Introduction to Skinner-Urban's Work on the Iwasawa Main Conjecture for GL_2

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1 Introduction

These notes are an introduction to the recent work of Christopher Skinner and Eric Urban [11] proving (one divisibility of) the Iwasawa main conjecture for $\mathrm{GL}_2/\mathbb{Q}$ (see Theorem 1). We give the necessary background materials and explain the proofs. We focus on the main ideas instead of the details and therefore will sometimes be brief and even imprecise.

These notes are organized as follows. In Section 2 we formulate various Iwasawa main conjectures for modular forms. We also explain an old result of Ribet to illustrate the rough idea of the strategy behind the later proofs. In Sections 3 and 4 we introduce the notions of automorphic forms and Eisenstein series on the unitary group $GU(2,2)$. Section 5 is devoted to explaining the Galois argument. Sections 6-9 give the tools used in the computation for the Fourier and Fourier-Jacobi coefficients of various Eisenstein series, which is a crucial ingredient in the argument. In Section 10 we give an example of a theorem of the author generalizing the Skinner-Urban work.

2 Main conjectures

We introduce the objects required to state the Iwasawa main conjectures for GL_2.

2.1 Families of Characters

Let p be an odd prime. Choose $\iota : \mathbb{C} \simeq \mathbb{C}_p$. Let $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $\mathbb{Q}_\infty \subset \mathbb{Q}(\mu_{p^\infty})$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. Let $\Lambda_{\mathbb{Q}} := \mathbb{Z}_p[[\Gamma_{\mathbb{Q}}]]$. We also define $\Lambda_A = \Lambda_{\mathbb{Q},A} = \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Z}_p} A$ for A a $\mathbb{Z}_p$-algebra. Let $\Psi = \Psi_{\mathbb{Q}} : G_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}^\times$ be the composition of $G_{\mathbb{Q}} \twoheadrightarrow \Gamma_{\mathbb{Q}}$ with $\Gamma_{\mathbb{Q}} \hookrightarrow \Lambda_{\mathbb{Q}}^\times$. Let $\varepsilon_{\mathbb{Q}}$ be a character of $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times$ which is the composition of $\Psi_{\mathbb{Q}}$ with the reciprocity map of class field theory (normalized using geometric Frobenius elements). Take $\gamma \in \Gamma$ to be the topological generator such that $\varepsilon(\gamma) = 1+p$ where ε is the cyclotomic character giving the canonical isomorphism $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_p^\times$. For each $\zeta \in \mu_{p^\infty}$ and integer k we let $\psi_{k,\zeta}$ be the finite order character of $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times$ that is the composition of $\Psi_{\mathbb{Q}}$ with the map $\Lambda_{\mathbb{Q}}^\times \to \mathbb{C}_p^\times$ that maps γ to $\zeta(1+p)^k$. We also write ψ_ζ for $\psi_{0,\zeta}$. We let ω be the Teichimuller character.

2.2 Characteristic Ideals and Fitting ideals

Let A be a Noetherian normal demain and X a finite A-module. The characteristic ideal $\mathrm{char}_A(X) \subset A$ is defined to be zero if X is not torsion and

$$\mathrm{char}_A(X) = \{x \in A | \mathrm{ord}_P x \geq \mathrm{length}_{A_P}(X_P), \text{ for all height one primes } P \subset A\}.$$

Now take any presentation

$$A^r \to A^s \to X \to 0$$

of X. The Fitting ideal is defined by the ideal of A generated by all the determinants of the $s \times s$ minors of the matrix representing the first arrow.

Remark 1. Fitting ideals respect any base change while characteristic ideals do not in general.

2.3 Selmer Groups for Modular Forms

Let $f=\sum_{n=1}^{\infty}a_nq^n\in S_k(N,\psi_0)$, $k\geq 2$, be a cuspidal eigenform with character ψ_0 of $(\mathbb{Z}/N\mathbb{Z})^{\times}$ and let $L/\mathbb{Q}_p$ be a finite extension containing all Fourier coefficients a_n of f. Let $\mathscr{O}_L$ be the ring of integers of L. Assume that f is ordinary, which means that a_p is a unit in $\mathscr{O}_L$. Let $\rho=\rho_f:G_{\mathbb{Q}}\to \mathrm{Aut}_L V_f$ be the usual two dimensional Galois representation associated to f. Then it is well known (by [14], for example) that there is a $G_{\mathbb{Q}_p}$-stable L-line $V_f^+\subset V_f$ such that V_f/V_f^+ is unramified. We fix a $G_{\mathbb{Q}}$-stable $\mathscr{O}_L$ lattice $T_f\subset V_f$ and let $T_f^+=T_f\cap V_f^+$.

Definition 1. (Selmer Groups) Let Σ be a finite set of primes.

$$\mathrm{Sel}_L^{\Sigma}(T_f):=\quad \ker\{H^1(\mathbb{Q},T_f\otimes_{\mathscr{O}_L}\Lambda^*_{\mathscr{O}_L}(\Psi^{-1}))\to H^1(I_p,(T_f/T_f^+)\otimes_{\mathscr{O}_L}\Lambda^*_{\mathscr{O}_L}(\Psi^{-1})) \times\prod_{\ell\neq p,\ell\notin\Sigma}H^1(I_\ell,T_f\otimes\Lambda^*_{\mathscr{O}_L}(\Psi^{-1}))\}$$

where $\Lambda_A^*=\mathrm{Hom}_{\mathbb{Z}_p}(\Lambda_A,\mathbb{Q}_p/\mathbb{Z}_p)$ is the Pontryagin dual and $\Lambda^*_{\mathscr{O}_L}(\Psi^{-1})$ means that the Galois action is given by the character Ψ^{-1}. Let

$$X_L^{\Sigma}(T_f):=\mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}_{\mathrm{L}}^{\Sigma}(T_f),\mathbb{Q}_p/\mathbb{Z}_p)$$

and

$$\mathrm{char}^{\Sigma}_{f,\mathbb{Q}}(f)=\mathrm{char}_{\Lambda_{\mathbb{Q},\mathscr{O}_L}}(X_L^{\Sigma}(T_f)).$$

2.4 p-adic L-functions

Let $0<n<k-2$ be an integer and $\zeta\neq 1$ a p^{t-1}th root of unity. Let Σ be a finite set of primes. We define the algebraic part of a special L-value for f by:

$$L^{\Sigma}_{alg}(f,\psi_\zeta^{-1}\omega^n,n+1):=a_p(f)^{-t}\frac{p^{t(n+1)}n!L^{\Sigma}(f,\psi_\zeta^{-1}\omega^n,n+1)}{(-2\pi i)^n\tau(\psi_\zeta^{-1}\omega^n)\Omega_f^{\mathrm{sgn}((-1)^m)}}$$

where $a_p(f)$ is the p-adic unit root of $x^2-a_px+p^{k-1}\psi_0=0$, $\tau(\psi)$ is a Gauss sum for ψ and $\Omega_f^{\pm}$ are Hida's canonical periods of f. (There is also a formula for $\zeta=1$ which is more complicated which we omit here). The p-adic L-function is a certain element $\mathscr{L}^{\Sigma}_{f,\mathbb{Q}}\in\Lambda_{\mathbb{Q},\mathscr{O}_L}$ characterized by the following interpolation property. Let $\phi_{n,\zeta}:\Lambda_{\mathscr{O}_L}\to\mathscr{O}_L(\zeta)$ be the $\mathscr{O}_L$ homomorphism sending γ to $\zeta(1+p)^n$. Then:

$$\phi_{n,\zeta}(\mathscr{L}^{\Sigma}_{f,\mathbb{Q}})=L^{\Sigma}_{alg}(f,\psi_\zeta^{-1}\omega^n,n+1), 0\leq n\leq k-2.$$

This was constructed in [1], and also [6].

2.5 The Main Conjecture

The Iwasawa main conjecture for f is the following

Conjecture 1. The module $X_L^{\Sigma}(T_f)$ is a finite torsion $\Lambda_{\mathbb{Q},\mathscr{O}_L}$-module and $\mathrm{char}^{\Sigma}_{f,\mathbb{Q}}$ is generated by $\mathscr{L}^{\Sigma}_{f,\mathbb{Q}}$.

The main result that we are going to prove in this lecture series is:

Theorem 1. *(Kato, Skinner-Urban) Suppose f has trivial character, weight 2 and good ordinary reduction at p. Suppose also that:*

- *The residual representation $\bar{\rho}_f$ is irreducible.*
- *For some $p\neq\ell||N$, $\bar{\rho}_f$ is ramified at ℓ.*

Then the Iwasawa main conjecture is true in $\Lambda_{\mathbb{Q},\mathscr{O}_L}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. If, moreover, there exists an $\mathscr{O}_L$-basis of T_f with respect to which the image of ρ_f contains $SL_2(\mathbb{Z}_p)$, then the equality holds in $\Lambda_{\mathbb{Q},\mathscr{O}_L}$.

The last condition is put by Kato [4] who proved "$\supseteq$". It is satisfied, for example, by the p-adic Tate modules of semistable elliptic curves if $p\geq 11$. We will focus on Skinner-Urban's proof for "$\subseteq$". The technical condition (ii) can be removed by working with forms over totally real fields and a base change trick .

2.6 Two and Three-Variable Main Conjectures

More p-adic characters: Let $\mathscr{K}/\mathbb{Q}$ be an imaginary quadratic extension such that p splits as $v_0\bar{v}_0$, where v_0 is determined by our chosen isomorphism $\iota : \mathbb{C} \simeq \mathbb{C}_p$. By class field theory there is a unique $\mathbb{Z}_p^2$-extension of $\mathscr{K}$ unramified outside p, which we denote by $\mathscr{K}_\infty$. Let $G_{\mathscr{K}} := \mathrm{Gal}(\bar{\mathscr{K}}/\mathscr{K})$ and $\Gamma_{\mathscr{K}} := \mathrm{Gal}(\mathscr{K}_\infty/\mathscr{K})$. There is an action of complex conjugation c on $\Gamma_{\mathscr{K}}$. We write $\Gamma_{\mathscr{K}}^{\pm}$ for the subgroup on which c acts by ± 1. For any $\mathbb{Z}_p$-algebra $A \subset \bar{\mathbb{Q}}_p$ we define: $\Lambda_{\mathscr{K},A} = A[[\Gamma_{\mathscr{K}}]]$, $\Lambda_{\mathscr{K},A}^{\pm} = A[[\Gamma_{\mathscr{K}}^{\pm}]]$ and generators $\gamma^{\pm}$ of $\Gamma_{\mathscr{K}}^{\pm}$ by requiring $\mathrm{rec}_{\mathscr{K}_p}((1+p)^{\frac{1}{2}}, (1+p)^{\pm\frac{1}{2}}) = \gamma^{\pm}$. Here $\mathrm{rec}_{\mathscr{K}_p}$ is the reciprocity map of class field theory. (Note that $\mathscr{K}_p \simeq \mathbb{Q}_p \times \mathbb{Q}_p$.) Let $\Psi_{\mathscr{K}}$ be the composition $G_{\mathscr{K}} \to \Gamma_{\mathscr{K}} \hookrightarrow \Lambda_{\mathscr{K},A}^{\times}$. We define $\Psi_{\mathscr{K}}^{\pm}$ similarly. We also define characters $\varepsilon_{\mathscr{K}}, \varepsilon_{\mathscr{K}}^{\pm}$ of $\mathscr{K}^{\times}\backslash\mathbb{A}_{\mathscr{K}}^{\times}$ by composing $\varepsilon_{\mathscr{K}}, \varepsilon_{\mathscr{K}}^{\pm}$ with the reciprocity map.

