{A}_F)$ then

$$H_{\mathcal{D}}^{m}(z_k, g) = \sum_{\beta \in S_{m+1}(F), \beta > 0} H_{\mathcal{D},\beta}^{(m)}(z_k, g)$$

.

Further more, if $\beta > 0$, $g_{\infty,i} = r(X_i)diag(Y_i, {}^t\bar{Y}_i^{-1})$ and $g_f = r(a)diag(u, {}^t\bar{u}^{-1}) \in G_{m+1}(\mathbb{A}_{F,f})$, then $H_{\mathcal{D},\beta}^{(m)}(z_k, g) = 0$ if $\beta \notin L_u^{(m)}$ and otherwise

$$H_{\mathcal{D},\beta}^{(m)}(z_k, g) = e(tr\beta a)\frac{(-2)^{-(m+1)d}(2\pi i)^{(m+1)dk}(2/\pi)^{m(m+1)d/2}\prod_j(det\beta_j^{k-(m+1)}, \det \bar{Y}_j^k)}{(\prod_{j=0}^m (k-j-1)!)^d \prod_{j=0}^m L^S(k-j, \bar{\tau}'\chi_{\mathcal{K}}^j)}$$

$$\times \prod_{j=1}^d e(Tr\beta(X_j + iY_j\bar{Y}_j^t)) \prod_{v \notin S} f_{\mathcal{D},\beta u,v}(z_k, 1)$$

$$\times \tau(\det u)|\det u\bar{u}|_F^{m+1-k/2} \prod_{v\notin S} H_{v,\beta}(\bar{\tau}'_v(\varpi_v)q_v^{-2z-n}).$$

where $\beta_u = {}^t\bar{u}\beta u$. $\beta_j = \iota_j(\beta)$, ι_j is the embedding $F \hookrightarrow \mathbb{R}$ for any finite set of places $S \supseteq \Sigma$ such that $g_v \in K_{m+1,v}$ if $v \notin S$.

Proof. See [SU]11.5.1. □

If $k \geq m+1$, define a function $H_{\mathcal{D}}^{m}(\mathcal{Z}, x)$ on $\mathbb{H}_{m+1} \times G_{m+1}(\mathbb{A}_f)$ by

$$H_{\mathcal{D}}^{(m)}(\mathcal{Z}, x) := \prod_{j=1}^d \mu_{m+1}(g_{\infty,j})^{(m+1)k/2} \prod_{j=1}^d J_{m+1}(g_{\infty,j}, i)^{-k} H_{\mathcal{D}}^{(m)}((k-m-1)/2, g_\infty x)$$

where $g_\infty \in G^+_{m+1}(\mathbb{R}), g_\infty(i) = \mathcal{Z}$.

Lemma 11.5.2. *Suppose $k \geq m+1$. Then $H^{(m)}_{\mathcal{D}}(\mathcal{Z}, x) \in M_k(K^{(m)}_{\mathcal{D}})$.*

Proof. See [SU]11.5.2. $\square$

Lemma 11.5.3. *Suppose $k \geq m+1$ and that $x = diag(u, \bar{u}^{-1}$, $u \in GL_{m+1}(\mathbb{A}_{F,f}$ with $u_v = diag(1_m, \bar{a}_v)$, $a_v \in \mathcal{O}_v^\times$, if $v \in \sum$. If $\beta \notin L^{(m)}_u$ or if $det\beta = 0$ then $A_{\mathcal{D}\beta}(x) = 0$, and for $\beta = (\beta_{i,j}) \in L^{(m)}_u$ with $det\beta > 0$*

$$\begin{aligned}A^{(m)}_{\mathcal{D},\beta}(x) = |\delta_{\mathcal{K}}|_{\mathcal{K}}^{m(m+1)/4}|\delta_F|_F^{(m+1)/2} & \frac{(-2)^{-(m+1)d}(2\pi i)^{(m+1)kd}(2/\pi)^{m(m+1)d/2}\prod_{v|p}(det\beta|\beta|_v)^{k-m-1}}{\prod_{j=0}^m (k-j-1)!^d \prod_{j=0}^m L^{\Sigma}(k-j, \bar{\tau}'\chi^j_{\mathcal{K}})} \\ & \times \prod_{v|p} \bar{\tau}_v(a_v det(\beta))\mathfrak{g}(\tau'_v)^{m+1}c(\bar{\tau}'_v, -(k-m-1)/2)e_v(Tr_{\mathcal{K}_v/\mathbf{Q}_v}(a_v b_{m+1,1}/x_v) \\ & \times \prod_{v \in \sum, v \nmid p} \tau^c_v(a_v)e_v(Tr_{\mathcal{K}_v/\mathbf{Q}_v}(a_v b_{m+1,1}/M_{\mathcal{D}})) \\ & \times \prod_{v \notin \sum} \tau_v(detu_v)|u_v\bar{u}_v|_v^{m+1-k/2} h_{v,\bar{u}^t_v\beta u_v}(\bar{\tau}_v(\varpi_v)q_v^{-k}).\end{aligned} \tag{11.18}$$

Proof. See [SU]11.5.3. $\square$

11.6 $E_{\mathcal{D}}$ via pull-back

Let φ_0 be defined by: $\varphi_0(g) = \varphi_\psi(gy)$ for

$$y_v = \begin{cases} 1, & v = \infty, v \notin \Sigma \\ \eta^{-1}\text{diag}(M_{\mathcal{D}}^{-1}, M_{\mathcal{D}})\eta & v \in \Sigma, v \not| p \\ \text{diag}(x_v, \bar{x}_v^{-1}), & v|p \end{cases}$$

Proposition 11.6.1. *Let $m = 1$ or 2. Suppose that for any $v|p$ $(x_v) = (p^{t_v})$ with $t_v > 0$ and that $x_v \in cond(\psi)$ and $x_v\bar{x}_v \in cond_{\pi_v}(\phi_v)$ where ϕ_v is defined by $\varphi = \otimes\varphi_v$. Let $g \in G_m(\mathbb{A}_F)$ and*

$h \in G_1(\mathbf{A}_F)$ *be such that* $\mu_1(h) = \mu_m(g)$*. If* $k \geq m+1$ *then*

$$\int_{U_1(F)/U_1(\mathbb{A}_F)} H_{\mathcal{D}}^{(m)}(z, \alpha(g, g'h))\bar{\tau}(\det g'h)\varphi_0(g'h)dg' \\ = [U_1(\hat{\mathcal{O}}_F) : U_{\mathcal{D}}]^{-1} \begin{cases} c_{\mathcal{D}}^{(1)}(z)\phi(g) & m=1 \\ c_{\mathcal{D}}^{(2)}(z)E_{\mathcal{D}}(z,g) & m=2 \end{cases} \tag{11.19}$$

where

$$c_{\mathcal{D}}(x) := \pi^d 2^{(-2z-m+1)d}|M_{\mathcal{D}}|_F^{(2z+m+1)} \prod_{v|p} |x_v\bar{x}_v|_v^{z-(m+1)/2}\bar{\tau}_v^c(x_v) \prod_{v\nmid p} \tau_v(M_{\mathcal{D}}) \\ \times \frac{\Gamma(z+(m-1+k)/2)^d L^{\Sigma}(\tilde{\pi}, \xi, z+m/2)}{\Gamma(z+(m+1+k)/2)^d \prod_{i=0}^{1} L^{\Sigma}(\bar{\tau}'\epsilon_{\mathcal{K}}^i, 2z+m+1-i)} \prod_{v|p} \gamma^{(m)}(\rho_{1,v}, -z) \tag{11.20}$$

Proposition 11.6.2. *Let* $m = 1$ *or* 2*. Suppose that for any* $v|p$ $(x_v) = (p^{t_p})$ *with* $t_p > 0$ *and that* $x_v \in cond(\psi)$ *and* $x_v\bar{x}_v \in cond_{\pi_v}(\phi_v)$*. Let* $g \in G_m(\mathbb{A}_F)$ *and* $h \in G_1(\mathbb{A}_F)$ *be such that* $\mu_1(h)\mu_m(g)$*. Let* $\beta \in S_m(F)$*. If* $k \geq m+1$ *then*

$$\int_{U_1(F)/U_1(\mathbb{A}_F)} H_{\mathcal{D},\beta}^{(m)}(z, \alpha(g, g'h))\bar{\tau}(\det g'h)\varphi_0(g'h)dg' \\ = [U_1(\hat{\mathcal{O}}_F) : U_{\mathcal{D}}]^{-1} \begin{cases} c_{\mathcal{D}}^{(1)}(z)\varphi_\beta(g) & m=1 \\ c_{\mathcal{D}}^{(2)}(z)\mu_{\mathcal{D}}(\beta, z, g) & m=2 \end{cases} \tag{11.21}$$

where $c_{\mathcal{D}}^{(m)}(z)$ *is as defined above.*

Recall that $a_1, ..., a_{h_{\mathcal{K}}} \in \hat{\mathcal{O}}_{\mathcal{K}}$ be representatives for the class group of $\mathcal{K}$. We assume that each $a_i = (\varpi_v, 1) \in \mathcal{O}_{\mathcal{K},v}$ for some prime $v \notin \Sigma$ that splits in $\mathcal{K}$. Let

$$\Gamma_{\mathcal{D}} := U_1(F) \cap U_{\mathcal{D}}, \Gamma_{\mathcal{D},i} := U_1(F) \cap \begin{pmatrix} a_i^{-1} & \\ & \bar{a}_i \end{pmatrix} U_{\mathcal{D}} \begin{pmatrix} a_i & \\ & \bar{a}_i^{-1} \end{pmatrix}.$$

We often write $\Gamma_{\mathcal{D}}$ for the GL_2 open compact with the same level as well. Also, we write $\Gamma_{\mathcal{D},0} \supseteq \Gamma_{\mathcal{D}}$ by removing the congruence conditions required for diagonal entries. (Similar to $\Gamma_0(N) \supset \Gamma_1(N)$ in the classical case) For any $v|p$

$$(p_{u_v}) := (x_v) \cap \mathcal{O}_{F,v}, (p^{r_v})_v := (x_v\bar{x}_v).$$

It follows easily from the strong approximation that if we let $\mathcal{Y} \in \hat{\mathcal{O}}$ be any set of representatives for $(\hat{\mathcal{O}}_{\mathcal{K}}/\prod_{v|p}\bar{x}_v M_{\mathcal{D}})^\times/(\hat{\mathcal{O}}_F/p^{u_p}M_{\mathcal{D}})^\times$. Then

$$U_1(\mathbf{A}_F) = \sqcup_{i=1}^{h_{\mathcal{K}}}\sqcup_{a\in\mathcal{Y}} U_1(F)U_1(F_\infty)\begin{pmatrix} a_i^{-1}a^{-1} & \\ & \bar{a}_i\bar{a}\end{pmatrix} U_{\mathcal{D}}.$$

with each element appearing exactly h_F times. Define:

$$\tilde{H}^{(m)}_{\mathcal{D},\beta}(z,g) := \sum_{a\in(\hat{\mathcal{O}}_{\mathcal{K}}/(\prod_{v|p}x_vM_{\mathcal{D}}))^\times}(\prod_{v|p}\mu_{1,v}^{-1})\xi^c\tau(a)H^{(m)}_{\mathcal{D},\beta}(z,g\alpha(1,diag(a^{-1},\bar{a}))),$$

Suppose h is a diagonal matrix, then the left hand side of 11.19 is

$$\begin{aligned}[]&[\mathcal{O}_{\mathcal{K}}^\times:\mathcal{O}_F^\times]^{-1}h_F^{-1}(\#(\mathcal{O}_F/p^{u_p}M_{\mathcal{D}})^\times)^{-1}[U_1(\hat{\mathcal{O}}_F):U_{\mathcal{D}}]^{-1}\\ &\times\sum_{i=1}^{h_{\mathcal{K}}}\bar{\tau}^c\tau(a_i)\int_{\Gamma_{\mathcal{D},i}\backslash U_1(F_\infty)}\tilde{H}^{(m)}_{\mathcal{D},\beta}(z,\alpha(g,g'diag(a_i^{-1},\bar{a}_i)h))\\ &\times\bar{\tau}(detg'h)\phi_0(g'diag(a_i^{-1},\bar{a}_i)h)dg'.\end{aligned} \tag{11.22}$$

11.7 Neben typus

In this section we discuss the relations between $U(1,1)$ automorphic forms and GL_2 automorphic forms. This will be useful later. In the [SU] case the situation is easier since they assumed the forms are newforms, i.e. invariant under the action of matrices: $\begin{pmatrix} * & * \\ & 1\end{pmatrix}$. Now since we are going to work with the full dimensional Hida family so we do not assume this anymore. A principle for this issue is: we assume the neben characters at places not dividing p and the torsion part at p-adic places to be similar to the new form and let the free part of the p-adic neben characters to vary arbitrarily. Let $\varepsilon' = \otimes_v\varepsilon'_v$ be a character of $T_{U(1,1)}(\hat{\mathcal{O}}_F)$. First look at the p-adic places. Note that $\mathbb{Z}_p = \Delta\times\Gamma$ for $\Delta\simeq\mathbb{F}_p^\times$ and $\Gamma = 1+p\mathbb{Z}_p$. $T_{U(1,1)}(\mathbb{Z}_p) = \{\begin{pmatrix}\bar{a}^{-1} & \\ & a\end{pmatrix}|a\in\mathcal{O}_{\mathcal{K},v}^\times\}$, $T_{GL_2}(\mathbb{Z}_p)\simeq\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times$. For $v|p$ ε'_v is a character of $T_{U(1,1)}(\mathcal{O}_{F,v})$ can be written as $\varepsilon'_{v,tor}.\varepsilon'_{v,fr}$ with respect to $\Delta\times\Gamma$. Let ψ be a Hecke character we can define $\psi_{v,tor}$ and $\psi_{v,fr}$ be characters of $\mathcal{O}_{\mathcal{K},v}^\times$ in the same way. Since $\varepsilon'_{v,fr}$ and $\psi_{v,fr}$ have order powers of p so there are unique square roots $\varepsilon'^{,\frac{1}{2}}_{v,fr}$ and $\psi^{\frac{1}{2}}_{v,fr}$ of them. Now suppose for each $v|p$ we have:

$$\varepsilon'_{v,tor}(\begin{pmatrix} \bar{a}^{-1} & \\ & a \end{pmatrix}) = \psi_{v,tor}(a)$$

for all $a \in \mathcal{O}_{\mathcal{K},v}^{\times}$ and that for all $v \nmid p$,

$$\varepsilon'_{v}(\begin{pmatrix} \bar{a}^{-1} & \\ & a \end{pmatrix}) = \psi_{v}(a)$$

for all $a \in \mathcal{O}_{\mathcal{K},v}^{\times}$. Then we define a neben character of $T_{GL_2}(\hat{\mathcal{O}}_F)$ by: for $v \nmid p$,

$$\varepsilon_{v}(\begin{pmatrix} a & \\ & b \end{pmatrix}) = \psi_{v}(b)$$

for $v|p$

$$\varepsilon_{v,tor}(\begin{pmatrix} a & \\ & b \end{pmatrix}) = \psi_{v,tor}(b)$$

and

$$\varepsilon_{v,fr}(\begin{pmatrix} a & \\ & b \end{pmatrix}) = \varepsilon_{v,fr}^{',\frac{1}{2}}(\begin{pmatrix} \frac{a}{b} & \\ & \frac{b}{a} \end{pmatrix})\psi_{v,fr}^{\frac{1}{2}}(ab)$$

and

$$\varepsilon = \otimes_v \varepsilon_v$$

. Now let ψ and ε' be as above and I be an ideal contained in the conductor of ε'. Let φ be an automorphic form on $U(1,1)(\mathbb{A}_F)$ such that the action of $k \in U_0(I)$ is given by $\varepsilon'(\begin{pmatrix} a_k & \\ & b_k \end{pmatrix})$.
Surippose moreover that it satisfies the condition that:
(*): for any totally positive global unit $b \in \mathcal{O}_F^{\times}$ we have:

$$\varphi(\begin{pmatrix} b & \\ & 1 \end{pmatrix} g \begin{pmatrix} b^{-1} & \\ & 1 \end{pmatrix}) = \varphi(g)\varepsilon'(\begin{pmatrix} b^{-1} & \\ & 1 \end{pmatrix})$$

This condition is necessary for a SL_2 Hilbert modular form to be able to extend to GL_2 with given neben typus)

Then we define a map α_ψ from φ''s as above to automorphic forms on $GL_2(\mathbb{A}_F)$.

Definition 11.7.1.

$$\alpha_\psi = \alpha_{\psi,\varepsilon,\varepsilon'}(\varphi)(g) = \sum_{j:a_j\bar{a}_j\sim g} \varphi(h_\infty \begin{pmatrix} \bar{a}_j & \\ & a_j^{-1} \end{pmatrix})\varepsilon(k)\psi(z_\infty a_j)$$

for $g = \gamma z_\infty h_\infty \begin{pmatrix} \bar{a}_j\bar{a}_j & \\ & 1 \end{pmatrix} k \in GL_2(\mathbf{A}_F)$ *where* $\gamma \in GL_2(F), h_\infty \in SL_2(F_\infty), z_\infty \in Z(F_\infty), k \in \Gamma_0(I)_{GL_2}$.

Lemma 11.7.1. *Assumptions are as above. Suppose* φ_1, φ_3 *are automorphic forms on* $GU(1,1)(\mathbb{A}_F)$, φ_2 *is an automorphic form on* $U(1,1)$. *Let* ψ_1, ψ_2, ψ_3 *be Hecke characters for* $\mathcal{K}$. *Suppose* $\psi_1\psi_2\bar{\psi}_3 = 1$ *and the central characters of* φ_1, φ_3 *are* ψ_1, ψ_3. *Suppose also that* $\varepsilon_1', \varepsilon_2', \varepsilon_3'$ *are neben typus of* $\alpha_1|_{U(1,1)}, \alpha_2, \alpha_3|_{U(1,1)}$. *Assume that* $\varepsilon_1'\varepsilon_2'\bar{\varepsilon}_3' = 1$ *and the* ε_i'*'s and* ψ_i*'s satisfy the assumptions above, then*

$$2^{u_F}[\mathcal{O}_{\mathcal{K}}^\times : \mathcal{O}_F^\times] < \varphi_1\varphi_2, \varphi_3 >_{U(1,1)} = < \varphi_1\alpha_{\psi_2}(\varphi_2), \varphi_3 >_{GL_2}$$

where u_F *is some number depending only on* F. *This factor comes out when considering* GL_2 *modulo the center.*

The proof is straightforward.

