IWASAWA Main Conjecture for Rankin-Selberg p-adic L-functions: Non-Ordinary Case

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Abstract

In this paper we prove that the p-adic L-function that interpolates the Rankin-Selberg product of a general weight two modular form which is unramified and non-ordinary at p, and an ordinary CM form of higher weight contains the characteristic ideal of the corresponding Selmer group. This is one divisibility of the Iwasawa-Greenberg main conjecture for the p-adic L-function. This generalizes an earlier work of the author to the non-ordinary case. The result of this paper plays a crucial role in the proof of Iwasawa main conjecture and refined Birch-Swinnerton-Dyer formula for supersingular elliptic curves.

1 Introduction

Let p be an odd prime. Let $\mathcal{K} \subset \mathbb{C}$ be an imaginary quadratic field such that p splits as $v_0\bar{v}_0$. We fix an embedding $\mathcal{K} \hookrightarrow \mathbb{C}$ and an isomorphism $\iota : \mathbb{C}_p \simeq \mathbb{C}$ and suppose v_0 is determined by this embedding. There is a unique $\mathbb{Z}_p^2$-extension $\mathcal{K}_\infty/\mathcal{K}$ unramified outside p. Let $\Gamma_\mathcal{K} := \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$. Suppose f is a weight two, level N cuspidal eigenform new outside p with coefficient ring $\mathcal{O}_L$ for some finite extension $L/\mathbb{Q}_p$ and $\mathcal{O}_L$ its integer ring. Suppose ξ is a Hecke character of $\mathbb{A}_\mathcal{K}^\times/\mathcal{K}^\times$ whose infinite type is $(-\frac{1}{2}, -\frac{1}{2})$. Suppose $\mathrm{ord}_{v_0}(\mathrm{cond}(\xi_{v_0})) \leq 1$ and $\mathrm{ord}_{\bar{v}_0}(\mathrm{cond}(\xi_{\bar{v}_0})) \leq 1$. Denote $\boldsymbol{\xi}$ as the $\mathcal{O}_L[[\Gamma_\mathcal{K}]]$-adic family of Hecke characters containing ξ as some specialization (will make it precise later). We are going to define a dual Selmer group $X_{f,\mathcal{K},\xi}$ which is a module over the ring $\mathcal{O}_L[[\Gamma_\mathcal{K}]]$. Let us point out that here the local Selmer condition at primes above p is: requiring the class to be unramified at $\bar{v}_0$ and put no restriction at v_0. On the other hand, there is a p-adic L-function $\mathcal{L}_{f,\mathcal{K},\xi} \in \mathrm{Frac}\mathcal{O}_L[[\Gamma_\mathcal{K}]]$ constructed in the text interpolating the algebraic parts of the special L-values $L_\mathcal{K}(\tilde{\pi}, \xi_\phi, 0)$, where ξ_ϕ's are specializations of $\boldsymbol{\xi}$ of infinite type $(-\kappa_\phi - \frac{1}{2}, -\frac{1}{2})$ with $\kappa_\phi > 6$. The Iwasawa main conjecture basically states that the characteristic ideal (to be defined later) of $X_{f,\mathcal{K},\xi}$ is generated by $\mathcal{L}_{f,\mathcal{K},\xi}$.

This conjecture was formulated by Greenberg in [8]. When the automorphic representation for f is unramified and ordinary at p, one divisibility of the conjecture has been proved by the author in [38]. The proof uses Eisenstein congruences on the unitary group $\mathrm{U}(3,1)$ and is influenced by the earlier work of Skinner-Urban [36] which proved another kind of main conjecture using $\mathrm{U}(2,2)$. C.Skinner has been able to use the result in [38] to deduce a converse of a theorem of Gross-Zagier and Kolyvagin ([34]). In fact no matter whether f is ordinary or not, the Selmer groups and Greenberg's main conjecture can be defined in the same manner. The reason is the Panchishikin condition in [8] is satisfied in both cases. Our main goal of this paper is to study this conjecture when f is non-ordinary at p.

To formulate our main result we need one more definition: suppose g is a cuspidal eigenform on $\mathrm{GL}_2/\mathbb{Q}$ which is nearly ordinary at p. We have a p-adic Galois representation $\rho_g : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_L)$ for some $L/\mathbb{Q}_p$ finite. We say g satisfies:
(irred) If the residual representation $\bar{\rho}_g$ is absolutely irreducible.
Also it is known that $\rho_g|_{G_p}$ is isomorphic to an upper triangular one. We say it satisfies:
(dist) If the Galois characters of G_p giving the diagonal actions are distinct modular the maximal ideal of $\mathcal{O}_L$.

We will see later that if the CM form g_{ξ_ϕ} associated to ξ_ϕ satisfies (irred) and (dist) then $\mathcal{L}_{f,\mathcal{K},\xi} \in \mathcal{O}_L[[\Gamma_{\mathcal{K}}]]$.

We prove one divisibility in this paper under certain conditions, following the strategy of [38]. More precisely, we have:

Theorem 1.1. *Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2/\mathbb{Q}$ of weight 2, square free level N and trivial character. Let ρ_π be the associated Galois representation. Assume π_p is good supersingular with distinct Satake paramters. Suppose also for some odd non-split q, $q||N$. Let ξ be a Hecke character of $\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times$ with infinite type $(-\frac{1}{2}, -\frac{1}{2})$. Suppose $(\xi|.|^{\frac{1}{2}})|_{\mathbb{A}_{\mathbb{Q}}^\times} = \omega \circ \mathrm{Nm}$ (ω is the Techimuller character).*
(1) Suppose the CM form g_ξ associated to the character ξ satisfies (dist) *and* (irred) *defined above and that for each inert or ramified prime v we have the conductor of ξ_v is not (ϖ_v) where ϖ_v is a uniformizer for $\mathcal{K}_v$ and that:*
$$\epsilon(\pi_v, \xi_v, \frac{1}{2}) = \chi_{\mathcal{K}/\mathbb{Q},v}(-1).$$
Then we have $\mathcal{L}_{f,\xi,\mathcal{K}} \in \mathcal{O}_L[[\Gamma_{\mathcal{K}}]]$ and $(\mathcal{L}_{f,\mathcal{K},\xi}) \supseteq \mathrm{char}_{\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]}(X_{f,\mathcal{K},\xi})$ as ideals of $\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]$. Here char *means the characteristic ideal to be defined later.*
(2) If we drop the conditions (irred) *and* (dist) *and the conditions on the local signs in (1), but assume that the p-adic avatar of $\xi|.|^{\frac{1}{2}}(\omega^{-1} \circ \mathrm{Nm})$ factors through $\Gamma_{\mathcal{K}}$, then*
$$(\mathcal{L}_{f,\mathcal{K},\xi}) \supseteq \mathrm{char}_{\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]\otimes_{\mathcal{O}_L} L}(X_{f,\mathcal{K},\xi})$$
is true as fractional ideals of $\mathcal{O}_L[[\Gamma_{\mathcal{K}}]] \otimes_{\mathcal{O}_L} L$.

This result generalizes the main conjecture proved in [38] to the non-ordinary case. Let us make a comparison between the situation here (and [38]) and the situation in [36]. The argument of [36] using the group $\mathrm{U}(2,2)$ is technically less complicated but there are essential difficulties to prove the corresponding main conjecture when f is non-ordinary along that line. In contrast, in our situation the ordinary and non-ordinary cases can be treated in a similar manner. One reason is the local Selmer conditions are defined in an uniform way. The other reason is that the theory for "partially ordinary" families of forms on $\mathrm{U}(3,1)$ we will develop later, when restricting to a two-dimensional subspace of the weight space is essentially a "Hida theory", enabling us to prove everything in almost the same way as ordinary case. The control theorems for such forms are first proved by Hida in [13]. That is why we can prove a main conjecture over the Iwasawa algebra. Another important new input for the proof here is the construction in the joint work with E.Eischen [6] which constructs families of vector-valued Klingen Eisenstein series using differential operators.

As in [38] our definition of Selmer groups is different from the Selmer group for π since the ordinary CM forms g_{ξ_ϕ} has weight higher than π. But it can be used to study the Selmer group for π by comparing Selmer groups in different contexts. Important applications along this line includes: a proof of the $\pm$-main conjecture and rank 0 refined Birch-Swinnerton-Dyer formula for supersingular elliptic curves by the author [41]; a proof of the rank 1 refined Birch-Swinnerton-Dyer formula in a joint work with Dimitar Jetchev and Christopher Skinner ([21]); a proof of the (anticyclotomic) $\pm$-main conjecture for Heegner points in a joint work with Francesc Castella [3].

In this paper we restrict to the case when π has weight two and trivial character. The higher weight case requires some local Fourier-Jacobi computations at Archimedean places which we do not touch at the moment. The proof goes along the same line as [38] and much of the important calculations are already carried out in [38], [6]. This paper is organized as follows: in Section 2 we recall some backgrounds for automorphic forms and p-adic automorphic forms. In Section 3 we develop the theory of partially ordinary forms and families, following ideas of [37] and arguments in [15, Section 4]. In Section 4 we construct the families of Klingen Eisenstein series using the calculations in [6] with some modifications. In Section 5 we develop a technique to interpolate the Fourier-Jacobi expansion, making use of the calculations in [38], and then deduce the main result.
<u>Notations</u>
Let $G_{\mathbb{Q}}$ and $G_{\mathcal{K}}$ be the absolute Galois groups of $\mathbb{Q}$ and $\mathcal{K}$. Let Σ be a finite set of primes containing all the primes at which $\mathcal{K}/\mathbb{Q}$ or π or ξ is ramified, the primes dividing $\mathfrak{s}$, and the primes such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is compact. Let Σ^1 and Σ^2, respectively be the set of non-split primes in Σ such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is non-compact, and compact. Let $\Gamma^\pm_{\mathcal{K}}$ be the subgroups of $\Gamma_{\mathcal{K}}$ such that the complex conjugation acts by ± 1. We also let Γ_{v_0} ($\Gamma_{\bar{v}_0}$) be one dimensional $\mathbb{Z}_p$-subspaces of $\Gamma_{\mathcal{K}}$ such that their fixed fields are maximal sub-extensions such that v_0 ($\bar{v}_0$) is unramified. We take topological generators $\gamma^\pm$ so that $\mathrm{rec}^{-1}(\gamma^+) = ((1+p)^{\frac{1}{2}}, (1+p)^{\frac{1}{2}})$ and $\mathrm{rec}^{-1}(\gamma^-) = ((1+p)^{\frac{1}{2}}, (1+p)^{-\frac{1}{2}})$ where $\mathrm{rec} : \mathbb{A}^\times_{\mathcal{K}} \to G^{\mathrm{ab}}_{\mathcal{K}}$ is the reciprocity map normalized by the geometric Frobenius. Let $\Psi = \Psi_{\mathcal{K}}$ be the composition $G_{\mathcal{K}} \twoheadrightarrow \Gamma_{\mathcal{K}} \hookrightarrow \mathbb{Z}_p[[\Gamma_{\mathcal{K}}]]^\times$ and $\Psi^\pm : G_{\mathcal{K}} \twoheadrightarrow \Gamma^\pm_{\mathcal{K}} \hookrightarrow \mathbb{Z}_p[[\Gamma^\pm_{\mathcal{K}}]]^\times$. We also let $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$ and let $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. Define $\Psi_{\mathbb{Q}}$ to be the composition $G_{\mathbb{Q}} \twoheadrightarrow \Gamma_{\mathbb{Q}} \hookrightarrow \mathbb{Z}_p[[\Gamma_{\mathbb{Q}}]]^\times$. We also define $\varepsilon_{\mathcal{K}}$ and $\varepsilon_{\mathbb{Q}}$ to be the compositions $\mathcal{K}^\times\backslash\mathbb{A}^\times_{\mathcal{K}} \stackrel{rec}{\to} G^{ab}_{\mathcal{K}} \to \mathbb{Z}_p[[\Gamma_{\mathcal{K}}]]^\times$ and $\mathbb{Q}^\times\backslash\mathbb{A}^\times_{\mathbb{Q}} \stackrel{rec}{\to} G^{ab}_{\mathbb{Q}} \to \mathbb{Z}_p[[\Gamma_{\mathbb{Q}}]]^\times$ where the second arrows are the $\Psi_{\mathcal{K}}$ and $\Psi_{\mathbb{Q}}$ defined above. Let ω and ϵ be the Techimuller character and the cyclotomic character. We will usually take a finite extension $L/\mathbb{Q}_p$ and write $\mathcal{O}_L$ for its integer ring and ϖ_L for a uniformizer. Let $\Lambda_{\mathcal{K},\mathcal{O}_L} = \mathcal{O}_L[[\Gamma_{\mathcal{K}}]]$ and similarly for $\Lambda^\pm_{\mathcal{K}}$. If D is a quaternion algebra, we will sometimes write $[D^\times]$ for $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_{\mathbb{Q}})$. We similarly write $[\mathrm{U}(2)]$, $[\mathrm{GU}(2,0)]$, etc. We also define $S_n(R)$ to be the set of $n\times n$ Hermitian matrices with entries in $\mathcal{O}_{\mathcal{K}}\otimes_{\mathbb{Z}} R$. Finally we define $G_n = \mathrm{GU}(n,n)$ for the unitary similitude group for the skew-Hermitian matrix $\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$ and $\mathrm{U}(n,n)$ for the corresponding unitary groups. We write $e_{\mathbb{A}} = \prod_v e_v$ where for each place v of $\mathbb{Q}$ and e_v is the usual exponential map at v. We refer to [15] for the discussion of the CM period Ω_∞ and the p-adic period Ω_p. For two automorphic forms f_1, f_2 on $\mathrm{U}(2)$ we write $\langle f_1, f_2\rangle = \int_{[\mathrm{U}(2)]} f_1(g)f_2(g)dg$ (we use Shimura's convention for the Harr measures).

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2 Backgrounds

2.1 Groups

We denote $G = \mathrm{GU}(3,1)$ as the unitary similitude group associated to the metric $\begin{pmatrix} & & 1 \\ & \zeta & \\ -1 & & \end{pmatrix}$ where $\zeta = \begin{pmatrix} \mathfrak{s}\delta & \\ & \delta \end{pmatrix}$ for $\delta \in \mathcal{K}$ a purely imaginary element such that $i^{-1}\delta > 0$ and $0 \neq \mathfrak{s} \in \mathbb{Z}_+$. Let $\mathrm{GU}(2,0)$ be the unitary similitude group with the metric ζ. We denote μ the similitude character in both cases. Let $\mathrm{U}(3,1) \subset G$ and $\mathrm{U}(2,0) \subset \mathrm{GU}(2,0)$ the corresponding unitary groups. Let W be the Hermitian space for $\mathrm{U}(2,0)$ and $V = W \oplus X_{\mathcal{K}} \oplus Y_{\mathcal{K}}$ be that for $\mathrm{U}(3,1)$ such that the metric is given by the above form $(X_{\mathcal{K}} \oplus Y_{\mathcal{K}})$ is a skew Hermitian space with the metric $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$). Let P be the stablizer of the flag $\{0\} \subset X_{\mathcal{K}} \subset V$ in G. It consists of matrices of the form $\begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & & & \times \end{pmatrix}$.