Now let f be a cuspidal eigenform of weight $k \geq 2$. We define the set of arithmetic points by:

$$\mathscr{X}_{f,\mathscr{K}}^{a} := \{\phi : \mathscr{O}_L \text{ homomorphism } \Lambda_{\mathscr{K},\mathscr{O}_L} \to \bar{\mathbb{Q}}_p : \phi(\gamma^+) = \zeta^+(1+p)^{k-2}, \phi(\gamma^-) = \zeta^-, \zeta^{\pm} \in \mu_{p^\infty}\}.$$

For $\phi \in \mathscr{X}_{f,\mathscr{K}}^{a}$, let $\theta_\phi := \omega^{2-k}\chi_f^{-1}\xi_\phi$ with $\xi_\phi := (\phi \circ \Psi_{\mathscr{K}})(\varepsilon^{2-k}\varepsilon^{k-2}.\chi_f)$ and let $\mathfrak{f}_{\theta_\phi}$ be the conductor of θ_ϕ. In particular we show that for any finite set Σ of primes containing all the bad primes (all the primes where f or $\mathscr{K}$ is ramified), we have the two variable p-adic L-function $\mathscr{L}_{f,\mathscr{K}}^{\Sigma} \in \Lambda_{\mathscr{K},\mathscr{O}_L}$ such that:

$$\phi(\mathscr{L}_{f,\mathscr{K}}^{\Sigma}) = u_f a_p(f)^{ord_p(Nm(\mathfrak{f}_{\theta_\phi}))}.\frac{((k-2)!)^2\mathfrak{g}(\theta_\phi)Nm(\mathfrak{f}_{\theta_\phi}\mathfrak{d})^{k-2}L_{\mathscr{K}}^{\Sigma}(f,\theta_\phi,k-1)}{(2\pi i)^{2k-2}\Omega_f^+\Omega_f^-}$$

for any sufficiently ramified $\phi \in \mathscr{X}_{f,\mathscr{K}}^{a}$, where $\mathfrak{d}$ is the different of $\mathscr{K}$ and u_f is a p-adic unit depending on f. We remark that if $\zeta^- = 1$ then our special L-value is just the product of the special L-values for f and $f \otimes \chi_{\mathscr{K}}$ twisted by some ψ such that $\psi \circ \mathrm{Nm} = \theta_\phi$. Here $\chi_{\mathscr{K}}$ is the quadratic character for $\mathscr{K}/\mathbb{Q}$.

We can also define Selmer groups $Sel_{f,\mathscr{K}}^{\Sigma}$ and $X_{f,\mathscr{K}}^{\Sigma}$, $\mathrm{char}_{f,\mathscr{K}}^{\Sigma}$ in the exact same way as in the one-variable case. We have the two-variable main conjecture:

Conjecture 2. $X_{f,\mathscr{K}}^{\Sigma}$ is a finite torsion $\Lambda_{\mathscr{K},\mathscr{O}_L}$-module. Furthermore $\mathrm{char}_{\mathscr{K},f}^{\Sigma}$ is principal and generated by $\mathscr{L}_{f,\mathscr{K}}^{\Sigma}$.

Additionally f can be embedded in a Hida family of ordinary cuspidal eigenforms $\mathbf{f}$ (we discuss these in the next section in more detail). We can form a three-variable p-adic L-function $\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma}$ and formulate a 3 variable main conjecture.

2.7 Comment on the Proof

The cyclotomic main conjecture for modular forms (conjecture 1) is deduced from a partial inclusion in three-variable main conjecture. Roughly speaking the inclusion $\mathrm{char}_{\mathbf{f},\mathscr{K}} \subseteq (\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma})$ in the three-variable main conjecture, when specialized to the cyclotomic $\mathbb{Z}_p$-extension of $\mathscr{K}$, implies that the inclusion $\mathrm{char}_{f,\mathbb{Q}}.\mathrm{char}_{f\otimes\chi_{\mathscr{K}},\mathbb{Q}} \subseteq (\mathscr{L}_{f,\mathbb{Q}}.\mathscr{L}_{f\otimes\chi_{\mathscr{K}},\mathbb{Q}})$ in the cyclotomic main conjecture for f and $f \otimes \chi_{\mathscr{K}}$. By Kato's work, this in turn implies that $\mathrm{char}_{f,\mathbb{Q}} = (\mathscr{L}_{f,\mathbb{Q}})$ and $\mathrm{char}_{f\otimes\chi_{\mathscr{K}},\mathbb{Q}} = (\mathscr{L}_{f\otimes\chi_{\mathscr{K}},\mathbb{Q}})$.

We will focus on proving the inclusion $\mathrm{char}_{\mathbf{f},\mathscr{K}}^{\Sigma} \subseteq (\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma})$ in the three variable main conjecture in the rest of the paper. The general strategy for proving this inclusion is: the product of $\mathscr{L}_{\mathscr{K}}^{\Sigma}$ and the Σ-imprimitive Kubota-Leopodlt p-adic L-function $\mathscr{L}_{1,\mathbb{Q}}^{\Sigma}$ attached to the trivial character gives the congruences between Eisenstein series and cusp forms on the unitary similitude group $GU(2,2) \Rightarrow$ the same congruences between reducible and irreducible Galois representations $\Rightarrow$ required extension class in $H^1(\mathscr{K},-)$. The first arrow is by the Langlands correspondence and the second is a Galois theoretic argument, the so-called "lattice construction".

2.8 A Theorem of Ribet

In this section we review Ribet's proof of the converse to Herbrand's theorem [8]. This illustrates the main ideas in the strategy. In this section we set $\mathscr{O}_L = \mathbb{Z}_p$, λ the maximal ideal of $\mathscr{O}_L$, $\kappa = \mathscr{O}_L/\lambda$.

Theorem 2. *Suppose $j \in [2, p-3]$ is an even number. If $p|\zeta(1-j)$ then $H_{ur}^1(G_{\mathbb{Q}}, \mathbb{F}(\omega^{1-j})) \neq 0$ (the group of everywhere unramified classes is non-zero).*

Proof. For $j \neq 2$, we make use of the level 1 weight j Eisenstein series:

$$E_j(q) = \frac{\zeta(1-j)}{2} + \sum_{n\geq 1} \sigma_{j-1}(n) q^n$$

where $\sigma_{j-1}(n) = \sum_{d|n} d^{j-1}$. If $p | \zeta(1-j)$, then E_j "looks" like a cusp form modulo p. We divide the proof into three steps:

<u>Step1</u>: Construct a cusp form $f' \in S_j(SL_2(\mathbb{Z}), \mathbb{Z}_p)$ such that $f' \equiv E_j(mod\ p)$ (in terms of q-expansion). This is a case by case study using the fact that the ring of modular forms of level 1 is $\mathbb{C}[E_4, E_6]$.

<u>Step 2</u>: Prove that f' can be replaced by an eigenform $f \in S_j(SL_2(\mathbb{Z}_p), \mathscr{O}_L)$ whose Hecke eigenvalues are the same as those of E_j modulo p. This can be proven by easy commutative algebra (essentially a lemma of Deligne and Serre).

<u>Step 3</u>: The lattice construction: construct the class by comparing the Galois representations of E_j and f. Note that the Galois representation for E_j is $\varepsilon^{j-1} \oplus 1$. It is easy to see that there is a $\sigma_0 \in I_p$ such that $\varepsilon^{j-1}(\sigma_0) \not\equiv 1(\text{mod } p)$. Since $a_p(f) \equiv \sigma_{j-1}(p) \equiv 1(\text{mod } p)$, f is ordinary. As we have noted before, $\rho_f|_{G_{\mathbb{Q}_p}} = \begin{pmatrix} \alpha\varepsilon^{j-1} & * \\ & \alpha \end{pmatrix}$ for some unramified character α. Take a basis $\{v_1, v_2\}$ such that

$$\rho_f(\sigma_0) = \begin{pmatrix} \varepsilon^{j-1}(\sigma_0) & \\ & 1 \end{pmatrix}.$$

Write $\rho = \rho_f$ and $\rho(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$ for $\sigma \in \mathscr{O}[G_{\mathbb{Q}}]$.
<u>Claim</u>:

(a) $a_\sigma, d_\sigma, b_\sigma c_\tau \in \mathscr{O}$ for $\sigma, \tau \in \mathscr{O}_L[G_{\mathbb{Q}}]$ and $a_\sigma \equiv \omega^{j-1}(\sigma), d_\sigma \equiv 1, b_\sigma c_\tau \equiv 0(\text{mod } p)$;
(b) $\mathscr{C} := \{c_\sigma : \sigma \in \mathscr{O}_L[G_{\mathbb{Q}}]\}$ is a non-zero fractional ideal.
(c) $c_\sigma = 0$ if $\sigma \in I_\ell$ for all ℓ.

<u>Proof of the claim</u>:

Let $\varepsilon_1 := \frac{1}{\varepsilon^{j-1}(\sigma_0)-1}(\sigma_0 - 1), \varepsilon_2 := \frac{1}{1-\varepsilon^{j-1}(\sigma_0)}(\sigma_0 - \varepsilon^{j-1}(\sigma_0))$; one can check: $\rho(\varepsilon_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\rho(\varepsilon_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Thus $a_\sigma = \text{trace}\rho(\varepsilon_1\sigma) \in \mathscr{O}_L$ and $\text{trace}\rho(\varepsilon_1\sigma) \equiv \text{trace}(\varepsilon^{j-1}+1)(\varepsilon_1\sigma) = \varepsilon^{j-1}(\sigma)$. The claim for d_σ is proven similarly. Also since $\rho(\sigma)\rho(\tau) = \rho(\sigma\tau)$, $b_\sigma c_\tau = a_{\sigma\tau} - a_\sigma a_\tau \equiv 0(\text{mod } p)$.

(b) follows from the irreducibility of ρ and (c) can be seen from the description for $\rho|_{G_{\mathbb{Q}_p}}$ above and the triviality of $\rho|_{I_\ell}$ for $\ell \neq p$.

Let $M_1 = \mathscr{O}_L v_1, M_2 = \mathscr{C} v_2, M = M_1 \oplus M_2$ (which is easily seen to be the $\mathscr{O}_L[G_{\mathbb{Q}}]$-submodule generated by v_1).