11.8 Formulas

Definition 11.8.1.

$$f^c(g) := \overline{s(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}$$

$$\tilde{f}^c := f^c \otimes \psi(det)$$

Definition 11.8.2.

$$\tilde{g}^{(m)}_{\mathcal{D},\beta}(-,x) = (\tilde{H}^{(m)}_{\mathcal{D},\beta}(\alpha(x,-)) \otimes \xi(\det -))$$

and

$$g^{(m)}_{\mathcal{D},\beta} = tr_{\Gamma_0(M)/\Gamma_0(M_{\mathcal{D}}^2)}\pi(\begin{pmatrix} 1 & \\ & \frac{M_{\mathcal{D}}^2}{M} \end{pmatrix})(\tilde{g}^{(m)}_{\mathcal{D},\beta})$$

Proposition 11.8.1. *Notations as above. Let* $\beta \in S_m(F)$.

(i) There exists a constant $C^{(m)}_{\mathfrak{D}}$ depending only on $\mathfrak{D}$ and m such that

$$< \tilde{g}^{(m)}_{\mathcal{D},\beta}(-,x), \rho(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f \begin{pmatrix} M_{\mathcal{D}}^2\prod_{v|p} x_v\bar{x}_v & \\ & 1\end{pmatrix}_f)\tilde{f}^c >_{\Gamma_{\mathcal{D},0}} = C^{(m)}_{\mathfrak{D}} \begin{cases} a_{\mathfrak{D}}(\beta,x) & m=1 \\ c_{\mathfrak{D}}(\beta,x) & m=2. \end{cases}$$

(the reason for the twisting ξ showing up here is the different of ψ which we twisted on f^c and τ in the pull back formula.) (ii) if $a(v,f)\neq 0$ for any $v|p$ and if $p|\mathfrak{f}_\chi$ and $p|f_{\chi^{-1}\xi}$ then

$$\begin{aligned} C^{(1)}_{\mathfrak{D}} = & \quad (-\pi 2^{2-k} i^{-k})^d \textstyle\prod_{v\nmid p, v\in\Sigma} \psi^c_v\tau_v(M_{\mathcal{D}})|M_{\mathcal{D}}|^{\kappa}_F \gamma^{(1)}(\rho_p,-z_\kappa) \\ & \times \textstyle\prod_{v|p}\xi^c_v(x_v)p^{r_v+n_v(\kappa-2)-\frac{\kappa}{2}r_v} \cdot \frac{\Gamma(k-1)^d L^\Sigma_{\mathcal{K}}(f,\chi^{-1}\xi,k-1)}{\Gamma(k)^d\prod_{j=0}^1 L^\Sigma(\chi^{-1}\xi'\chi^j_{\mathcal{K}},k-j)} \end{aligned}$$

where

$$\gamma^{(1)}(\rho_p,z_\kappa) = \bar{\psi}_p(-1)\prod_{v|p} c_2(\bar{\tau}'_{0,v}, 1-k/2)\bar{\xi}^c_v(x_v)\mathfrak{g}(\tau'_{0,v})^2\prod_{v|p}\mu_{1,v}(p)^{r_v-n_v}\mathfrak{g}(\mu^{-1}_{1,v}\xi^c_v,x_v).\chi_v\bar{\xi}_v(y_v)\mathfrak{g}(\mu_{1,v}\bar{\chi}_v\xi^c_v,y_v)$$

$(y_v) := cond(\bar{\chi}_v\xi^c_v)$ and $(p^{n_v}) := (y_v\bar{y}_v)$.
(iii) if $\mathcal{O}_v\neq cond(\xi_v\psi_v^{-2}\psi^c_v)\supseteq cond(\bar{\chi}_v\xi_v\xi^c_v)$ for any $v|p$ then $C^{(2)}_{\mathfrak{D}} = C^{(1)}_{\mathfrak{D}}\prod_{v|p}p^{r_v}$.

Proof. One argues similarly [SU]11.7.1 and the end of [SU]11.6. □

We also state here some formulas for Fourier coefficient which follow from section 11.5 in the same way as in [SU] 11.5 and 11.7.

Proposition 11.8.2. *Suppose $y = diag(u,\bar{u}^{t^{-1}}), u\in GL_m(\mathbb{A}^\Sigma_{\mathcal{K},f})$. For $i=1,2,\cdots,h_{\mathcal{K}}$ let $v_i := diag(u,\bar{a}_i)$. For $\beta\in S_m(F), \beta'\geq 0$ and $n\in F_{>>0}$ or 0, let:*

$$L^{(m)}_{v_j}(\beta,n) := \{T = \begin{pmatrix}\beta & c\\ \bar{c} & n\end{pmatrix}\in L^{(m)}_{v_j}, T>0\}.$$

This is a finite set

(i)

$$\tilde{b}^{(m)}_{\mathcal{D},\beta}(n,i,y) = \sum_{j:\bar{a}_j a_j\sim\bar{a}_i a_i}\ \sum_{T\in L^{(m)}_{v_j}(\beta,n)}\ \sum_{a\in(\hat{\mathcal{O}}/x_pM_{\mathcal{D}})^\times}(\prod_{v|p}\mu^{-1}_{1,v})\xi^c\tau(a)A^{(m)}_{\mathcal{D},T}(y_{j,a}),$$

where $y_{j,a} := diag(u,\bar{a}_j\bar{a},\bar{u}^{-1},a_j^{-1}a^{-1})$.

(ii) Let $(x_v) := cond(\xi_v^c)$. *Suppose* $x_v = p^{t_v}$ *with* $t_v > 0$ *for any* $v|p$. *If* $T \in L_{v_j}^{(m)}(\beta, n)$ *then* $A_{\mathcal{D},T}(y_{j,a}) = 0$ *unless for any* $v|p$, $T_v^* := T_{m+1,1} \in \mathcal{O}_{\mathcal{K},v}^\times$ *and* $T_v^* := T_{m+1,1}x_v\delta_\mathcal{K}/M_\mathcal{D} \in \mathcal{O}_{\mathcal{K},v}$ *for all* $v|\Sigma \setminus \{p\}$, *in which case;*

$$\begin{aligned}\sum_{a\in(\hat{\mathcal{O}}/(x_pM_\mathcal{D}))^\times}(\prod_{v|p}\mu_{1,v}^{-1})\xi^c\tau(a)A_{\mathcal{D},T}(y_{j,a})\\
&= |\delta_\mathcal{K}|_\mathcal{K}^{m(m+1)/4}|\delta_F|_F^{(m+1)/2}\frac{(-2)^{-(m+1)d}(2\pi i)^{(m+1)kd}(2/\pi)^{m(m+1)d/2}}{\prod_{j=0}^m(k-j-1)!^d\prod_{j=0}^m L^\Sigma(k-j,\bar{\tau}'\chi_\mathcal{K}^j)}\\
&\times\prod_{v|p}(detT|T|_v)^{k-m-1}\prod_{v|p}\bar{\tau}_v(det(\beta))\mathfrak{g}(\tau_v')^{m+1}c(\bar{\tau}_v', -(k-m-1)/2)\\
&\times\mu_{1,v}\bar{\xi}_v^c(T_v^*)\prod_{v|p}\mathfrak{g}(\mu_{1,v}^{-1}\xi_v^c, x_v)\\
&\times|M_\mathcal{D}|_F^2\prod_{v\in\Sigma, v\nmid p}|x_v|_\mathcal{K}\begin{cases}\bar{\xi}_v^c(T_v^*)\mathfrak{g}(\xi_v^c, x_v\delta_\mathcal{K}) & (x_v)\neq\mathcal{O}_{\mathcal{K},v}, T_v^*\in\mathcal{O}_{\mathcal{K},v}^\times\\ 0 & (x_v)\neq\mathcal{O}_{\mathcal{K},v}, T_v^*\notin\mathcal{O}_{\mathcal{K},v}^\times\\ 1 & (x_v)=\mathcal{O}_{\mathcal{K},v}\end{cases}\\
&\times\prod_{v\notin\Sigma}\tau_v(detu_v\bar{a}_j)|\bar{a}_ju_v\bar{u}_v|_v^{m+1-k/2}h_{v,\bar{u}_v^t\beta u_v}(\bar{\tau}_v(\varpi_v)q_v^{-k}).\end{aligned}$$

Proposition 11.8.3. *Suppose* $y = diag(u, \bar{u}^{-1}), u \in GL_m(\mathbb{A}_{\mathcal{K},f}^\Sigma)$. *let* $v_i = diag(u, \bar{a}_i)$. *Suppose* $x_v = p^{t_v}$ *with* $t_v > 0$ *for any* $v|p$. *Then for* $n \in F_{>>0}$ *or* 0.

$$\rho_{\mathcal{D},\beta}^{(m)}(n,y) = (-i2^{m(m+1)-1})^d|\delta_\mathcal{K}|_\mathcal{K}^{m(m+1)/4}|\delta_F|_F^{(m+1)/2}\sum_{j:\bar{a}_ja_j\sim\bar{a}_ia_i}\sum_{T\in L_{v_j}^{(m)}(\beta,n)}R_{\mathcal{D},T}^{(m)}.$$

Where $R_{\mathcal{D},T} = 0$ *unless* $T_p^* \in \mathcal{O}_{\mathcal{K},p}^\times$ *and* $T_v^* \in \mathcal{O}_{\mathcal{K},p}$ *for all* $v|\Sigma \setminus \{p\}$ *in which case*

$$R_{\mathcal{D},T}^{(m)} = (|(detT)|_{F_\infty}|detT|_p)^{\kappa-m-1}\bar{\xi}_p^c(T_p^*)\prod_{v\nmid p}\psi\bar{\xi}_v(detT)$$

$$\times \psi_p(-1)\chi_p\bar{\xi}_p^c(M_{\mathcal{D}})$$

$$\times \prod_{v\in\Sigma, v\nmid p}(x_v)_{\mathcal{K}}\begin{cases}\bar{\xi}_v^c(T_v^*)\mathfrak{g}(\xi_v^c, x_v\delta_{\mathcal{K}}) & (x_v)\neq \mathcal{O}_{\mathcal{K},v}, T_v^*\in\mathcal{O}_{\mathcal{K},v}^\times \\ 0 & (x_v)\neq\mathcal{O}_{\mathcal{K},v}, T_v^*\notin\mathcal{O}_{\mathcal{K},v}^\times \\ 1 & (x_v)=\mathcal{O}_{\mathcal{K},v}\end{cases}$$

$$\times \prod_{v\in\Sigma, v\nmid p}\chi_v\bar{\xi}_v^c(y_v\delta_{\mathcal{K}})|y_v\delta_{\mathcal{K}}|_{\mathcal{K}}^{2-\kappa}\mathfrak{g}(\bar{\chi}_v, \xi_v^c, y_v, \delta_m athcal K)$$

$$\times \psi\bar{\xi}(a_j^m detu)|detu|_{\mathcal{K}}^{-\kappa/2}\prod_{v\notin\Sigma}h_{v,\bar{u}_v^t\beta u_v}(\bar{\tau}_v(\varpi_v)q_v^{-k})$$

$$\times\begin{cases}\prod_{v|\sum/\{p\}}\chi_v\bar{\xi}_v'(\varpi_v^{e_v})q_v^{e_v(k-2)}\mathfrak{g}(\chi_v\bar{\xi}_v')^{-1} & m=2\\ 1 & m=1\end{cases}$$

We define a normalization constant:

$$B_{\mathfrak{D}}^{(m)} := \frac{|M_{\mathcal{D}}|_{\mathcal{K}}^{\frac{\kappa}{2}}\prod_{j=0}^m(k-j-1)!^d\prod_{j=0}^m L^{\Sigma}(k-j,\chi\xi'\epsilon_{\mathcal{K}}^j)\prod_{v\in\Sigma,v\nmid p}\chi_v\bar{\xi}_v^c(y_v\delta_{\mathcal{K}})\mathfrak{g}(\bar{\chi}_v\xi_v^c, y_v\delta_{\mathcal{K}})|y_v\delta_{\mathcal{K}}|_{\mathcal{K},v}^{2-k}}{\prod_{v\nmid p,v\in\Sigma}\psi_v^c\tau_v(M_{\mathcal{D}})\bar{\psi}_p(-1)\prod_{v|p}c_{m+1}(\bar{\tau}_{0,v}', -(k-m-1)/2)\mathfrak{g}(\tau_{0,v}')^{m+1}\mathfrak{g}(\xi_v^c, x_v)}$$

$$\times i^d(-1)^m d2^{m(m+2)d}(2\pi i)^{-(m+1)dk}(\pi/2)^{m(m+2)d/2}$$

$$\times \begin{cases}\prod_{v\in\Sigma,v\nmid p}\chi_v\bar{\xi}_v'(\varpi_v^{e_v})q_v^{e_v(k-2)}\mathfrak{g}(\chi_v\bar{\xi}_v')^{-1} & m=2\\ 1 & m=1\end{cases} \tag{11.23}$$

Now for $m=1$ or 2 we define:

$$L_{\mathcal{D}}^{(m)} = \frac{2^{-3d}(2i)^{d(\kappa+1)}}{\prod_{v|p}p^{r_v(1-\kappa/2)}}B_{\mathcal{D}}^{(m)}C_{\mathcal{D}}^{(m)}.$$

and

$$S(f) := \prod_{v|p}\mu_{1,v}(p)^{-r^v}p^{r^v(\kappa/2-1)}W'(f)$$

where $W'(f)$ is the prime to p part of the root number of f with $|W'(f)|_p = 1$ (See [SU]11.7.3.) and recall in the section for notations we defined r^v is such that $p^{r^v}\|N_v$ for $v|p$.

Proposition 11.8.4. *Assumptions are as before. Suppose* $\kappa \geq 2$ *if* $m=1$ *and* $\kappa > 6$ *if* $m =$

2. *Suppose* $x = diag(u, {}^t\bar{u}^{-1})$ *with* $GL_m(\mathbf{A}_{\mathcal{K},f})$. *Suppose* $p|\mathfrak{f}_{\bar{\chi}\xi}$ *and* $p^r|Nm(\mathfrak{f}_\xi)$. *Suppose also* $cond(\psi_p)|\mathfrak{f}_\xi^c\mathcal{O}_{\mathcal{K},p}$..

(i)

$$\frac{< (f^{(m)}_{\mathcal{D},\beta,x}\otimes\xi), \rho(\begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix}_f)\tilde{f}^c >_{\Gamma_{\mathcal{D},0}}}{< f, \rho(\begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix}_f)\tilde{f}^c >_{GL_2,\Gamma_{\mathcal{D},0}}} = \frac{L^{(m)}_{\mathcal{D}}}{2^{-3d}(2i)^{d(\kappa+1)}S(f) < f, \rho(\begin{pmatrix} & -1 \\ N & \end{pmatrix}_f)\tilde{f}^c >_{GL_2,\Gamma_0(N)}}$$
$$\times W'(f)^{-1}\begin{cases} a_{\mathcal{D}}(T,x) & m=1 \\ c_{\mathcal{D}}(T,x) & m=2. \end{cases}$$

(ii)
$$L^{(1)}_{\mathcal{D}} = \prod_{v|p} a(v,f)^{-ord_v(Nm(\mathfrak{f}_{\bar{\chi}\xi}))}(\frac{(\kappa-2)!}{(-2\pi i)^{\kappa-1}})^{2d}\mathfrak{g}(\bar{\chi}\xi)Nm(\mathfrak{f}_{\bar{\chi}\xi}\delta_{\mathcal{K}})^{\kappa-2}L^{\Sigma}_{\mathcal{K}}(f,\bar{\chi}\xi,\kappa-1).$$

(iii) Under the hypotheses of Proposition 11.4.1 (iii)

$$L^{(2)}_{\mathcal{D}} = \prod_v p^{r_v}\times L(3-\kappa,\chi\bar{\xi}')\prod_{v\in Sigma}(1-\bar{\chi}\xi'(\varpi_v)q_v^{2-\kappa})L^{(1)}_{\mathcal{D}}.$$

Corollary 11.8.1. *Under the hypotheses above*

$$\frac{< (f^{(1)}_{\mathcal{D},1,x_M}\otimes\xi), \rho(\begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix}_f)\tilde{f}^c >_{\Gamma_{\mathcal{D},0}}}{< f, \rho(\begin{pmatrix} & -1 \\ p^{r_p}M & \end{pmatrix}_f)\tilde{f}^c >_{GL_2,\Gamma_{\mathcal{D},0}}} = \frac{L^{(1)}_{\mathcal{D}}}{(2^{-3}(2i)^{\kappa+1})^dS(f) < f, \rho(\begin{pmatrix} & -1 \\ N & \end{pmatrix}_f)\tilde{f}^c >_{GL_2,U_0(N)}}.$$

For any $x\in G(\mathbb{A}_{F,f})$ let

$$G_{\mathcal{D}}(Z,x) := W'(f)^{-1}L^{(2)}_{\mathcal{D}}|\mu(x)|_F^{-\kappa}E_{\mathcal{D}}(Z,x). \tag{11.24}$$

Corollary 11.8.2. *Under the hypotheses as above,*

$$\frac{< (f^{(2)}_{\mathcal{D},\beta,x}\otimes\xi), \rho(\begin{pmatrix} & -1\\ p^{r_p}M & \end{pmatrix}_f)\tilde{f}^c >_{\Gamma_{\mathcal{D},0}}}{< f, \rho(\begin{pmatrix} & -1\\ p^{r_p}M & \end{pmatrix}_f)\tilde{f}^c >_{GL_2,\Gamma_{\mathcal{D},0}}} = \frac{C_{\mathcal{D}}(\beta,x)}{(2^{-3}(2i)^{\kappa+1})^d S(f) < f, \rho(\begin{pmatrix} & -1\\ N & \end{pmatrix}_f)\tilde{f}^c >_{GL_2,\Gamma_0(N)}}.$$

11.9 A formula for Fourier Coefficients

Now we express certain fourier coefficients of $G_{\mathcal{D}}(Z,x)$ as essentially Rankin-Selberg convolutions of f and sums of theta functions. This is used later to prove various p-adic properties of these coefficients.