Let N_P be the unipotent radical of P and let

$$M_P := \mathrm{GL}(X_{\mathcal{K}}) \times \mathrm{GU}(W) \hookrightarrow GU(V), (a, g_1) \mapsto \mathrm{diag}(a, g_1, \mu(g_1)\bar{a}^{-1})$$

be the levi subgroup and Let $G_P := \mathrm{GU}(W) \hookrightarrow \mathrm{diag}(1, g_1, \mu(g))$. Let δ_P be the modulus character for P. We usually use a more convenient character δ such that $\delta^3 = \delta_P$.

The group $\mathrm{GU}(2,0)$ is closely related to a division algebra. Put

$$D = \{g \in M_2(\mathcal{K}) | gg^* = \det(g)\},$$

then D is a definite quaternion algebra over $\mathbb{Q}$ with local invariants $\mathrm{inv}_v(D) = (-\mathfrak{s}, -D_{\mathcal{K}/\mathbb{Q}})_v$. For each finite place v we write D_v^1 for the set of elements $g_v \in D_v^\times$ such that $|\mathrm{Nm}(g_v)|_v = 1$ where Nm is the reduced norm.
Since p splits as $v_0\bar{v}_0$ in $\mathcal{K}$, $\mathrm{U}(3,1)(\mathbb{Z}_p) \simeq \mathrm{GL}_4(\mathbb{Z}_p)$ using the projection onto the factor of $\mathcal{O}_{v_0} \simeq \mathbb{Z}_p$. Let B and N be the upper triangular Borel and the unipotent radical of B. Let $K_p = \mathrm{GU}(3,1)(\mathbb{Z}_p)$ and K_0^n be the subgroups of K consisting of matrices whose projection onto the $\mathcal{O}_{v_0}$ factor is upper triangular modulo p^n. Let $K_1^n \subset K_0^n$ consist of matrices whose diagonal elements are 1 modulo p^n.

2.2 Hermitian Spaces and Automorphic Forms

Suppose $(r,s) = (3,3)$ or $(3,1)$ or $(2,0)$, then the unbounded Hermitian symmetric domain for $\mathrm{GU}(r,s)$ is

$$X^+ = X_{r,s} = \{\tau = \begin{pmatrix} x \\ y \end{pmatrix} | x \in M_s(\mathbb{C}), y \in M_{(r-s)\times s}(\mathbb{C}), i(x^* - x) > iy^*\zeta^{-1}y\}.$$

We use x_0 to denote the Hermitian symmetric domain for $\mathrm{GU}(2)$, which is just a point. We have the following embedding of Hermitian symmetric domains:

$$\iota : X_{3,1} \times X_{2,0} \hookrightarrow X_{3,3}$$

$$(\tau, x_0) \hookrightarrow Z_\tau,$$

where $Z_\tau = \begin{pmatrix} x & 0 \\ y & \frac{\zeta}{2} \end{pmatrix}$ for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $G = \mathrm{GU}(r,s)$ and $H = \mathrm{GL}_r \times \mathrm{GL}_s$. If $s \neq 0$ we define a cocycle: $J : R_{F/\mathbb{Q}}G(\mathbb{R})^+ \times X^+ \to \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_s(\mathbb{C}) := H(\mathbb{C})$ by $J(\alpha,\tau) = (\kappa(\alpha,\tau), \mu(\alpha,\tau))$ where for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\alpha = \begin{pmatrix} a & b & c \\ g & e & f \\ h & l & d \end{pmatrix}$,

$$\kappa(\alpha,\tau) = \begin{pmatrix} \bar{h}^t x + \bar{d} & \bar{h}^t y + l\bar{\zeta} \\ -\bar{\zeta}^{-1}(\bar{g}^t x + \bar{f}) & -\bar{\zeta}^{-1}\bar{g}^t y + \bar{\zeta}^{-1}\bar{e}\bar{\zeta} \end{pmatrix},\ \mu(\alpha,\tau) = hx + ly + d$$

in the $\mathrm{GU}(3,1)$ case and

$$\kappa(\alpha,\tau) = \bar{h}^t x + \bar{d},\ \ \mu(\alpha,\tau) = hx + d$$

in the $\mathrm{GU}(3,3)$ case. Let i be the point $\begin{pmatrix} i1_s \\ 0 \end{pmatrix}$ on the Hermitian symmetric domain for $\mathrm{GU}(r,s)$ (here 0 means the $(r-s)\times s$ 0-matrix). Let $\mathrm{GU}(r,s)(\mathbb{R})^+$ be the subgroup of elements of $\mathrm{GU}(r,s)(\mathbb{R})$ whose similitude factors are positive. Let K_∞^+ be the compact subgroup of $\mathrm{U}(r,s)(\mathbb{R})$ stabilizing i and let K_∞ be the groups generated by K_∞^+ and $\mathrm{diag}(1_{r+s}, -1_s)$. Then $J : K_\infty^+ \to H(\mathbb{C}), k_\infty \mapsto J(k_\infty, i)$ defines an algebraic representation of K_∞^+.

Definition 2.1. *A weight $\underline{k}$ is defined by a set $\{\underline{k}\}$ where each*

$$\underline{k} = (a_1, \cdots, a_r; b_1, \cdots, b_s)$$

with $a_1 \geq \cdots \geq a_r \geq -b_1 + r + s \geq \cdots \geq -b_s + r + s$ for the a_i, b_i's in $\mathbb{Z}$

As in [15], we define some rational representations of GL_r. Let R be an $\mathbb{Z}$ algebra. For a weight $\underline{k}$, we define the representation with minimal weight $-\underline{k}$ by

$$L_{\underline{k}}(R) = \{f \in \mathcal{O}_{\mathrm{GL}_r} | f(tn_+g) = k^{-1}(t)f(g), t \in T_r \times T_s, n_+ \in N_r \times {}^tN_s\}.$$

We define the functional $l_{\underline{k}}$ on $L_{\underline{k}}$ by evaluating at the identity. We define a model $L^{\underline{k}}(\mathbb{C})$ of the representation $H(\mathbb{C})$ with the highest weight $\underline{k}$ as follows. The underlying space of $L^{\underline{k}}(\mathbb{C})$ is $L_{\underline{k}}(\mathbb{C})$ and the group action is defined by

$$\rho^{\underline{k}}(h) = \rho_{\underline{k}}({}^th^{-1}), h \in H(\mathbb{C}).$$

For a weight $\underline{k}$, define $\|\underline{k}\| = \{\|\underline{k}\|\}$ by:

$$\|\underline{k}\| := a_1 + \cdots + a_r + b_1 + \cdots + b_s$$

and $|\underline{k}|$ by:

$$|\underline{k}| = (b_1 + \cdots + b_s)\cdot\sigma + (a_1 + \cdots + a_r)\cdot\sigma c.$$

Here σ is the Archimedean place of $\mathcal{K}$ determined by our fixed embedding $\mathcal{K} \hookrightarrow \mathbb{C}$. Let χ be a Hecke character of $\mathcal{K}$ with infinite type $|\underline{k}|$, i.e. the Archimedean part of χ is given by:

$$\chi(z_\infty) = (z^{(b_1+\cdots+b_s)} \cdot \bar{z}^{(a_1+\cdots+a_r)}).$$

Definition 2.2. *Let U be an open compact subgroup in $G(\mathbb{A}_{F,f})$. We denote by $M_{\underline{k}}(U,\mathbb{C})$ the space of holomorphic $L^{\underline{k}}(\mathbb{C})$-valued functions f on $X^+ \times G(\mathbb{A}_{F,f})$ such that for $\tau \in X^+$, $\alpha \in G(F)^+$ and $u \in U$ we have:*

$$f(\alpha\tau, \alpha g u) = \mu(\alpha)^{-\|\underline{k}\|}\rho^{\underline{k}}(J(\alpha,\tau))f(\tau,g).$$

If $r = s = 1$ then there is also a moderate growth condition.

Now we consider automorphic forms on unitary groups in the adelic language. The space of automorphic forms of weight $\underline{k}$ and level U with central character χ consists of smooth and slowly increasing functions $F : G(\mathbb{A}_F) \to L_{\underline{k}}(\mathbb{C})$ such that for every $(\alpha, k_\infty, u, z) \in G(F) \times K_\infty^+ \times U \times Z(\mathbb{A}_F)$,

$$F(z\alpha g k_\infty u) = \rho^{\underline{k}}(J(k_\infty, \boldsymbol{i})^{-1})F(g)\chi^{-1}(z).$$

2.3 Galois representations Associated to Cuspidal Representations

In this section we follow [35] to state the result of associating Galois representations to cuspidal automorphic representations on $\mathrm{GU}(r,s)(\mathbb{A}_F)$. Let $n = r + s$. First of all let us fix the notations. Let $\bar{\mathcal{K}}$ be the algebraic closure of $\mathcal{K}$ and let $G_{\mathcal{K}} := \mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$. For each finite place v of $\mathcal{K}$ let $\bar{\mathcal{K}}_v$ be an algebraic closure of $\mathcal{K}_v$ and fix an embedding $\bar{\mathcal{K}} \hookrightarrow \bar{\mathcal{K}}_v$. The latter identifies $G_{\mathcal{K}_v} := \mathrm{Gal}(\bar{\mathcal{K}}_v/\mathcal{K}_v)$ with a decomposition group for v in $G_{\mathcal{K}}$ and hence the Weil group $W_{\mathcal{K}_v} \subset G_{\mathcal{K}_v}$ with a subgroup of $G_{\mathcal{K}}$. Let π be a holomorphic cuspidal irreducible representation of $\mathrm{GU}(r,s)(\mathbb{A}_F)$ with weight $\underline{k} = (a_1, \cdots, a_r; b_1, \cdots, b_s)$ and central character χ_π. Let $\Sigma(\pi)$ be a finite set of primes of F containing all the primes at which π is unramified and all the primes dividing p. Then for some L finite over $\mathbb{Q}_p$, there is a Galois representation (by [33], [28] and [35]):

$$R_p(\pi) : G_{\mathcal{K}} \to \mathrm{GL}_n(L)$$

such that:
(a)$R_p(\pi)^c \simeq R_p(\pi)^\vee \otimes \rho_{p,\chi_\pi^{1+c}}\epsilon^{1-n}$ where χ_π is the central character of π, $\rho_{p,\chi_\pi^{1+c}}$ denotes the associated Galois character by class field theory and ϵ is the cyclotomic character.
(b)$R_p(\pi)$ is unramified at all finite places not above primes in $\Sigma(\pi) \cup \{$ primes dividing p), and for such a place w:

$$\det(1 - R_p(\pi)(\mathrm{frob}_w q_w^{-s})) = L(BC(\pi)_w \otimes \chi^c_{\pi,w}, s + \frac{1-n}{2})^{-1}$$

Here the frob_w is the geometric Frobenius and BC means the base change from $\mathrm{U}(r,s)$ to GL_{r+s}. We write V for the representation space and it is possible to take a Galois stable $\mathcal{O}_L$ lattice which we denote as T.

2.4 The Main Conjecture

Before formulating the main conjecture we first define the characteristic ideals and the Fitting ideals. We let A be a Noetherian ring. We write $\mathrm{Fitt}_A(X)$ for the Fitting ideal in A of a finitely generated A-module X. This is the ideal generated by the determinant of the $r \times r$ minors of the matrix giving the first arrow in a given presentation of X:

$$A^s \to A^r \to X \to 0.$$

If X is not a torsion A-module then $\mathrm{Fitt}(X) = 0$.

Fitting ideals behave well with respect to base change. For $I \subset A$ an ideal, then:

$$\mathrm{Fitt}_{A/I}(X/IX) = \mathrm{Fitt}_A(X) \bmod I$$

Now suppose A is a Krull domain (a domain which is Noetherian and normal), then the characteristic ideal is defined by:

$$\mathrm{char}_A(X) := \{x \in A : \mathrm{ord}_Q(x) \geq \mathrm{length}_Q(X) \text{ for any } Q \text{ a height one prime of } A\},$$

Again if X is not torsion then we define $\mathrm{char}_A(X) = 0$.