- $\bar{M_2} := M_2/\lambda M_2 \simeq \kappa$ (note that $\mathscr{C}$ is non-zero by (b)) is a $G_{\mathbb{Q}}$ stable submodule of $\bar{M} := M/\lambda M$. This is because for any $m_2 = cv_2 \in M_2$, $\rho(\sigma)m_2 = b_\sigma c v_1 + d_\sigma c v_2 \in \lambda v_1 + \mathscr{C} v_2$ by (a);
- by (a), $G_{\mathbb{Q}}$ acts by ω^{j-1} and 1 on $\bar{M_1} = \bar{M}/\bar{M_2}$ and $\bar{M_2}$ respectively;
- the extension: $0 \to \bar{M_2} \to \bar{M} \to \bar{M_1} \to 0$ is non split since M is generated by v_1 over $\mathscr{O}_L[G_{\mathbb{Q}}]$.

Thus $\bar{M}$ gives a nontrivial extension class and it actually in $H^1_{ur}(\mathbb{Q}, \kappa(\omega^{1-j}))$ by claim (c).

If $j = 2$ the Eisenstein series E_2 is not holomorphic and we use E_{p+1} in the place of E_2.

3 Hermitian Modular Forms on $GU(n,n)$

3.1 Hermitian Half Space and Automorphic Forms

Let $\mathcal{K}/\mathbb{Q}$ be an imaginary quadratic extension and let $\mathcal{O}$ be the ring of integers of $\mathcal{K}$. Let $GU(n,n)$ be the unitary similitude group associated to the pairing $\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix} = \omega_n$ on $\mathcal{K}^{2n}$:

$$G := GU(n,n)(A) = \{g \in GL_{2n}(\mathcal{O}\otimes A) : g\omega_n {}^t\bar{g} = \lambda_g\omega_n, \lambda_g \in A^\times\}.$$

Here $\mu(g) := \lambda_g$ is the similitude character, and we write $U := U(n,n) \subset G$ for the kernel of μ. We define $Q = Q_n$ to be the Siegel parabolic subgroup of G consisting of block matrices of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $C = 0$. Let

$$\mathbf{H}_n := \{Z \in M_n(\mathbb{C}) : -i(Z - {}^t\bar{Z}) > 0\}.$$

(Note that $\mathbf{H}_1$ is the usual upper half plane).

Let $Z \in \mathbf{H}_n$. For $\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(\mathbb{R})$ with A,B,C,D $n\times n$ block matrices. Let $\mu_\alpha(Z) := CZ + D, \kappa_\alpha(Z) = \bar{C}{}^tZ + \bar{D}$. We define the automorphy factor:

$$J(\alpha, Z) := (\mu_\alpha(Z), \kappa_\alpha(Z)).$$

Let $G(\mathbb{R})^+ = \{g \in G(\mathbb{R}), \mu(g) > 0\}$ then $G(\mathbb{R})^+$ acts on $\mathbf{H}_n$ by

$$g(Z) := (A_gZ + B_g)(C_gZ + D_g)^{-1},\ g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}.$$

Let $K_\infty^+ = \{g \in U(\mathbb{R}) : g(i) = i\}$ (we write i for the matrix $i1_n \in \mathbf{H}_n$) and Z_∞ be the center of $G(\mathbb{R})$. We define $C_\infty := Z_\infty K_\infty^+$. Then $k_\infty \mapsto J(k_\infty, i)$ defines a homomorphism from C_∞ to $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$.

Definition 2. A weight $\underline{k}$ is a set of integers $(k_{n+1},...,k_{2n}; k_n,...,k_1)$ such that $k_1 \geq k_2 \geq ... \geq k_{2n}$ and $k_n \geq k_{n+1} + 2n$. A weight $\underline{k}$ defines an algebraic representation of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ by

$$\rho_{\underline{k}}(g_+, g_-) := \rho_{(k_n,...,k_1)}(g_+) \otimes \rho_{(-k_{n+1},...,-k_{2n})}(g_-)$$

where $\rho_{(a_1,...,a_n)}$ is the dual of the usual irreducible algebraic representation of GL_n with highest weight $(a_1,...,a_n)$. Let $V_{\underline{k}}(\mathbb{C})$ be the representation of C_∞ given by

$$k_\infty \to \rho_{\underline{k}} \circ J(k_\infty, i).$$

Fix K an open compact of $G(\mathbb{A}_f)$. We let

$$Sh_K(G) = G(\mathbb{Q})^+ \backslash \mathbf{H}_n \times G(\mathbb{A}_f)/KC_\infty.$$

The automorphic sheaf $\omega_{\underline{k}}$ is the sheaf of holomorphic sections of

$$G(\mathbb{Q})^+\backslash \mathbf{H}_n \times G(\mathbb{A}_f) \times V_{\underline{k}}(\mathbb{C})/KC_\infty \to G(\mathbb{Q})^+\backslash \mathbf{H}_n^+ \times G(\mathbb{A}_f)/KC_\infty.$$

One can also define these Shimura varieties and automorphic sheaves in terms of moduli of abelian varieties. We omit these here.

The global sections of $\omega_{\underline{k}}$ is the space of modular forms consisting of holomorphic functions:

$$f : \mathbf{H}_n \times G(\mathbb{A}_f) \to V_{\underline{k}}(\mathbb{C})$$

which are invariant by some open compact K of the second variable, and satisfy:

$$\mu(\gamma)^{\frac{k_1+...+k_{2n}}{2}} \rho_{\underline{k}}(J(\gamma, Z))^{-1} f(\gamma(Z), g) = f(Z, g)$$

for all $\gamma \in gKg^{-1} \cap G^+(\mathbb{Q})$. Also, when $n = 1$ we require a moderate growth condition.

Remark 2. We will be mainly interested in the scalar-valued forms. In this case $V_{\underline{k}}(\mathbb{C})$ is 1-dimensional of weight $\underline{k} = (0,...,0; \kappa,...,\kappa)$ for some integer $\kappa \geq 2$.

3.2 Hida Theory

Hida Theory $GL_2/\mathbb{Q}$

We choose a quick way to present Hida theory. Let M be prime to p and χ a character of $(\mathbb{Z}/pM\mathbb{Z})^\times$. The *weight space* is SpecΛ for $\Lambda := \mathbb{Z}_p[[T]]$. Let $\mathbb{I}$ be a domain finite over Λ. A point $\phi \in$ Spec$\mathbb{I}$ is called arithmetic if the image of ϕ in SpecΛ is given by the $\mathbb{Z}_p$-homomorphism sending $(1+T) \mapsto \zeta(1+p)^{\kappa-2}$ for some $\kappa \geq 2$ and ζ a p-power root of unity. We usually write κ_ϕ for this κ, called the weight of ϕ. We also define χ_ϕ to be the character of $\mathbb{Z}_p^\times \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times (1+p\mathbb{Z}_p)$ that is trivial on the first factor and is given by $(1+p) \mapsto \zeta$ on the second factor.

Definition 3. An $\mathbb{I}$-family of forms of tame level M and character χ is a formal q-expansion $\mathbf{f} = \sum_{n=0}^\infty a_n q^n, a_n \in \mathbb{I}$, such that for a Zariski dense set of arithmetic points ϕ the specialization $f_\phi = \sum_{n=0}^\infty \phi(a_n)q^n$ of $\mathbf{f}$ at ϕ is the q-expansion of a modular form of weight κ_ϕ, character $\chi\chi_\phi\omega^{2-\kappa_\phi}$ where ω is the Techimuller character, and level M times some power of p.

Definition 4. The U_p operator is defined on both the spaces of modular forms and families. It is given by:

$$U_p(\sum_{n=0}^\infty a_n q^n) = \sum_{n=0}^\infty a_{pn} q^n.$$

Hida's ordinary idempotent e_p is defined by $e_p := \lim_{n\to\infty} U_p^{n!}$. A form f or family $\mathbf{f}$ is called ordinary if $e_p f = f$ or $e_p \mathbf{f} = \mathbf{f}$.

FACT The space of ordinary families is finite and free over the ring $\mathbb{I}$.

Remark 3. For Hilbert modular forms the analogues space is still finite but not free in general. The subspace of ordinary cuspidal families is both finite and free.

Hida Theory for $GU(2,2)$

For simplicity let us restrict to the case when the prime to p part of the nebentypus is trivial. Fix some prime to p level group K of $G(\hat{\mathbb{Z}})$. Let T be the diagonal torus of $U = U(2,2)$. Let χ be a character of $T(\mathbb{Z}/p\mathbb{Z})$. The weight space is SpecΛ_2 where Λ_2 is defined to be the completed group algebra of $T(1+p\mathbb{Z}) = (1+p\mathbb{Z})^4$. Let A be any domain finite over Λ_2.

Definition 5. A weight $\underline{k} = (k_1,k_2;k_3,k_4)$ is a set of integers k_i such that $k_1 \geq k_2+2 \geq k_3+4 \geq k_4+4$.

Definition 6. A point $\phi \in$ Spec$A$ is called arithmetic if its image in SpecΛ_2 is given by the character $[\underline{k}]\chi_\phi.\chi$ where $\underline{k}$ is a weight and $[\underline{k}]$ is given by:

$$\mathrm{diag}(t_1,t_2,t_3,t_4) \mapsto t_1^{k_3}t_2^{k_4}t_3^{k_2}t_4^{k_1}$$

(We identify $U(\mathbb{Z}_p) \simeq GL_4(\mathbb{Z}_p)$ by the first projection of $\mathscr{K}_p \simeq \mathscr{K}_{v_0} \times \mathscr{K}_{\bar{v}_0}$) and χ_ϕ is a finite order character of $T(1+p\mathbb{Z})$.

We are going to define Hida families by a finite number of q-expansions: Let $K \subset G(\mathbb{A}_f)$ be a level group $X(K)$ be a finite set of representatives x of $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ with $x_p \in Q(\mathbb{Z}_p)$. For any $g \in GU(2,2)(\mathbb{A}_\mathbb{Q})$ let $S_{[g]}^+$ comprise those positive semi-definite Hermitian matrices h in $M_n(\mathscr{K})$ such that Tr$hh' \in \mathbb{Z}$ for all Hermitian matrices h' such that $\begin{pmatrix} 1 & h' \\ & 1 \end{pmatrix} \in N_Q(\mathbb{Q}) \cap gKg^{-1}$.

Definition 7. For any ring A finite over Λ_2 we define space of A-adic forms with tame level $K \subset G(\mathbb{A}_f)$ and coefficient ring A to be the elements:

$$F := \{F_x\}_{x\in X(K)} \in \oplus_{x\in X(K)} A[[q^{S_{[x]}^+}]]$$

such that for a Zariski dense set of arithmetic points $\phi \in$ A the specialization F_ϕ of F at ϕ is the q-expansion of the matrix coefficient of the highest weight vector of holomorphic modular forms of weight $\underline{k}_\phi$ and nebentypus $\chi\chi_\phi\omega(t_1^{-k_3}t_2^{-k_4}t_3^{-k_2}t_4^{-k_1})$.

(In the Skinner-Urban case the interpolated points ϕ are of scalar weights and thus do not need to take the highest weight vector.)