11.9.1 The formula

Let $\mathcal{D}=(f,\psi,\xi,\sum)$ be an Eisenstein datum. We assume:

$$\text{for any } v|p, \pi_v, , \phi_v, \psi_v, \tau_v \text{are in the Generic Case .} \tag{11.25}$$

Let λ be an idele class character of $\mathbb{A}_{\mathcal{K}}^\times$ such that

- $\lambda|_{\mathbb{A}_F^\times}=1$;
- $\lambda_\infty(x)=(x/|x|)^{-2}$;
- λ_v is unramified if $v\nmid\Sigma\setminus\{v|p\}$.

Let $a_1,..,a_{h_{\mathcal{K}}}\in\mathbb{A}_{\mathcal{K}}^\times$ be representatives of the class group of $\mathcal{K}$ as in the previous sections; so $a_i=(\varpi_{v_i},1)$ for some place v_i of F splitting in $\mathcal{K}$. Also for $i\in I_1$, then $a_i\bar{a}_i$ is trivial in the narrow class group, we assume $\varpi_v=q_i$ for some totally positive $q_i\in F$. Let $\mathcal{Q}=\{v_i\}_{i\in I_1}$.

Let $\beta \in S_2(F), \beta > 0$, and $u \in GL_2(\mathbb{A}_{\mathcal{K},f})$ be such that

- $u_v \in GL_2(\mathcal{O}_{\mathcal{K},v})$for $v \notin \mathcal{Q}$;
- ${}^t\bar{u}\beta u \in S_2(\mathcal{O}_{F,v})^*$for all primes v;
- ${}^t\bar{u}\beta u$ is v-primitive for all $v \notin \Sigma \setminus \{v|p\}$;
- if $u^{-1}\beta^{-1}{}^t\bar{u}^{-1} = \begin{pmatrix} * & * \\ & d \end{pmatrix}$ then $d_v \in \mathcal{O}_F$for all $v \in \Sigma \setminus \{v|p\}$.

Let $M_{\mathcal{D}}$ be as before and also satisfying:

$$cond(\lambda)|M_{\mathcal{D}} \text{ and } D_{\mathcal{K}} \det {}^t\bar{u}\beta u|M_{\mathcal{D}}. \tag{11.26}$$

All Weil representations that show up are defined using the splitting determined by the character λ. By our choice of $\mathcal{K}$, there is an idele $\mathfrak{d}_1$ of $\mathbb{A}_{\mathcal{K}}$ so that $\mathfrak{d}_1\bar{\mathfrak{d}}_1 = \mathfrak{d}$

Later we are going to choose u and β such that they do not belong to $GL_2(\mathcal{O}_v)$ only when $v = v_i$ for some v_i above. Recall that we have proved:

for $v = v_i$

$$\begin{aligned} & FJ_{\beta,v}(f; z_k, x, g_v, y) \\ = & \frac{\tau(\det r_v y_v)|\det r_v y_v|_{\mathcal{K}}^{-z+\frac{1}{2}}}{\prod_{j=0}^{1} L(2z+3-j, \bar{\tau}'_{0,v}\chi_{\mathcal{K},v}^j)} f_1^{sph}(g_v)\omega_\beta(r_v, g_v)\Phi_{0,y_v}(x) \end{aligned}$$

For $v|\Sigma\backslash\{v|p\}$, then (notice that we have restricted ourselves to the case when the local characters ψ_v, τ_v are trivial):

$$\begin{aligned} & \sum_{a \in (\mathcal{O}_{\mathcal{K},v}/M_{\mathcal{D}})^\times} F_{\beta,v}(z; x, g_v \begin{pmatrix} a^{-1} & \\ & \bar{a} \end{pmatrix}, r_v u_v) \\ = & \sum_{b \in \mathcal{O}_{F,v}/\tilde{D}_v\mathfrak{d}} f_b(z, g_v\eta)\omega_\beta(r_v, g_v \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix})\Phi_{1,M_{\mathcal{D}},u_v}(x) \end{aligned}$$

for $v|p$, then

$$\sum_{a\in(\mathcal{O}_{\mathcal{K},v}/x_v)^\times} \xi_V^c\tau(a)\mu_{1,v}^{-1}F_{\beta,v}(-z;x,g_v\begin{pmatrix}a^{-1} & \\ & \bar{a}\end{pmatrix},r_vu_v)$$

$$=\psi_v(-1)\bar{\tau}_v'(deth)\mathfrak{g}(\tau_v')^2\bar{\tau}_v'(p^{2u_v})p^{-4u_vz-5u_v}\tau(\det r_vy_v)|\det r_v|_{\mathcal{K}}^{-z+1/2}|\det h|_v^{2z+1}$$

$$\times f_{u_v,1}(z,g_v\eta)\omega_\beta(r_v,g_v)\Phi_{\xi^c\mu_{1,v}^{-1},x_v,u_v(x)}(x),$$

(sorry for the bad notation u_v, the last one is of different meaning.)

Now for $v|\Sigma\setminus\{v|p\}$, we have

$$\begin{aligned} f_{-b,v}(\begin{pmatrix}1 & \\ n & 1\end{pmatrix}g'\eta) &= \frac{\tau_v}{\lambda_v}(-1)f_{-b,v}(\begin{pmatrix}1 & \\ -n & 1\end{pmatrix}g_v\eta) \\ &= \frac{\tau}{\lambda}(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)^{-1}|\delta_{\mathcal{K}}\bar{\mathfrak{d}}_1|^{\frac{\kappa}{2}}f_v^\dagger(\begin{pmatrix}1 & \\ -n & 1\end{pmatrix}g_v\eta\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\begin{pmatrix}\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1 & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1}\bar{\mathfrak{d}}_1^{-1}\end{pmatrix}) \end{aligned}$$

For $h\in U_\beta(\mathbb{A}_F),u\in GL_2(\mathbb{A}_{\mathcal{K}})$, we define

$$\tilde{\Phi}_{\mathcal{D},\beta,u}=\otimes\Phi_{\beta,\infty}\prod_{v|p}\Phi_{\beta,\xi_v^c\mu_{1,v}^{-1},x_v,u_v}\prod_{v\nmid p}\Phi_{\beta,1,M_{\mathcal{D}},u_v}\prod_{v\notin\Sigma}\Phi_{0,u_v}$$

and $\Phi_{\mathcal{D},\beta,u} = \lambda(\mathfrak{d}_1\delta_{\mathcal{K}}^{-1})^{-1}|\mathfrak{d}_1\delta_{\mathcal{K}}^{-1}|_{\mathcal{K}}^{-1}\omega((\begin{pmatrix}\mathfrak{d}_1\delta_{\mathcal{K}}^{-1} & \\ & \bar{\mathfrak{d}}_1^{-1}\bar{\delta}_{\mathcal{K}}\end{pmatrix}\eta^{-1})\tilde{\Phi}_{\mathcal{D},\beta,u}$ and define $\Theta_{\mathcal{D},\beta}(h,g;u):=\Theta_\beta(h,g;\Phi_{\mathcal{D},\beta,u})$

$$\begin{aligned} &\omega_{\beta\mathfrak{d}^{-1},v}(-\eta\begin{pmatrix}1 & n\\ & 1\end{pmatrix}g_v^\eta\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\eta)\tilde{\Phi}(v\mathfrak{d}_1) \\ =\ &\omega_{\beta,v}(-\begin{pmatrix}\delta_1 & \\ & \bar{\delta}_1^{-1}\end{pmatrix}\begin{pmatrix}\mathfrak{d}^{-1} & \\ & 1\end{pmatrix}\eta\begin{pmatrix}1 & n\\ & 1\end{pmatrix}g_v^\eta\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\eta^{-1}\begin{pmatrix}\mathfrak{d} & \\ & 1\end{pmatrix})\tilde{\Phi}(v) \\ =\ &\lambda(\mathfrak{d}_1\delta_{\mathcal{K}}^{-1})|\mathfrak{d}_1\delta_{\mathcal{K}}^{-1}|_{\mathcal{K}}\omega_{\beta,v}(-\begin{pmatrix}\bar{\delta}_1^{-1} & \\ & \bar{\delta}_1^{-1}\end{pmatrix}\eta\begin{pmatrix}1 & n\\ & 1\end{pmatrix}g_v^\eta\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ & \mathfrak{d}\end{pmatrix}\begin{pmatrix}\delta_{\mathcal{K}}\mathfrak{d}_1^{-1} & \\ & \bar{\delta}_{\mathcal{K}}^{-1}\bar{\mathfrak{d}_1}\end{pmatrix})\Phi(v) \\ =\ &\lambda(\mathfrak{d}_1\delta_{\mathcal{K}}^{-1})|\mathfrak{d}_1\delta_{\mathcal{K}}^{-1}|_{\mathcal{K}}\omega_{\beta,v}(-\eta\begin{pmatrix}1 & n\\ & 1\end{pmatrix}g_v^\eta\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\begin{pmatrix}\tilde{\delta_{\mathcal{K}}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1}\end{pmatrix})\Phi(v) \end{aligned}$$

To see this, observe that

$$\tilde{\Phi} = \omega_\beta(\eta \begin{pmatrix} \delta_{\mathcal{K}}\mathfrak{d}_1^{-1} & \\ & \bar{\delta}_{\mathcal{K}}^{-1}\bar{\mathfrak{d}}_1 \end{pmatrix})\Phi$$

and

$$\omega_{\beta\mathfrak{d}^{-1}}(g) = \omega_\beta(\begin{pmatrix} \mathfrak{d}^{-1} & \\ & 1 \end{pmatrix} g \begin{pmatrix} \mathfrak{d} & \\ & 1 \end{pmatrix})$$

Definition 11.9.1.

$$\otimes f_v = f := \prod_{v|\infty} f_\kappa \prod_{v|p} f_{u_v} \prod_{v\in\Sigma, v\nmid p} f_v^\dagger \prod_{v\notin\Sigma} f_v^{sph}$$

and define $\mathcal{E}_{\mathcal{D}}$ to be the corresponding Eisenstein series on $U(1,1)(\mathbb{A}_F)$.

Let $g^\eta = \eta^{-1}g\eta$ and $g' = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, if $x = \begin{pmatrix} hu & \\ & {}^t(\bar{hu})^{-1} \end{pmatrix}$ for $u \in GL_2(\mathbb{A}_{\mathcal{K}}), h \in U_\beta(\mathbb{A}_F)$ satisfying the assumptions at the beginning of this section then:

$$\begin{aligned}
\tilde{H}_{\mathcal{D},\beta} \quad & (z_k, \alpha(x,g')\mathrm{diag}(\mathfrak{d}_1^{-1}\mathfrak{d}_1^{-1}, 1, \bar{\mathfrak{d}}_1, \bar{\mathfrak{d}}_1, 1)) \\
&= \sum_{a\in(\hat{\mathcal{O}}/x_pM_{\mathcal{D}})^\times} \xi^c\tau(a)H_{\mathcal{D},\beta}(\alpha(x,g' \begin{pmatrix} a^{-1} & \\ & \bar{a} \end{pmatrix})\mathrm{diag}(\mathfrak{d}_1^{-1}\mathfrak{d}_1^{-1}, 1, \bar{\mathfrak{d}}_1, \bar{\mathfrak{d}}_1, 1)) \\
&= C_{\mathcal{D}}(\beta, r, u)\sum_n\sum_v\sum_b\prod_v f_{-b,v}(\begin{pmatrix} 1 & \\ -n & 1 \end{pmatrix} g_v\eta)\omega_{\beta\mathfrak{d}^{-1},v}(h, \begin{pmatrix} 1 & \\ n & 1 \end{pmatrix}' g_v \begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix})\tilde{\Phi}_{\mathcal{D},\beta,u}(v\mathfrak{d}_1) \\
&= |\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\tau(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)^{-1}C_{\mathcal{D}}(\beta,r,u)\sum_n\sum_v\sum_b\prod_v f_v(\begin{pmatrix} 1 & \\ -n & 1 \end{pmatrix} g_v\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} \tilde{\delta}_{\mathcal{K}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} \end{pmatrix}\begin{pmatrix} \bar{\mathfrak{d}}_1 & \\ & \mathfrak{d}_1^{-1} \end{pmatrix}) \\
&\quad \times\omega_{\beta,v}(\eta \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} g_v^\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} \tilde{\delta_{\mathcal{K}}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} \end{pmatrix})\Phi_{\mathcal{D},\beta,u}(v) \\
&= |\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\tau(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)^{-1}C_{\mathcal{D}}(\beta,r,u)\sum_b \rho(\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} \tilde{\delta_{\mathcal{K}}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} \end{pmatrix})\Theta_{\mathcal{D},\beta}(h,g;u) \\
&\quad \times\rho(\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} \tilde{\delta}_{\mathcal{K}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} \end{pmatrix}\begin{pmatrix} \bar{\mathfrak{d}}_1 & \\ & \mathfrak{d}_1^{-1} \end{pmatrix})\mathcal{E}_{\mathcal{D}}(g)
\end{aligned}$$

The last step is because Θ_β is an automorphic form.

Now let $x = \begin{pmatrix} hu\mathfrak{d}_1^{-1} & \\ & {}^t\bar{hu\mathfrak{d}_1}^{-1} \end{pmatrix}$, then:

$$< \tilde{g}^{(m)}_{\mathcal{D},\beta}(-,x), \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_f \begin{pmatrix} 1 & \\ & M_{\mathcal{D}}^2\prod_{v|p}(p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f)\tilde{f}^c >_{\Gamma_{\mathcal{D}}}$$

$$= |\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\tau(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)^{-1} < \sum_b \rho(\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} \end{pmatrix})(\Theta_{\mathcal{D},\beta}\otimes\xi)(h,-,u)\cdot$$

$$\rho(\eta \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1 & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1}\mathfrak{d}_1{}^{-1} \end{pmatrix})\mathcal{E}_{\mathcal{D}}, \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_f \begin{pmatrix} 1 & \\ & M_{\mathcal{D}}^2\prod_{v|p}(p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f)\tilde{f}^c >$$

$$= |\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\tau(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)^{-1} < \sum_b \rho(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} \end{pmatrix})(\Theta_{\mathcal{D},\beta}\otimes\xi)(h,-;u)\cdot$$

$$\rho(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1 & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1}\mathfrak{d}_1{}^{-1} \end{pmatrix})\mathcal{E}_{\mathcal{D}}, \rho(\begin{pmatrix} 1 & \\ & M_{\mathcal{D}}^2\prod_{v|p}(p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f)\tilde{f}^c >$$

$$= |\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\tau(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)^{-1} < \sum_b \rho(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}} & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1} \end{pmatrix})(\Theta_{\mathcal{D},\beta}\otimes\xi)(h,-;u)\cdot$$

$$\rho(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1 & \\ & \bar{\tilde{\delta}}_{\mathcal{K}}^{-1}\mathfrak{d}_1{}^{-1} \end{pmatrix})\mathcal{E}_{\mathcal{D}}, \rho(\begin{pmatrix} 1 & \\ & M_{\mathcal{D}}^2\prod_{v|p}(p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f)\tilde{f}^c >$$

$$= |\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\tau(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)^{-1} < A'_\beta(h,-;u)\cdot\mathcal{E}_{\mathcal{D}},$$

$$\rho(\begin{pmatrix} 1 & \\ & \tilde{D}_{\mathcal{K}}\mathfrak{d} \end{pmatrix} \begin{pmatrix} 1 & \\ & M_{\mathcal{D}}^2\prod_{v|p}(p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f)\tilde{f}^c >_{\Gamma_0(M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}\prod_{v|p}p^{r_v})}$$

$$= |\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\xi(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1) < A'_\beta(h,-;u)\cdot\mathcal{E}_{\mathcal{D}},$$

$$\rho(\begin{pmatrix} 1 & \\ & \tilde{D}_{\mathcal{K}}\mathfrak{d} \end{pmatrix} \begin{pmatrix} 1 & \\ & M_{\mathcal{D}}^2\prod_{v|p}(p^{r_v})_v \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}_f)\tilde{f}^c >_{\Gamma_0(M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}\prod_{v|p}p^{r_v})}$$

where $A'_\beta = (\rho(\begin{pmatrix} \mathfrak{d}_1^{-1} & \\ & \bar{\mathfrak{d}}_1 \end{pmatrix})(\Theta_{\mathcal{D},\beta}\otimes\xi)(h,-;u))$. Now for those v_i for $i \in I_1$, by definition we have $v_i\bar{v}_i$ is an ideal of F generated by a totally positive global element. Let us take such a generator q_i. Also we take a representative $\{b_j\}_j$ of the coset:

$$\{b: \text{ totally positive units in } \mathcal{O}_F^\times\}/\{c\bar{c} \text{ for } c \text{ a unit in } \mathcal{O}_{\mathcal{K}}^\times\}$$

Then we for

$$A' := \sum_{i,j,k} \Theta_{\mathcal{D},\beta_{ijk}} \otimes \xi(h,-,u)$$

where $\beta_{ijk} = \begin{pmatrix} b_j & \\ & q_i b_k \end{pmatrix}$.