We consider the Galois representation:

$$V_{f,\mathcal{K},\xi} := \rho_f \sigma_{\bar{\xi}^c} \epsilon^{\frac{4-\kappa}{2}} \otimes \Lambda_{\mathcal{K}}(\Psi_{\mathcal{K}}^{-c}).$$

Define the Selmer group to be:

$$\mathrm{Sel}_{f,\mathcal{K},\xi} := \ker\{H^1(\mathcal{K}, T_{f,\mathcal{K},\xi} \otimes \mathcal{O}_L[[\Gamma_{\mathcal{K}}]]^*) \to H^1(I_{\bar{v}_0}, T_{f,\mathcal{K},\xi} \otimes \mathcal{O}_L[[\Gamma_{\mathcal{K}}]]^*) \times \prod_{v \nmid p} H^1(I_v, T_{f,\mathcal{K},\xi} \otimes \mathcal{O}_L[[\Gamma_{\mathcal{K}}]]^*)\}$$

where $*$ means Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(-, \mathbb{Q}_p/\mathbb{Z}_p)$ and the Σ-primitive Selmer groups:

$$\mathrm{Sel}^{\Sigma}_{f,\mathcal{K},\xi} := \ker\{H^1(\mathcal{K}, T_{f,\mathcal{K},\xi} \otimes \mathcal{O}_L[[\Gamma_{\mathcal{K}}]]^*) \to H^1(I_{\bar{v}_0}, T_{f,\mathcal{K},\xi} \otimes \mathcal{O}_L[[\Gamma_{\mathcal{K}}]]^*) \times \prod_{v \notin \Sigma} H^1(I_v, T_{f,\mathcal{K},\xi} \otimes \mathcal{O}_L[[\Gamma_{\mathcal{K}}]]^*)\}$$

and

$$X^{\Sigma}_{f,\mathcal{K},\xi} := (\mathrm{Sel}^{\Sigma}_{f,\mathcal{K},\xi})^*.$$

We are going to define the p-adic L-functions $\mathcal{L}_{f,\mathcal{K},\xi}$ and $\mathcal{L}^{\Sigma}_{f,\mathcal{K},\xi}$ in section 6. The two-variable Iwasawa main conjecture and its Σ-imprimitive version state that (see [8]):

$$\mathrm{char}_{\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]} X_{f,\mathcal{K},\xi} = (\mathcal{L}_{f,\mathcal{K},\xi}),$$

$$\mathrm{char}_{\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]} X^{\Sigma}_{f,\mathcal{K},\xi} = (\mathcal{L}^{\Sigma}_{f,\mathcal{K},\xi}).$$

3 Hida Theory for Partially-Ordinary Forms

3.1 Shimura varieties and Igusa varieties

For Unitary Similitude Groups

We will be brief in the following and refer the details to [15, Section 2, 3] (see also [6, Section 2]). We consider the group $\mathrm{GU}(3,1)$. For any level group $K = \prod_v K_v$ of $\mathrm{GU}(3,1)(\mathbb{A}_f)$ whose p-component is $\mathrm{GU}(3,1)(\mathbb{Z}_p)$, we refer to [15] for the definitions and arithmetic models of Shimura varieties which we denote as $S_G(K)$ over a ring $\mathcal{O}_{v_0}$ which is the localization of the integer ring of the reflex field at the ideal v_0. It parameterizes quadruples $\underline{A} = (A, \bar{\lambda}, \iota, \bar{\eta}^p)$ where A is an abelian scheme with CM by $\mathcal{O}_{\mathcal{K}}$ given by ι, $\bar{\lambda}$ is an orbit of prime to p polarizations, $\bar{\eta}^p$ is an orbit of prime to p level structure. Let $\mathcal{O}_m := \mathcal{O}_{v_0}/p^m\mathcal{O}_{v_0}$. We define the set of cusp labels by:

$$C(K) := (\mathrm{GL}(X_{\mathcal{K}}) \times G_P(\mathbb{A}_f))N_P(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K.$$

This is a finite set. We denote $[g]$ for the class represented by $g \in G(\mathbb{A}_f)$. For each such g whose p-component is 1 we define $K_P^g = G_P(\mathbb{A}_f) \cap gKg^{-1}$ and denote $S_{[g]} := S_{G_P}(K_P^g)$ the corresponding Shimura variety for the group G_P with level group K_P^g. By the strong approximation we can choose a set $\underline{C}(K)$ of representatives of $C(K)$ consisting of elements $g = pk^0$ for $p \in P(\mathbb{A}_f^{(pN_0)})$ and $k^0 \in K^0$ for K^0 the maximal compact defined in [15, Section 2]. There is also a theory of compactifications of $S_G(K)$ developed in [26]. We denote $\bar{S}_G(K)$ the toroidal compactification and $S_G^*(K)$ the minimal compactification. We omit the details.

Now we recall briefly the notion of Igusa varieties in [15] section 2. Let M be a standard lattice of V and $M_p = M \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let $\mathrm{Pol}_p = \{N^{-1}, N^0\}$ be a polarization of M_p. Recall that this is a polarization if N^{-1} and N^0 are maximal isotropic submodules in M_p and they are dual to each other with respect to the Hermitian metric on V and also that:

$$\mathrm{rank} N_{v_0}^{-1} = \mathrm{rank} N_{\bar{v}_o}^{0} = 3, \mathrm{rank} N_{\bar{v}_0}^{-1} = \mathrm{rank} N_{v_0}^{0} = 1.$$

The Igusa variety of level p^n is the scheme representing the usual quadruple for Shimura variety together with a

$$j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$$

where A is the abelian variety in the quadruple we use to define the arithmetic model of the Shimura variety. Note that the existence of j implies that if p is nilpotent in the base ring then A must be ordinary. There is also a definition for Igusa varieties over $\bar{S}_G(K)$ (see [15, 2.7.6]). Let K^n, K_0^n and K_1^n be the subset of $\mathbf{H}$ consisting of matrices which are in $\{\mathrm{Id}\}$, $B_3 \times {}^tB_1$ or $N_3 \times {}^tN_1$ modulo p^n. (These notations are already used for level groups of automorphic forms. The reason for using same notation is p-adic automorphic forms with level group $K_\bullet^n$ correspond to automorphic forms of level group $K_\bullet^n$). We denote $I_G(K^n)$, $I_G(K_1^n)$ and $I_G(K_0^n)$ the Igusa varieties with the corresponding level groups over $\bar{S}_G(K)$. We can define the Igusa varieties for G_P as well. There also defined $\mathcal{Z}_{[g]}$ a group scheme over $S_{[g]}$ and $\mathcal{Z}_{[g]}^\circ$ the connected component of $\mathcal{Z}_{[g]}$ (over $S_{[g]}$). For any β in a sub-lattice of $\mathbb{Q}$ (depending on K) there is a line bundle $\mathcal{L}(\beta)$ on $\mathcal{Z}_{[g]}^\circ$. Let $\mathbf{H} := \mathrm{GL}_3(\mathbb{Z}_p) \times \mathrm{GL}_1(\mathbb{Z}_p)$ be the Galois group of the Igusa tower over the ordinary locus of the Shimura variety.

Definition 3.1. *We define the p-adic cusps to be the set of pairs $([g], h)$ for $[g]$ being a cusp label and $h \in \mathbf{H}$. They can be thought of as cusps on the Igusa tower.*

For $\bullet = 0, 1$ we let $K_{P,\bullet}^{g,n} := gK_\bullet^n g^{-1} \cap G_P(\mathbb{A}_f)$ and let $I_{[g]}(K_\bullet^n) := I_{G_P}(K_{P,\bullet}^{g,n})$ be the corresponding Igusa variety over $S_{[g]}$. We denote $A_{[g]}^n$ the coordinate ring of $I_{[g]}(K_1^n)$. Let $A_{[g]}^\infty = \varinjlim_n A_{[g]}^n$ and let $\hat{A}_{[g]}^\infty$ be the p-adic completion of $A_{[g]}^\infty$. This is the space of p-adic automorphic forms for the group $\mathrm{GU}(2,0)$.

<u>For Unitary Groups</u>

Assume the tame level group K is neat. For any c an element in $\mathbb{Q}_+\backslash\mathbb{A}^\times_{\mathbb{Q},f}/\mu(K)$. We refer to [15, 2.5] for the notion of c-Igusa Schemes $I^0_{\mathrm{U}(2)}(K,c)$ for the unitary groups $\mathrm{U}(2,0)$ (not the similitude group). It parameterizes quintuples $(A,\lambda,\iota,\bar{\eta}^{(p)},j)_{/S}$ similar to the Igusa Schemes for unitary similitude groups but requiring λ to be a prime to p c-polarization (see *loc.cit*) of A such that $(A,\bar{\lambda},\iota,\bar{\eta}^{(p)},j)$ is a quintuple in the definition for Shimura varieties for $\mathrm{GU}(2)$. For g_c in the class of c and let ${}^cK = g_cKg_c^{-1}\cap\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q},f})$. Then the space of forms on $I^0_{\mathrm{U}(2)}(K,c)$ is isomorphic to the space of forms on $I^0_{\mathrm{U}(2)}({}^cK,1)$.

Pullbacks

In order to use the pullback formula algebraically we need a map of Igusa schemes given by:

$$i([(A_1,\lambda_1,\iota_1,\eta_1^pK_1,j_1)],[(A_2,\lambda_2,\iota_2,\eta_2^pK_2,j_2)]) = [(A_1\times A_2,\lambda_1\times\lambda_2,\iota_1,\iota_2,(\eta_1^p\times\eta_2^p)K_3,j_1\times j_2)].$$

We define an element $\Upsilon\in\mathrm{U}(3,3)(\mathbb{Q}_p)$ such that $\Upsilon_{v_0}=S^{-1}$ and $\Upsilon'_{v_0}=S'^{-1}$ where S and S' will be defined in section 4.3. Similar to [15], we know that taking the change of polarization into consideration the above map is given by

$$i([\tau,g],[x_0,h]) = [Z_\tau,(g,h)\Upsilon].$$

Fourier-Jacobi Expansions

Define $N_H^1 := \{\begin{pmatrix}1 & 0\\ * & 1_2\end{pmatrix}\}\times\{1\}\subset H$. For an automorphic form or p-adic automorphic form F on $\mathrm{GU}(3,1)$ we refer to [6, Section 2.8] for the notion of analytic Fourier-Jacobi expansions

$$FJ_P(g,f) = a_0(g,f)+\sum_\beta a_\beta(y,g,f)q^\beta$$

at $g\in\mathrm{GU}(3,1)(\mathbb{A}_\mathbb{Q})$ for $a_\beta(-,g,f):\mathbb{C}^2\to L_{\underline{k}}(\mathbb{C})$ being theta functions with complex multiplication, and algebraic Fourier-Jacobi expansion

$$FJ^h_{[g]}(f)_{N_H^1} = \sum_\beta a^h_{[g]}(\beta,f)q^\beta,$$

at a p-adic cusp $([g],h)$, and $a^h_{[g]}(\beta,f)\in L_{\underline{k}}(A^\infty_{[g]})_{N_H^1}\otimes_{A_{[g]}}H^0(\mathcal{Z}^\circ_{[g]},\mathcal{L}(\beta))$. We define the Siegel operator to be taking the 0-th Fourier-Jacobi coefficient as in *loc.cit.* Over $\mathbb{C}$ the analytic Fourier-Jacobi expansion for a holomorphic automorphic form f is given by:

$$FJ_\beta(f,g) = \int_{\mathbb{Q}\backslash\mathbb{A}} f(\begin{pmatrix}1 & & n\\ & 1_2 & \\ & & 1\end{pmatrix}g)e_\mathbb{A}(-\beta n)dn.$$

3.2 p-adic modular forms

As in [15] let $\bar{H}_{p-1}$ be the Hasse invariant $H^0(S_G(K)_{/\bar{\mathbb{F}}},\det(\underline{\omega})^{p-1})$. Over the minimal compactification some power (say the tth) of the Hasse invariant can be lifted to $\mathcal{O}_{v_0}$, which we denote as E. By the Koecher principle we can regard it as in $H^0(\bar{S}_G(K),\det(\underline{\omega}^{t(p-1)})$. Set $T_{0,m} := \bar{S}_G(K)[1/E]_{/\mathcal{O}_m}$.

For any positive integer n define $T_{n,m} := I_G(K^n)_{/\mathcal{O}_m}$ and $T_{\infty,m} = \varprojlim_n T_{n,m}$. Then $T_{\infty,m}$ is a Galois cover over $T_{0,m}$ with Galois group $\mathbf{H} \simeq \mathrm{GL}_3(\mathbb{Z}_p) \times \mathrm{GL}_1(\mathbb{Z}_p)$ and $\mathbf{N} \subset \mathbf{H}$ the upper triangular unipotent radical. Define:

$$V_{n,m} = H^0(T_{n,m}, \mathcal{O}_{T_{n,m}}),$$

Let $V_{\infty,m} = \varinjlim_n V_{n,m}$ and $V_{\infty,\infty} = \varprojlim_m V_{\infty,m}$. We also define $W_{n,m} = V_{n,m}^{\mathbf{N}}$, $W_{\infty,m} = V_{\infty,m}^{\mathbf{N}}$ and $\mathcal{W} = \varinjlim_n \varinjlim_m W_{n,m}$. We define $V_{n,m}^0$, etc, to be the cuspidal part of the corresponding spaces.

We can do similar thing for the definite unitary similitude groups G_P as well and define $V_{n,m,P}$,$V_{\infty,m,P}$, $V_{\infty,\infty,P}$, $V_{n,m,P}^N$, $\mathcal{W}_P$, etc.

3.3 Partially Ordinary Forms

3.3.1 Definitions

In this subsection we develop a theory for families of "partially ordinary" forms over a two dimensional weight space (the whole weight space for $\mathrm{U}(3,1)$ is three dimensional). The idea goes back to the work of Hida [13] (also [37]) where they defined the concept of being ordinary with respect to different parabolic subgroups (the usual definition of ordinary is with respect to the Borel subgroup), except that we are working with coherent cohomology while Hida and Tilouine-Urban used group cohomology. The crucial point is, our families are over the two dimensional Iwasawa algebra, which is similar to Hida theory for ordinary forms (instead of Coleman-Mazur theory for finite slope forms). Our argument here will mostly be an adaption of the argument in the ordinary case in [15] and we will sometimes be brief and refer to *loc.cit* for some computations so as not to introduce too many notations.

We always use the identification $\mathrm{U}(3,1)(\mathbb{Q}_v) \simeq \mathrm{GL}_4(\mathbb{Q}_p)$. We define $\alpha_i = \mathrm{diag}(1_{4-i}, p \cdot 1_i)$. We let $\alpha = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p^2 \end{pmatrix}$ and refer to [15, 3.7, 3.8] for the notion of Hida's U_α and U_{α_i} operators associated to α or α_i. We define $e_\alpha = \lim_{n\to\infty} U_\alpha^{n!}$. We are going to study forms and families invariant under e_α and call them "partially ordinary" forms. Suppose π is an irreducible automorphic representation on $\mathrm{U}(3,1)$ with weight $\underline{k}$ and suppose that π_p is an unramified principal series representation. If we write $\kappa_1 = b_1$ and $\kappa_i = -a_{5-i} + 5 - i$ for $2 \le i \le 4$, then there is a partially ordinary vector in π if and only if we can re-order the Satake parameters as $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that

$$\mathrm{val}_p(\lambda_3) = \kappa_3 - \frac{3}{2}, \mathrm{val}_p(\lambda_4) = \kappa_4 - \frac{3}{2}.$$

Galois Representations

The Galois representations associated to cuspidal automorphic representation π in subsection 2.3 which is unramified and partially ordinary at p for e_α has the following description when restricting to G_{v_0}:

$$R_p(\pi)|_{G_{v_0}} \simeq \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & \xi_{2,v}\epsilon^{-\kappa_2} & * \\ & & & \xi_{1,v}\epsilon^{-\kappa_1} \end{pmatrix} \quad (1)$$

where $\xi_{1,v}$ and $\xi_{2,v}$ are unramified characters and also

$$R_p(\pi)|_{G_{\bar{v}_0}} \simeq \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}.$$

This can be proved by noting that the Newton Polygon and the Hodge Polygon have four out of five vertices coincide (see [37, Proposition7.1]).