Definition 8. Some U_p operators: for $t^+ = \mathrm{diag}(t_1,t_2,t_3,t_4) \in T(\mathbb{Q}_p)$ such that $t_2/t_1, t_3/t_2, t_3/t_4 \in p\mathbb{Z}_p$. We define an operator U_{t^+} on the space of Hermitian modular forms by: $U_{t^+}.f = |[\underline{k}^*](t)|_p^{-1} f|_{\underline{k}} u_{t^+}$ where $[\underline{k}^*] = [\underline{k} + (2,2;-2,-2)]$ and $f|_{\underline{k}} u_{t^+}$ is the usual Hecke operator defined by double coset decomposition (with no normalization factors).

Hida proved that this u_{t^+} preserves integrality of modular forms and defined an idempotent:

$$e^{ord} := \lim_{n\to\infty} u_{t^+}^{n!}.$$

A form or family F is called nearly ordinary if $e^{ord}F = F$. Again, we have that the space of nearly ordinary Hida families with coefficient ring A is finite and free over A. This is called the *Hida's control theorem* for ordinary forms.

Remark 4. In order to prove the finiteness and freeness (both in the GL_2 and unitary group case) we need to go back to the notion of p-adic modular forms using the Igusa tower, which we omit here.

Another important input of Hida theory is the fundamental exact sequence proved [11, Chapter 6]. We let $C_1(K)$ be the set of cusp labels of genus 2 and label K ([11, 5.4.3]). Write Λ_1 for the weight ring of $U(1,1)/\mathbb{Q}$. Then Skinner-Urban proved the following

Theorem 3. *For any Λ_2-algebra A there is a short exact sequence*

$$0 \to \mathscr{M}^0_{ord}(K^p,A) \to \mathscr{M}^1_{ord}(K^p,A) \to \oplus_{[g]\in C_1(K)} \mathscr{M}^0_{ord}(K^p_{1,g},\Lambda_1) \otimes_{\Lambda_1} A \to 0.$$

Here $\mathscr{M}^0_{ord}(K^p,A)$ is the space of A-valued families of ordinary cusp forms on $GU(2,2)$, $\mathscr{M}^1_{ord}(K^p,A)$ is the space of ordinary forms taking 0 at all genus 0 cusps ([11, 5.4]). The $\mathscr{M}^0_{ord}(K^p_{1,g},\Lambda_1)$ is the space of ordinary cusp forms on $U(1,1)$ with tame level group $K^p_{1,g}$ for $K^p_{1,g} = GU(1,1)(\mathbb{A}_f) \cap gKg^{-1}$ and $GU(1,1)$ is embedded as the levi subgroup of the Klingen Parabolic subgroup of $GU(2,2)$. The Φ is the "Siegel operator" giving the restricting to boundary map. The Λ_1-algebra structure for Λ_2 is given by the embedding $T_1 \hookrightarrow T_2 : (t_1,t_2) \to (t_1,1,t_2,1)$.

The proof is a careful study of the geometry of the boundary of the Igusa varieties ([11, 6.2,6.3]). This theorem is used to construct a cuspidal Hida family on $GU(2,2)$ that is congruent to the Klingen Eisenstein series modulo the p-adic L-function.

One more important property of ordinary families is that the specialization of a nearly ordinary family to a very regular weight is classical. This will be used to ensure that the $\Lambda_{\mathscr{D}}$-adic Hecke algebra of ordinary $\Lambda_{\mathscr{D}}$-adic form can not have CAP components.

4 Eisenstein Series on GU(2,2)

4.1 Klingen Eisenstein Series

Let P be the Klingen Parabolic subgroup of $GU(2,2)$ consisting of matrices of the form

$$\begin{pmatrix} \times & 0 & \times & \times \\ \times & \times & \times & \times \\ \times & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{pmatrix}.$$

Let M_P be the levi subgroup of P defined by

$$M_P \simeq GU(1,1) \times \mathrm{Res}_{\mathscr{O}_{\mathscr{K}}/\mathbb{Z}} \mathbb{G}_m, (g,x) \mapsto \begin{pmatrix} A_g & & B_g & \\ & \mu(g)\bar{x}^{-1} & & \\ C_g & & D_g & \\ & & & x \end{pmatrix}.$$

Let N_P be the unipotent radical of P.

Observe that if π is an automorphic representation of GL_2 and ψ is a Hecke character of $\mathbb{A}_{\mathcal{K}}^{\times}$ which restricts to the central character χ_π of π on $\mathbb{A}_{\mathbb{Q}}^{\times}$, then these uniquely determine an automorphic representation π_ψ of $GU(1,1)$ with central character ψ. Now suppose we have a triple (π,ψ,τ) where π is an irreducible cuspidal automorphic representation of GL_2 and ψ and τ are Hecke characters of $\mathbb{A}_{\mathcal{K}}^{\times}$ such that $\psi|_{\mathbb{A}_{\mathbb{Q}}^{\times}} = \chi_\pi$. Then $\pi_\psi \boxtimes \tau$ is an automorphic representation of M_P. We extend this to a representation of P by requiring that N_P act trivially. Then Klingen Eisenstein series are forms on $GU(2,2)$ which are induced the above representation of P. In fact we need to first work locally for each place v (say, finite) of $\mathbb{Q}$. Let (π_v,ψ_v,τ_v) be the local triple then $(\pi_\psi)_v \boxtimes \tau_v$ is a representation of $M_P(\mathbb{Q}_v)$. We extend it to a representation ρ_v of $P(\mathbb{Q}_v)$ by requiring that $N_P(\mathbb{Q}_v)$ acts trivially. Then we form the induced representation $I(\rho_v) = \mathrm{Ind}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)} \rho_v$. When everything is unramified and ϕ_v is a spherical vector of π_v, there is a unique vector $f_{\phi_v}^0 \in I(\rho_v)$ which is invariant under $G(\mathbb{Z}_v)$ and $f_{\phi_v}^0(1) = \phi_v$. The Archimedean picture is slightly different (see [11, section 9.1]).

Let $\phi = \otimes_v \phi_v \in \pi$ and let $I(\rho)$ be the restricted product of the $I(\rho_v)$'s with respect to the unramified vectors above. If $f \in I(\rho)$ we let $f_z(g) = \delta(m)^{\frac{3}{2}+z}\rho(m)f(k)$ for $g = mnk \in M_P N_P K$. Here we let K be $G(\hat{\mathbb{Z}})$. Note that the f_z takes values in the representation space V of π. However π can be embedded to the space of automorphic forms on $GL_2(\mathbb{A}_{\mathbb{Q}})$. We also write $f_z(g)$ for the function on $GU(2,2)(\mathbb{A}_{\mathbb{Q}})$ given by $f_z(g)(1)$.

The *Klingen Eisenstein Series* is defined by:

$$E(f;z,g) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f_z(\gamma g).$$

This is absolutely convergent for $\mathrm{Re} z >> 0$ and can be meromorphically continued to all $z \in \mathbb{C}$.

4.2 p-adic Families

Let $\mathbb{I}$ be a normal domain finite over $\mathbb{Z}_p[[W]]$. (W is a variable) and $\mathbf{f}$ is a normalized ordinary eigenform with coefficient ring $\mathbb{I}$. In section 8 we are going to define the "Eisenstein Datum" $\mathcal{D}$ which contains the information of $\mathbf{f}, \mathbb{I}, \mathcal{K}$, etc. Define $\Lambda_{\mathcal{D}} := \mathbb{I}[[\Gamma_{\mathcal{K}}]][[\Gamma_{\mathcal{K}}^{-}]]$. We are going to define the set of arithmetic points $\phi \in \mathrm{Spec}\Lambda_{\mathcal{D}}$ and this $\Lambda_{\mathcal{D}}$ p-adically parameterizes triples $(f_\phi,\psi_\phi,\tau_\phi)$ to which we associate the Klingen Eisenstein series. Later we will also give $\Lambda_{\mathcal{D}}$ the structure of a finite Λ_2-algebra and construct a $\Lambda_{\mathcal{D}}$-adic nearly ordinary Klingen Eisenstein family, which we denote by $\mathbf{E}_{\mathcal{D}}$.

Now we consider $\Lambda_{\mathcal{D}}$-adic cusp forms on $GU(2,2)$. Let $h_{\mathcal{D}} := h_{ord}^{\Sigma,0}(K,\Lambda_{\mathcal{D}})$ be the Hecke algebra for the space of $\Lambda_{\mathcal{D}}$-coefficient nearly ordinary cuspidal forms with respect to some level group K. It is generated by Hecke operators at primes outside Σ and the prime p.

Definition 9. Let $I_{\mathcal{D}}$ be the ideal of $h_{\mathcal{D}}$ generated by $\{T - \lambda_{\mathbf{E}_{\mathcal{D}}}(T)\}$'s for T elements in the abstract algebra. Here $\lambda_{\mathbf{E}_{\mathcal{D}}}(T)$ is the Hecke eigenvalue of T on $\mathbf{E}_{\mathcal{D}}$. The structure map $\Lambda_{\mathcal{D}} \to h_{\mathcal{D}}/I_{\mathcal{D}}$ is easily seen to be surjective. Thus there is an ideal $\mathcal{E}_{\mathcal{D}}$ of $\Lambda_{\mathcal{D}}$ such that $\Lambda_{\mathcal{D}}/\mathcal{E}_{\mathcal{D}} \simeq h_{\mathcal{D}}/I_{\mathcal{D}}$. This $\mathcal{E}_{\mathcal{D}}$ is called the Klingen Eisenstein ideal.

The motivation to define this ideal will be more clear after we have discussed the Galois representations.

5 Galois Representations and Lattice construction

5.1 Galois Representations

We first recall the following theorem (due to Harris-Taylor, S.W Shin, S. Morel, C.Skinner *et al.*) attaching Galois representations to automorphic representations on $GU(n,n)$.

Theorem 4. *Let π be an irreducible cuspidal representation of $GU(n,n)(\mathbb{A}_{\mathbb{Q}})$ and let χ_π be its central character. Let $\Sigma(\pi)$ be the finite set of primes ℓ such that either π_ℓ or $\mathscr{K}$ is ramified. Suppose π_∞ is the regular holomorphic discrete series of weight $\underline{k} := (k_{n+1},...,k_{2n};k_1,...,k_n)$ such that*

$$k_1 \geq ... \geq k_n, k_n \geq k_{n+1} + 2n, k_{n+1} \geq ... \geq k_{2n}.$$

Then there is a continuous semisimple representation:

$$R_p(\pi) : G_{\mathscr{K}} \to GL_n(\bar{\mathbb{Q}}_p)$$

such that:

(i) $R_p(\pi)^\vee(1-2n) \otimes \sigma_{\chi_\pi}^{1+c} \simeq R_p(\pi)^c$.
(ii) $R_p(\pi)$ *is unramified outside primes above those in* $\Sigma(\pi) \cup \{p\}$ *and for such primes* w *we have*

$$\det(1 - R_p(\pi)(\mathrm{frob}_w)q_w^{-s}) = L(BC(\pi)_w \otimes \psi_w, s + \frac{1}{2} - n)^{-1}.$$

(iii) If π *is nearly ordinary at* p*, then:*

$$R_p(\pi)_{G_{\mathscr{K}_{v_0}}} \simeq \begin{pmatrix} \xi_{2n,v_0}\varepsilon^{-\kappa_{2n}} & * & * \\ 0 & ... & * \\ 0 & 0 & \xi_{1,v_0}\varepsilon^{-\kappa_1} \end{pmatrix}$$

and

$$R_p(\pi)|_{G_{\mathscr{K},\bar{v}_0}} \simeq \begin{pmatrix} \xi_{1,\bar{v}_0}\varepsilon^{\kappa_1+1-2n-|\underline{k}|} & * & * \\ 0 & ... & * \\ 0 & 0 & \xi_{2n,\bar{v}_0}\varepsilon^{\kappa_{2n}+1-2n-|\underline{k}|} \end{pmatrix}.$$

Here ξ_{i,v_0} *and* $\xi_{i,\bar{v}_0}$ *are unramified characters and* ε *is the cyclotomic character,* $|\underline{k}| = k_1 + ... + k_{2n}$, $\kappa_i = k_i + n - i$ *for* $1 \leq i \leq n$ *and* $\kappa_i = k_i + 3n - i$ *for* $n+1 \leq i \leq 2n$.