Remark 11.9.1. *The reason for introducing such b_j is to make sure that the A' satisfy $(*)$ in the section for neben typus (see also [Hida91] on top of page 324 for the q-expansion) and can be identified later with some theta functions on GL_2.*

Definition 11.9.2. *Let $\alpha_{\xi\lambda}$ be the operator defined in the section for neben-typus, we define*

$$A := \alpha_{\xi\lambda} A'.$$

Then we are in a position to state our formula for the Fourier coefficients for Klingen Eisenstein series. Before this let us do some normalizations:

$$\begin{aligned} C_{\mathcal{D}}(\beta, r, u) = & \frac{(2\pi i)^{2kd}(2/\pi)^d |\delta_{\mathcal{K}}|_{\mathcal{K}}^{-1/2} |\delta_F|_F^{-1} \chi\bar{\xi}(\det ru)|\det ru|_{\mathcal{K}}^{-\frac{k}{2}+2} \prod_{v_i|\infty}(\det \beta_{v_i}^{k-2})^d}{(\prod_{j=0}^{1}(k-1-j)!)^d L^{\Sigma}(k-j, \bar{\chi}\xi'\chi_{\mathcal{K}}^j)} \\ & \times \Psi_p(-1)\bar{\chi}_p\xi'_p(\det\beta)|\det\beta|_p^{k-2} \prod_{v|p} \mathfrak{g}^2(\chi_v\bar{\xi}'_v, p)\bar{\chi}_p\xi_p(p^{2u_p})p^{u_p(1-2k)} \end{aligned} \tag{11.27}$$

$$B_{\mathfrak{D},1} = \frac{(k-3)!^d L^{\Sigma}(k-2, \bar{\chi}\xi')}{(-2)^d (2\pi i)^{(k-2)d} \prod_{v|p}(\mathfrak{g}(\chi_v\bar{\xi}'_v, p))\bar{\chi}_v\xi'_v(p^{u_p})p^{(2-k)u_p}}$$

$$B_{\mathfrak{D},2} := \frac{|M_{\mathcal{D}}^2|_{\mathcal{K}}^{\frac{\kappa}{2}} 2^{3d} i^{-2d} |\delta_{\mathcal{K}}|_{\mathcal{K}}^{-1/2} |\delta_F|_F^{-1} \prod_{v|\Sigma/\{v|p\}} \chi_v\bar{\xi}_v^c(y_v\delta_{\mathcal{K}})\mathfrak{g}(\bar{\chi}_v\xi_v^c, y_v\delta_{\mathcal{K}})|y_v\delta_{\mathcal{K}}|_{\mathcal{K},v}^{2-k}}{\bar{\chi}_p\xi_p^c(M_{\mathcal{D}})\mathfrak{g}(\xi_p^c, x_p)}$$

$$B_{\mathfrak{D}}(\beta, r, u) := \frac{\psi\xi(\det ru)|\det ru|_{\mathcal{K}}^{\frac{\kappa}{2}+2} \bar{\chi}_p\xi'_p(\det\beta)|\det\beta|_p^k \det\beta^{k-2}}{\prod_{v|\Sigma/\{v|p\}} \bar{\chi}_v\xi'_v(\varpi_v^{e_v})q_v^{e_v(2-k)}\mathfrak{g}(\chi_v\bar{\xi}'_v)}.$$

thus

$$B_{\mathfrak{D}}^{(2)} C_{\mathcal{D}}(\beta, r, u) = B_{\mathfrak{D}}(\beta, r, u) B_{\mathfrak{D},1} B_{\mathfrak{D},2}.$$

Proposition 11.9.1. *With the assumptions at the beginning of this section. Let $\beta \in S_2(F), \beta > 0$,*

u, h, x *as before, then:*

$$\frac{C_{\mathcal{D}}(\beta, x)}{2^{-3d}(2i)^{(\kappa-1)d}S(f) < f, \rho(\begin{pmatrix} & -1 \\ N & \end{pmatrix})\tilde{f}^c >_{GL_2,\Gamma_0(N)}}$$

$$= \quad |\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\xi(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)2^{u_F}[\mathcal{O}_{\mathcal{K}}^{\times} : \mathcal{O}_F^{\times}]^{-1}B_{\mathcal{D}}(\beta, h, u)$$

$$\times \frac{< B_{\mathcal{D},1}\mathcal{E}_{\mathcal{D}}(-)B_{\mathcal{D},2}A_{\beta}(h, -; u), \rho(\begin{pmatrix} & -1 \\ M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}\mathfrak{d} & \end{pmatrix}\begin{pmatrix} & -1 \\ \prod_{v|p}p^{r_v} & \end{pmatrix})\tilde{f}^c >_{GL_2,\Gamma_0(\prod_{v|p}p^{r_v}M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}})}}{< f, \rho(\begin{pmatrix} & -1 \\ \prod_{v|p}p^{r_v} & \end{pmatrix}\begin{pmatrix} & -1 \\ M & \end{pmatrix})\tilde{f}^c >_{GL_2,\Gamma_{\mathcal{D}}}}$$

Now let us make some choices for the u and β and record some formulas for the Theta kernel functions:

Let $\gamma_0 \in GL_2(\mathbf{A}_{\mathcal{K},f})$ be such that $\gamma_{0,v} = (\eta, 1)$ for $v|p$ and $\gamma_{0,v} = 1$ otherwise. For $1 \leq i \leq h_1$, we let $\beta_i := \begin{pmatrix} 1 & \\ & q_i \end{pmatrix}$, and $u_i = \gamma_0 \begin{pmatrix} 1 & \\ & a_i^{-1} \end{pmatrix}$. Then β_i, u_i satisfy the assumptions at the beginning of the section.

for $v|p$

$$\Phi_{\beta_{ijk},\xi_v^c\mu_{1,v}^{-1},x_v,\gamma_{0,p}}(x) = \begin{cases} \bar{\xi}_{v,2}\mu_{1,v}^{-1}(b_kq_ix_2'')g(\xi_{v,2})\bar{\xi}_{v,1}\mu_{1,v}^{-1}(x_1'b_j)g(\xi_{v,1}) & x_1 = (x_1', x_2'') \in \mathbb{Z}_p^{\times} \times \mathbb{Z}_p \\ & x_2 = (x_2', x_2'') \in \mathbb{Z}_p^{\times} \times \mathbb{Z}_p \\ 0 & otherwise \end{cases} \tag{11.28}$$

for $s = 1, \ldots, d$

$$\omega_{\beta_{ijk}}(g_{\infty_s})\Phi_{\beta_{ijk},\infty_s}(x) = e(Nm(x_1)b_jw)e(Nm(x_2)b_kq_iw)j(g_{\infty_s}, i)^{-2}$$

also if $v \nmid p$,

$$\Phi_{\beta_{ijk},1,M_{\mathcal{D}},1}(x) = |D_v|^{-1}\lambda_v(-1)|M_{\mathcal{D}}^2|_v^{-1}\begin{cases} 1 - \frac{1}{q_v'}, & x_1 \in \frac{M_{\mathcal{D}}\mathcal{O}_v}{\mathfrak{d}_1}, x_2 \in \frac{\mathcal{O}_v}{\mathfrak{d}_1} \\ -\frac{1}{q_v'}, & x_2 \in \frac{\mathcal{O}_v}{\mathfrak{d}_1}, x_1 \in \frac{M_{\mathcal{D}}}{\varpi_v\mathfrak{d}_1}\mathcal{O}_v^{\times} \\ 0, & otherwise. \end{cases} \tag{11.29}$$

for v non split

$$\Phi_{\beta_{ijk},1,M_{\mathcal{D}},1}(x) = |D_v|^{-1}\lambda_v(-1)|M_{\mathcal{D}}|_v^{-1}\begin{cases}(1-\frac{1}{q_v})^2, & x_1 \in M_{\mathcal{D}}\mathcal{O}_v, x_2 \in \mathcal{O}_v. \\ -\frac{1}{q_v}(1-\frac{1}{q_v}), & x_2 \in \mathcal{O}_v, x_1 \in (\frac{M_{\mathcal{D}}}{\varpi_{v'}}\mathcal{O}_{F_v}^\times \times \mathcal{O}_{F_v}^\times) \\ & or(\mathcal{O}_{F_v}^\times \times \frac{M_{\mathcal{D}}}{\varpi_{v'}}\mathcal{O}_{F_v}^\times) \\ \frac{1}{q_v^2}, & x_2 \in \mathcal{O}_v, x_1 \in \frac{M_{\mathcal{D}}}{\varpi_v}\mathcal{O}_v^\times \times \frac{M_{\mathcal{D}}}{\varpi_v}\mathcal{O}_v^\times \\ 0 & otherwise.\end{cases} \quad (11.30)$$

for v split

if $v = v_i$:

$$\Phi_{0,u_i,v}(x) = \begin{cases}1 & x_1 \in \mathcal{O}_{\mathcal{K},v}, x_2 \in a_i^{-1}\mathcal{O}_{\mathcal{K},v} \\ 0 & otherwise.\end{cases} \quad (11.31)$$

11.10 Identify with Rankin Serberg Convolutions

From now on we assume that all characters are unramified outside p.

Let $\alpha \in GL_2(\mathbb{A}_{F,f})$ be defined by $a_v = \begin{pmatrix} & -1 \\ M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}} & \end{pmatrix}$ of $v|\sum \setminus \{p\}$ and $a_v = 1$ otherwise. For $m \geq 0$ let $b_m \in GL_2(\mathbb{A}_{F,f})$ be defined by $b_{m,v} = \begin{pmatrix} & -1 \\ p^m & \end{pmatrix}$ and $b_{m,v} = 1$ if $v \not| p$ Then

$$\rho(\alpha)\mathcal{E}_{\mathcal{D}} = E(\mathcal{F}'_{\mathcal{D}}, z_\kappa; \gamma_\infty).$$

where $\mathcal{F}(z,g) := \mathcal{F}_{\mathcal{D}}(z, g\alpha_f^{-1}) \in I_1(\tau/\lambda)$. It follows that $\mathcal{F}'_{\mathcal{D}}(z,g)$ is supported on

$$B_1(\mathbb{A}_F)\eta K_{1,\infty}^+ N_{B_1}(\hat{\mathcal{O}}_F)\alpha = B_1(\mathbb{A}_F)K_{1,\infty}^+ K_1(p^{u_p}M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}})$$

and that for $g = bk_\infty k_f$ in the support we have :

$$\mathcal{F}'_{\mathcal{D}}(z,g) = (M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}})^{d(\kappa/2-1)}\tau\bar{\lambda}(d_b d_{k_f})|a_b/d_b|_{\mathbb{A}_F}^{z+1/2}J_1(k_\infty, i)^{2-\kappa}.$$

Now we recall the notion of Rankin Selberg convolution for Hilbert modular forms, following [Hida91]. Given two Hilbert automorphic forms f and g (as functions on $GL_2(\mathbb{A}_F)$). For simplicity, we assume that both f and g have unitary central characters χ and ξ, having parallel weight k and κ such that $k > \kappa$. Letting $\tau = \chi/\xi$, as in [Hida91] p341(4.5), consider the following integral:

$$Z(s, f^c, g, \tau) = \int_{F_{\mathbb{A}_+}^{\times}/F_+} \int_{F_{\mathbb{A}}/F} \Phi(f^c, g)(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) dx \tau(y) |y|_{\mathbb{A}}^s d^{\times} y$$

where $\Phi(f^c, g)(x) = \overline{f^u(x)} g^u(x) |j(x_\infty, z_0)^{-k-\kappa}|$, $f^u(x) = D^{-1} f(x) j(x_\infty, z_0)^k$ and $g^u(x) = D^{-1} g(x) j(x_\infty, z_0)^\kappa$, D is the discriminant of $F/\mathbb{Q}$. Note that there are miner differences between the notations here and there, and the m and μ there are 0 in our case. Then:

$$Z(s, f^c, g, \tau) = D^{(1+2s)/2} \tau(\mathfrak{d})^{-1} (4\pi)^{-d(s+(k+\kappa)/2)} \Gamma(s + k/2 + \kappa/2)^d D(s, f^c, g, \tau)$$

By (4.7) in *loc.cit*,

$$\begin{aligned} Z(s, f, g, \tau) =& D^{-2} \int_{X_0} \bar{f} g(x) \\ & \times \mathscr{E}(x; s+1) j(x_\infty, z_0)^{\kappa - k} |j(x_\infty, z_0)^{k-\kappa}| dx, \end{aligned}$$

where

$$\mathscr{E}(x; s) = \sum_{\gamma} \tau(\gamma x) \eta(\gamma x)^s |j(\gamma, x_\infty(z_0))^{k-\kappa}| j(\gamma, x_\infty(z_0))|^{\kappa - k}.$$

Suppose $h \in S_2(p^{r_p} M_{\mathcal{D}}^2 \tilde{D}_{\mathcal{K}})$ such that the neben typus of $\mathcal{E}_{\mathcal{D}}.h$ is the same as f (this satisfies [Hida91]4.5), then:

$$
\begin{aligned}
&B_{\mathcal{D},1} < \mathcal{E}_{\mathcal{D}} \cdot \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_{v\nmid,\infty} \begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d}\prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f)h, \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_f \begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d}\prod_{v\nmid\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f \tilde{f}^c > \\
&= B_{\mathcal{D},1} < \rho(\begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d}\prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f^{-1} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_{v\nmid p,\infty}^{-1} \mathcal{E}_{\mathcal{D}} \cdot h, \rho(\begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d}\prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f^{-1} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_{v\nmid p,\infty}^{-1} \\
&\quad \times \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_f \begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d}\prod_{v\nmid\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f)\tilde{f}^c > \\
&= B_{\mathcal{D},1} < \rho(\begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d}\prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_{v\nmid p,\infty})\mathcal{E}_{\mathcal{D}} \cdot h, \rho(\begin{pmatrix} & -1 \\ p^{r_p} & \end{pmatrix}_p)\tilde{f}^c > \\
&= B_{\mathcal{D},1} < \rho(\begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d}\prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f)\mathcal{E}_{\mathcal{D}} \cdot h, (\chi_p(p)a_p(\tilde{f}_p^c))^{r_p-u_p}\rho(\begin{pmatrix} & -1 \\ p^{u_p} & \end{pmatrix}_p)\tilde{f}^c > \\
&= |M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}|_F^{\frac{\kappa}{2}-1}(\bar{\chi}_p(p)a(f_p))^{r_p-u_p}B_{\mathcal{D},1}(4\pi)^{(1-k)d}\Gamma(k-1)^d D(\rho(\begin{pmatrix} & -1 \\ p^{u_p} & \end{pmatrix}_p)\tilde{f}^c, h; k-1) \\
&= |M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}|_F^{\frac{\kappa}{2}-1}(\bar{\chi}_p(p)a_p(f_p))^{r_p-u_p}B_{\mathcal{D},1}c(\bar{f})(\xi_p(p)a_p(h_p))^{u_p-r}(4\pi)^{(1-k)d}\Gamma(k-1)^d \\
&\times L^{\Sigma}(k-2,\bar{\chi}\xi')^{-1}L(f_1^c \times h, k-1)
\end{aligned}
\tag{11.32}
$$

Lemma 11.10.1. *Assumptions are as above. Suppose $h \in S_2(p^{r_p}M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}})$ is a normalized eigen form on $GL_2(\mathbb{A}_F)$ then*

$$
\begin{aligned}
&< B_{\mathcal{D},1}\mathcal{E}_{\mathcal{D}} \cdot \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_{v\nmid p,\infty} \begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d}\prod_{v\nmid p,\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f)h, \rho(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_f \begin{pmatrix} \tilde{D}_{\mathcal{K}}\mathfrak{d}\prod_{v\nmid\infty} x_v\bar{x}_v & \\ & 1 \end{pmatrix}_f \tilde{f}^c > \\
&\qquad = B_{\mathcal{D},3}L(f_1^c \times h, k-1)
\end{aligned}
$$

where:

$$B_{\mathcal{D},3} = |M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}|_F^{\frac{\kappa}{2}-1}(\bar{\chi}_p(p)a_p(f_p))^{r_p-u_p}B_{\mathfrak{D},1}c(\bar{f})(\xi_p(p)a_p(h_p))^{u_p-r}(4\pi)^{(1-k)d}\Gamma(k-1)^d \times L^{\Sigma}(k-2,\bar{\chi}\xi')^{-1}$$

Chapter 12

p-adic Interpolations

12.1 p-adic Eisenstein datum

As in [SU] Chapter 12 we define the p-adic Eisenstein datum to be $\mathcal{D} = (A, \mathbb{I}, \mathbf{f}, \psi, \Sigma)$ consists of:

- The integer ring A of a finite extension of $\mathbb{Q}_p$.
- $\mathbb{I}$ a finite integral domain over $\Lambda_{W,A}$.
- A nearly ordinary $\mathbb{I}$-adic form $\mathbf{f}$ which is new at all $v \nmid p$ and has the tame part of the character $\equiv 1$.
- A finite order Hecke character ψ of $\mathbb{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ and cond$\psi|p$ and $\psi|_{\mathbb{A}_F^\times} \equiv 1$.
- A finite set Σ of primes containing all primes dividing $N\delta_{\mathcal{K}}$

Remark 12.1.1. *For simplicity we have assumed ψ is unramified outside p and that the $\boldsymbol{\chi}_{\mathbf{f}}$ and ξ in [SU]12.1 are trivial.*

Recall also that we have defined in section 7.1 the maps α and β. Let $\boldsymbol{\psi} := \alpha \circ \omega\psi\Psi_{\mathcal{K}}^{-1}$ and $\boldsymbol{\xi} := \beta \circ \Psi_{\mathcal{K}}$. For $\phi \in \mathcal{X}^a$ we define:

$$\psi_\phi(x) := \prod_{\sigma \in \Sigma_{F,\infty}} x_\infty^{-\kappa_\phi} x_{v_\sigma}^{\kappa_\phi} (\phi \circ \boldsymbol{\psi}(x)|.|_F^{-\kappa_\phi}.$$

Here v_σ is the p-adic place corresponding to σ under $\iota : \mathbb{C} \simeq \mathbb{C}_p$. We also define:

$$\xi_\phi := \phi \circ \boldsymbol{\xi}.$$

We always assume (irred) and (dist) holds for our Hilbert modular form f or Hida family $\mathbf{f}$.

12.2 Interpolation

12.2.1 congruence module and the canonical period

We denote $\mathbf{T}_\kappa^{ord}(Mp^r,\chi;R)$ ($\mathbf{T}_\kappa^{0,ord}(Mp^r,\chi;R)$)be the R-sub-algebra of $End_R(M_\kappa^{ord}(Mp^r,\chi;R))$ (respectively, $End_R(S_\kappa^{ord}(Mp^r,\chi;R))$) generated by the Hecke operators T_v. For any $f\in S_\kappa^{ord}(Mp^r,\chi;R)$ is a p-stabilized eigenform and F the fractional field of R. Then we have $1_f\in \mathbf{T}_\kappa^{0,ord}(Mp^r,\chi;R)\otimes_R F$ the idempotent associated to f.