3.3.2 Control Theorems

We define $K_0(p,p^n)$ to be the level group with the same components at primes outside p as K outside p and, at p, consists of matrices which are of the form $\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$ modulo p and are of the form $\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$ modulo p^n. We are going to prove some control theorems for the level group $K_0(p,p^n)$. These will be enough to show that the Eisenstein series constructed in [6] do give families in the sense here. (See Section 4.) We refer the definition of the automorphic sheaves $\omega_{\underline{k}}$ and the subsheaf to [15, section 3.2]. There also defined a $\omega_{\underline{k}}^\flat$ in Section 4.1 of *loc.cit* as follows. Let $\mathcal{D} = \bar{S}_G(K) - S_G(K)$ be the boundary of the toroidal compactification and $\underline{\omega}$ the pullback to identity of the relative differential of the Raynaud extension of the universal Abelian variety. Let $\underline{k}'' = (a_1 - a_3, a_2 - a_3)$. Let $\mathcal{B}$ be the abelian part of the Mumford family of the boundary. Its relative differential is identified with a subsheaf of $\underline{\omega}|_{\mathcal{D}}$. The $\omega_{\underline{k}}^\flat \subset \omega_{\underline{k}}$ is defined to be $\{s \in \omega_{\underline{k}} | s|_{\mathcal{D}} \in \mathscr{F}_{\mathcal{D}}\}$ for $\mathscr{F}_{\mathcal{D}} := \det(\underline{\omega}|_{\mathcal{D}})^{a_3} \otimes \underline{\omega}_{\mathcal{B}}^{\underline{k}''}$, where the last term means the automorphic sheaf of weight $\underline{k}''$ for $\mathrm{GU}(2,0)$.

Weight Space

Let $H = \mathrm{GL}_3 \times \mathrm{GL}_1$ and T be the diagonal torus. Then $\mathbf{H} = H(\mathbb{Z}_p)$. We let $\Lambda_{3,1} = \Lambda$ be the completed group algebra $\mathbb{Z}_p[[T(1+\mathbb{Z}_p)]]$. This is a formal power series ring with four variables. There is an action of $T(\mathbb{Z}_p)$ given by the action on the $j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$. (see [15, 3.4]) This gives the space of p-adic modular forms a structure of Λ-algebra. A $\bar{\mathbb{Q}}_p$-point ϕ of SpecΛ is call arithmetic if it is determined by a character $[\underline{k}].[\zeta]$ of $T(1+p\mathbb{Z}_p)$ where $\underline{k}$ is a weight and $\zeta = (\zeta_1, \zeta_2, \zeta_3; \zeta_4)$ for $\zeta_i \in \mu_{p^\infty}$. Here $[\underline{k}]$ is the character by regarding $\underline{k}$ as a character of $T(1+\mathbb{Z}_p)$ by $[\underline{k}](t_1,t_2,t_3,t_4) = (t_1^{a_1}t_2^{a_2}t_3^{a_3}t_4^{-b_1})$ and $[\zeta]$ is the finite order character given by mapping $(1+p\mathbb{Z}_p)$ to ζ_i at the corresponding entry t_i of $T(\mathbb{Z}_p)$. We often write this point $\underline{k}_\zeta$. We also define $\omega^{[\underline{k}]}$ a character of the torsion part of $T(\mathbb{Z}_p)$ (isomorphic to $(\mathbb{F}_p^\times)^4$) given by $\omega^{[\underline{k}]}(t_1,t_2,t_3,t_4) = \omega(t_1^{a_1}t_2^{a_2}t_3^{a_3}t_4^{-b_1})$.

Definition 3.2. *We fix $\underline{k}' = (a_1, a_2)$ and $\rho = L_{\underline{k}'}$. Let $\mathcal{X}_\rho$ be the set of arithmetic points $\phi \in$ $\mathrm{Spec}\Lambda_{3,1}$ corresponding to weight $(a_1, a_2, a_3; b_1)$ such that $a_2 \geq a_3 \geq -b_1 + 4$. (The ζ-part being trivial). Let $\mathrm{Spec}\tilde{\Lambda} = \mathrm{Spec}\tilde{\Lambda}_{(a_1,a_2)}$ be the Zariski closure of $\mathcal{X}_\rho$.*

We define for $q = 0, \flat$

$$V^q_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m) := \{f \in H^0(T_{n,m}, \omega^q_{\underline{k}}), g \cdot f = [\underline{k}]\omega^{[\underline{k}]}\}.$$

(Note the "ω"-part of the nebentypus).
As in [15, 3.3] we have a canonical isomorphism given by taking the "p-adic avartar"

$$H^0(T_{n,m}, \omega_{\underline{k}}) \simeq V_{n,m} \otimes L_{\underline{k}}, f \mapsto \hat{f}$$

and $\beta_{\underline{k}} : V_{\underline{k}}(K^n_1, \mathcal{O}_m) \to V^{\mathbf{N}}_{n,m}$ by $f \mapsto \beta_{\underline{k}}(f) := l_{\underline{k}}(\hat{f})$. The following lemma is [15, lemma 4.2].

Lemma 3.3. *Let $q \in \{0, \flat\}$ and let $V^q_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m) := H^0(T_{n,m}, \omega^q_{\underline{k}})^{K_0(p,p^n)}$. Then we have*

$$H^0(I_{\mathcal{S}}, \omega^q_{\underline{k}}) \otimes \mathcal{O}_m = V^q_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m).$$

We record a contraction property for the operator U_α.

Lemma 3.4. *If $n > 1$, then we have*

$$U_\alpha \cdot V_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m) \subset V_{\underline{k}}(K_0(p, p^{n-1}), \mathcal{O}_m).$$

The proof is the same as [15, Proposition 4.4]. The following proposition follows from the contraction property for e_α:

Proposition 3.5.

$$e_\alpha V^q_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m) = e_\alpha V_{\underline{k}}(K_0(p), \mathcal{O}_m).$$

The following lemma tells us that to study partially ordinary forms one only needs to look at the sheaf $\omega^\flat_{\underline{k}}$.

Lemma 3.6. *Let $n \geq m > 0$, then*

$$e_\alpha.V^\flat_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m) = e_\alpha \cdot V^q_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m).$$

Proof. Same as [15, lemma 4.10]. □

Similar to the $\beta_{\underline{k}}$ we define a more general $\beta_{\underline{k},\rho}$ as follows: Let ρ be the algebraic representation $L_\rho = L_{\underline{k}'}$ of GL_2 with weight $\underline{k}' = (a_1, a_2)$. We identify $L_{\underline{k}}$ with the algebraically induced representation $\mathrm{Ind}^{\mathrm{GL}_3 \times \mathrm{GL}_1}_{\mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1} \rho \otimes \chi_{a_3} \otimes \chi_{b_1}$ (χ_a means the algebraic character defined by taking the $(-a)$-th power). We define the functional $l_{\underline{k},\rho}$ taking values in $L_{\underline{k}'}$ by evaluating at identity (similar to the definition of $l_{\underline{k}}$). We define $\beta_{\underline{k},\rho}$ similar to $\beta_{\underline{k}}$ but replacing $l_{\underline{k}}$ by $l_{\underline{k},\rho}$.

Proposition 3.7. *If $n \geq m > 0$, then the morphism*

$$\beta_{\underline{k},\rho} : V_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m) \to (V_{n,m} \otimes L_\rho)^{K_0(p,p^n)}$$

is U_α-equivariant, and there is a Hecke-equivariant homomorphism $s_{\underline{k},\rho} : (V_{n,m} \otimes L_\rho)^{K_0(p,p^n)} \to$ $V_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m)$ such that $\beta_{\underline{k},\rho} \circ s_{\underline{k},\rho} = U^m_\alpha$ and $s_{\underline{k},\rho} \circ \beta_{\underline{k},\rho} = U^m_\alpha$. So the kernel and the cokernel of $\beta_{\underline{k},\rho}$ are annihilated by U^m_α.

Proof. Similar to [15, proposition 4.7]. Our $s_{\underline{k},\rho}$ is defined by for $(\underline{A},\bar{j})$ over a $\mathcal{O}_m$-algebra R,

$$s_{\underline{k},\rho}(\alpha)(\underline{A},\bar{j}) := \sum_{v_{\chi'}\in\rho\otimes\chi_{a_3}\otimes\chi_{b_1}} \sum_{u} \frac{1}{\chi_{r,1}(\alpha)} \cdot \mathrm{Tr}_{R_0^\alpha/R}(f(\underline{A}_{\alpha u}.j_{\alpha u}))\rho_{\underline{k}}(u)v_{\chi'}.$$

Here the character $\chi_{r,1}$ is defined by

$$\chi_{r,1}(\mathrm{diag}(a_1,a_2,a_3;d)) := (a_1a_2a_3)^{-1}d.$$

The $v_{\chi'}$'s form a basis of the representation $\rho\otimes\chi_{a_3}\otimes\chi_{b_1}$ which are eigenvectors for the diagonal action with eigenvalues χ''s (apparently the eigenvalues appear with multiplicity so we use the subscript χ' to denote the corresponding vector). The u runs over a set of representatives of $\alpha^{-1}N_H(\mathbb{Z}_p)\alpha\cap N_H(\mathbb{Z}_p)\backslash N_H(\mathbb{Z}_p)$. The $(\underline{A}_{\alpha u}, j_{\alpha u})$ is a certain pair with $\underline{A}_{\alpha u}$ an abelian variety admitting an isogeny to $\underline{A}$ of type α (see [15, 3.7.1] for details) and $R_0^{\alpha u}/R$ being the coordinate ring for $(\underline{A}_{\alpha u}, j_{\alpha u})$ (see 3.8.1 of *loc.cit*). Note that the twisted action of [15, Remark 3.1] $\tilde{\rho}_{\underline{k}}(\alpha^{-1})v_{\chi'}=1$ for all the χ' above. Write χ for $\chi_{a_3}\boxtimes\chi_{b_1}$. Note also that for any eigenvector $v_{\chi'}\in\mathrm{Ind}_{\mathrm{GL}_2\times\mathrm{GL}_1\times\mathrm{GL}_1}^{\mathrm{GL}_3\times\mathrm{GL}_1}\rho\otimes\chi$ for the torus action such that $v_{\chi'}\notin\rho\otimes\chi$, if we write $\alpha=\mu(p)$ for $\mu\in X_*(T)$ (the co-character group), then we have $\langle\mu,\underline{k}+\chi'\rangle<0$. So the argument of *loc.cit* works through. $\square$

The following proposition follows from the above one as [15, Proposition 4.9]. Let $\underline{k}$ and ρ be as before.

Proposition 3.8. *If $n\geq m>0$, then*

$$\beta_{\underline{k},\rho}: e_\alpha\cdot V_{\underline{k}}(K_0(p,p^n),\mathcal{O}_m)\simeq e_\alpha(V_{n,m}\otimes L_\rho)^{K_0(p,p^n)}[\underline{k}].$$

We are going to prove some control theorems and fundamental exact sequence for partially-ordinary forms along this smaller two-dimensional weight space $\mathrm{Spec}\tilde{\Lambda}$. The following proposition follows from Lemma 3.3 and Proposition 3.5 in the same way as [15, Lemma 4.10, Proposition 4.11], noting that by the contraction property the level group is actually in $K_0(p)$.

Proposition 3.9. *Let $e_\alpha.\mathcal{V}_{\underline{k}}(K_0(p,p^n)) := \varinjlim_m e_\alpha\cdot V_{\underline{k}}(K_0(p,p^n),\mathcal{O}_m)$. Then $e_\alpha.\mathcal{V}(K_0(p,p^n))$ is p-divisible and*

$$e_\alpha\cdot\mathcal{V}_{\underline{k}}(K_0(p,p^n))[p^m] = e\cdot V_{\underline{k}}(K_0(p,p^n),\mathcal{O}_m) = e_\alpha\cdot H^0(\mathcal{I}_S,\omega_{\underline{k}})\otimes\mathcal{O}_m.$$

The following proposition is crucial to prove control theorems for partially ordinary forms along the weight space $\mathrm{Spec}\tilde{\Lambda}$.

Proposition 3.10. *The dimension of $e_\alpha M_{\underline{k}}(K_0(p,p^n),\mathbb{C})$'s are uniformly bounded for all $\underline{k}\in\mathcal{X}_\rho$.*

Proof. The uniform bound for group cohomology is proved in [13, Theorem 5.1] and the bound for coherent cohomology follows by the Eichler-Shimura isomorphism. See [15, Theorem 4.18]. $\square$

The following theorem says that all partially-ordinary forms of sufficiently regular weights are classical, and can be proved in the same way as [15, Theorem 4.19] using Proposition 3.10.

Theorem 3.11. *For each weight $\underline{k}=(a_1,a_2,a_3;b_1)\in\mathcal{X}_\rho$, there is a positive integer $A(\underline{a})$ depending on $\underline{a}=(a_1,a_2,a_3)$ such that if $b_1>A(\underline{a},n)$ then the natural restriction map*

$$e_\alpha M_{\underline{k}}(K_0(p),\mathcal{O})\otimes\mathbb{Q}_p/\mathbb{Z}_p\simeq e_\alpha\cdot\mathcal{V}_{\underline{k}}(K_0(p))$$

is an isomorphism.

For $q=0,\phi$ define

$$V^q_{po}:=\mathrm{Hom}(e_\alpha.\mathcal{W}^q,\mathbb{Q}_p/\mathbb{Z}_p)\otimes_{\Lambda_{3,1}}\tilde{\Lambda}$$

$$\mathcal{M}^q_{po}(K,\tilde{\Lambda}):=\mathrm{Hom}_{\tilde{\Lambda}}(V^q_{po},\tilde{\Lambda}).$$

Thus from the finiteness results and the p-divisibility of the space of semi-ordinary p-adic modular forms, we get the Hida's control theorem

Theorem 3.12. *Let $q=0$ or ϕ. Then*

(1) V^q_{po} is a free $\tilde{\Lambda}$-module of finite rank.

(2) For any $k\in\mathcal{X}_\rho$ we have $\mathcal{M}^q_{po}(K,\tilde{\Lambda})\otimes\tilde{\Lambda}/P_{\underline{k}}\simeq e_\alpha\cdot M^q_{\underline{k}}(K,\mathcal{O})$.

The proof is same as [15, Theorem 4.21] using Proposition 3.5, 3.8, Theorem 3.11 and Proposition 3.9.