Returning to the $GU(2,2)$ case, it is formal to patch the Galois representations attached to cuspidal nearly ordinary forms to a Galois pseudo-character $R_{\mathscr{D}}$ of $G_{\mathscr{K}}$ with values in $h_{\mathscr{D}}$. (Pseudo characters are firstly introduced by Wiles [14]. They are function on $G_{\mathscr{K}}$ satisfying the relations that should be satisfied by the trace of a representation. However it does not necessarily come from a representation. We omit the definitions.) We can associate a Galois representation $\rho_{\mathbf{E}_{\mathscr{D}}}$ to the Klingen Eisenstein family $\mathbf{E}_{\mathscr{D}}$ with coefficient ring $\Lambda_{\mathscr{D}}$ by a similar recipe. It is essentially the direct sum of the Galois representation $\rho_{\mathbf{f}}$ associated to the Hida family $\mathbf{f}$ with two $\Lambda_{\mathscr{D}}$-adic characters.

The *motivation* for the Klingen Eisenstein ideal is:

$$R_{\mathscr{D}}(\mathrm{mod} I_{\mathscr{D}}) = \mathrm{tr}\rho_{\mathbf{E}_{\mathscr{D}}}(\mathrm{mod}\mathscr{E}_{\mathscr{D}}).$$

(Recall that $h_{\mathscr{D}}/I_{\mathscr{D}} \simeq \Lambda_{\mathscr{D}}/\mathscr{E}_{\mathscr{D}}$.) This relation follows from the congruences for the corresponding Hecke eigenvalues. Also, $R_{\mathscr{D}}$ is generically "more irreducible" than $\rho_{\mathbf{E}_{\mathscr{D}}}$ in the sense that it can be written as the sum of at most two "generically irreducible" pseudo-characters while $\rho_{\mathbf{E}_{\mathscr{D}}}$ has three pieces. (This is proven in [11, 7.3.1] using a result of M.Harris on non-existence of CAP forms of very regular weights.)

The next thing to do is use the "lattice construction" to get the Galois cohomology classes from the congruences between irreducible and reducible Galois representations.

Recall in the last section we have:

$$\Lambda_{\mathscr{D}}/\mathscr{E}_{\mathscr{D}} \xrightarrow{\sim} \mathbf{h}_{\mathscr{D}}/I_{\mathscr{D}}$$

$$\mathrm{trace}\rho_{\mathbf{E}_{\mathscr{D}}}(\mathrm{mod}\mathscr{E}_{\mathscr{D}}) = R_{\mathscr{D}}(\mathrm{mod} I_{\mathscr{D}}).$$

Our goal is to prove:

$$(\mathscr{L}_{1,\mathbb{Q}}^{\Sigma}\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma}) \supset \mathscr{E}_{\mathscr{D}} \supset \mathrm{char}_{\mathbf{f},\mathscr{K}}^{\Sigma}.$$

Now we are going to prove the second inclusion using the lattice construction. The first one will be proved at the end of section 9.

5.2 Galois Argument: Lattice Construction

The lattice construction in [11] involves 3 irreducible pieces and is complicated. Instead we are going to give the lattice construction which involves only 2 pieces (the case in [15]) and briefly mention the difference at the end. We partly follow the treatment of C.Skinner's CMI lecture notes [10]. Let us axiomize the situation: let Λ be the weight algebra and $\mathbb{I}$ a reduced ring which is a finite Λ-algebra. Let ρ be a Galois representation of $G_{\mathbb{Q}}$ on $\mathbb{I}^2$. Let J and I be nonzero ideals of Λ and $\mathbb{I}$ such that the structure map induces $\Lambda/J \simeq \mathbb{I}/I$. Let P be a height one prime of Λ such that $\mathrm{ord}_P(J) = t > 0$. Then there is a unique height one prime P' of $\mathbb{I}$ containing (I,P). Since $\mathbb{I}$ is reduced we can talk about its total fraction ring $K = \prod_i F_{\mathbb{J}_i}$ where the $\mathbb{J}_i$'s are domains finite over $\mathbb{I}$ and the $F_{\mathbb{J}_i}$'s are the fraction fields of the $\mathbb{J}_i$'s.

Suppose:

1 Each representation $\rho_{\mathbb{J}_i}$ on $F_{\mathbb{J}_i}^2$ induced from ρ via projection to $F_{\mathbb{J}_i}$ is irreducible.
2 There are $\Lambda^\times$-valued characters χ_1 and χ_2 of $G_{\mathbb{Q}}$ such that:
$$\mathrm{tr}\rho(\sigma) \equiv \chi_1(\sigma) + \chi_2(\sigma) (\mathrm{mod} I)$$
for each $\sigma \in G_{\mathbb{Q}}$.
3 There are $\mathbb{I}^\times$-valued characters χ_1' and χ_2' of $G_{\mathbb{Q}_p}$ such that
$$\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \chi_1' & * \\ & \chi_2' \end{pmatrix}$$
and there is a $\sigma_0 \in G_{\mathbb{Q}_p}$ such that $\chi_1'(\sigma_0) \not\equiv \chi_2'(\sigma_0) (\mathrm{mod} P')$.
4 $\chi_1(\sigma) \equiv \chi_1'(\sigma) (\mathrm{mod}\ I)$, $\chi_2(\sigma) \equiv \chi_2'(\sigma) (\mathrm{mod}\ I)$ for each $\sigma \in \mathbb{I}[G_{\mathbb{Q}_p}]$.
5 ρ is unramified outisde p.

We define the dual Selmer group $X := H^1_{ur}(\mathbb{Q}, \Lambda^*(\chi_1^{-1}\chi_2))^*$. Here "ur" means extensions unramified everywhere and $*$ means Pontryagin dual.

Definition 10. Let $\mathrm{char}_\Lambda(X)$ be the characteristic ideal of X as a Λ module.

We are going to prove:

Proposition 1. *Under the assumptions above,* $\mathrm{ord}_P(\mathrm{char}_\Lambda(X)) \geq \mathrm{ord}_P(J)$.

Proof. Suppose $t = \mathrm{ord}_P(J) > 0$. We take the σ_0 in assumption (3) and a basis (v_1, v_2) so that $\rho(\sigma_0)$ has the form $\begin{pmatrix} \chi_1(\sigma_0) & \\ & \chi_2(\sigma_0) \end{pmatrix}$. We write $\rho(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \in M_2(K)$ for each $\sigma \in K[G_{\mathbb{Q}}]$ with respect to this basis. Then we claim the following. Let $r := \chi_1(\sigma_0) - \chi_2(\sigma_0)$, (so $r \notin P$)

a $ra_\sigma, rd_\sigma, r^2 b_\sigma c_\tau \in \mathbb{I}$ for all $\sigma, \tau \in \mathbb{I}[G_{\mathbb{Q}}]$ and $ra_\sigma \equiv r\chi_1(\sigma) (\mathrm{mod} I)$, $rd_\sigma \equiv r\chi_2(\sigma) (\mathrm{mod} I)$, $r^2 b_\sigma c_\tau \equiv 0 (\mathrm{mod} I)$.
b $\mathscr{C} := \{c_\sigma : \sigma \in \mathbb{I}[G_{\mathbb{Q}}]\}$ is a finite faithful A-module;
c $c_\sigma = 0$ for $\sigma \in I_p$.

(c) is by (3) and (b) follows easily from assumption (1) and $G_{\mathbb{Q}}$ being compact. (a) is by calculation: e.g. set $\delta_1 := \sigma_0 - \chi_2(\sigma_0)$ then $ra_\sigma = \mathrm{trace}\rho(\delta_1\sigma) \equiv r\chi_1(\sigma) (\mathrm{mod} I)$ by assumption (2).

Now we deduce the proposition using these claims. We write Λ_P, $\mathbb{I}_P$, etc for the localizations at P and define $\mathscr{M} := \mathbb{I}_P[G_{\mathbb{Q}}]v_1 \subset V$. Then it is easy to check that $\mathscr{M} = \mathbb{I}_P v_1 \oplus \mathscr{C}_P v_2$. Define $\mathscr{M}_2 := \mathscr{C}_P v_2$. Then (a) implies $\bar{\mathscr{M}}_2 := \mathscr{M}_2 / I\mathscr{M}_2 \subset \bar{\mathscr{M}} := \mathscr{M}/I\mathscr{M}$ is a direct summand that is $G_{\mathbb{Q}}$ stable. Define $\mathscr{M}_1 = \mathbb{I}_P v_1$ and $\bar{\mathscr{M}}_1 = \mathscr{M}_1 / I\mathscr{M}_1$ then we have
$$0 \to \bar{\mathscr{M}}_2 \to \bar{\mathscr{M}} \to \bar{\mathscr{M}}_1 \to 0.$$

Now we return to the lattice construction without localization at P. We will find a finite Λ-module $\mathfrak{m}_2 \subseteq \bar{\mathscr{M}}_2$ such that

i $\mathfrak{m}_{2,P} = \bar{\mathscr{M}}_2$;

ii there exists a Λ-map $X \to \mathfrak{m}_2$ that is a surjection after localizing at P.

Then $\mathrm{ord}_P(\mathrm{char}_\Lambda(X)) = \mathrm{ord}_P(\mathrm{char}_{\Lambda_P}(X_P)) = \mathrm{ord}_P(\mathrm{Fitt}_{\Lambda_P}(\Lambda_P/\mathrm{char}_{\Lambda_P}(X_P))) \geq \mathrm{ord}_P(\mathrm{Fitt}_{\Lambda_P}\mathfrak{m}_{2,P}) = \mathrm{ord}_P(\mathrm{Fitt}_{\Lambda_P}\bar{\mathscr{M}}_2)$. But $\mathrm{Fitt}_{\Lambda_P}\bar{\mathscr{M}}_2(\mathrm{mod}\ J) = \mathrm{Fitt}_{\Lambda_P/J\Lambda_P}\bar{\mathscr{M}}_2 = \mathrm{Fitt}_{\mathbb{I}_P/I\mathbb{I}_P}\bar{\mathscr{M}}_2 = \mathrm{Fitt}_{\mathbb{I}_P}\bar{\mathscr{M}}_2(\mathrm{mod} I) = \mathrm{Fitt}_{\mathbb{I}_P}\mathscr{M}_2(\mathrm{mod} I) = 0$, the last equality follows from the fact that $\mathscr{M}_2$ is a faithful submodule of $\mathrm{Frac}\mathbb{I}$ in view of the generic irreducibility of the Galois representation ρ. So $\mathrm{Fitt}_{\Lambda_P}\bar{M}_2 \subseteq J$ and $\mathrm{ord}_P(\mathrm{char}_\Lambda(X)) \geq \mathrm{ord}_P J$.