Suppose $\mathbf{f}\in\mathcal{M}^{ord}(M,\chi;\mathbb{I})$ is an ordinary $\mathbb{I}$-adic cuspidal newform. Then as above $\mathbb{T}^{ord,0}(M,\chi;\mathbb{I})\otimes F_{\mathbb{I}}\simeq \mathbf{T}'\otimes F_{\mathbb{I}}$, $F_{\mathbb{I}}$ beign the fraction field of $\mathbb{I}$ where projection onto the second factor gives the eigenvalues for the actions on $\mathbf{f}$. $1_{\mathbf{f}}$ be the idempotent corresponding to the second factor. Then for an $\mathbf{g}\in S^{ord}(M,\chi;\mathbb{I})\otimes_{\mathbb{I}}F_{\mathbb{I}}$, $1_{\mathbf{f}}\mathbf{g}=c\mathbf{f}$ for some $c\in F_{\mathbb{I}}$

Suppose $(irred)_f$ and $(dist)_f$ hold for $\mathbf{f}$ and that the localization of the Hecke algebra at $m_{\mathbf{f}}$ satisfies the Gorenstein property, then $T^{ord,0}(M,\chi_f;\mathbb{I})_{\mathfrak{m}_f}$ is a Gorenstein R-algebra, So $T^{ord,0}(M,\chi_f;\mathbb{I})\cap(0\otimes F_{\mathbb{I}})$ is a rank one $\mathbb{I}$-module. We let ℓ_f be a generator; so $\ell_f=\eta_f 1_f$ for some $\eta_f\in R$.

Definition 12.2.1. *For a classical point f_ϕ of $\mathbf{f}$ the canonical period of f_ϕ is defined by*

$$\Omega_{can}:=< f_\phi, f_\phi^c >_{\Gamma_0(N)}/\eta_{f_\phi}.$$

Remark 12.2.1. *This canonical period is not quite canonical since it depends on the generator $\ell_{\mathbf{f}}$.*

Now we define $\mathcal{M}_{\mathcal{X}}(B,\Lambda_{\mathcal{D}})$ to be the space of formal q expansions which when specializing to $\phi\in\mathcal{X}$ is a classical modular form with the neben typus determined by ϕ. Lemma 12.2.4 in [SU] is true as well for the Hilbert modular forms: (the character θ there is assumed to be trivial in our situation.)

Lemma 12.2.1. *There exists an idempotent $e\in End_{\Lambda_{\boldsymbol{D}}}(\mathcal{M}_{\mathcal{X}}(B;\Lambda_{\boldsymbol{D}}))$ such that for any $\mathbf{g}\in\mathcal{M}_{\mathcal{X}}(B;\Lambda_{\boldsymbol{D}}),(e\mathbf{g})_\phi=e\mathbf{g}_\phi\in M_{\kappa_\phi}^{ord}(Bp_\phi^t,\omega^{\kappa_\phi-2}\chi_\phi;\phi(\Lambda_{\boldsymbol{D}}))$ for all $\phi\in\mathcal{X}$*

Lemma 12.2.2. *Let $\mathbf{f}\in S^{ord}(M,\chi_{\mathbf{f}};\mathbb{I})$ be an ordinary newform. Let R be any integral extension of $\mathbb{I}$. Let $\mathbf{g}\in\mathcal{M}^{ord}(M,\chi;\mathbb{I})\otimes_{\mathbb{I}}R$. Suppose also $(irred)_{\mathbf{f}}$ and $(dist)_{\mathbf{f}}$ hold, then there exists an element $\mathcal{N}_{\mathbf{g}}\in R$ such that for any sufficiently regular arithmetic weight ϕ.*

Proof. The lemma follows in the same way as lemma 12.2.7 of [SU]. □

12.3 p-adic L-functions

Theorem 12.3.1. *Let $A, \mathbb{I}, \mathbf{f}, \xi$, and Σ as above. Suppose that there exists a finite A-valued idele class character ψ of $\mathbb{A}_{\mathcal{K}}^\times$ such that $\psi|_{\mathbb{A}_F^\times} = \chi_{\mathbf{f}}$ and ψ is unramified outside Σ.*
(i)There exists $\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi} \in F_{\mathbb{I}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{K}}$ such that for any $\phi \in \mathcal{X}_{\mathbb{I}_{\mathcal{K}},A}(\mathbf{f},\psi,\xi)$,(so k_ϕ is a parallel weight) $\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi} \in F_{\mathbb{I}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{K}}$ is finite at ϕ and

$$\begin{aligned}&\phi(\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi})\\ &= \prod_{v|p}\mu_{1,v,\phi}(p)^{-ord_v(Nm(\mathfrak{f}_{\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi}))}\frac{((k_\phi-2)!)^{2d}\mathfrak{g}(\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi)Nm(\mathfrak{f}_{\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi}\delta_{\mathcal{K}})^{k_\phi-2}L^\Sigma_{\mathcal{K}}(\mathbf{f}_\phi,\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi,k_\phi-1)}{(-2\pi i)^{2d(k_\phi-1)}2^{-3d}(2i)^{d(k_\phi+1)}S(\mathbf{f}_\phi)<\mathbf{f}_\phi,\mathbf{f}^c_\phi>}.\end{aligned}$$

(ii) Suppose that the localization of the Hecke algebra at $\mathfrak{m}_f$ is Gorenstein. Then There exists $\tilde{\mathcal{L}}^\Sigma_{\mathbf{f},\mathcal{K},\xi} \in F_{\mathbb{I}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{K}}$ such that for any $\phi \in \mathcal{X}_{\mathbb{I}_{\mathcal{K}},A}(\mathbf{f},\psi,\xi)$,(so k_ϕ is a parallel weight) $\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi} \in F_{\mathbb{I}} \otimes_{\mathbb{I}} \mathbb{I}_{\mathcal{K}}$ is finite at ϕ and

$$\begin{aligned}&\phi(\mathcal{L}^\Sigma_{\mathbf{f},\mathcal{K},\xi})\\ &= \prod_{v|p}\mu_{1,v,\phi}(p)^{-ord_v(Nm(\mathfrak{f}_{\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi}))}\frac{((k_\phi-2)!)^{2d}\mathfrak{g}(\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi)Nm(\mathfrak{f}_{\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi}\delta_{\mathcal{K}})^{k_\phi-2}L^\Sigma_{\mathcal{K}}(\mathbf{f}_\phi,\bar{\chi}_{\mathbf{f}_\phi}\xi_\phi,k_\phi-1)}{(-2\pi i)^{2d(k_\phi-1)}\Omega_{can}}.\end{aligned}$$

Recall that the $\mu_{1,v}$ are defined by $\pi_v \simeq \pi(\mu_{1,v},\mu_{2,v})$ and $\mu_{1,v}(p)$ has lower p-adic valuation than $\mu_{2,v}(p)$.

Proof. See [SU]12.3.1. The point is the Fourier coefficients of the normalized Siegel Eisenstein series constructed in the last chapter are elements in $\Lambda_{\mathcal{D}}$ and thus the fourier jacobi coefficients (the $g^{(2)}_{\mathcal{D},\beta}(-,x)$)'s there) are elements in $\mathcal{M}_{\mathcal{X}}(B,\Lambda_{\mathcal{D}})$. A difference is that: the Fourier Jacobi coefficients are only forms on $U(1,1)$, which we do not know how to compare the unitary group inner product with the GL_2 unless it satisfies $(*)$ as defined in the section for neben typus. So we use

$$\sum_j g^{(2)}_{\mathcal{D},\beta}(\begin{pmatrix} b_j^{-1} & \\ & 1\end{pmatrix} g \begin{pmatrix} b_j & \\ & 1\end{pmatrix})\varepsilon'(\begin{pmatrix} b_j & \\ & 1\end{pmatrix})$$

instead, where b_j's are defined in the last chapter and ε' is some neben character. Then one can apply the constructions in the last section. ☐

Remark 12.3.1 (Hida91)**.** *also constructed a full dimensional p-adic L-functions for Hilbert modular Hida families. In fact his p-adic L-function corresponds to our $\tilde{\mathcal{L}}$ except for local Euler factors at Σ. Our interpolation points are not quite the same as his. In fact he used the differential operators to* *get the whole family while we instead allowed more general neben typus at p. (Recall that he used the Rankin-Selberg method and required the difference of the p-parts of the neben typus of* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *comes from a global character.) Hida is able to interpolate more general critical values. In particular, the points* ϕ_0 *corresponding to the special value* $L(f_2, 1)$ *where* f_2 *is the element in* $\mathbf{f}$ *with parallel weight* 2 *and trivial neben typus is an interpolation point. Our* $\tilde{\mathcal{L}}^{\Sigma}_{\boldsymbol{f},\mathcal{K},1}$ *coincides with his along a subfamily containing the cyclotomic* 1*-dimensional family containing* ϕ_0*. This is very useful in proving some characteristic* 0 *results for Selmer groups.*

We also have the Σ primitive p-adic L-functions $\tilde{L}^{\Sigma}_{f,\mathcal{K},\xi}$ and $L^{\Sigma}_{f,\mathcal{K},\xi}$ for a single f by specializing the one for $\mathbf{f}$ to f. (See [SU]12.3.2)

12.3.1 connections with anticyclotomic p-adic L-functions

Let $\beta : \Lambda_{\mathcal{K},A} \to \Lambda^{-}_{\mathcal{K},A}$ be the homomorphism induced by the canonical projection $\Gamma_{\mathcal{K}} \to \Gamma^{-}_{\mathcal{K}}$. For A reduced, β extends to $F_A \otimes_A \Lambda_{\mathcal{K},A} \to F_A \otimes_A \Lambda^{-}_{\mathcal{K},A}$, F_A the ring of fractions of A.
Now for any A and $f \in S^{ord}_2(Mp^t, \chi; A)$ such that (irred) and (dist) are satisfied, we define the anticyclotomic p-adic L-function:

$$\mathcal{L}^{\Sigma,-}_{f,\mathcal{K},\xi} := \beta(\mathcal{L}^{\Sigma}_{f,\mathcal{K},\xi}) \in \Lambda^{-}_{\mathcal{K},A}$$

and

$$\tilde{\mathcal{L}}^{\Sigma,-}_{f,\mathcal{K},\xi} := \beta(\tilde{\mathcal{L}}^{\Sigma}_{f,\mathcal{K},\xi}) \in \Lambda^{-}_{\mathcal{K},A} \otimes_A F_A$$

For $v|p$ we can further specialize $\gamma_{v'} = 1$ for all $v' \neq v$ to get $\mathcal{L}^{\Sigma,-}_{f,\mathcal{K},\xi,v}$.

We define two notions concerning the anticyclotomic p-adic L-function which would be useful.

Definition 12.3.1. *For some* $v|p$*, writing* $\tilde{\mathcal{L}}^{\Sigma,-}_{f,\mathcal{K},\xi,v} = \tilde{a}_0 + \tilde{a}_1(\gamma_{-,v} - 1) + \cdots, \tilde{a}_i \in F_A$ *(the fraction field), and* $\mathcal{L}^{\Sigma,-}_{f,\mathcal{K},\xi} = a_0 + a_1(\gamma_{-,v} - 1) + \cdots, a_i \in A$*, then we say* f *satisfies*
(NV1) if at least one of the $\tilde{a}_i$ *is non-zero.*
(NV2) if at least one of the a_i *is a p-adic unit.*

We denote f_2 to be the ordinary form in the family $\boldsymbol{f}$ of parallel weight 2 and trivial neben typus and characters. Also let ϕ_0 be the arithmetic points corresponding to the special L-value $L(f_2, 1)$.

Theorem 12.3.2. *Let* $A, \mathbb{I}, \boldsymbol{f}, \xi$*, and* Σ *be as before and assume the hypotheses there and (irred) and (dist) holds for* $\boldsymbol{f}$*. For simplicity we assume that* $\xi = 1$*.*

(i) If f_2 satisfies (NV1), then $\tilde{\mathcal{L}}^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi}$ is not contained in any prime of $F_{\mathbb{I}}\otimes\mathbb{II}_{\mathcal{K}}$ passing the point ϕ_0 and of the form $QF_{\mathbb{I}}\otimes\mathbb{II}_{\mathcal{K}}$ for some height one prime $Q\subset F_{\mathbb{I}}[[\Gamma^+_{\mathcal{K}}]]$.
(ii) Assume that the localization of the Hecke algebra at the maximal ideal of $\boldsymbol{f}$ is Gorenstein. If one member of the family $\boldsymbol{f}$ satisfies (NV2), then$\mathcal{L}^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi}$ is not contained in any prime of $\mathbb{I}_{\mathcal{K}}$ of the form $Q\mathbb{I}_{\mathcal{K}}$ for some height one prime $Q\subset\mathbb{I}[[\Gamma^+_{\mathcal{K}}]]$.

Proof. Same as [SU]12.3.2. □

Now we state two theorems giving sufficient condition for that (NV1) and (NV2) to be satisfied.

Theorem 12.3.3. *([Vastal04] f is a Hilbert modular form of parallel weight 2 and trivial Neben typus and character. If the conductor of $\chi_{\mathcal{K}/F}$ and f are disjoint and the $S(1)$ defined in [Vatsal07] p123 has even number of primes, then picking any $v|p$ we have (NV1) is satisfied for f.*

Also Jeanine Van-Order constructed an anti-cyclotomic p-adic L-function $\mathcal{L}_{f,\mathcal{K},1}$. We state the following theorem of [VAN]:

Theorem 12.3.4. *(Jeanine Van-Order) Suppose the level of f_2 is $M=M^+M^-$ where M^+ and M^- are products of split and inert primes respectively. Suppose:*
(1) M^- is square free with the number of prime factors $\equiv d(\mathrm{mod}2)$;
(2)$\bar{\rho}_f$ is ramified at all $v|M^-$.
then for any $v|p$ the anti-cyclotomic μ invariant at v defined by her is 0.

In fact in her paper the result is not stated this way. First of all her formula is stated in an implicit say since she is using [YZZ]. However she informed the author that in our situation it is not hard to get the above theorem using the special value formula in [Zh04] instead. Note also that her period is not our canonical period. However the difference of the periods is a p-adic unit under the second hypothesis above. So we can relate our Σ-primitive anticylotomic L-function to hers similar to [SU]12.3.5. Thus by the argument in *loc.cit* the μ invariant of our p-adic L-function is also 0. Thus (NV2) is OK for f_2.

12.4 p-adic Eisenstein series

We state some theorems which are straight generalizations of the section 12.4 of [SU].

Theorem 12.4.1. *Assumptions as in theorem 12.3.1 (ii). Let $\mathcal{D}=(A,\mathbb{I},\boldsymbol{f},\psi,\xi,\Sigma)$ be a p-adic Eisenstein datum. Suppose that (irred) and (dist) hold. Then for each $x=diag(u,{}^t\bar{u}^{-1})\in G(\mathbb{A}^{\Sigma}_{F,f})$*

there exists a formal q-expansion

$$E_{\mathcal{D}}(x) := \sum_{\beta \in S(F), \beta \geq 0} c_{\mathcal{D}}(\beta, x) q^{\beta}$$

$c_{\mathcal{D}}(\beta, z) \in \Lambda_{\mathcal{D}}$*, with the property that for each* $\phi \in \mathcal{X}_{\mathcal{D}}^{gen}$*:*

$$E_{\mathcal{D},\phi}(x) := \sum_{\beta \in S(F), \beta \geq 0} \phi(c_{\mathcal{D}}(\beta, x)) e(Tr\beta Z)$$

is the q expansion at x for $\frac{G_{\mathcal{D}_\phi}}{\Omega_{can}}$ *with* $G_{\mathcal{D}_\phi}$ *being as in the last chapter.*

Remark 12.4.1. *There is also a* $\tilde{E}$ *version of the above theorem under the hypothesis of theorem 12.3.1 (i). We omit it here.*

Chapter 13

p-adic Properties of Fourier coefficients of $E_{\mathcal{D}}$

In this chapter, following [SU]chapter 13, using the theta correspondence between different unitary groups, we prove that certain Fourier coefficients of $E_{\mathcal{D}}$ is not divisible by certain hight one prime P.

13.1 Automorphic forms on some definite unitary groups

13.1.1 generalities

Let $\beta \in S_2(F)$, $\beta > 0$. Let H_β be the unitary group of the pairing determined by β. We write H for H_β sometimes for simplicity.

For an open compact subgroup $U \subseteq H(\mathbb{A}_{F,f})$ and any Z-algebra R we let:

$$\mathcal{A}(U, R) := \{f : H(\mathbb{A}_F) \to R : f(\gamma h k u) = f(h), \gamma \in H(F), k \in H(F_\infty), u \in U\}.$$

This is identified with the set of functions $f : H(\mathbb{A}_{F,f}) \to R$ such that $f(\gamma h u) = f(h)$ for all $\gamma \in H(F)$ and $u \in U$. For any subgroup $K \subseteq (\mathbb{A}_{F,f})$ let

$$\mathcal{A}_H(K; R) := \varinjlim_{U \supseteq K} \mathcal{A}_H(U; R),$$

13.1.2 Hecke operators.

Let $U, U' \subset H(\mathbb{A}_{F,f})$ be open compact subgroups and let $h \in H(\mathbb{A}_{F,f})$. We define a hecke operator $[UhU'] : \mathcal{A}(U, R) \to \mathcal{A}_H(U'; R)$ by

$$[U'hU]f(x) = \sum f(xh_i), U'hU = \sqcup_i h_i U.$$

We will be interested in two cases:

Case 1. The unramified case. Suppose v splits in $\mathcal{K}$. The identification $GL_2(\mathcal{K}_v) = GL_2(F_v) \times GL_2(F_v)$ yields an identification of $H(F_v)$ with $GL_2(F_v)$ via projection ont the first factor: $H(F_v) = \{(A, \beta^{-1}{}^tA^{-1}\beta) \in GL_2(\mathcal{K}_v)\}$. We let $H_v \subset H(F_v)$ be the subgroup identified with $GL_2(\hat{\mathcal{O}}_{F,v})$.