Descent to Prime to p-Level

Proposition 3.13. *Suppose $\underline{k}$ is such that $a_1=a_2=0$, $a_3\equiv b_1\equiv 0(\mathrm{mod}\ p-1)$, $a_2-a_3>>0, a_3+b_1>>0$, . Suppose $F\in e_\alpha M^0_{\underline{k}}(K_0(p),\mathbb{C})$ is an eigenform with trivial nebentypus at p whose mod p Galois representation (semi-simple) is the same as our Klingen Eisenstein series constructed in section 4. Let π_F be the associated automorphic representation. Then $\pi_{F,p}$ is unramified principal series representation.*

Proof. Similar to [15, proposition 4.17]. Let f be the GL_2 cusp form having good supersingular reduction at p in the introduction. Note that $\pi_{F,p}$ has a fixed vector for $K_0(p)$ and $\bar{\rho}_{\pi_f}|_{G_{\mathbb{Q}_p}}$ is irreducible by [4]. By the classification of admissible representations with $K_0(p)$-fixed vector we know $\pi_{F,p}$ has to be a subquotient of $\mathrm{Ind}_B^{\mathrm{GL}_4}\chi$ for χ an unramified character of $T_n(\mathbb{Q}_p)$. If this induced representation is irreducible then we are done. Otherwise recall that $\bar{\rho}^{ss}_F\simeq\bar{\rho}_f\oplus\chi_1\oplus\chi_2$ for χ_1 and χ_2 two characters. By considering the Galois representation in equation (1), if $a_2-a_3>>0, a_3+b_1>>0$, we have the upper left $\begin{pmatrix}\times&\times\\\times&\times\end{pmatrix}$ has to be reducible modulo p by the local-global compatibility at p (in the sense of Weil-Deligne representation given by D_{pst}, see e.g. [1] for details, which implies F is actually ordinary). This contradicts that $\rho_{\pi_f}|_{G_{\mathbb{Q}_p}}$ is irreducible. Thus $\pi_{F,p}$ must by unramified. $\square$

A Definition Using Fourier-Jacobi Expansion

We can define a $\tilde{\Lambda}$-adic Fourier-Jacobi expansion map for families of partially ordinary families as in [15, 4.6.1] by taking the $\tilde{\Lambda}$-dual of the Pontryagin dual of the usual Fourier-Jacobi expansion map (replacing the e's in *loc.cit* by e_α's). We also define the Siegel operators $\Phi^h_{[g]}$'s by taking the 0-th Fourier-Jacobi coefficient.

Definition 3.14. *Let A be a finite torsion free Λ-algebra. Let $\mathcal{N}_{po}(K,A)$ be the set of formal Fourier-Jacobi expansions:*

$$F=\{\sum_{\beta\in\mathscr{S}_{[g]}}a(\beta,F)q^\beta, a(\beta,F)\in A\hat{\otimes}\hat{A}^\infty_{[g]}\otimes H^0(\mathcal{Z}^\circ_{[g]},\mathcal{L}(\beta))\}_{g\in X(K)}$$

such that for a Zariski dense set $\mathcal{X}_F\subseteq\mathcal{X}_\rho$ of points $\phi\in\mathrm{Spec}A$ such that the induced point in $\mathrm{Spec}\Lambda$ is some arithmetic weight $\underline{k}_\zeta$, the specialization F_ϕ of F is the highest weight vector of the Fourier-Jacobi expansion of a partially ordinary modular form with tame level $K^{(p)}$, weight $\underline{k}$ and nebentype at p given by $[\underline{k}][\underline{\zeta}]\omega^{-[\underline{k}]}$ as a character of $K_0(p)$.

Then we have the following

Theorem 3.15.

$$\mathcal{M}_{po}(K,A)=\mathcal{N}_{po}(K,A).$$

The proof is the same as [15, Theorem 4.25]. This theorem is used to show that the construction in [6] recalled later does give a partially ordinary family in the sense of this section.

<u>Fundamental Exact Sequence</u>

Now we prove a fundamental exact sequence for partially-ordinary forms. Let $w_3'=\begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1\end{pmatrix}$.

Lemma 3.16. *Let $\underline{k}\in\mathcal{X}_\rho$ and $F\in e_\alpha M_{\underline{k}}(K_0(p,p^n),R)$ and $R\subset\mathbb{C}$. Let $W_2=\begin{pmatrix}1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1\end{pmatrix}\cup\mathrm{Id}$ be the Weyl group for $G_P(\mathbb{Q}_p)$. There is a constant A such that for any $\underline{k}\in\mathcal{X}_\rho$ such that $a_2-a_3>A, a_3+b_1>A$, for each $g\in G(\mathbb{A}_f^{(p)})$, $\Phi_{P,wg}(F)=0$ for any $w\notin W_2w_3'$.*

The lemma can be proved using the computations in the proof of [15, lemma 4.14]. Note that by partial-ordinarity and the contraction property the level group for F is actually $K_0(p)$.

The following is a partially ordinary version of [15, Theorem 4.16]. The proof is also similar (even easier since the level is in fact in $K_0(p)$ by the contraction property).

Theorem 3.17. *, noting that e_α induces identity after the Siegel operator $\hat{\Phi}^{w_3'}$. For $\underline{k}\in\mathcal{X}_\rho$, we have*

$$0\to e_\alpha\mathcal{M}^0_{\underline{k}}(K,A)\to e_\alpha\mathcal{M}_{\underline{k}}(K,A)\xrightarrow{\hat{\Phi}^{w_3'}=\oplus\hat{\Phi}^{w_3'}_{[g]}}\oplus_{g\in C(K)}\mathcal{M}_{\underline{k}'}(K^g_{P,0}(p),A)$$

is exact.

The family version of the fundamental exact sequence can be deduced from Theorem 3.11, 3.12, 3.17, as well as the affine-ness of $S^*_G(K)(1/E)$ (See [15, Theorem 4.16]).

Theorem 3.18.

$$0\to e_\alpha\mathcal{M}^0(K,A)\to e_\alpha\mathcal{M}(K,A)\xrightarrow{\hat{\Phi}^{w_3'}=\oplus\hat{\Phi}^{w_3'}_{[g]}}\oplus_{g\in C(K)}\mathcal{M}(K^g_{P,0}(p),A)\to 0.$$

4 Eisenstein Series and Families

4.1 Klingen Einstein Series

Archimedean Places

Let (π_∞, V_∞) be a finite dimensional representation of $D_\infty^\times$. Let ψ_∞ and τ_∞ be characters of $\mathbb{C}^\times$ such that $\psi_\infty|_{\mathbb{R}^\times}$ is the central character of π_∞. Then there is a unique representation π_ψ of $\mathrm{GU}(2)(\mathbb{R})$ determined by π_∞ and ψ_∞ such that the central character is ψ_∞. These determine a representation $\pi_\psi \times \tau$ of $M_P(\mathbb{R}) \simeq \mathrm{GU}(2)(\mathbb{R}) \times \mathbb{C}^\times$. We extend this to a representation ρ_∞ of $P(\mathbb{R})$ by requiring $N_P(\mathbb{R})$ acts trivially. Let $I(V_\infty) = \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\rho_\infty$ (smooth induction) and $I(\rho_\infty) \subset I(V_\infty)$ be the subspace of K_∞ -finite vectors. Note that elements of $I(V_\infty)$ can be realized as functions on K_∞. For any $f \in I(V)$ and $z \in \mathbb{C}^\times$ we define a function f_z on $G(\mathbb{R})$ by

$$f_z(g) := \delta(m)^{\frac{3}{2}+z}\rho(m)f(k), g = mnk \in P(\mathbb{R})K_\infty.$$

There is an action $\sigma(\rho, z)$ on $I(V_\infty)$ by

$$(\sigma(\rho, z)(g))(k) = f_z(kg).$$

Non-Archimedean Places

Let (π_ℓ, V_ℓ) be an irreducible admissible representation of $D^\times(\mathbb{Q}_\ell)$ and π_ℓ is unitary and tempered if D is split at ℓ. Let ψ and τ be characters of $\mathcal{K}_\ell^\times$ such that $\psi|_{\mathbb{Q}_\ell^\times}$ is the central character of π_ℓ. Then there is a unique irreducible admissible representation π_ψ of $\mathrm{GU}(2)(\mathbb{Q}_\ell)$ determined by π_ℓ and ψ_ℓ. As before we have a representation $\pi_\psi \times \tau$ of $M_P(\mathbb{Q}_\ell)$ and extend it to a representation ρ_ℓ of $P(\mathbb{Q}_\ell)$ by requiring $N_P(\mathbb{Q}_\ell)$ acts trivially. Let $I(\rho_\ell) = \mathrm{Ind}_{P(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}\rho_\ell$ be the admissible induction. We similarly define f_z for $f \in I(\rho_\ell)$ and $\rho_\ell^\vee, I(\rho_\ell^\vee), A(\rho_\ell, z, f)$, etc. For $v \notin \Sigma$ we have $D^\times(\mathbb{Q}_\ell) \simeq \mathrm{GL}_2(\mathbb{Q}_\ell)$.

Global Picture

Let $(\pi = \otimes_v \pi_v, V)$ be an irreducible unitary cuspidal automorphic representation of $D^\times(\mathbb{A}_\mathbb{Q})$ we define $I(\rho)$ to be the restricted tensor product of $\otimes_v I(\rho_v)$ with respect to the unramified vectors $f_{\varphi_\ell}^0$ for some $\varphi = \otimes_v \phi_v \in \pi$. We can define f_z, $I(\rho^\vee)$ and $A(\rho, z, f)$ similar to the local case. f_z takes values in V which can be realized as automorphic forms on $D^\times(\mathbb{A}_\mathbb{Q})$. We also write f_z for the scalar-valued functions $f_z(g) := f_z(g)(1)$ and define the Klingen Eisenstein series:

$$E(f, z, g) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f_z(\gamma g).$$

This is absolutely convergent if $\mathrm{Re} z >> 0$ and has meromorphic continuation to all $z \in \mathbb{C}$.

4.2 Siegel Eisenstein Series

Local Picture:

Our discussion in this section follows [36, 11.1-11.3] closely. Let $Q = Q_n$ be the Siegel parabolic subgroup of GU_n consisting of matrices $\begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$. It consists of matrices whose lower-left $n \times n$ block is zero. For a place v of $\mathbb{Q}$ and a character τ of $\mathcal{K}_v^\times$ we let $I_n(\tau_v)$ be the space of smooth $K_{n,v}$-finite functions (here $K_{n,v}$ means the maximal compact subgroup $G_n(\mathbb{Z}_v)$) $f : K_{n,v} \to \mathbb{C}$ such that $f(qk) = \tau_v(\det D_q)f(k)$ for all $q \in Q_n(\mathbb{Q}_v) \cap K_{n,v}$ (we write q as block matrix $q =$

$\begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$). For $z \in \mathbb{C}$ and $f \in I(\tau)$ we also define a function $f(z,-) : G_n(\mathbb{Q}_v) \to \mathbb{C}$ by $f(z,qk) := \chi(\det D_q))|\det A_q D_q^{-1}|_v^{z+n/2} f(k), q \in Q_n(\mathbb{Q}_v)$ and $k \in K_{n,v}$.

For $f \in I_n(\tau_v), z \in \mathbb{C}$, and $k \in K_{n,v}$, the intertwining integral is defined by:

$$M(z,f)(k) := \bar{\tau}_v^n(\mu_n(k)) \int_{N_{Q_n}(F_v)} f(z, w_n r k) dr.$$

For z in compact subsets of $\{\mathrm{Re}(z) > n/2\}$ this integral converges absolutely and uniformly, with the convergence being uniform in k. In this case it is easy to see that $M(z,f) \in I_n(\bar{\tau}_v^c)$. A standard fact from the theory of Eisenstein series says that this has a continuation to a meromorphic section on all of $\mathbb{C}$.
Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set. By a meromorphic section of $I_n(\tau_v)$ on $\mathcal{U}$ we mean a function $\varphi : \mathcal{U} \mapsto I_n(\tau_v)$ taking values in a finite dimensional subspace $V \subset I_n(\tau_v)$ and such that $\varphi : \mathcal{U} \to V$ is meromorphic.

Global Picture
For an idele class character $\tau = \otimes \tau_v$ of $\mathbb{A}_{\mathcal{K}}^\times$ we define a space $I_n(\tau)$ to be the restricted tensor product defined using the spherical vectors $f_v^{sph} \in I_n(\tau_v)$ (invariant under $K_{n,v}$) such that $f_v^{sph}(K_{n,v}) = 1$, at the finite places v where τ_v is unramified.

For $f \in I_n(\tau)$ we consider the Eisenstein series

$$E(f; z, g) := \sum_{\gamma \in Q_n(\mathbb{Q}) \backslash G_n(\mathbb{Q})} f(z, \gamma g).$$

This series converges absolutely and uniformly for (z,g) in compact subsets of $\{\mathrm{Re}(z) > n/2\} \times G_n(\mathbb{A}_{\mathbb{Q}})$. The defined automorphic form is called Siegel Eisenstein series.

The Eisenstein series $E(f; z, g)$ has a meromorphic continuation in z to all of $\mathbb{C}$ in the following sense. If $\varphi : \mathcal{U} \to I_n(\tau)$ is a meromorphic section, then we put $E(\varphi; z, g) = E(\varphi(z); z, g)$. This is defined at least on the region of absolute convergence and it is well known that it can be meromorphically continued to all $z \in \mathbb{C}$.

4.3 Pullback Formula

We define some embeddings of a subgroup of $\mathrm{GU}(3,1) \times \mathrm{GU}(0,2)$ into $\mathrm{GU}(3,3)$. This will be used in the doubling method. First we define $G(3,3)'$ to be the unitary similitude group associated to:

$$\begin{pmatrix} & & 1 & \\ & \zeta & & \\ -1 & & & \\ & & & -\zeta \end{pmatrix}$$

and $G(2,2)'$ to be associated to

$$\begin{pmatrix} \zeta & \\ & -\zeta \end{pmatrix}.$$

We define an embedding

$$\alpha : \{g_1 \times g_2 \in \mathrm{GU}(3,1) \times \mathrm{GU}(0,2), \mu(g_1) = \mu(g_2)\} \to GU(3,3)'$$

and

$$\alpha' : \{g_1 \times g_2 \in \mathrm{GU}(2,0) \times \mathrm{GU}(0,2), \mu(g_1) = \mu(g_2)\} \to GU(2,2)'$$

as $\alpha(g_1, g_2) = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$ and $\alpha'(g_1, g_2) = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$. We also define isomorphisms:

$$\beta : \mathrm{GU}(3,3)' \xrightarrow{\sim} \mathrm{GU}(3,3), (\beta' : \mathrm{GU}(2,2)' \xrightarrow{\sim} \mathrm{GU}(2,2))$$

by:

$$g \mapsto S^{-1}gS, (g \mapsto S'^{-1}gS')$$

where

$$S = \begin{pmatrix} 1 & & & \\ & 1 & & -\frac{\zeta}{2} \\ & & 1 & \\ & -1 & & -\frac{\zeta}{2} \end{pmatrix}, S' = \begin{pmatrix} 1 & -\frac{\zeta}{2} \\ -1 & -\frac{\zeta}{2} \end{pmatrix}.$$

We define

$$i(g_1, g_2) = S^{-1}\alpha(g_1, g_2)S, i'(g_1, g_2) = S'^{-1}\alpha(g_1, g_2)S'.$$

We recall the pullback formula of Shimura (see [38] for details). Let τ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^\times$. Given a cuspform φ on GU(2) we consider

$$F_\varphi(f; z, g) := \int_{\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})} f(z, S^{-1}\alpha(g, g_1h)S)\bar{\tau}(\det g_1 g)\varphi(g_1 h)dg_1,$$

$$f \in I_3(\tau), g \in \mathrm{GU}(3,1)(\mathbb{A}_\mathbb{Q}), h \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q}), \mu(g) = \mu(h)$$

or

$$F'_\varphi(f'; z, g) = \int_{\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})} f'(z, S'^{-1}\alpha'(g, g_1h)S')\bar{\tau}(\det g_1 g)\varphi(g_1 h)dg_1$$

$$f' \in I_2(\tau), g \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q}), h \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q}), \mu(g) = \mu(h)$$

This is independent of h. The pullback formulas are the identities in the following proposition.