Now let $\mathfrak{m} \subset \mathscr{M}$ be the $\mathbb{I}[G_\mathbb{Q}]$-module generated by $\bar{v}_1$, $\mathfrak{m}_2 := \mathfrak{m} \cap \bar{\mathscr{M}}_2$, $\mathfrak{m}_1 := \mathfrak{m}/\mathfrak{m}_2 \subset \bar{\mathscr{M}}_1$. Note that $\mathfrak{m}_{2,P} = \bar{\mathscr{M}}_2$. Then we have:

$$0 \to \mathfrak{m}_2 \to \mathfrak{m} \to \mathfrak{m}_1 \to 0. \qquad (*)$$

- By assumption (5) and (c) above this extension is everywhere unramified.
- $\bar{\mathscr{M}}_1 \simeq \Lambda/P^t\Lambda$ as Λ-module by definition. So it is easy to see $\mathfrak{m}_1 \simeq \Lambda/P^t\Lambda$ as well.
- By (a) the $G_\mathbb{Q}$-action on $\mathfrak{m}_2$ and $\mathfrak{m}_1$ are given by χ_2 and χ_1 respectively.

We expect the $(*)$ in the matrix to give the desired extension. More precisely let $[\mathfrak{m}] \in H^1(\mathbb{Q}, \mathfrak{m}_2(\chi_1^{-1}\chi_2))$ be the class defined by $(*)$. Then we get a canonical map $\theta : \mathrm{Hom}_\Lambda(\mathfrak{m}_1, \Lambda^*) \to H^1(\mathbb{Q}, \Lambda^*(\chi_1^{-1}\chi_2))$. Taking the Pontryagin dual $\theta^* : H^1(\mathbb{Q}, \Lambda^*(\chi_1^{-1}\chi_2))^* \to \mathfrak{m}_2$. We claim that θ^* becomes surjective after taking localization at P. (As in Section 2 this is basically because $\mathfrak{m}$ is generated by $\bar{v}_1$ over $\mathbb{I}[G_\mathbb{Q}]$.)

Proof of the claim
Let $\mathfrak{R} = \ker(\theta)$ and let $S \subset \mathfrak{R}$ be any finite subset, $\mathfrak{m}_S := \cap_{\phi\in S}\ker\phi$. Then we have:

$$0 \to \mathfrak{m}_2/\mathfrak{m}_S \to \prod_{\phi\in S}\Lambda^* \to \prod_{\phi}\Lambda^*/(\mathfrak{m}_2/\mathfrak{m}_S) \to 0. \qquad (**)$$

Equip each module with the $G_\mathbb{Q}$ action $\chi_1^{-1}\chi_2$ and take the cohomology long exact sequence. By the definition of $\mathfrak{R}$ the image of $[\mathfrak{m}]$ in $H^1(\mathbb{Q}, \mathfrak{m}_2/\mathfrak{m}_S(\chi_1^{-1}\chi_2))$ is in the kernel of the map $H^1(\mathbb{Q}, \mathfrak{m}_2/\mathfrak{m}_S(\chi_1^{-1}\chi_2)) \to H^1(\mathbb{Q}, \prod_{\phi\in S}\Lambda^*(\chi_1^{-1}\chi_2))$ which is a quotient of $\prod_{\phi\in S}\Lambda^*(\chi_1^{-1}\chi_2)^{G_\mathbb{Q}}$ which is killed by $r = \chi_1(\sigma_0) - \chi_2(\sigma_0) \notin P$. Thus the exact sequence

$$0 \to (\mathfrak{m}_2/\mathfrak{m}_S)_P \to (\mathfrak{m}/\mathfrak{m}_S)_P \to \mathfrak{m}_{1,P} \to 0$$

is split. If $(\mathfrak{m}_2/\mathfrak{m}_S)_P \neq 0$ then this contradicts the fact that $\mathfrak{m}$ is generated by $\bar{v}_1$ over $\mathbb{I}[G_\mathbb{Q}]$. Thus $\mathfrak{m}_{2,P} = \mathfrak{m}_{S,P}$. By the arbitrariness of S we get $\mathfrak{R}_P = 0$. This proves the claim.

Now we compare with the [11] case. There we have 3 irreducible pieces and the matrix is like $\begin{pmatrix}\chi_1 & & \\ * & \rho_f & \\ * & & \chi_2\end{pmatrix}$. We expect the upper $*$ in the matrix to give the required extension. However we are not able to distinguish the contribution of $(*)$ to $H^1_f(\mathscr{K}, \chi_1^{-1}\rho_f)$ and $H^1_f(\mathscr{K}, \chi_1^{-1}\chi_2)^{c=1} = H^1_f(\mathbb{Q}, \tau)$ where τ is the composition of the transfer map $V : G_\mathbb{Q}^{ab} \to G_\mathscr{K}^{ab}$ and $\chi_1^{-1}\chi_2$. But by the Iwasawa main conjecture for Hecke characters proved in [15], this $H^1_f(\mathscr{K}, \chi^{-1}\chi_2)^{c=1}$ is controlled by the p-adic L-function for the trivial character, which is a unit.

6 Doubling Methods

6.1 Siegel Eisenstein Series on $GU(n,n)$

Let Q_n be the Siegel parabolic consists of block matrices $\begin{pmatrix}\times & \times \\ & \times\end{pmatrix}$. Let v be a finite prime of $\mathbb{Q}$, write $K_{n,v}$ for $GU(n,n)(\mathbb{Z}_v)$. Fix χ a character of $\mathscr{K}_v^\times$. Let $I_n(\chi)$ be the space of smooth and K_n-finite functions $f : K_{n,v} \to \mathbb{C}$ such that $f(qk) = \chi(\det D_q)f(k)$ for $q = \begin{pmatrix}A_q & B_q \\ & D_q\end{pmatrix} \in Q_n$ from such f we define

$$f(z,-):G_n(\mathbb{Q}_v)\to\mathbb{C}$$

by

$$f(z,qk):=\chi(\det D_q)|\det A_qD_q^{-1}|_v^{z+\frac{n}{2}}f(k).$$

Suppose $\mathscr{K}_v$ is unramified over $\mathbb{Q}_v$ and χ is unramified, then there is a unique vector $f^0\in I(\chi)$ which is invariant under $K_{n,v}$ and $f^0(1)=1$. There is an Archimedean picture as well (See [11, 11.4.1]).

Now let $\chi=\otimes_v\chi_v$ be a Hecke character of $\mathbb{A}_{\mathscr{K}}^\times/\mathscr{K}^\times$. Then we define $I(\chi)$ as a restricted product of local $I(\chi_v)$'s as above with respect to the above unramified vectors. For any $f\in I(\chi)$ we define the Siegel Eisenstein series

$$E(f;z,g):=\sum_{\gamma\in Q_n(\mathbb{Q})\backslash G_n(\mathbb{Q})}f(z,\gamma g).$$

This is absolutely convergent if $\mathrm{Re}z>>0$ and has a meromorphic continuation to all $z\in\mathbb{C}$.

6.2 Some Embedding

The Klingen Eisenstein series are difficult to compute, while Siegel Eisenstein series are much easier. The point of doubling method is to reduce the computation of the former to the latter. We are going to introduce some important embeddings that are used in the Pullback formulas. Let (V_1,ω_1) be the Hermitian space for $U(1,1)$ and (V_1^-) another Hermitian space whose metric is $(-\omega_1)$. Elements of V_1 and V_1^- are denoted (v_1,v_2) and (u_1,u_2) for $v_i,u_i\in\mathscr{K}$. Let $V_2=V_1\oplus X\oplus Y$ be the Hermitian space for $U(2,2)$ where $X\oplus Y$ is a 2-dimensional Hermitian space for the metric $\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$ and elements are written as (x,y) for $x,y\in\mathscr{K}$ with respect to this basis. Let $W=V_2\oplus V_1^-$ be the Hermitian space for $U(3,3)$. Let $R=\begin{pmatrix}1&&&&&\\&1&&&&\\&&&1&&\\&&&&1&\\&&&&&1\\&&1&&&\end{pmatrix}$ and $S=\begin{pmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&-1&1&&\\&&&&1&\\-1&&&&&1\end{pmatrix}$ These give maps:

$$(v_1,x,v_2,y,u_1,u_2)\mapsto(v_1,x,u_2,v_2,y,u_1)$$

$$(v_1,x,u_2,v_2,y,u_1)\mapsto(v_1-u_1,x,u_2-v_2,v_2,y,u_1).$$

Now we define the embedding:

$$\gamma_3:G_{2,1}:=\{(g,g')\in GU(2,2,)\times GU(1,1),\mu(g)=\mu(g')\}\hookrightarrow GU(3,3)$$

by

$$\begin{pmatrix}g&\\&g'\end{pmatrix}\mapsto S^{-1}R^{-1}\begin{pmatrix}g&\\&g'\end{pmatrix}RS.$$

Let V^d be the image of V_1 in $V_1\oplus V_1^-$ by the diagonal embedding. Let τ_1 be any element of $U(3,3)(\mathbb{Q})$ which maps the maximal isotrophic subspace $V^d\oplus X$ to $\mathscr{K}v_1\oplus\mathscr{K}u_1\oplus X$, then one can check that: $\tau_1^{-1}Q_3\tau_1\cap\gamma_3(U(2,2)\times U(1,1))=\gamma_3(Q_2\times B_1)$. An important property of such embedding is:

$$\{(m(g,x)n,g):g\in GU(1,1),x\in\mathrm{Res}_{\mathscr{K}/\mathbb{Q}}\mathbb{G}_m,n\in N_P\}\subset Q_3.$$

6.3 Pullback Formula

Let χ be a unitary Hecke character as before and $f \in I(\chi)$. Given a cusp form φ on G_1, define the pullback section by:

$$F_\varphi(f,z,g) := \int_{U(1,1)(\mathbb{A})} f(z,\gamma_3(g,g_1h))\bar{\chi}(\det g_1 h)\varphi(g_1h)dg_1$$

where $h \in GU(1,1)(\mathbb{A})$ is any element such that $\mu(h) = \mu(g)$. This is absolutely convergent if Re$z >> 0$. It is easy to see that F_ϕ does not depend on the choice of h. Note that this is a Klingen Eisenstein section. Then

Proposition 2. *For z in the region of absolute convergence and h as above, we have:*

$$\int_{U(1,1)(\mathbb{Q})\backslash U(1,1)(\mathbb{A})} E(f;z,\gamma_3(g,g_1h))\bar{\chi}(\det g_1 h)\varphi(g_1,h)dg_1 = \sum_{P(\mathbb{Q})\backslash GU(2,2)(\mathbb{Q})} F_\varphi(f;z,\gamma g).$$

Remark 5. The right hand side is nothing but the expression of the Klingen Eisenstein series.