For $U = H_v U', U' \subset H(\mathbb{A}^v_{F,f})$, we write T^H_v for the Hecke operator $[Uh_vU], h_v := \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix} \in GL_2(F_v) = H(F_v)$, where ϖ_v is a uniformizer at v. (in the unramified case this does not depend on the choice of ϖ_v)

Case 2. Hecke operators at p. If $v|p$, for a positive integer n we let $I_{n,v} \subset H_v$ be the subgroup identified with the set of $g \in GL_2(Z_p)$ such that g *modulo* p^n belongs to $N_{B'}(Z/p^nZ_p)$. For $U = I_nU', U' \subset H(\mathbb{A}^{\{v\}}_{F,f})$, we write U^H_v for the Hecke operator $[Uh_pU]$. This operator respects variation in n and U' and commutes with the T^H_v's for $v \nmid p$. Let $U_p := \prod_{v|p} U_v$.

13.1.3 The nearly ordinary projector.

Let R be either a p-adic ring or of the form $R = R_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with R_0 a p-adic ring. Then for $U = \prod_{v|p} I_{n,v}U', U' \subset H(\mathbb{A}^{\{p\}}_{F,f})$,

$$e_H := \varinjlim_m U^{H,m!}_p \in End_R(\mathcal{A}_H(U; R))$$

exists and is an idempotent. By identifying $\mathbb{C}_p \simeq \mathbb{C}$, e_H is defined on $\mathcal{A}'_H := \lim_{n\to\infty} \mathcal{A}_H(\prod_{v|p} I_{n,v})$.

13.2 Applications to fourier coefficients

13.2.1 Forms on $H \times U_1$

If v splits in $\mathcal{K}$ then we view representations of $H(F_v)$ via the respective identifications of these groups with $GL_2(F_v)$ (projection onto the first factor of $GL_2(\mathcal{K}_v) = GL_2(F_v) \times GL_2(F_v)$). Let λ be a character of $\mathbb{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ such that $\lambda_\infty(z) = (z/|z|)^{-2}$ and $\lambda|_{\mathbb{A}_F^\times} = 1$. Let $(\pi, \mathcal{V}), \mathcal{V} \subseteq \mathcal{A}_H$, be an irreducible representation of $H(\mathbb{A}_{F,f})$ and let $(\sigma, \mathcal{W}), \mathcal{W} \subseteq \mathcal{A}(U_1)$, be an irreducible representation of $U_1(\mathbb{A}_{F,f})$. Let χ_π and χ_σ be their respective central characters. We assume that:

- $\chi_\sigma = \lambda\chi_\pi$;
- if v splits in $\mathcal{K}$ then $\sigma_v \simeq \pi_v \otimes \lambda_{v,1}$ as representations of $GL_2(F_v)$.

(13.1)

We also assume that we are given:

- a finite set S of primes outside of which λ is unramified
- a finite order character θ of $\mathbb{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ extending χ_π and unramified outside S

(13.2)

Let $\varphi \in \mathcal{V} \otimes \mathcal{W}$. We assume that

- if $v \notin S$ then $\varphi(hu, g) = \varphi(h, g)$for $u \in H_v$
- there is a character ε of $T_{U(1,1)}(\hat{\mathcal{O}}_F)$ and an ideal N divisible only by primes in S such that

$\varphi(h, gk) = \varepsilon(k)\varphi(h, g)$for all $k \in U_1(\hat{\mathcal{O}}_F)$ satisfying $N|c_k$. ($k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$)

(13.3)

Now suppose there is a ε' on $T_{GL_2}(\hat{\mathcal{O}}_F)$ which coincides with ε on $T_{SL_1}(\hat{\mathcal{O}}_F)$ then it makes sense to define $\alpha_{\lambda\theta,\varepsilon,\varepsilon'}\varphi$.

Lemma 13.2.1. *Suppose above assumptions are valid, then for any $v \notin S$ that splits in $\mathcal{K}$.*

$$\alpha_{\lambda\theta}(\theta_{v,1}T_v^H\varphi)(h, u) = T_v\alpha_{\lambda\theta}(\varphi)(h, u),$$

where $\theta_v = (\theta_{v,1}, \theta_{v,2})$ as a character of $\mathcal{K}_v^\times = F_v^\times \times F_v^\times$.

Proof. Same as [SU] lemma 13.2.2. □

Now we consider the p-adic ordinary idempotents e_H and e. Suppose additionally that

$$
\begin{aligned}
&\bullet \quad cond(\theta_{v,1}) = (p^r).cond(\theta_{v,2} = (p^s), r > s \text{ for any } v|p \\
&\bullet \quad p^r||N; \\
&\bullet \quad \phi((hk, g) = \theta_{p,1}^{-1}\theta_{p,2}(a_{k_2})\theta_{p,1}(detk_2)\varphi(h, g) \text{ for } k = (k_1, k_2) \in H_p,\ p^r|c_{k_1}.
\end{aligned}
\tag{13.4}
$$

Lemma 13.2.2. *Assumptions all above assumptions. Then*

$$\alpha_{\lambda\theta}(e_H\varphi)(h, u) = e\alpha_{\lambda\theta}(\varphi)(h, u).$$

Her, e is the usual ordinary idempotent action on $\varphi(h, -) \in M_2(N, \theta')$.

Proof. Completely the same as [SU]13.2.3. □

13.2.2 consequences for fourier coefficients

We return to the notation and setup of chapter 11. In particular $\mathcal{D} = (\varphi, \psi, \tau, \Sigma)$is a Eisenstein datum. Letting $\Theta_i(h, g) := \Theta_{\beta_{ijk}}(h, g; \Phi_{\mathcal{D},\beta_{ijk},u_i})$. From the definition of $\Phi_{\mathcal{D},\beta_{ijk},u_i}$:

$$
\begin{aligned}
&\bullet \quad \text{if } v \nmid \Sigma \cup \mathcal{Q}, \text{ then } \Theta_{ijk}(hu, gk) = \Theta_i(h, g) \text{ for } u \in H_{ijk,v} \text{ and } k \in U_1(\hat{\mathcal{O}}_F); \\
&\bullet \quad \text{if } v|\Sigma \cup \mathcal{Q}, v \nmid p \text{ then } \Theta_{ijk}(h, gk) = \lambda(d_k)\Theta_{ijk}(h, g) \text{ for } k \in K_{1,v}(M_{\mathcal{D}}^2\mathfrak{d}\tilde{D}_{\mathcal{K}}\textstyle\prod_{q\in\mathcal{Q}} q); \\
&\bullet \quad \text{if } v|p, \Theta_{ijk}(hu, gk) = \xi_1^{-1}\xi_2^{-1}(a_{u_2})\xi_1(\det u_2)\lambda\xi^c(d_k)\Theta_i(h, g). \\
&\qquad \text{for } u = (u_1, u_2) \in H_{ijk,v} \text{ with } p^{u_v}|c_{u_2} \text{ and } k \in U_1(\mathcal{O}_{F,v}) \text{ with } p^{u_v}|c_k.
\end{aligned}
\tag{13.5}
$$

Now we decompose each $\Theta_{ijk}(h, g)$ with respect to irreducible automorphic representations π_H of $H_{ijk}(\mathbf{A}_{F,f})$:

$$\Theta_{ijk}(h, g) = \sum_{\pi_H} \varphi_{\pi_H}^{(ijk)}(h, g).$$

Then, as in [SU] p202, using general consequences of theta correspondences in the split case we may decompose:

$$\Theta_{ijk}(h, g) = \sum_{(\pi_H,\sigma)} \varphi_{(\pi_H,\sigma)}^{(ijk)}(h, g), \varphi_{(\pi_H,\sigma)}^{(ijk)} \in \pi_H \otimes \sigma,$$

$$\sigma_v \simeq \pi_{H,v} \otimes \lambda_{v,1} \text{ as representations of } GL_2(F_v) \text{ for all } v \text{ splits in } \mathcal{K},$$

and such $\varphi_{(\pi_H,\sigma)}^{(ijk)}(h, g)$ satisfies the assumptions of the last subsection.

For $i \in I_1$, let

$$C_{\mathcal{D},ijk}(h) := \bar{\tau}(\det h) C_{\mathcal{D}}(\beta_{ijk}, diag(u_i, {}^t\bar{u}_i^{-1}); h) \in \mathcal{A}_{H_{ijk}},$$

Recall that we have defined $A := \alpha_{\xi\lambda} \sum_{ijk} A'_{\beta_{ijk}}$.

Proposition 13.2.1. *Let $\mathcal{L} = \{v_1, v_2, \cdots, v_m\}$ be a set of primes that split in $\mathcal{K}$ and do not belong to $\sum \cup \mathcal{Q}$. Let $P \in \mathbf{C}[X_1, ..., X_m]$. Let $P_{H_i} := P(\xi_{v_1,1}(\varpi_{v_1}) T_{v_1}^{H_i}, \cdots, \xi_{v_m,1}(\varpi_{v_m}) T_{v_m}^{H_i})$ and $P_1 := P(T_{v_1}, \cdots, T_{v_m})$. Then:*

$$\frac{\sum_{ijk} e_{H_i} P_{H_{ijk}} C_{\mathcal{D},ijk}(h) B_{\mathcal{D}}(\beta_i, h, u_i)^{-1} . 2^{u_F} [\mathcal{O}_{\mathcal{K}}^{\times} : \mathcal{O}_F^{\times}]}{2^{-3d}(2i)^{d(k+1)} S(f) < f, \rho(\begin{pmatrix} & N \\ -1 & \end{pmatrix}) f^c >}$$

$$= \quad \bar{\tau}(\det h) \frac{< \mathcal{E}_{\mathcal{D}} \cdot \rho(\begin{pmatrix} 1 & \\ & \mathfrak{d} \end{pmatrix}) e P_1, \rho(\begin{pmatrix} & -1 \\ M_{\mathcal{D}}^2 \tilde{D}_{\mathcal{K}} \mathfrak{d} & \end{pmatrix} \prod_{v|p} \begin{pmatrix} & -1 \\ p^{r_v} & \end{pmatrix}) f^c >_{GL_2}}{< f, \rho(\begin{pmatrix} & -1 \\ M & \end{pmatrix} \prod_{v|p} \begin{pmatrix} & -1 \\ p^{r_v} & \end{pmatrix}_f) f^c >_{GL_2}}$$

Proof. Same as [SU]13.2.4. and 13.2.5. Observe that $\rho(\begin{pmatrix} 1 & \\ & \mathfrak{d} \end{pmatrix})$ commutes with eP_1. □

13.3 p-adic properties of fourier coefficients

In this section we put the operations above in p-adic families. Let $\mathcal{D} = (A, \mathbb{I}, \boldsymbol{f}, \psi, \xi, \sum)$ be a p-adic Eisenstein datum as in the last chapter. Let $\boldsymbol{E}_{\mathcal{D}} \in \mathcal{M}_{\underline{a},ord}(K'_{\mathcal{D}}, \Lambda_{\mathcal{D}})$ be as there. For $x \in G(\mathbb{A}_{F,f})$ with $x \in Q(\mathcal{O}_{F,p})$ we let $\boldsymbol{c}_{\mathcal{D}}(\beta, x) \in \Lambda_{\mathcal{D}}$ be the β-fourier coefficient of $\boldsymbol{E}_{\mathcal{D}}$ at x. So for $\phi \in \mathcal{X}_{\mathcal{D}}^a$, $c_{\mathcal{D},\phi}(\beta, x) ; = \phi(\boldsymbol{c}_{\mathcal{D}}(\beta, x))$ is the β-fourier expansion at x of a holomorphic hermitian modular form $E_{\mathcal{D},\phi}(Z, x)$ Define

$$\varphi_{\mathcal{D},\beta,x,\phi}(h) := \chi_{\boldsymbol{f}} \psi_{\phi}^{-1} \xi_{\phi}(\det h) c_{\mathcal{D},\phi}(\beta, \begin{pmatrix} h & \\ & {}^t\bar{h}^{-1} \end{pmatrix} x)$$

This belongs to $\mathcal{A}_{H_\beta}(\phi(\Lambda_{\mathcal{D}}))$ when restricted to $H_\beta(\mathbb{A}_{F,f})$. As in [SU]13.3.1, recall that $\beta_{ijk} = \begin{pmatrix} b_j & \\ & q_i b_k \end{pmatrix}$ and $u_i = \gamma_0 \begin{pmatrix} 1 & \\ & a_i^{-1} \end{pmatrix}$. For $h \in GL_2(\mathbb{A}_{\mathcal{K},f})$ with $h_p \in GL_2(\mathcal{O}_{\mathcal{K},p})$ let

$$\varphi_{\mathcal{D},ijk} := 2^{u_F}[\mathcal{O}_{\mathcal{K}}^\times : \mathcal{O}_F^\times]\chi_f \boldsymbol{\psi}^{-1}\boldsymbol{\xi}^{-1}(\det h) c_{\mathcal{D}}(\beta_{ijk}, \begin{pmatrix} hu_i\mathfrak{d}_1^{-1} & \\ & {}^t\bar{h}^{-1}\,{}^t\bar{u}_i\bar{\mathfrak{d}}_1 \end{pmatrix}) B_{\mathcal{D}}(\beta_{ijk}, h, u_i)^{-1} \in \Lambda_{\mathcal{D}}.$$

(Note that by our choices $B_{\mathcal{D}}(\beta_{ijk}, h, u_i)^{-1}$ moves as a unit in $\Lambda_{\mathcal{D}}$.)

and for $\phi \in \mathcal{X}_{\mathcal{D}}^a$ and $h \in GL_2(\mathbb{A}_{\mathcal{K},f})$ let

$$\varphi_{\mathcal{D},ijk,\phi}(h) := \varphi_{\mathcal{D},\beta_{ijk},diag(u_i\mathfrak{d}_1^{-1},\bar{u}_i^{t^{-1}}\bar{\mathfrak{d}}_1),\phi}(h).$$

If $h_p \in GL_2(\mathcal{O}_{\mathcal{K},p})$, then $\phi(\varphi_{\mathcal{D},ijk,\phi}(h) = \varphi_{\mathcal{D},ijk,\phi}(h)$. Now we have the following lemma interpolating the Hecke operators, completely as in [SU]13.3.2.

Lemma 13.3.1. *Let $\mathcal{L} := \{v_1, \cdots, v_m\}$ be a finite set of prime that split in $\mathcal{K}$ and do not belong to $\Sigma \cup \mathcal{Q}$. Let $P \in \Lambda_{\mathcal{D}}[X_1, \cdots, X_m]$. For $h \in H_i(\mathbf{A}_{F,f})$ with $h_p \in H_{i,p}$, there exists $\varphi_{\mathcal{D},i}(\mathcal{L}, P; h) \in \Lambda_{\mathcal{D}}$ such that:*
(a) for all $\phi \in \mathcal{X}_{\mathcal{D}}^a$,

$$\phi(\varphi_{\mathcal{D},ijk}(\mathcal{L}, P; h)) = P_\phi(\xi_{\phi,v_1,1}(\varphi_{v_1})T_{v_1}^{H_{ijk}}, \cdots, \xi_{\phi,v_m,1}(\varphi_{v_m})T_{v_m}^{H_{ijk}})e_{H_{ijk}}\varphi_{\boldsymbol{D},ijk,\phi}(h),$$

where P_ϕ is the polynomial obtained by applying ϕ to the coefficients of P.
(b) if $M \subseteq \Lambda_{\mathcal{D}}$ is a closed $\Lambda_{\mathcal{D}}$-submodule and $\varphi_{\mathcal{D},ijk}(h) \in M$ for all h with $h_p \in H_{i,p}$, then $\varphi_{\mathcal{D},ijk}(\mathcal{L}, P; h) \in M$.

Observe that the neben typus of $\alpha_{\xi\lambda}(A)$ at $v|p$ are given by:

$$\varepsilon'(\begin{pmatrix} a_v & \\ & d_v \end{pmatrix}) \to \mu_{1,v}(a_v)\mu_{2,v}(d_v)\tau_{1,v}^{-1}\tau_{2,v}^{-1}(d_v).$$

for any $a_v, d_v \in \mathcal{O}_{F_v}^\times$. From the definition of the theta functions (q-expansion) we know that $\alpha_{\xi\lambda}(A)$ is a $\Lambda_{\mathcal{D}}$ adic form. Also for each arithmetic weight ϕwe consider the resulting form at ϕ is a form of parallel weight 2 and neben-typus at $v|p$ only depend on $\phi|R^+$.

Now let $\boldsymbol{g} \in M^{ord}(M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}, 1; \Lambda_{W,A})$ be a Hida family of forms which are new at primes not dividing

p and such that $\boldsymbol{g}\otimes\chi_{\mathcal{K}}=\boldsymbol{g}$. Suppose also that the localization of the Hecke algebra at the maximal ideal corresponding the $\boldsymbol{g}$ is Gorenstein so that $\ell_{\boldsymbol{g}}$ makes sense. Now following the remark of [SU] before 13.3.4, one can change the weight homomorphism and view $\boldsymbol{g}$ as an element of $M^{ord}(M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}},1;R^+)$ such that at any ϕ we consider it is a normalized nearly ordinary form of parallel weight 2 and neben-typus at $v|p$ the same as $\alpha_{\xi\lambda}(A)$. Also as in *loc.cit* one can find a polynomial of the Hecke actions $P_{\boldsymbol{g}}:=P(T_{v_1},...,T_{v_m})\in\mathbb{T}^{ord}(M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}},1;R^+)$ such that $P_{\boldsymbol{g}}=a_{\boldsymbol{g}}\ell_{\boldsymbol{g}}$ with $0\neq a_{\boldsymbol{g}}\in R^+$.

With these preparations we can prove the following proposition in the same way as [SU]13.3.4.