Proposition 4.1. *Let τ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^\times$.*
(i) If $f' \in I_2(\tau)$, then $F'_\varphi(f'; z, g)$ converges absolutely and uniformly for (z, g) in compact sets of $\{\mathrm{Re}(z) > 1\} \times \mathrm{GU}(2,0)(\mathbb{A}_\mathbb{Q})$, and for any $h \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q})$ such that $\mu(h) = \mu(g)$

$$\int_{\mathrm{U}(2)(\mathbb{Q})\backslash \mathrm{U}(2)(\mathbb{A}_\mathbb{Q})} E(f'; z, S'^{-1}\alpha'(g, g_1h)S')\bar{\tau}(\det g_1 h)\phi(g_1 h)dg_1 = F'_\phi(f'; z, g).$$

(ii) If $f \in I_3(\tau)$, then $F_\varphi(f; z, g)$ converges absolutely and uniformly for (z, g) in compact sets of $\{Re(z) > 3/2\} \times \mathrm{GU}(3,1)(\mathbb{A}_\mathbb{Q})$ such that $\mu(h) = \mu(g)$

$$\int_{\mathrm{U}(2)(\mathbb{Q})\backslash \mathrm{U}(2)(\mathbb{A}_\mathbb{Q})} E(f; z, S^{-1}\alpha(g, g_1h)S)\bar{\tau}(\det g_1 h)\varphi(g_1 h)dg_1$$

$$= \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{GU}(3,1)(\mathbb{Q})} F_\varphi(f; z, \gamma g),$$

with the series converging absolutely and uniformly for (z, g) in compact subsets of $\{\mathrm{Re}(z) > 3/2\} \times \mathrm{GU}(3,1)(\mathbb{A}_\mathbb{Q})$.

4.4 p-adic Interpolation

We recall our notations in [6, Section 5.1] and correct some errors in the formulas for parameterization in *loc.cit.* We define an "Eisenstein datum" $\mathcal{D}$ to be a pair (φ, ξ_0) consisting of a cuspidal eigenform φ of prime to p level, trivial character and weight $\underline{k} = (a_1, a_2), a_1 \geq a_2 \geq 0$ on $\mathrm{GU}(r, 0)$ and a Hecke character ξ_0 of $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times$ such that $\xi_0 |\cdot|^{\frac{1}{2}}$ is a finite order character. Let σ be the reciprocity map of class field theory $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times \to G_\mathcal{K}^{ab}$ normalized by the geometric Frobenius. Note $\Gamma_\mathcal{K} = \Gamma^+ \oplus \Gamma_{\bar{v}_0}$. Let $\Psi_1 : G_\mathcal{K} \twoheadrightarrow \Gamma_\mathcal{K} \to \Gamma^+ \hookrightarrow \mathbb{Z}_p[[\Gamma_\mathcal{K}^+]]^\times$ and $\Psi_2 : G_\mathcal{K} \twoheadrightarrow \Gamma_\mathcal{K} \to \Gamma^{\bar{v}_0} \hookrightarrow \mathbb{Z}_p[[\Gamma_{\bar{v}_0}]]^\times$ where the middle arrows are projections with respect to the above direct sum. Then $\Psi_\mathcal{K} = \Psi_1 \cdot \Psi_2$. We define

$$\begin{aligned} \tau_0 &:= \overline{(\xi_0|\cdot|^{\frac{1}{2}})}^c, \\ \boldsymbol{\xi} &:= \xi_0 \cdot (\Psi \circ \sigma), \\ \boldsymbol{\tau} &:= \tau_0 \cdot (\Psi_1^{-c} \circ \sigma), \\ \psi_\mathcal{K} &:= \Psi_2. \end{aligned}$$

We define $\mathcal{X}^{pb}$ to be the set of $\bar{\mathbb{Q}}_p$-points $\phi \in \mathrm{Spec}\Lambda_{\mathcal{K},\mathcal{O}_L}$ such that $\phi \circ \boldsymbol{\tau}((1+p, 1)) = \tau_0((1+p, 1))$,

$$\phi \circ \boldsymbol{\tau}((1, 1+p)) = (1+p)^{\kappa_\phi} \tau_0((1, 1+p))$$

for some integer $\kappa_\phi > 6$, $\kappa_\phi \equiv 0 (\mathrm{mod}(p-2))$ and such that the weight $(a_1, a_2, 0; \kappa_\phi)$ is in the absolutely convergent range for P in the sense of Harris [11], and such that

$$\phi \circ \psi_\mathcal{K}(\gamma^-) = (1+p)^{\frac{m_\phi}{2}}$$

for some non-negative integer m_ϕ, and such that the τ_ϕ (to be defined in a moment) is such that, under the identification $\tau_\phi = (\tau_1, \tau_2)$ for $\mathcal{K}_p^\times \simeq \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$, we have τ_1, τ_2, $\tau_1\tau_2$ all have conductor (p).

We denote by $\mathcal{X}$ the set of $\bar{\mathbb{Q}}_p$-points ϕ in $\mathrm{Spec}\Lambda_{\mathcal{K},\mathcal{O}_L}$ such that

$$\phi \circ \boldsymbol{\tau}((1, 1+p)) = (1+p)^{\kappa_\phi} \zeta_1 \tau_0((1, 1+p)), \phi \circ \boldsymbol{\tau}((p+1, 1)) = \tau_0((p+1, 1))$$

and $\phi \circ \psi_\mathcal{K}(\gamma^-) = \zeta_2$ with ζ_1 and ζ_2 being p-power roots of unity. Let $\mathcal{X}^{gen}$ be the subset of points such that the ζ_1 and ζ_2 above are all primitive p^t roots of unity for some $t \geq 2$.

Remark 4.2. *We will use the points in $\mathcal{X}^{pb}$ for p-adic interpolation of special L-values and Klingen Eisenstein series, and we will use the points in $\mathcal{X}$ to construct a Siegel Eisenstein measure.*

For each $\phi \in \mathcal{X}^{pb}$, we define Hecke characters ψ_ϕ and τ_ϕ of $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times$ by

$$\begin{aligned} \bar{\tau}_\phi^c(x) &:= \bar{x}_\infty^{\kappa_\phi} (\phi \circ \boldsymbol{\tau})(x) x_{\bar{v}}^{-\kappa_\phi} \cdot |\cdot|^{-\frac{\kappa_\phi}{2}}, \\ \psi_\phi(x) &:= x_\infty^{\frac{m_\phi}{2}} \bar{x}_\infty^{-\frac{m_\phi}{2}} (\phi \circ \psi_\mathcal{K} \circ \sigma) x_v^{-\frac{m_\phi}{2}} x_{\bar{v}}^{\frac{m_\phi}{2}}. \end{aligned}$$

Let

$$\begin{aligned} \xi_\phi &= |\cdot|^{\frac{\kappa_\phi - 1}{2}} \bar{\tau}_\phi^c \psi_\phi, \\ \varphi_\phi &= \varphi \otimes \psi_\phi^{-1}. \end{aligned}$$

The weight $\underline{k}_\phi$ for φ_ϕ at the arithmetic point ϕ is $(a_1 + m_\phi, a_2 + m_\phi)$.

4.5 Explicit Sections

Now we make explicit sections for the Siegel and Klingen Eisenstein series. Recall that we defined $g_1, g_3 \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q}), g_2', g_4' \in \mathrm{U}(2)(\mathbb{A}_\mathbb{Q})$ in [38, subsection 7.4]. Moreover their p-component are 1. We use a slight modification of the sections constructed in [6]. For the Siegel section we use the construction $f_{sieg} = \prod_v f_v$ in [6, Section 5.1]. Recall that the f_∞ is a vector valued section. In *loc.cit* we pullback this section under the embedding γ^{-1} and take the corresponding component for the representation $L^{(\underline{k}_\phi,0)} \boxtimes L^{(\kappa)} \boxtimes (L^{(\underline{k}_\phi)} \otimes \det \kappa)$ (notations as in *loc.cit* Section 4). Recall that in [38, section 7] we constructed a character ϑ of $\mathbb{A}_\mathbb{Q}^\times$ and elements $g_1 \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Recall we start with a eigenform $f \in \pi$ new outside p and is an eigenvector for the U_p-operator with eigenvalue α_1. We extend it to a form on $\mathrm{GU}(2)(\mathbb{A}_\mathbb{Q})$ using the central character ψ and as in [38, 5.10] define

$$f_\Sigma = (\prod_{v\in\Sigma, v\nmid N} \pi(\begin{pmatrix} & 1 \\ \varpi_v & \end{pmatrix}) - \chi_{1,v}(\varpi_v) q_v^{\frac{1}{2}})f,$$

$$f_\vartheta(g) = \prod_{v \text{ split } \in\Sigma, v\nmid p} \sum_{\{a_v \in \frac{\varpi_v \mathbb{Z}_v^\times}{\varpi_v^{1+s_v}\mathbb{Z}_v}\}_v} \vartheta(\frac{-a_v}{\varpi_v}) f_\Sigma(g \prod_v \begin{pmatrix} 1 & \\ a & 1 \end{pmatrix}_v \begin{pmatrix} \varpi_v^{-s_v} & \\ & 1 \end{pmatrix}_v)$$

where $\varpi_v^{s_v}$ is the conductor of ϑ at v, $\pi_{f,v} = \pi(\chi_{1,v}, \chi_{2,v})$ (choose any order) and define our φ to be $\pi(g_1)f_\vartheta$. When we are varying our datum in p-adic families we write $\mathcal{E}_{sieg}$ for the Siegel Eisenstein measure on $\Gamma_\mathcal{K}$ obtained.

4.6 Construction of A Measure

We first recall the notion of p-adic L-functions for Dirichlet characters which is needed in the proposition below. There is an element $\mathcal{L}_{\bar{\tau}'}$ in $\Lambda_{\mathcal{K},\mathcal{O}_L}$ such that at each arithmetic point $\phi \in \mathcal{X}^{pb}$, $\phi(\mathcal{L}_{\bar{\tau}'}) = L(\bar{\tau}_\phi', \kappa_\phi - 2).\tau_\phi'(p^{-1})p^{\kappa_\phi - 2}\mathfrak{g}(\bar{\tau}_\phi')^{-1}$. For more details see [36, 3.4.3].

Constructing Families

The following theorem is proved in [6]. The construction in *loc.cit* is for unitary groups but one can easily obtain the construction for unitary similitude groups using the central characters $\psi_\phi \cdot \tau_\phi$. In the statement we use f (extension of the GL_2 form f to $\mathrm{GU}(2)$ by the trivial central character). in the place of the vector φ to keep consistent with the notations in the introduction.

Proposition 4.3. *Let π be an irreducible cuspidal automorphic representation of $GU(2,0)$ of weight $\underline{k} = (a_1, a_2), a_1 \geq a_2 \geq 0$ such that π_p is unramified and is an irreducible induced representation $\pi(\chi_1, \chi_2)$ from the Borel subgroup such that $\chi_1 \neq \chi_2$. Let $f \in \pi^{K_0(p)}$ be an eigenvector for the U_p operators at p. Let $\tilde{\pi}$ be the dual representation of π.*

(i) There is a constant $C_{\pi,p}$ and an element $\mathcal{L}_{f,\mathcal{K},\xi_0}^\Sigma \in \Lambda_{\mathcal{K},\mathcal{O}_L}$ such that for a Zariski dense set of arithmetic points $\phi \in \mathrm{Spec}\Lambda_{\mathcal{K},\mathcal{O}_L}$ (to be specified in the text) we have

$$\phi(\mathcal{L}_{f,\mathcal{K},\xi_0}^\Sigma) = C_{\varphi,p}.\frac{L^\Sigma(\tilde{\pi}, \xi_\phi, 0)}{\Omega_\infty^{d_\phi + r\kappa_\phi}} c'_{\underline{k}_\phi,0,\kappa_\phi}.p^{\frac{2\kappa_\phi}{2}-3}\mathfrak{g}(\tau_{1,\phi}^{-1})^r \prod_{i=1}^2 (\chi_{i,\phi}\tau_{1,\phi})(p) \prod_{i=1}^2 (\chi_{i,\phi}^{-1}\tau_{2,\phi})(p)\bar{\tau}_\phi^c((p^2,1))$$

where $d_\phi = 2(a_{1,\phi} + a_{2,\phi})$, $\tau_{\phi,p} = (\tau_{1,\phi}, \tau_{2,\phi})$, $c_{\underline{k}_\phi,0,\kappa_\phi}$ is an algebraic constant coming from an Archimedean integral and $C_{\varphi,p}$ is a product of local constant coming from the pullback integrals.

(ii) There is a set of formal q-expansions $\mathbf{E}_{f,\xi_0} := \{\sum_\beta a^t_{[g]}(\beta)q^\beta\}_{([g],t)}$ *for* $\sum_\beta a^t_{[g]}(\beta)q^\beta \in \Lambda_{\mathcal{K},\mathcal{O}_L}\otimes_{\mathbb{Z}_p}\mathcal{R}_{[g],\infty}$ *where* $\mathcal{R}_{[g],\infty}$ *is some ring to be defined later,* $([g],t)$ *are p-adic cusp labels, such that for a Zariski dense set of arithmetic points* $\phi \in \mathrm{Spec}_{\mathcal{K},\mathcal{O}_L}$, $\phi(\mathbf{E}_{f,\xi_0})$ *is the Fourier-Jacobi expansion of the highest weight vector of the holomorphic Klingen Eisenstein series constructed by pullback formula which is an eigenvector for* U_{t^+} *with non-zero eigenvalue. The weight for* $\phi(\mathbf{E}_{f,\xi_0})$ *is* $(m_\phi + a_1, m_\phi + a_2, 0; \kappa_\phi)$.