Proof. This is proven by Shimura [9]. There Shimura proved the following double coset decomposision in (2.4) and (2.7) in *loc.cit*:

$$U(3,3) = Q_3\gamma_3(U(2,2)\times U(1,1)) \cup Q_3\tau_1\gamma_3(U(2,2)\times U(1,1))$$

and

$$Q_3\gamma_3(U(2,2)\times U(1,1)) = \cup_{\beta\in U(2,2),\xi\in U(1,1)} Q_3\gamma_3((\beta,\xi)),$$
$$Q_3\tau_1\gamma_3(U(2,2)\times U(1,1)) = \cup_{\beta\in Q_2\backslash U(2,2),\gamma\in B_1\backslash U(1,1)} Q_3\tau_1\gamma_3((\beta,\gamma)).$$

Thus by unfolding the Siegel Eisenstein series we write the integration into two parts. We claim that the integration for the part involving terms with τ_1 is 0. We first fix β and sum over the γ's, this equals

$$\int_{B_1(\mathbb{Q})\backslash U(1,1)(\mathbb{A})} f(z;\tau_1\gamma_3((\beta g,g_1h)))\varphi(g_1h)dg_1.$$

Recall we have noted that $\tau_1\gamma_3(1,B_1)\tau_1^{-1} \subseteq Q_3$. Since φ is cuspidal, $\int_{B_1(\mathbb{Q})\backslash B_1(\mathbb{A}_\mathbb{Q})} \phi(bg_1h)db = 0$ for all g_1. Thus the integration is 0. This proves the claim. The proposition then follows from our description for $Q_3\gamma_3(U(2,2)\times U(1,1))$.

7 Constant Terms

Suppose ϕ is of weight κ and let $z_\kappa = \frac{\kappa-3}{2}$. Let P be the Klingen parabolic and R any standard $\mathbb{Q}$ parabolic of $GU(2,2)$. We are going to compute the constant terms $E(f,z,g)_R$ of the Klingen Eisenstein series $E(f,z,g)$ along R. We write N_R for the unipotent radical of P. The constant term along R is given by:

$$E(f,z,g)_R = \int_{N_R(\mathbb{Q})\backslash N_R(\mathbb{A})} E(f,z,ng)dn.$$

A famous computation of Langlands tells us that: if $R \neq P$ then $E(f,z,g)_R = 0$. For $R = P$ we first define the intertwining operator:

$$A(\rho,z,f)(g) := \int_{N_P(\mathbb{A})} f_z(wng)dn.$$

This is absolutely convergent for Re$z >> 0$ and is defined for all $z \in \mathbb{C}$ by meromorphic continuation. It is a product of local integrals. This intertwins the representations $I(\rho)$ and some $I(\rho_1)$ where ρ_1 is defined similar to ρ but replacing (π,ψ,τ) by $(\pi\times(\tau\circ\det),\psi\tau\tau^c,\bar{\tau}^c)$. Then $E(f,z,g)_P = f_z(g) + A(\rho,z,f)(g)$. It turns out that under our choices $z = z_\kappa$ and $\kappa > 6$, $A(\rho,z,f)$ is absolutely convergent and the Archimedean component is 0. Thus $A(\rho,z_\kappa,f)$ equals 0. Thus

$$E(f,z_\kappa,g)_P = f_{z_\kappa}(g).$$

Let us explain how the special L-values that we are interested in show up in the constant term of the Klingen Eisenstein series. The Klingen section f is realized as the pullback section of some Siegel Eisenstein series on $GU(3,3)$. At the unramified places a computation of Lapid and Rallis [5] tells us that if the Siegel section is f_v^0 then the pullback section is $f_{\phi_v}^0 L(\tilde{\pi},\psi/\tau,z+1)L(\bar{\chi}_\pi(\psi/\tau)',2z+1)$ Here the first L-factor is the local Euler factor for the base change of the dual $\tilde{\pi}$ twisted by ψ/τ and the second is a Dirichlet L-factor. Taking the product over all good primes, the special L-values we are interested in show up as the constant term of the Klingen Eisenstein series obtained by pullback.

8 p-adic Interpolation

Definition 11. An Eisenstein datum is a sextuple $\mathscr{D} := (A, \mathbb{I}, \mathbf{f}, \psi, \xi, \Sigma)$ where

- A is a finite $\mathbb{Z}_p$-algebra and $\mathbb{I}$ is a normal domain finite over $A[[W]]$.
- $\mathbf{f}$ is a Hida family of cuspidal newforms with coefficient ring $\mathbb{I}$.
- ψ is an A-valued finite order character which restricts to the tame part of the central character of $\mathbf{f}$ on $\mathbb{A}_{\mathbb{Q}}^{\times}$.
- ξ is another A-valued finite order Hecke character of $\mathbb{A}_{\mathscr{K}}^{\times}$.
- Σ is a finite set of primes containing all the bad primes.

Recall that we have defined a ring $\Lambda_{\mathscr{D}} = \mathbb{I}[[\Gamma_{\mathscr{K}}]][[\Gamma_{\mathscr{K}}^{-}]]$. We use $\Lambda_{\mathscr{D}}$ to interpolate triples (f, ψ, τ) that are used to construct Klingen Eisenstein series. Recall that we have defined a weight ring $\Lambda_2 \simeq \mathbb{Z}_p[[\Gamma_2]]$ for $\Gamma_2 \simeq (1+p\mathbb{Z}_p)^4$. We first give $\Lambda_{\mathscr{D}}$ a Λ_2-algebra structure. We define homomorphisms $\alpha : A[[\Gamma_{\mathscr{K}}]] \to \mathbb{I}[[\Gamma_{\mathscr{K}}^{-}]]$ and $\beta : A[[\Gamma_{\mathscr{K}}]] \to \mathbb{I}[[\Gamma_{\mathscr{K}}]]$ (we omit the formulas). Then the Λ_2-algebra structure map is given by composing $\alpha \otimes \beta : A[[\Gamma_{\mathscr{K}} \times \Gamma_{\mathscr{K}}]] \to \Lambda_{\mathscr{D}}$ with the map $\Gamma_2 \to \Gamma_{\mathscr{K}} \times \Gamma_{\mathscr{K}}$ given by:

$$(t_1, t_2, t_3, t_4) \mapsto \mathrm{rec}_{\mathscr{K}}(t_3 t_4, t_1^{-1} t_2^{-1}) \times \mathrm{rec}_{\mathscr{K}}(t_4, t_2^{-1}),$$

where $\mathrm{rec}_{\mathscr{K}}$ is the reciprocity map in class field theory normalized by the geometric Frobenius. Let $\boldsymbol{\psi} := \alpha \circ \omega^{-1} \psi \Psi_{\mathscr{K}}^{-1}$ and $\boldsymbol{\xi} := \beta \circ \chi_{\mathbf{f}} \xi \Psi_{\mathscr{K}}$.

Definition 12. A point $\phi \in \mathrm{Spec}\Lambda_{\mathscr{D}}$ is called arithmetic if $\phi|_{\mathbb{I}}$ is arithmetic with some weight $\kappa_\phi \geq 2$ and there are $\zeta_{\pm}, \zeta'_{-} \in \mu_{p^\infty}$ such that $\phi(\gamma^{+}) = \zeta_{+}(1+p)^{\kappa_\phi - 2}, \phi(\gamma^{-}) = \zeta_{-}$ for $\gamma^{\pm} \in \Gamma_{\mathscr{K}}^{\pm}$ and $\phi(\gamma^{-'}) = \zeta'_{-}$ for $\gamma^{-'}$ the topological generator of $\Gamma_{\mathscr{K}}^{-}$.

For every such ϕ we define Hecke characters. Let $p = v_0 \bar{v}_0$ be the decomposition in $\mathscr{K}$ and let

$$\psi_\phi(x) := x_\infty^{-\kappa_\phi} x_{v_0}^{\kappa_\phi} (\phi \circ \boldsymbol{\psi})(x),\ \xi_\phi = \phi \circ \boldsymbol{\xi}.$$

Then we can construct a $\Lambda_{\mathscr{D}}$-coefficient formal q-expansion E_{sieg} that, when specialize to a Zariski dense set of arithmetic points ϕ, is the nearly ordinary Klingen Eisenstein series $E_{Kling,\phi}$ we constructed using the triple: ($\mathbf{f}_\phi$, $\psi_\phi |.|^{\frac{\kappa_\phi}{2}}, \tau_\phi = \psi_\phi \xi_\phi^{-1} |.|^{\frac{\kappa_\phi}{2}}$). This is achieved by first constructing a $\Lambda_{\mathscr{D}}$-adic Siegel Eisenstein series on $GU(3,3)$ and using the pullback formula to construct the Klingen Eisenstein family on $GU(2,2)$. To do this we need to choose a Siegel section f_ϕ at each arithmetic point ϕ so that

1 f_ϕ depends p-adic analytically on ϕ;
2 the pulls back of f_ϕ to (a multiple of) the nearly ordinary Klingen Eisenstein section.

The hardest part is the computations at the primes dividing p ([11, 11.4.14,15,19]). It turns out that certain Siegel-Weil Eisenstein sections work well. In fact in [11], the section is not given in terms of the Siegel-Weil section. However it indeed provided the idea of how the section given in *loc.cit* is figured out. Let us briefly explain the idea.

Let Φ be the Schwartz function on $M_{(3,6)}(Q_p)$ defined by:

$$\Phi(X,Y) := \Phi_1(X)\hat{\Phi}_2(Y),$$

where X and Y are 3×3 matrices and define a Siegel-Weil section by:

$$f^{\Phi}(g) = \chi_2^{-1}(\det g)|\det g|_p^{-s+\frac{3}{2}} \times \int_{GL_3(Q_p)} \Phi((0,X)g)\chi_1^{-1}\chi_2^{-1}(\det X)|\det X|_p^{-2s+3} d^{\times}X$$

for $\chi_p = (\chi_1, \chi_2)$. The $\hat{\Phi}_2$ means the Fourier transform of Φ_2. We let Φ_1 be a Schwartz function supported on the set of matrices X such that the X_{13} and X_{31} are in $\mathbb{Z}_p^{\times}$ and the values on it is given by the product of two characters of X_{13} and X_{31}. Choosing Φ_2 properly and unfolding the formula for the β-th local Fourier coefficients, we can make sure that it is essentially given by $\Phi_1({}^t\beta)$ (up to some easier constant depending on β). Thus the first requirement is ensured. This Siegel Weil section is explicitly given by

$$f_p(g) = \sum_{a\in(\mathscr{O}_p/x)^\times} f_z^{0,(2)}(g\begin{pmatrix} a^{-1} & \\ & \bar{a}\end{pmatrix})$$

where $(x) = \mathrm{cond}(\xi^c)$, τ is our χ defining the Siegel Eisenstein section, $\xi = \psi/\tau$ and $f_z^{0,(2)}$ in *loc.cit* lemma 11.4.20.
How to interpolate the Klingen Eisenstein series?