Proposition 13.3.1. *Under the above hypotheses,*

$$\sum_{i,j,k}\varphi_{\mathcal{D},ijk}(\mathcal{L},P_{\boldsymbol{g}};1)=\mathcal{A}_{\mathcal{D},\boldsymbol{g}}\mathcal{B}_{\mathcal{D},\boldsymbol{g}}.$$

with $\mathcal{A}_{\mathcal{D},\boldsymbol{g}}\in\mathbb{I}[[\Gamma_{\mathcal{K}}^+]]$ *and* $\mathcal{B}_{\mathcal{D},\boldsymbol{g}}\in\mathbb{I}[[\Gamma_{\mathcal{K}}]]$ *such that for all* $\phi\in\mathcal{X}'_{\mathcal{D}}$*:*

$$\phi(\mathcal{A}_{\mathcal{D},\boldsymbol{g}})=\frac{|\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1|_{\mathcal{K}}^{\frac{\kappa}{2}-1}\xi(\tilde{\delta}_{\mathcal{K}}\bar{\mathfrak{d}}_1)\phi(a_{\boldsymbol{g}})\eta_{\boldsymbol{f}_\phi}\frac{<B_{\mathcal{D},1}\mathcal{E}_{\mathcal{D}_\phi}\rho(\begin{pmatrix}1&\\&\mathfrak{d}\end{pmatrix})\boldsymbol{g}_\phi,\rho(\begin{pmatrix}&-1\\M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}&\end{pmatrix}\prod_{v|p}\begin{pmatrix}&-1\\p^{r_{v,\phi}}&\end{pmatrix})\boldsymbol{f}_\phi^c>_{GL_2}}{<f_\phi,\rho(\begin{pmatrix}&-1\\M&\end{pmatrix}\prod_{v|p}\begin{pmatrix}&-1\\p^{r_{v,\phi}}&\end{pmatrix})f_\phi^c>_{GL_2}}}{}$$

and for $\phi\in\mathcal{X}^a$*,*

$$\phi(\mathcal{B}_{\mathcal{D},\boldsymbol{g}})=\eta_{\boldsymbol{g}_\phi}\frac{<B_{\mathcal{D},2}\alpha_{\xi\lambda}(\sum_{ijk}\Theta_{\mathcal{D},\beta_{ijk}}\otimes\xi)_\phi,\rho(\begin{pmatrix}&-1\\M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}&\end{pmatrix}\prod_{v|p}\begin{pmatrix}&-1\\p^{r_{v,\phi}}&\end{pmatrix})\boldsymbol{g}_\phi^c>_{GL_2}}{<\boldsymbol{g}_\phi,\rho(\begin{pmatrix}&-1\\M_{\mathcal{D}}^2\tilde{D}_{\mathcal{K}}&\end{pmatrix}\prod_{v|p}\begin{pmatrix}&-1\\p^{r_{v,\phi}}&\end{pmatrix})\boldsymbol{g}_\phi^c>_{GL_2}}.$$

Furthermore, $\mathcal{A}_{\mathcal{D},\boldsymbol{g}}\neq0$.

Definition 13.3.1. *Suppose we have a Hida family* $\boldsymbol{f}$ *of ordinary Hilbert modular forms and* $\mathcal{K}$ *is a CM extension of* F *as before. Let* f_2 *be the element in* $\boldsymbol{f}$ *of parallel weight* 2 *and trivial character. We also denote* ϕ_0 *to be the point on the weight space corresponding to the special* L*-value* $L(f_2,1)$.

Now we prove the following key proposition:

Proposition 13.3.2. *Let A be the integer ring of a finite extension of $\mathbf{Q}_p$, $\mathbb{I}$ a domain and a finite $\Lambda_{W,A}$-algebra, and $\boldsymbol{f} \in \mathcal{M}^{ord}(M, \mathbf{1}; \mathbb{I})$ an $\mathbb{I}$-adic newform such that (irred) and **(dist)** hold. There is a Σ and a p-adic Eisenstein datum $\mathcal{D}$ such that contains an odd prime not dividing p that splits completely in $\mathcal{K}/\mathbb{Q}$. There exists an integer $M_{\mathcal{D}}$ as before and divisible by all primes dividing Σ such that the following hold for the associated $\Lambda_{\mathcal{D}}$-adic Eisenstein series $\boldsymbol{E}_{\mathcal{D}}$ and the set $\mathcal{C}_{\mathcal{D}} = \{c_{\mathcal{D}}(\beta_i, x); i \in I_0, x \in G(\mathbb{A}_{F,f}) \cap Q(F_p)\}$ of fourier coefficients of $\boldsymbol{E}_{\mathcal{D}}$.*

(i) If $R \subseteq \Lambda_{\mathcal{D}}$ is any height-one prime containing $\mathcal{C}_{\mathcal{D}}$, then $R = P\Lambda_{\mathcal{D}}$ for some height-one prime $P \subset \mathbb{I}[[\Gamma_{\mathcal{K}}^+]]$.
(ii) if f satisfies (NV2) then there are no height-one primes of $\Lambda_{\mathcal{D}}$ containing $\mathcal{L}^{\Sigma}_{\boldsymbol{f},\mathcal{K},1}$ and $\mathcal{C}_{\mathcal{D}}$. If f satisfies (NV1), then there are no height-one primes passing through ϕ_0 containing $\mathcal{L}^{\Sigma}_{f,\mathcal{K},1}$ and $\mathcal{C}_{\mathcal{D}}$.

Proof. we give a brief summary of the proof following [SU]13.4.1 closely.
As in loc.cit, we only need to find an $M_{\mathcal{D}}$ so that there is an $\boldsymbol{g}$ with $\mathcal{B}_{\mathcal{D},\boldsymbol{g}}$ is a p-adic unit.
First we find an idele class character θ of $\mathbb{A}_{\mathcal{K}}^{\times}$ such that:

- $\theta_\infty(z) = \prod_{v\in\Sigma} z_v^{-1}$;
- $\theta|_{\mathbf{A}_F^{\times}} = |\cdot|_F \chi_{\mathcal{K}/F}$;
- $Nm(\mathfrak{f}_\theta) = M_\theta^2$for some $M_\theta \in F^{\times}$ prime to p and such that $D_{\mathcal{K}}M|M_\theta$

  and $v|M_\theta$ for all $v \in \Sigma \setminus \{p\}$;
- for some $v|\tilde{D}_{\mathcal{K}}$, the anticyclotomic part of $\theta|^{\times}_{\mathcal{O}_{\mathcal{K},q}}$ has order divisible by q_v.
- $\Omega_\infty^{-\Sigma} L(1, \theta)$ is a p-adic unit, where Ω_∞ is the CM period defined in [HAnti];
- $\theta_{v,2}(p) - 1$ is a p-adic unit for any $v|p$.
- ψ has order prime to p.
- the local character ψ is nontrivial over $\mathcal{K}_{\mathfrak{P}}^{\times}$ for all $\mathfrak{P} \in \Sigma_p$
- the restriction of ψ to $Gal(\bar{F}/\mathcal{K}[\sqrt{p^*}])$ is nontrivial.

Here ψ is the "torsion part" (as defined in [Hida05]) of the anticyclotomic part of $\theta^a := \theta^c/\theta$, p^* is $(-1)^{(p-1)/2}p$.

The existence is proven in a similar way as in [SU] 13.4.1, using the main theorem of [Hsieh11] instead of [Fi06]. (The result in [Hsieh11] is not stated in the generality we need since he put a condition (C) there requiring that the non split part of the CM character is square free. But M-L Hsieh informed the author that, as mentioned in that paper, this condition is removed later.) Now using the main result of [Hida 06] and [Hida09], (we thank Hida for informing us his results in loc.cit). under the last three conditions above, we have

$$\eta_{g_\theta}|\frac{(g_\theta, g_\theta)}{\Omega_\infty^{2\Sigma}}$$

Thus

$$L(1,\theta)^2/\Omega_{can}|L(1,\theta)^2/\Omega_\infty^{2\Sigma} \tag{13.6}$$

where Ω_{can} is the canonical period associated to g_θ.

If g_θ is the CM newform associated with θ. It has parallel weight 2, level $M_\theta^2\tilde{D}_{\mathcal{K}}$, and trivial neben character. Similarly as in [SU] p210, we see that it satisfies ***(irred)*** and ***(dist)***. Let $\boldsymbol{g} \in \mathcal{M}^{ord}(M_\theta^2\tilde{D}_{\mathcal{K}}, \mathbf{1}; R^+)$ be the ordinary CM newform associated with θ. (this is constructed in [Hida-Tilouine]p133-134. one need to first construct the automorphic representation generated by some theta series and then pick up the nearly ordinary vector inside that representation space.) The Gorenstein properties are also true as remarked by [Hida06]. Recall that we have defined $A := \alpha_{\xi\lambda}(\sum_{i,j,k}\Theta_{\beta_{ijk}}\otimes\xi)$. Now we evaluate $B_{\mathcal{D},g}$ at the ϕ which restricts trivially to $W_{i,v}$'s and $\Gamma_{\mathcal{K}}$. In this case the argument in [SU] 11.9.3 gives that:

$$\alpha_{\xi\lambda}(A)_\phi = (B_{\mathcal{D},4})_\phi E'(\chi_{\mathcal{K}})\rho(\begin{pmatrix} & -1 \\ M_{\mathcal{D}} & \end{pmatrix})E'(\chi_{\mathcal{K}})$$

where $(B_{\mathcal{D},4})_\phi = |M_{\mathcal{D}}^2|_F^{-1}|\delta_{\mathcal{K}}|_{\mathcal{K}}2^{3d}i^{-2d}|\delta_{\mathcal{K}}|_{\mathcal{K}}^{\frac{1}{2}}$ which is a p-adic unit. Here $E' = \prod_{v|p}(1-p^{\frac{1}{2}}(\rho(\begin{pmatrix} 1 & \\ & p \end{pmatrix}_v))E(\chi_{\mathcal{K}})$ for $E(\chi_{\mathcal{K}})$ being the weight 1 Eisenstein series whose L-function is $L(F,s).L(F,\chi_{\mathcal{K}},s)$. We write

$$h = E'(\chi_{\mathcal{K}})\rho(\begin{pmatrix} & -1 \\ M_{\mathcal{D}} & \end{pmatrix})E'(\chi_{\mathcal{K}})$$

Then the argument in [SU]13.4.1 tells us that:

$$< h, \rho(\prod_{v|p}\begin{pmatrix} & -1 \\ p & \end{pmatrix}_v \begin{pmatrix} & -1 \\ M_{\mathcal{D}}\tilde{D}_{\mathcal{K}} & \end{pmatrix})g^c >= \frac{\pm|\tilde{D}_{\mathcal{K}}|_F\prod_{v|p}\theta_{v,2}(p)^{-2}}{i^d(-2\pi i)^{2d}\mathfrak{g}(\chi_{\mathcal{K}})}L(1,\theta)^2\prod_{v|p}(1-\theta_{v,2}(p))^3$$

Thus

$$\phi(\mathcal{B}_{\boldsymbol{D},\boldsymbol{g}}) = \frac{\pm|\tilde{D}_{\mathcal{K}}|_F \prod_{v|p} \theta_{v,2}(p)^{-2}}{i^d(-2\pi i)^{2d}\mathfrak{g}(\chi_{\mathcal{K}})\Omega_{\mathrm{can}}} L(1,\theta)^2 \prod_{v|p}(1-\theta_{v,2}(p))^3.$$

By definition $\phi(\mathcal{B}_{\boldsymbol{D},\boldsymbol{g}})$ is p-integral. But as noted before, $\frac{L(1,\theta)^2}{\Omega_{can}}$ divides a p-adic unit, thus itself must also be a p-adic unit. Therefore, $\mathcal{B}_{\boldsymbol{D},\boldsymbol{g}}$ is a unit. This proves (i). (ii) is just an easy consequence of (i). $\square$

Chapter 14

Construction of the cuspidal family

In this chapter we construct a Λ_D-adic cusp form which is prime to the p-adic L-function by explicitly writing down some Λ_D-adic forms with the same boundary restriction as the Klingen Eisenstein family constructed before.

14.1 Certain Eisenstein series on GU(2,2)

14.1.1 Siegel Eisenstein Series

In this chapter we use $\mathbf{P}$ instead of P to denote the Klingen parabolic and save the letter P for the height one prime. Consider the p-adic family of CM characters of $\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times$. In the component containing the trivial character, there is one element τ_{κ_0} which is unramified everywhere and has infinite types $(\frac{\kappa_0}{2},-\frac{\kappa_0}{2})$ at all infinite places for some $\kappa > 6$ divisible by $(p-1)$. Define a Siegel Eisenstein series E_{κ_0} on $GU(2,2)$ by choosing the local sections as follows: Let $f_v^\dagger$ be the section supported on $Qw_2K_Q(\varpi_v)$ and equals 1 on $K_Q(\varpi_v)$. (Here $K_Q(\varpi_v)$ means matrices with v-integral entries that belong to $Q(\mathcal{O}_{F,v})$ modulo ϖ_v.) If $v|p$ Let

$$f_v'(g)=\begin{cases}\tau_0(\det D_1)|A_qD_q^{-1}|^s & \text{if } g=qw_{13}k\in Qw_{13}K_Q(\varpi_v)\\ 0 & \text{otherwise}\end{cases}$$

where $w_{13} = \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix}$.

We define $f_v = f_{\kappa_0}$ for $v|\infty$, $f_v = f_v^\dagger$ for $v \in \Sigma, v \nmid p$, and $f_v = f_v'$ for $v|p$. Now we want to compute the constant terms of E_{κ_0} along $\mathbf{P}$ at g as an automorphic form on $M_{\mathbf{P}} = \{m(a,x)\}$.

Now we compute the constant term of E_{κ_0} along the Klingen parabolic subgroup $\mathbf{P}$. First note that

$$G(F) = Q(F)\mathbf{P}(F) \sqcup Q(F)w_2\mathbf{P}(F).$$

thus

$$\begin{aligned} E_{\kappa_0}(g) &= \sum_{\gamma \in Q(F)\backslash G(F)} f(\gamma g) \\ &= \sum_{\gamma \in Q(F)\backslash Q(F)\mathbf{P}(F)} f(\gamma g) + \sum_{\gamma \in Q(F)\backslash (F)w_2\mathbf{P}(F)} f(\gamma g) \end{aligned}$$

Suppose the above summation is in the absolute convergent region.

$$\begin{aligned} E_{\kappa_0,\mathbf{P}}(z,g) &= \int_{N_{\mathbf{P}}(F)\backslash N_{\mathbf{P}}(\mathbf{A}_f)} E_{\kappa_0}(ng)dn \\ &= \int_{N_{\mathbf{P}}(F)\backslash N_{\mathbf{P}}(\mathbf{A}_F)} \sum_{\gamma \in Q(F)\backslash Q(F)\mathbf{P}(F)} f_z(\gamma ng)dn \\ &+ \int_{N_{\mathbf{P}}(F)\backslash N_{\mathbf{P}}(\mathbf{A}_F)} \sum_{\gamma \in Q(F)\backslash Q(F)w_2\mathbf{P}(F)} f_z(\gamma ng)dn \\ &= \mathbb{I}_1 + \mathbb{I}_2 \end{aligned}$$

we claim that $\mathbb{I}_2(g) = 0$. By [MW] we have

$$\begin{aligned} \mathbb{I}_2(g) &= \sum_{m' \in M_{\mathbf{P}}(F) \cap w^{-1}Q(F)w^{-1} \backslash M_{\mathbf{P}}(F)} \int_{N_{\mathbf{P}}(F)\backslash N_{\mathbf{P}}(\mathbf{A}_F)} \sum_{n' \in N_{\mathbf{P}}(F) \cap m'^{-1}w^{-1}Q(F)wm'} f_z(wm'n'ng)dn \\ &= \sum_{m'} \int_{N_{\mathbf{P}}(F)\cap w^{-1}Q(F)w \backslash N_{\mathbf{P}}(\mathbf{A}_F)} f_z(wnm'g)dn \\ &= \sum_{m'} \int_{N_1(F)\backslash(\mathbf{A}_F)} \int_{N_1(\mathbf{A}_F)\backslash N_{\mathbf{P}}(\mathbf{A})} f_z(wnm'g)dn \\ &= \sum_{m'} \int_{N_2(\mathbf{A}_F)} f_z(wnm'g)dn \end{aligned}$$

For any g_0 such that $g_{0,\infty}$ is of the form $\begin{pmatrix} y & \\ & \bar{y}^t \end{pmatrix}$, $\mathbb{I}_2(gg_0)$ as a function of g is an automorphic form on $M_{\mathbf{P}}$, in order to prove this is zero, we show that all the Fourier coefficients of $\mathbb{I}_2$ along the unipotent group $\begin{pmatrix} 1 & & \times & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ are zero. But this is nothing but

$$\int_{N_Q(\mathbf{A}_F)} f_z(w \begin{pmatrix} 1 & S \\ & 1 \end{pmatrix} g_0)e(-tr\beta S)dS$$

for $\beta = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}$, which we have proven to be 0 at $z = z_\kappa = \frac{\kappa-2}{2}$ for all g_0 with the required ∞ part.

Next we consider $\mathbb{I}_1$. We define a Siegel Eisenstein series $E^1_{\kappa_0}$ on $GU(1,1)$ by chossing the local sections by $f_v = f_v^\dagger$ for v finite and $f_v = f_{\kappa_0}$ for $v|\infty$. Then it is easy to see that if g is such that:

$$g_v \in \begin{cases} K_v & \text{if } v|p \\ w_2K_v & \text{if } v \nmid p, \infty \\ 1 & \text{if } v|\infty \end{cases}$$

(Here K_v are the level groups for some Klingen Eisenstein series we constructed before at some weight ϕ.) One can check that $\mathbb{I}$ is

$$E^1_{\kappa_0}(a \prod_{v\nmid p,\infty} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_v)$$

14.2 Hecke operators

In this section we study the relations between the $GU(2,2)$ and $GU(1,1)$ Hecke operators via the restriction to the boundary. For any automorphic form F on $G(\mathbf{A}_F)$ and some $g_0 \in G(\mathbf{A}_F)$ we consider $F_{\mathbf{P}}$ as an automorphic form on $GU(1,1)$: the value at $g' \in GU(1,1)(\mathbf{A}_F)$ is given by $F_{\mathbf{P}}(m(g',1)g_0)$.