(iii) The $a^t_{[g]}(0)$*'s are divisible by* $\mathcal{L}^\Sigma_{f,\mathcal{K},\xi_0}.\mathcal{L}^\Sigma_{\bar{\tau}'}$ *where* $\mathcal{L}^\Sigma_{\bar{\tau}'}$ *is the p-adic L-function of a Dirichlet character above.*

We will write $\mathbf{E}_{Kling}$ later on for this Klingen Eisenstein measure. Here at ϕ the weight of the Klingen Eisenstein series constructed is $(a_1 + m_\phi, a_2 + m_\phi, 0; \kappa)$. To adapt to the situation of section 3, we multiply the family constructed in (ii) above by $\boldsymbol{\psi}(\det -)$ (so that we fix the weight a_1, a_2 and allow a_3, b_1 to vary). According to the control theorems proved in section 3 and Theorem 3.15 the family constructed thereby comes from a partially ordinary family defined there. By an appropriate weight map $\tilde{\Lambda} \to \mathcal{O}_L[[\Gamma_\mathcal{K}]]$ (we omit the precise formula) this gives a $\mathcal{O}_L[[\Gamma_\mathcal{K}]]$-coefficients family in the sense of section 3.

The interpolation formula for the p-adic L-function considered above is not satisfying since it involves non-explicit Archimedean constants. But in fact it also has the following interpolation property. If $a_1 = a_2 = 0$ then for a Zariski dense set of arithmetic points $\phi \in \mathrm{Spec}\Lambda_\mathcal{K}$ such that $\phi \circ \boldsymbol{\xi}$ is the p-adic avatar of a Hecke character ξ_ϕ of $\mathcal{K}^\times\backslash\mathbb{A}^\times_\mathcal{K}$ of infinite type $(-\kappa - \frac{1}{2}, -\frac{1}{2})$ for some $\kappa \geq 6$, of conductor (p^t, p^t) $(t > 0)$ at p, then:

$$\phi(\mathcal{L}^\Sigma_{f,\mathcal{K}}) = \frac{p^{(\kappa-3)t}\xi^2_{1,p}(p^{-t})\mathfrak{g}(\xi_{1,p}\chi^{-1}_{1,p})\mathfrak{g}(\xi_{1,p}\chi^{-1}_{2,p})L^\Sigma(\tilde{\pi}, \xi_\phi, 0)(\kappa-1)!(\kappa-2)!}{(2\pi i)^{2\kappa-1}\Omega^{2\kappa}_\infty}. \tag{2}$$

Here $\mathfrak{g}$ is the Gauss sum and $\chi_{1,p}, \chi_{2,p}$ are characters such that $\pi(\chi_{1,p}, \chi_{2,p}) \simeq \pi_{f,p}$. This can be seen as follows. The Siegel Eisenstein measure considered in [6] is just the specialization of the measure considered in [40] to our two-dimensional weight space. We can employ the computations in [40] to get equation 2. (Note that the Archimedean weights in equation (2) are nothing but the weights considered in [40]. Also the restrictions in [40] on conductors of π and ξ are put to prove the pullback formulas for Klingen Eisenstein series and has nothing to do with computation for p-adic L-functions. This computation is also done in the forthcoming work [5].) We also remark that in our situation it is possible to determine the constants $c'_{\underline{k}_\phi,0,\kappa_\phi}$ by taking an auxiliary eigenform ordinary at p and comparing our construction with Hida's (although we do not need it in this paper).

We can also construct the complete p-adic L-function $\mathcal{L}_{f,\mathcal{K},\xi}$ by putting back all the local Euler factors at primes in Σ. By doing this we only get elements in $\mathrm{Frac}\mathcal{O}_L[[\Gamma_\mathcal{K}]]$. In some cases we can study the integrality of it by comparing with other constructions. There is another way of constructing this p-adic L-function using Rankin-Selberg method by adapting the construction in [12]. We refer to [38, Section 6.4] for the discussion. Note that although Hida's construction assumes both forms are nearly ordinary, however, it works out in the same way in our situation since in the Rankin-Selberg product the form with higher weight is the CM form which is ordinary by our assumption that p splits in $\mathcal{K}$. The p-adic L-functions of Hida are not integral since he used Petersson inner product as the period. Under assumption (1) of the main theorem in the introduction, we know the local Hecke algebra corresponding to the CM form $\mathbf{g}$ is Gorenstein. The work of Hida-Tilouine [18] shows that the congruence module for $\mathbf{g}$ is generated by the corresponding Katz p-adic L-function (where the CM period Ω_∞ is used). If we multiply Hida's p-adic L-function by this Katz p-adic L-function then we recover our p-adic L-function in Proposition 4.3. So under assumption (1) of Theorem 1.1 the $\mathcal{L}_{f,\mathcal{K},\xi}$ is in $\mathcal{O}_L[[\Gamma_\mathcal{K}]]$. By our discussion in [38, Section 6.4] we know that under the assumption of part (1) of the main theorem $\mathcal{L}_{f,\mathcal{K},\xi}$ is co-prime to any height one prime of $\mathcal{O}_L[[\Gamma_\mathcal{K}]]$ which is not a pullback of a height one prime of $\mathcal{O}_L[[\Gamma^+]]$. Under assumption (2) of Theorem 1.1 we only know $\mathcal{L}_{f,\mathcal{K},\xi}$ is in $\mathrm{Frac}\mathcal{O}_L[[\Gamma_\mathcal{K}]]$ and we call the fractional ideal generated by $\mathcal{L}_{f,\mathcal{K},\xi}$ to be $\mathcal{O}_L[[\Gamma_\mathcal{K}]]\cdot\mathcal{L}_{f,\mathcal{K},\xi}\subset\mathrm{Frac}\mathcal{O}_L[[\Gamma_\mathcal{K}]]$.

4.7 Galois Representations for Klingen Eisenstein Series

We can also associate a reducible Galois representation to the holomorphic Klingen Eisenstein series constructed with the same recipe as in subsection 2.3. The resulting Galois representation is:

$$\sigma_{\tau'}\sigma_{\psi^c}\epsilon^{-\kappa}\oplus\sigma_{\psi^c}\epsilon^{-3}\oplus\rho_f.\sigma_{\tau^c}\epsilon^{-\frac{\kappa+2}{2}}.$$

5 Proof of Main Results

In this section we assume the π we start with has weight two so that the Jacquet-Langlands correspondence is trivial representation at ∞. This is because we can not study the Fourier-Jacobi coefficients in the higher weight case at the moment. The Klingen Eisenstein measure we construct is interpolating forms of weight $(0,0,a_3;b_1)$.

5.1 p-adic Properties of Fourier-Jacobi Coefficients

Our discussion on Fourier-Jacobi coefficients here will refer a lot to the argument in [38, Section 7].
Interpolating Petersson Inner Products
Recall that in [38, section 6] we made an additional construction for interpolating Petersson inner products of forms on definite unitary groups: For a compact open subgroup $K=\prod_v K_v$ of $\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})$ which is $\mathrm{U}(2)(\mathbb{Z}_p)$ at p we take $\{g_i^\triangle\}_i$ a set of representatives for $\mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})/K_0(p)$ where we write $K_0(p)$ also for the level group $\prod_{v\nmid p}K_v\times K_0(p)$. Suppose K is sufficiently small so that for all i we have $\mathrm{U}(2)(\mathbb{Q})\cap g_i^\triangle K g_i^{\triangle-1}=1$. For an ordinary Hida family $\mathbf{f}^\vee$ of eigenforms with some coefficient ring $\mathbb{I}$ (whose p-part of level group is the lower diagonal Borel congruent to powers of p) we construct a set of bounded $\mathbb{I}$-valued measure μ_i on $N^-(p\mathbb{Z}_p)$ as follows. We only need to specify the measure for sets of the form $t^-N^-(\mathbb{Z}_p)(t^-)^{-1}n$ where $n\in N^-(\mathbb{Z}_p)$ and t^- a matrix of the form $\begin{pmatrix}p^{t_1}&\\&p^{t_2}\end{pmatrix}$ with $t_2>t_1$. We assign $\mathbf{f}^\vee(g_int^-)\lambda(t^-)^{-1}$ as its measure where $\lambda(t^-)$ is the Hecke eigenvalue of $\mathbf{f}^\vee$ for U_{t^-}. This measure is well defined by the expression for Hecke operators U_{t^-}. Let $\boldsymbol{\chi}$ be the central character of $\mathbf{f}$ we write $\mathbf{f}^\vee$ for the family $\pi(\begin{pmatrix}&1\\1&\end{pmatrix}_p)\mathbf{f}\otimes\boldsymbol{\chi}^{-1}(\det -)$. The above set $\{\mu_i\}_i$ can be viewed as a measure on $\mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})/K^{(p)}$ by requiring it to be invariant under $B(\mathbb{Z}_p)$, which we denote as $\mu_{\mathbf{f}^\vee}$.

Recall that in [38] we constructed measures of Siegel-Eisenstein series on the three-variable $\mathbb{Z}_p$-space $\Gamma_\mathcal{K}\times\mathbb{Z}_p$. (The $\mathbb{Z}_p$ here actually gives the variable for the Hida family in *loc.cit* on GL_2 by

$\mathcal{O}_L[[W]] \simeq \mathcal{O}_L[[\mathbb{Z}_p]], W \mapsto [1]-1$ where W is a variable and $[1]$ means the group-like element.) We still denote it as $\mathcal{E}_{sieg}$ and the $\mathcal{E}_{sieg}$ we construct in this paper is the specialization of the three-variable one by $W \to 1$. (The maps the weight spaces differ by a linear isomorphism of $\Gamma_\mathcal{K}$, which will not affect our later argument). Let $\Lambda_3 := \mathcal{O}_L[[\Gamma_\mathcal{K} \times \mathbb{Z}_p]]$. We first make a construction which will be useful later on. Let $\mathbf{h}$ and $\boldsymbol{\theta}$ be two Λ_3-valued Hida families (the p-part of level group for $\mathbf{h}$ is with respect to the upper-triangular Borel while $\boldsymbol{\theta}$ is lower triangular Borel) of forms on $\mathrm{U}(2)$ and $\mu_{\boldsymbol{\theta}}$ the measure associated to it as above (we are a little vague by not specifying the map to the weight space of $\mathrm{U}(2)$). Let $\{g_j^\triangle\}$ be a set of representatives of $\mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})/K_0(p)$ such that the p-component of $g_j^\triangle$ is 1. We are going to define a continuous Λ_3-valued function $\mathrm{tr}(\mathbf{h})d\mu_{\boldsymbol{\theta}}$ on $\mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})$ as follows (recall that we equip the latter set the topology induced from the p-adic Lie group $\mathrm{U}(2)(\mathbb{Q}_p)$). For any $k = 0, 1, ..., p-1$ and $g \in \mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})/K^{(p)}$, we let $g\begin{pmatrix} 1 & \\ k & 1\end{pmatrix}_p = g_j^\triangle g_0$ for $g_0 \in \Gamma_0(p) \subset \mathrm{U}(2)(\mathbb{Z}_p)$ (in an unique way). Suppose $g_0 = \begin{pmatrix} a & b \\ & d\end{pmatrix}\begin{pmatrix} 1 & \\ c & 1\end{pmatrix}$ for $a, d \in \mathbb{Z}_p^\times, c \in p\mathbb{Z}_p, b \in \mathbb{Z}_p$. We first calculate that

$$\begin{pmatrix} a & b \\ & d\end{pmatrix}\begin{pmatrix} 1 & \\ c & 1\end{pmatrix}\begin{pmatrix} 1 & \\ m & 1\end{pmatrix} = \begin{pmatrix} a+b(c+m) & b \\ d(c+m) & d\end{pmatrix} = \begin{pmatrix} 1 & \\ \frac{d(c+m)}{a+b(c+m)} & 1\end{pmatrix}\begin{pmatrix} a+b(c+m) & b \\ 0 & \frac{ad}{a+b(c+m)}\end{pmatrix}.$$

We define

$$(\mathrm{tr}\mathbf{h}d_{\mu_{\boldsymbol{\theta}}})(g) = \sum_{0\le k\le p-1}\int_{m\in p\mathbb{Z}_p}\chi_{\boldsymbol{\theta},1}\chi_{\mathbf{h},1}(a+b(c+m))\chi_{\boldsymbol{\theta},2}\chi_{\mathbf{h},2}(\frac{ad}{a+b(c+m)})(\pi(g_2')\mathbf{h})(g_j^\triangle\begin{pmatrix} 1 & \\ \frac{d(c+m)}{a+b(c+m)} & 1\end{pmatrix}_p)d_{\mu_{j,\theta,g}} \tag{3}$$

where $\mu_{j,\theta,g}$ is the measure on $p\mathbb{Z}_p$ obtained by composing $d\mu_{j,\theta}$ with the map

$$m \mapsto \frac{d(c+m)}{a+b(c+m)}$$

and $\chi_{\boldsymbol{\theta},1}, \chi_{\boldsymbol{\theta},2}$ is such that it gives the right action of $\begin{pmatrix}\mathbb{Z}_p^\times & \\ & \mathbb{Z}_p^\times\end{pmatrix}$ on $\boldsymbol{\theta}$ is given by $\begin{pmatrix}\chi_{\boldsymbol{\theta},1} & \\ & \chi_{\boldsymbol{\theta},2}\end{pmatrix}$ and similarly for $\chi_{\mathbf{h},1}, \chi_{\mathbf{h},2}$.(This notation is different from [38]). Maybe a bit more explanation will help the reader understand this construction. Taking a point on the weight space such that for some t, p^t is the conductor of the specializations h and θ of $\mathbf{h}$, $\boldsymbol{\theta}$ (in our applications such points form a Zariski dense subset of the parameter space). One can apply certain operator in $\mathbb{Q}_p[\mathrm{U}(\mathbb{Q}_p)]$ to θ and obtain a θ^{ss} which is invariant under $\Gamma_1(p^t) \subset \mathrm{U}(\mathbb{Z}_p)$ and

$$\sum_{0\le k\le p-1}\pi(\begin{pmatrix} 1 & \\ k & 1\end{pmatrix}_p)\theta^{ss} = \theta.$$

Then $\mathrm{tr}\mathbf{h}d_{\mu_{\boldsymbol{\theta}}}(g)$ specializes to

$$\sum_{0\le k\le p-1} h\theta^{ss}(g\begin{pmatrix} 1 & \\ k & 1\end{pmatrix}_p).$$

(This construction implies the above expression interpolates p-adic analytically.)