14.2.1 unramified cases

Suppose v is a place unramified in $\mathcal{K}/F$.

<u>*split case*</u>

if v splits in $\mathcal{K}/F$, then $U(\mathcal{O}_{F_v}) \simeq GL_4(\mathcal{O}_{F_v})$. We write $\tau_v = (\tau_1, \tau_2)$ and $\tau_{0,v} = (\tau_1^\circ, \tau_2^\circ)$ with respect to $\mathcal{K}_v = F_v \times F_v$. Recall in this cse we have defined

$$d_v := diag((\varpi_v, 1), 1, (1, \varpi_v^{-1}), 1).$$

via projection onto the first component, $t_2^{(1)} = \begin{pmatrix} \varpi_v & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $B(F_v)$ is identified with the matrices $\begin{pmatrix} \times & & \times & \times \\ \times & \times & \times & \times \\ & & \times & \times \\ & & & \times \end{pmatrix}$. Let K be identified with $G(\mathcal{O}_{F_v})$

$$Kt_2^{(1)}K = \sqcup n_{1i} d_1 K \sqcup n_{2j} d_2 K \sqcup n_{3k} d_3 K \sqcup d_4 K$$

where n_{1i} goes through $\begin{pmatrix} 1 & & \times & \times \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, n_{2j} goes through $\begin{pmatrix} 1 & & & \\ \times & 1 & \times & \times \\ & & 1 & \\ & & & 1 \end{pmatrix}$ n_{3k} goes through $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \times \\ & & & 1 \end{pmatrix}$, and d_1, d_2, d_3, d_4 are $\begin{pmatrix} \varpi_v & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ & \varpi_v & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varpi_v & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi_v \end{pmatrix}$ respectively. If $g = m(g', 1)$ for some $g' \in U(1,1)$, then

$$\begin{aligned} E_{\phi,\mathbf{P}}(gn_{2j}d_2) =& E_{\phi,\mathbf{P}}(gd_2) \\ =& E_{\phi,\mathbf{P}}(g)\tau_2^{-1}(\varpi_v). \end{aligned}$$

$$E_{\phi,\mathbf{P}}(gd_4) = E_{\phi,\mathbf{P}}(g)\tau_1(\varpi_v)$$

$$E_{\kappa_0,\mathbf{P}}(gn_{2j}d_2) = E_{\kappa_0,\mathbf{P}}(g,d_2) = (\tau_2^\circ)^{-1}(\varpi_v)$$

$$E_{\kappa_0,\mathbf{P}}(gd_4) = \tau_1^\circ(\varpi_v)E_{\kappa_0,\mathbf{P}}(g).$$

thus one sees:

Lemma 14.2.1. *For g such that $g_v = m(g_{1,v},1)$ for some $g_{1,v} \in U(1,1)(F_v)$ we have:*

$$\begin{aligned}&(T_v(t_2^{(1)})(E_\phi \cdot E_{\kappa_0}))_{\mathbf{P}}(g)\\ =\ &(q_v^3(\tau_2^{-1}(\varpi_v)\cdot(\tau_2^\circ)^{-1}(\varpi_v)) + \tau_1\tau_1^\circ(\varpi_v))(E_\phi \cdot E_{\kappa_0})_{\mathbf{P}}(g)\\ &+ q_vT_v(\begin{pmatrix}\varpi_v & \\ & 1\end{pmatrix})((E_\phi \cdot E_{\kappa_0})_{\mathbf{P}})(g)\end{aligned}$$

where $T_v(\begin{pmatrix}\varpi_v & \\ & 1\end{pmatrix})$ is as a Hecke action on $GU(1,1)$ and we consider $(E_\phi \cdot E_{\kappa_0})_{\mathbf{P}}$ as an automorphic form on $U(1,1)$ using g_0 by the remark at the beginning of this section.

<u>unramified inertial case</u>

Suppose v is inertial in $\mathcal{K}/F$ and take K to be $G(\mathcal{O}_{F_v})$, define: $d_v = \begin{pmatrix}\varpi_v &&&\\ & \varpi_v &&\\ &&1&\\ &&&1\end{pmatrix}$. Then:

$$Kd_vK = \sqcup_i n_{1i}d_1K \sqcup n_{2j}d_2K \sqcup_k n_{3k}d_3K \sqcup \{d_4\}K$$

where

$$d_1 = \begin{pmatrix}\varpi_v &&&\\ & \varpi_v &&\\ &&1&\\ &&&1\end{pmatrix}, d_2 = \begin{pmatrix}1 &&&\\ & \varpi_v &&\\ &&\varpi_v&\\ &&&1\end{pmatrix}$$

$$d_3=\begin{pmatrix}\varpi_v&&&\\&1&&\\&&1&\\&&&\varpi_v\end{pmatrix}, d_4=\begin{pmatrix}1&&&\\&1&&\\&&\varpi_v&\\&&&\varpi_v\end{pmatrix}$$

where n_{1i} runs over matrices of the form $\begin{pmatrix}1&&\times&\times\\&1&\times&\times\\&&1&\\&&&1\end{pmatrix}$, n_{2j} over $\begin{pmatrix}1&&&\\\times&1&&\times\\&&1&\times\\&&&1\end{pmatrix}$, n_{3k} over $\begin{pmatrix}1&&\times&\\&1&&\\&&1&\\&&&1\end{pmatrix}$. As in the split case (actually even simpler), we see that

Lemma 14.2.2. *For g such that $g_v=m(g_{1,v},1)$ for some $g_{1,v}\in U(1,1)(F_v)$:*

$$(T_{d_v}\cdot(E_\phi\cdot E_{k_0}))_{\mathbf{P}}(g)=(q_v^3+1)T_{\begin{pmatrix}\varpi_v&\\&1\end{pmatrix}}((E_\phi\cdot E_{k_0})_{\mathbf{P}})(g)$$

where $T_v(\begin{pmatrix}\varpi_v&\\&1\end{pmatrix})$ is the Hecke action on $GU(1,1)$.

<u>*$p-case$*</u>: $v|p$.

Suppose $d=d_v=(\begin{pmatrix}p^3&&&\\&p^2&&\\&&1&\\&&&p\end{pmatrix},1)$ we study $(T_{d_v}(E_\phi\cdot E_{\kappa_0}))_{\mathbf{P}}(g)$. Using the decomposition $KdK=\sqcup n_idK$ where n_i running over

$$\begin{pmatrix}1&x&&\\&1&&\\&&1&\\&&-\bar{x}&1\end{pmatrix}\begin{pmatrix}1&&\alpha&\beta\\&1&\bar{\beta}&\gamma\\&&1&\\&&&1\end{pmatrix}$$

where x,α,β,γ runs over congruence classes modulo: $\mathcal{O}_v/(p,p)$, $\mathbb{Z}_v/p^3,\mathcal{O}_v/(p^2,p^2),\mathbb{Z}_v/p$ respectively.

First notice that if $x \neq 0$, $E_{\phi,\mathbf{P}}(gn_id) = 0$, so we may ignore such terms while summing up,so

$$\begin{aligned}&(T_{d_v}(E_\phi \cdot E_{\kappa_0}))_{\mathbf{P}}(g)\\ =&\sum_{\alpha,\beta,\gamma} E_{\phi,\mathbf{P}}(gn_id)E_{\kappa_0,\mathbf{P}}(gn_id)\end{aligned}$$

Observe that for all choices of β, γ, the above expression does not change. Therefore the summation is essentially only over α's.

If $g = m(g_1, 1)$ for some $g_1 \in U(1,1)$ then an easy computation taking into account Hida's normalization factors for $U(2,2)$ and $U(1,1)$ gives:

$$(U_{d_v}(E_\phi.E_{\kappa_0}))_{\mathbf{P}}(g) = p^{-\frac{3}{2}(\kappa_\phi+\kappa_0)}\tau_{\phi,v}((p,p^{-2}))\tau_{\kappa_0,v}((p,p^{-2}))U_{p^3}((E_\phi.E_{\kappa_0})_{\mathbf{P}})(g).$$

Here U_{p^3} is the $U(1,1)$ normalized Hecke operator associated to $(\begin{pmatrix} p^3 & \\ & 1 \end{pmatrix}, 1)$.

Recall if we define $U_p^{(2,2)} := \prod_{v|p} U_{d_v}$, then $e^{ord}_{U(2,2)} = \lim_n U_p^{n!}$.
The above calculation told us that

Lemma 14.2.3. *For g such that $g_v = m(g_{1,v}, 1)$ for some $g_{1,v} \in U(1,1)(F_v)$, then:*

$$(e^{ord}_{(2,2)}(\tilde{E}_\phi \cdot E_{\kappa_0}))_{\mathbf{P}}(g) = e^{ord}_{(1,1)}((\tilde{E}_\phi \cdot E_{\kappa_0})_{\mathbf{P}})(g)$$

14.2.2 construction of the family

Now we define an automorphism $\gamma : \Lambda_W \to \Lambda_W$ such that for any arithmetic weight ϕ , $\gamma \circ \phi$ is an arithmetic weight with the same neben typus at p but $\kappa_{\gamma\circ\phi} = \kappa_\phi + \kappa_0$. The formula is given by: $(\gamma(1 + W_{1,v}) = (1 + W_{1,v}), \gamma(1 + W_{2,v}) = (1 + W_{2,v})(1 + p)^{\kappa_0}$. Then we consider $\mathbb{I} \otimes_\gamma \mathbb{I}$. We choose a reduced irreducible component $\mathbb{J}'$ whose spectrum maps surjectively onto SpecΛ_W. Then it is easy to see that both $\mathbb{I}$'s inject to $\mathbb{J}'$. We define $\mathbb{J}$ to be the normalization of $\mathbb{J}'$. (Intuitively $\mathbb{J}$ is parameterizing pairs of forms with weight κ_ϕ and $\kappa_{\gamma\circ\phi}$). We write $j_1 : \mathbb{I} \to \mathbb{J}$ and $j_2 : \mathbb{I} \to \mathbb{J}$ for the two embeddings. We have $j_2 \circ \gamma = j_1$. We also define an automorphism $\Lambda_{\mathcal{D}} \to \Lambda_{\mathcal{D}}$ which we again denote as γ such that the Eisenstein date we get by ϕ and $\phi \circ \gamma$ have the same "finite order part" and $\kappa_{\phi\circ\gamma} = \kappa_\phi - \kappa_0$. If we denote $\Lambda_{\mathcal{D},\mathbb{J}}$ then both $j_1(\mathbf{E}_{\mathcal{D}})$ and $j_2(\mathbf{E}_{\mathcal{D}}.E_{\kappa_0})$ are $\Lambda_{\mathcal{D},\mathbb{J}}$-adic forms. Considering $\mathbf{f}$ as a $\mathbb{J}$-adic Hida family.

Let $P_{\mathbf{f}} := P_{\mathbf{f}}(T_{v_1}, \cdots, T_{v_m})$ be a polynomial in $\mathbb{J}(X_1, \cdots, X_m)$ such that $P_{\mathbf{f}} = \mathbf{a}_{\mathbf{f}}\ell_{\mathbf{f}}$ for some

$0 \neq a_{\mathbf{f}} \in \mathbb{J}$. We define a Λ_D-coefficient formal q-expansion:

$$\mathbf{E}^\circ := (P_{\mathbf{f}}(\tilde{T}_{v_1,d_{v_1}},\cdots,\tilde{T}_{v_m,d_{v_m}})e^{ord}_{U(2,2)}(\mathbf{E}\cdot j_2(E_{\kappa_0})))$$

where

$$\tilde{T}_v(d_v) := \begin{cases} (\frac{1}{q_v}(T_v(d_v) - q_v^3(\bar{\tilde{\tau}}_2)(\varpi_v)\cdot\bar{\tau}_2^\circ(\varpi_v)) - \tilde{\tau}_1\tau_1^\circ(\varpi_v))\Psi_{2,v}(\varpi_v) & \text{if } v \text{ splits in } \mathcal{K}/F \\ \frac{1}{q_v^3+1}T_v(d_v) & \text{if } v \text{ is inertial in } \mathcal{K}/F \end{cases}$$

Here d_v's are defined as before and $\Psi_{2,v}$ is to take care of the difference between Hecke eigenvalues between $U(1,1)$ and GL_2. We should justify the operator $e^{\mathrm{ord}}_{U(2,2)}$ acting on $\Lambda_{\mathcal{D},\mathbb{J}}$-adic forms. This could be done in the same way as [SU]12.2.4 (i).

We have already computed that if g is such that $g_v = 1$ for $v|p$ and $g_v = w$ for $v \nmid p, v \in \Sigma$ and $g_v = 1$ for $v|\infty$ then the constant term $j_1(\mathbf{E}_{\mathcal{D}})_{\mathbf{P}}$ at g is given by $j_1(\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\xi}'}\mathcal{L}^{\Sigma}_{\mathcal{D}})(\pi(\prod_{v\nmid p,\infty}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}_v \mathbf{f})$.

Therefore the constant terms $j_2(\mathbf{E}_{\mathcal{D}}.E_{\kappa_0})_{\mathbf{P}}$ at g is given by:

$$j_2(\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\xi}'}\mathcal{L}^{\Sigma}_{\mathcal{D}})(\pi(\prod_{v\nmid p,\infty}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}_v \mathbf{f})E^1_{\kappa_0}(a\prod_{v\nmid p,\infty}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}_v).$$

Now we consider a 1-dimensional subspace of $\Lambda_{\mathcal{D}}$ defined by the closure of the arithmetic ϕ's such that ξ_ϕ is trivial and f_ϕ has trivial neben typus at p. Observe that $j_1(\mathcal{L}^{\Sigma}_{\mathcal{D}})$ is not identically 0 along this family by the interpolation properties and the temperedness of f_ϕ. (recall also that we do not know a priory that these points are interpolations points. But by comparing with Hida we know that his and our constructions coincide along a subfamily containing the 1-dimensional family above and we can use Hida's interpolation formula.) So we can choose κ_0 properly so that $j_2(\mathcal{L}^{\Sigma}_{\mathcal{D}})$ does not pass through ϕ_0. (only need to avoid a finite number of points). (Note that $j_2(\mathcal{L}^{\Sigma}_{\mathcal{D}})$ does not interpolate any classical L-values since the weight is $2-\kappa_0$.) Let P be any height one prime of $\Lambda_{\mathcal{D},\mathbb{J}}$ passing through ϕ_0 then P is prime to $j_2(\mathcal{L}^{\Sigma}_{\mathcal{D}})$.

Sum up, for $\prod_v g_v = g$, if:

$$g_v \in \begin{cases} \mathbf{P}(F_v)K_v. & v|p \\ \mathbf{P}(F_v)wK_v, & v \in \Sigma \setminus \{v|p\}. \\ Q(F_v), & v|\infty \\ 1, & \text{otherwise} \end{cases}$$

then there is an $\mathbf{a} \in \mathbb{J}$ satisfying:

$$(j_1(\mathcal{L}^{\Sigma}_{\mathcal{D}}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'}\mathbf{E}^{\circ}))_{\phi,\mathbf{P}} = (\mathbf{a}\cdot\mathbf{a_f})_{\phi}(j_2(\mathcal{L}^{\Sigma}_{\mathcal{D}}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'}\mathbf{E}))_{\phi,\mathbf{P}}(g)$$

and $\mathbf{a}_{\phi} = \eta_{f_{\phi}} \frac{<\mathbf{f}_{\phi\circ\gamma^{-1}}\cdot E^1_{\kappa_0}, \rho(\begin{pmatrix} & -1 \\ M & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_{\phi}} & \end{pmatrix})\mathbf{f}^c_{\phi}>}{<\mathbf{f}_{\phi\circ\gamma^{-1}}, \rho(\begin{pmatrix} & -1 \\ M & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_{\phi}} & \end{pmatrix})\mathbf{f}^c_{\phi}>}$ In the case when $\kappa_{\phi} >> \kappa_0$ our previous computations on Rankin-Selberg convolutions told us that $\mathbf{a}_{\phi} \neq 0$ by the temperedness of $\mathbf{f}_{\phi\circ\gamma^{-1}}$ and $\rho(\begin{pmatrix} & -1 \\ M & \end{pmatrix}\prod_{v|p}\begin{pmatrix} & -1 \\ p^{r_{\phi}} & \end{pmatrix})\mathbf{f}_{\phi}$.

Theorem 14.2.1. *There is a $\Lambda_{\mathcal{D},\mathbb{J}}$-coefficients formal q expansion $\boldsymbol{F}$, such that:*

(i) for a Zarisi dense set of arithmetic points ϕ, $\boldsymbol{F}_{\phi}$ is an ordinary cusp form on $GU(2,2)(\mathbb{A}_F)$.

(ii) $\boldsymbol{F} \equiv \boldsymbol{a}\boldsymbol{a_f}j_2(\mathcal{L}^{\Sigma}_{\mathcal{D}}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'})\boldsymbol{E_D}(\mathrm{mod} j_1(\mathcal{L}^{\Sigma}_{\mathcal{D}}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'}))$ for some $0 \neq \boldsymbol{a}\boldsymbol{a_f} \in \mathbb{J}(\Gamma^+)$.

(iii) for any height 1 prime P of $\Lambda_{\boldsymbol{D}}$ containing $j_1(\mathcal{L}^{\Sigma}_{\mathcal{D}})$ passing through ϕ_0 which is not a pull back of a height 1 primes prime of $\mathbb{J}[[\Gamma^+_{\mathcal{K}}]]$, there is a coefficient of $\boldsymbol{F}$ outside P.

Proof. Taking $\mathbf{F} := j_1(\mathcal{L}^{\Sigma}_{\mathcal{D}}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'})\mathbf{E}^{\circ} - \mathbf{a}\mathbf{a_f}j_2(\mathcal{L}^{\Sigma}_{\mathcal{D}}\mathcal{L}^{\Sigma}_{\boldsymbol{\chi}\bar{\boldsymbol{\xi}}'})\mathbf{E}_{\mathcal{D}}$.

□

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