Recall that we defined an element $g_2' \in \mathbf{U}(2)(\mathbb{A}_{\mathbb{Q}})$ in [38, end of Section 7.4]. We refer to [38] for the definition of the theta function θ_1 and a functional l_{θ_1} on the space of p-adic automorphic forms on $\mathrm{U}(3,1)$ defined by taking Fourier-Jacobi coefficients (viewed as a a form on $P(\mathbb{A}_{\mathbb{Q}})$) and pair with the theta function θ_1.

In [38, Section 7.3] we constructed families of CM forms $\mathbf{h}$ and $\boldsymbol{\theta}$ on $\mathrm{U}(2)$ associated two Λ_3-valued CM characters $\chi_{\mathbf{h}}$ and $\chi_{\boldsymbol{\theta}}$ and we write their two-variable specializations still using the same symbols. Recall that our $\mathcal{E}_{sieg}$ is realized as the specialization of a three-variable $\mathcal{E}_{sieg}$ to a two-variable one by taking $W \mapsto 1$. We want to study $\int l_{\theta_1}(\mathbf{E}_{Kling}) d\mu_{(\pi(g_2')\mathbf{h})}$. Since $\mathbf{E}_{Kling}$ is realized as $\langle \int i^{-1}(\mathcal{E}_{sieg}), \varphi\rangle_{low}$ ($i : \mathrm{U}(3,1)\times \mathrm{U}(0,2) \hookrightarrow \mathrm{U}(3,3)$ and $\langle , \rangle_{low}$ means taking inner product with respect to the $\mathrm{U}(0,2)$-factor), we need first to study

$$A_2 := \int l_{\theta_1}^{up} i^{-1}(\mathcal{E}_{sieg}) d^{up}_{\mu(\pi(g_2')\mathbf{h})}$$

regarded as a family of p-adic automorphic forms on $\mathrm{U}(2)$. Here $i^{-1}(\mathcal{E}_{sieg})$ is a measure of forms on $\mathrm{U}(3,1)\times \mathrm{U}(2)$ and the $l_{\theta_1}^{up}, d^{up}$ means the functional and integration on the $\mathrm{U}(3,1)$ factor in $\mathrm{U}(3,1)\times \mathrm{U}(0,2)$. Let A_2' be defined in a similar way to A_2 but using the three-variable family. Let $K_{A_2'} \subseteq \mathrm{U}(\mathbb{A}_f)$ be a level group under whose components outside p is A_2' invariant and is the maximal compact subgroup at p. Let $K_{\mathrm{U}(1)}$ be $\mathrm{U}(1)(\mathbb{A}_{\mathbb{Q}}) \cap K_{A_2'}$ ($\mathrm{U}(1)$ is regarded as the central elements of $\mathrm{U}(2)$). Take a representative $\{u_i\}_i$ of $\mathrm{U}(1)(\mathbb{Q})\backslash \mathrm{U}(1)(\mathbb{A}_{\mathbb{Q}})/K_{\mathrm{U}(1)}$. Define $A_1(g) := \sum_i \pi(u_i)A_2$ and $A := \langle A_1, \varphi\rangle$. Let A_1' be defined similar to A_1 but using the three-variable family. We remark that A_1 is invariant under ${}^tK_0(p)$ by using the expression for A_2 and noting that $\chi_{\mathbf{h},1}\cdot\chi_{\boldsymbol{\theta},1} = \chi_{\mathbf{h},2}\cdot\chi_{\boldsymbol{\theta},2} = 1$ by construction.

The Fourier-Jacobi coefficients calculations in [38], in particular Proposition 5.28 and Corollary 5.29 there shows that A_1' is $\mathrm{tr}(\pi(g_2')\mathbf{h})d_{\mu_{\boldsymbol{\theta}}}$. So A is

$$\mathcal{L}_5^{\Sigma}\mathcal{L}_6^{\Sigma}\cdot < \mathrm{tr}(\pi(g_2')\mathbf{h})d_{\mu_{\boldsymbol{\theta}}}, \varphi >$$

times some element in $\bar{\mathbb{Q}}_p^\times$. Here $\mathcal{L}_5^{\Sigma}$ and $\mathcal{L}_6^{\Sigma}$ are defined in [38, subsection 7.5] which are Σ-primitive p-adic L-functions for certain CM characters. They come from the pullback integral for $\mathbf{h}$ under $\mathrm{U}(2)\times\mathrm{U}(2) \hookrightarrow \mathrm{U}(2,2)$. By our choices of characters they are some $\bar{\mathbb{Q}}_p^\times$ multiples of a unit in $\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]$. In [38] we also constructed three-variable families $\tilde{\mathbf{h}}, \tilde{\boldsymbol{\theta}}$ in the dual representations for $\mathbf{h}$ and $\boldsymbol{\theta}$. We still use $\tilde{\mathbf{h}}$ and $\tilde{\boldsymbol{\theta}}$ for their specializations to our two-variable families. Let $\tilde{f}_{\tilde{\vartheta}} \in \tilde{\pi}$ be chosen the same as as in [38, Section 7.5] at primes outside p. But at p we take it as the stabilization with U_p-eigenvalue α_1^{-1} (recall α_1 is the eigenvalue for the U_p action on f_ϑ). We consider the expression at arithmetic point ϕ

$$\tilde{A}_\phi := p^t \int_{[\mathrm{U}(2)]} \pi(g_4')\tilde{\mathbf{h}}_\phi(g)\tilde{\boldsymbol{\theta}}_\phi(g)\pi(g_3)\tilde{f}_{\tilde{\vartheta}}(g)dg.$$

From our discussions before (in particular the construction of $\mathrm{tr}\mathbf{h}d_{\mu_{\boldsymbol{\theta}}}$) they are interpolated by an element $\mathcal{O}_L[[\Gamma_{\mathcal{K}}]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. We are going to calculate $A\tilde{A}$ using Ichino's triple product formula. We do this by calculating it at arithmetic points in $\mathcal{X}^{gen}$. This is enough since these points are Zariski dense. We refer to [38, subsection 7.4] for a summary of the backgrounds of Ichino's formula. The local calculations are the same as *loc.cit* except at the p-adic places where we have different assumptions for ramification. We give a lemma for our situation.

Lemma 5.1. *Let $\chi_{h,1},\chi_{h,2},\chi_{\theta,1},\chi_{\theta,2},\chi_{f,1},\chi_{f,2}$ be character of $\mathbb{Q}_p^\times$ whose product is the trivial character and such that $\chi_{h,1},\chi_{\theta,1},\chi_{f,1},\chi_{f,2}$ are unramifed and $\chi_{h,2}\cdot\chi_{\theta,2}$ is unramified. Let $f_p\in\pi(\chi_{f,2},\chi_{f,1})$ and by using the induced representation model f is the characteristic function of K_1wK_1. Similarly we define $\tilde{f}_p\in\pi(\chi_{f,2}^{-1},\chi_{f,1}^{-1})$. So f is a Hecke eigenvector for T_p with eigenvalue $\chi_{f,1}(p)$. Let $h_p\pi(\chi_{h,1},\chi_{h,2}),\theta_p\in\pi(\chi_{\theta,1},\chi_{\theta,2}),\tilde{h}_p\in\pi(\chi_{h,1}^{-1},\chi_{h,2}^{-1}),\tilde{\theta}_p(\chi_{\theta,1}^{-1},\chi_{\theta,2}^{-1})$ be the f_{χ_h}, f_{χ_θ}, $\tilde{f}_{\tilde{\chi}_h}$, $\tilde{f}_{\tilde{\chi}_\theta}$ defined in [38, lemma 7.4]. Then the local triple product integral (defined at the beginning of [38, subsection 7.4])*

$$\frac{I_p(h_p\otimes\theta_p\otimes f_p,\tilde{h}_p\otimes\tilde{\theta}_p\otimes\tilde{f}_p)}{\langle h_p,\tilde{h}_p\rangle\langle\theta_p,\tilde{\theta}_p\rangle\langle f_p,\tilde{f}_p\rangle}$$

is

$$\frac{p^{-t}(1-p)}{1+p}\cdot\frac{1}{1-\chi_{h,1}(p)\chi_{\theta,1}(p)\chi_{f,1}(p)p^{-\frac{1}{2}}}\cdot\frac{1}{1-\chi_{h,1}(p)\chi_{\theta,2}(p)\chi_{f,1}(p)p^{-\frac{1}{2}}}.$$

Proof. This is an easy consequence of [38, lemma 7.4] and [44, Proposition 3.2]. □

Now as in [38, Section 7.5] by computing at arithmetic points $\phi\in\mathcal{X}^{gen}$ and applying Ichino's formula, the local integrals at finite primes are non-zero constants in $\bar{\mathbb{Q}}_p$ (fixed throughout the family). We conclude that up to multiplying by an element in $\bar{\mathbb{Q}}_p^\times$ the $A\cdot\tilde{A}$ equals $\mathcal{L}_5^\Sigma\mathcal{L}_6^\Sigma\mathcal{L}_1\mathcal{L}_2$ where $\mathcal{L}_1$ is the p-adic L-function interpolating the algebraic part of $L(\lambda^2(\chi_{\boldsymbol{\theta}}\chi_{\mathbf{h}})_\phi,\frac{1}{2})$ (λ is the splitting character of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ we use to define theta functions, see [38, Section 3]) which we can choose the Hecke characters properly so that it is a unit in $\mathcal{O}_L[[\Gamma_\mathcal{K}]]$. (Note that since the CM character λ^2 has weight higher than f the result cited in [38, subsection 7.2] of M. Hsieh does not assume that f is ordinary). The $\mathcal{L}_2$ is the algebraic part of $L(f,\chi_\theta^c\chi_h,\frac{1}{2})\in\bar{\mathbb{Q}}_p$ (fixed throughout the family) which we can choose to be non-zero. (See the calculations in [38, subsection 7.5].)

To sum up we get the following proposition in the same way as in *loc.cit.*

Proposition 5.2. *Any height one prime of $\mathcal{O}_L[[\Gamma_\mathcal{K}]]$ containing $\int l_{\theta_1}(\mathbf{E}_{kling})d\mu_{\pi(g_2')\mathbf{h}}$ must be (p).*

We also remark that the assumption made in [38] that 2 splits can be removed. It was used there because the Howe duality in characteristic two was unknown, which is needed when studying the Fourier-Jacobi coefficients at non-split primes. But this is recently proved by Gan-Takeda ([9]).

5.2 Proof of Main Theorem

Now we prove our main theorem in the introduction, following [38, Section 8]. We refer to [38, section 8.1] for the definitions for Hecke operators for $\mathrm{U}(3,1)$ at unramified primes. Let $K_\mathcal{D}$ be an open compact subgroup of $\mathrm{U}(3,1)(\mathbb{A}_\mathbb{Q})$ maximal at p and all primes outside Σ such that the Klingen Eisenstein series we construct is invariant under $K_\mathcal{D}$. We let $\mathbb{T}_\mathcal{D}$ be the reduced Hecke algebra generated by the Hecke operators at unramified primes space of the two variable family of partially ordinary cusp forms with level group $K_\mathcal{D}$, the U_i operator at p, and then take the reduced quotient. Let the Eisenstein ideal $I_\mathcal{D}$ of $\mathbb{T}_\mathcal{D}$ to be generated by $\{t-\lambda(t)\}_t$ for t in the abstract Hecke algebra and $\lambda(t)$ is the Hecke eigenvalue of t acting on $\mathbf{E}_{Kling}$ and let $\mathcal{E}_\mathcal{D}$ be the inverse image of $I_\mathcal{D}$ in $\mathcal{O}_L[[\Gamma_\mathcal{K}]]\subset\mathbb{T}_\mathcal{D}$.

Now the main theorem can be proven in almost the same way as [38, Section 8], using Proposition 5.2 and Theorem 4.3. One uses the fundamental exact sequence Theorem 3.18 to show that

$(\mathcal{L}^\Sigma) \supseteq \mathcal{E}_\mathcal{D}$ as in Lemma 8.4 of *loc.cit.* Then use the lattice construction (Proposition 8.2 there, due to E. Urban) to show that $\mathcal{E}_\mathcal{D}$ contains the characteristic ideal of the dual Selmer group. Note also that to prove part (2) of the main theorem we need to use Lemma 8.3 of *loc.cit.* The only difference is to check the condition (9) in Section 8.3 of *loc.cit*: We suppose our pseudo-character $R = R_1 + R_2 + R_3$ where R_1 and R_2 are 1-dimensional and R_3 is 2-dimensional. Then by residual irreducibility we can associate a 2-dimensional $\mathbb{T}_\mathcal{D}$-coefficient Galois representation. Take an arithmetic point x in the absolute convergence region for Eisenstein series such that $a_2 - a_3 >> 0$ and $a_3 + b_1 >> 0$ and consider the specialization of the Galois representation to x. First of all as in [36, Theorem 7.3.1] a twist of this descends to a Galois representation of $G_\mathbb{Q}$ which we denote as $R_{3,x}$. By our description for the local Galois representations for partially ordinary forms at p we know that $R_{3,x}$ has Hodge-Tate weight $0, 1$ and is crystalline (by the corresponding property for $R_x = R_1 + R_2 + R_3$, note that R_x corresponds to a Galois representation for a classical form unramified at p by Theorem 3.11, 3.12 and Proposition 3.13). By [22] it must be modular unless the residual representation were induced from a Galois character for $\mathbb{Q}(\sqrt{(-1)^{\frac{p-1}{2}}p})$. But if it is such an induced representation then it must be reducible restricting to the decomposition group for p since this quadratic field is ramified in p. But as we noted before $\bar{\rho}_f|_{G_p}$ is irreducible by [4], a contradiction. These implies R_x is CAP, contradicting the result of [11, Theorem 2.5.6].

Once we get one divisibility for $\mathcal{L}^\Sigma_{f,\mathcal{K},\xi}$, up to height one primes which are pullbacks of height one primes of $\mathcal{O}_L[[\Gamma^+_\mathcal{K}]]$ (coming from local Euler factors at non-split primes in Σ, by our discussion in [38, Section 6.4] on μ-invariants), the corresponding result for $\mathcal{L}_{f,\mathcal{K},\xi}$ also follows by using [10, Proposition 2.4] as in [38, End of 8.3] (note that $\mathcal{K}_\infty$ contains the cyclotomic $\mathbb{Z}_p$-extension).

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