Hida proved the existence of a Hecke operator $1_f \in \mathbb{T}_\kappa^{ord}(N,\chi_f,A)\otimes_A F_A$ on the space $S_\kappa^{ord}(N,\chi_f)$ of ordinary cusp forms with weight κ level N and character χ_f. such that

$$1_f.g = \frac{< g, f^c|_\kappa \begin{pmatrix} & -1\\ N & \end{pmatrix} >}{< f, f^c|_\kappa \begin{pmatrix} & -1\\ N & \end{pmatrix} >} f.$$

This 1_f is not necessarily p-adically integral ([3]). The congruence number η_f is defined (up to a p-adic unit) to be the minimally divisible by p number such that $\ell_f := \eta_f 1_f$ is in $\mathbb{T}_\kappa^{ord}(N,\chi_f,A)$. The candidate that we choose for the Klingen Eisenstein series $E_{Kling,\phi}$ at the arithmetic point ϕ is the one such that:

$$\ell_f^{U(1,1)} e^{U(1,1)} E_{sieg,\phi}|_{U(2,2)\times U(1,1)} = E_{Kling,\phi} \boxtimes f.$$

Here the superscript means the Hecke operators are applied to the forms considered as a form on $U(1,1)$. If we replace f by a Hida family $\mathbf{f}$ and suppose the local Hecke algebra $\mathbb{T}_{\mathfrak{m}_f}$ (the localization of the Hecke algebra at the maximal ideal $\mathfrak{m}_f$ corresponding to f) is Gorenstein, then we can similarly define $1_\mathbf{f}$ and $\eta_\mathbf{f}$, $\ell_\mathbf{f}$, thus interpolating everything in p-adic families.

In particular, we get the Klingen-Eisenstein series interpolating $E_{Kling,\phi}$ whose constant terms are divisible by $\mathscr{L}^\Sigma_{\mathbf{f},\mathscr{K}}.\mathscr{L}^\Sigma_1$ in light of the discussion at the end of the last section.

9 Fourier-Jacobi Coefficients

Recall that we have seen that the constant terms of the Klingen Eisenstein family are divisible by the p-adic L-function. In order to show that the Eisenstein ideal is contained in the principal ideal $(\mathscr{L}^\Sigma_{\mathbf{f},\mathscr{K}})$, we still need to show that some Fourier coefficient is co-prime to the p-adic L-function.

9.1 Generalities

We are going to compute the Fourier-Jacobi coefficient of the Siegel Eisenstein serie E_{sieg} as a function on $U(1,1)(\mathbb{A})$ via the embedding $\gamma_3 : U(2,2)\times U(1,1) \hookrightarrow U(3,3)$. The purpose is, by the pullback formula introduced in the previous section, to express the Fourier coefficients of the Klingen Eisenstein series in terms of the Petersson inner product with the cusp form we start with. For $\mathscr{Z} \in \mathbf{H}_3$

$$E_{sieg}(\mathscr{Z}) = \sum_{T\geq 0} a_T e(\mathrm{Tr} T\mathscr{Z}).$$

Write $S_2(\mathbb{Q})$ or $S_2(\mathbb{Q}_v)$ for the set of 2×2 Hermitian matrices over $\mathbb{Q}$ or $\mathbb{Q}_v$. For β a 2×2 Hermitian matrix the βth Fourier-Jacobi coefficient is

$$\sum_{T=\begin{pmatrix}\beta & *\\ * & *\end{pmatrix}} a_T e(\mathrm{Tr} T\mathscr{Z}).$$

We have an integral representation for the Fourier-Jacobi coefficients:

$$E_{sieg,\beta}(g) = \int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} E^{sieg}(\begin{pmatrix} 1_3 & \begin{matrix} S & 0 \\ 0 & 0 \end{matrix} \\ & 1_3 \end{pmatrix} g) e_{\mathbb{A}}(-\mathrm{Tr}\beta S) dS.$$

Here $e_{\mathbb{A}} = \otimes e_v$ and $e_v(x_v) = e^{-2\pi i\{x_v\}}$ for v a finite primes and $e_v(x_v) = e^{2\pi x_v}$ for $x \in \infty$. The following lemma gives a way to compute the Fourier-Jacobi coefficients of E_{sieg}.

Lemma 1. *Let $f \in I_3(\chi)$, $\beta \in S_2(\mathbb{Q})$. Suppose $\beta > 0$. Let V be the 2-dimensional $\mathscr{K}$-space of column vectors. If* $\mathrm{Re}(z) > \frac{3}{2}$*. Then:*

$$E_{sieg,\beta}(f; z, g) = \sum_{\gamma \in Q_1(\mathbb{Q})\backslash G_1(\mathbb{Q})} \sum_{x \in V} \int_{S_2(\mathbb{A})} f(w_3 \begin{pmatrix} 1_3 & \begin{matrix} S & x \\ {}^t\bar{x} & 0 \end{matrix} \\ & 1_3 \end{pmatrix} \alpha_1(1,\gamma) g) e_{\mathbb{A}}(-\mathrm{Tr}\beta S) dS.$$

Proof. We omit it here. See [11, 11.3]

The integrand in the lemma is a product of local integrals. We are mainly interested in evaluating the Fourier Jacobi coefficients at $\alpha_1(\mathrm{diag}(y, {}^t\bar{y}^{-1}), g)$ for $y \in \mathrm{GL}_2(\mathbb{A}_{\mathscr{K}})$ and $g \in U_1(\mathbb{A}_{\mathbb{Q}})$.

Definition 13. For each prime v of $\mathbb{Q}$ and $f \in I_3(\chi_v)$, set

$$FJ_{\beta}(f; z, x, g, y) = \int_{S_2(\mathbb{Q}_v)} f(z, w_3 \begin{pmatrix} 1_3 & \begin{matrix} S & x \\ {}^t\bar{x} & 0 \end{matrix} \\ & 1_3 \end{pmatrix} \alpha_1(\mathrm{diag}(y, {}^t\bar{y}^{-1}), g)) e_v(-\mathrm{Tr}\beta S) dS.$$

We are going to identify the Fourier Jacobi coefficients with some forms that we are more familiar with.

9.2 Backgrounds for Theta Functions

Local Picture
Let v be a prime of $\mathbb{Q}$ and $h \in S_2(\mathbb{Q}_v)$, $\det h \neq 0$. Then h defines a two-dimensional Hermitian space V_v. Let U_h be the corresponding unitary group. Let λ_v be a character of $\mathscr{K}_v^\times$ whose restriction to $\mathbb{Q}_v^\times$ is trivial. One can define the Weil representation $\omega_{h,\lambda}$ of $U_h(\mathbb{Q}_v) \times U(1,1)(\mathbb{Q}_v)$ on the space $S(V_v)$ of Schwartz functions on V_v (we omit the formulas).

Global Picture
Now let $h \in S_2(\mathbb{Q}), h > 0$ and a Hecke character $\lambda = \otimes \lambda_v$ of $\mathbb{A}_{\mathscr{K}}^\times / \mathscr{K}^\times$ such that $\lambda|_{\mathbb{A}_{\mathbb{Q}}^\times} = 1$. Then we define a Weil representation $\omega_{h,\lambda}$ of $U_h(\mathbb{A}_{\mathbb{Q}}) \times U(1,1)(\mathbb{A}_{\mathbb{Q}})$ on $S(V \otimes \mathbb{A})$ by tensoring the local representations.

Theta Functions
Given $\Phi \in S(V \otimes \mathbb{A}_{\mathbb{Q}})$ we define

$$\Theta_h(u, g; \Phi) := \sum_{x \in V} \omega_{h,\lambda}(u, g) \Phi(x)$$

which is an automorphic form on $U_h(\mathbb{A}_{\mathbb{Q}}) \times U_1(\mathbb{A}_{\mathbb{Q}})$ and gives the theta correspondence between U_h and $U(1,1)$.

9.3 Coprime to the p-adic L-function

Now let us return to the Fourier Jacobi coefficients. It turns out that by some local computations, for each v, $FJ_{\beta}(f; z, x, g, y)$ has the form $f_1(g)(\omega_{\beta,\lambda_v}(y, g)\Phi_v)(0)$ where we have chosen a Hecke character λ as above, $f_1 \in I_1(\chi_v/\lambda_v)$ and Φ_v is a Schwartz function on $\mathscr{K}_v^2$, ω_{β,λ_v} is defined using the character λ_v. Thus from lemma 1 the Fourier Jacobi coefficient is the product of an Eisenstein series $E_1(g)$ and a theta series $\Theta_{\beta}(y, g)$.

Now we prove that the Klingen Eisenstein series is coprime to the p-adic L-function. Let us take an auxiliary Hida family $\mathbf{g}$ of cuspidal eigenforms. Using the functorial property of the theta correspondence we can find some linear combinations of $E_\beta(f;z_\kappa,\alpha_1(\mathrm{diag}(y,{}^t\bar{y}^{-1}),g))$'s which "picks up" the $\mathbf{g}$-eigencomponent of $\Theta_\beta(y,g)$ (as a function of g). By pairing this with the original $\phi\in\pi$ we started with, we find certain linear combinations of the Fourier coefficients of the Klingen Eisenstein family which can be expressed in the form $\mathscr{A}_{\mathbf{g}}\mathscr{B}_{\mathbf{g}}$ where $\mathscr{B}_{\mathbf{g}}$ is the "multiple" of $\mathbf{g}$ showing up in $\Theta_\beta(y,g)$. By choosing $\mathbf{g}$ properly $\mathscr{B}_{\mathbf{g}}$ can be made a unit in $\Lambda_{\mathscr{D}}$ ($\mathbf{g}$ is chosen to be a Hida family of theta series from the quadratic imaginary field $\mathscr{K}$. $\mathscr{B}_{\mathbf{g}}$ interpolates a square of central critical values of Hecke L-functions of CM characters. One needs to use a result of Finis [2] on the non-vanishing modulo p of anticyclotomic Hecke L-values to conclude $\mathscr{B}_{\mathbf{g}}$ can be chosen to be a unit). The factor $\mathscr{A}_{\mathbf{g}}$ is interpolating $<E_1(g).\mathbf{g},\mathbf{f}_\phi>$, essentially the Rankin Selberg L-values of $\mathbf{g}$ with $\mathbf{f}$. By checking the nebentypus we find that $\mathscr{A}_{\mathbf{g}}$ only involves $\mathbb{I}[[\Gamma_{\mathscr{K}}^+]]$ and is non-zero by the temperedness of $\mathbf{f}$ and $\mathbf{g}$.

Now we make the following assumption: $N=N^+N^-$ where N^+ is a product of primes split in $\mathscr{K}$ and N^- is a square-free product of an odd number of primes inert in $\mathscr{K}$. Furthermore we assume that for each $\ell|N^-$, $\bar{\rho}_{\mathbf{f}}$ is ramified at ℓ. Under this assumption Vatsal [12] proved if we expand the p-adic L-function as:

$$\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma}=a_0+a_1(\gamma^- -1)+a_2(\gamma^- -1)^2+\dots\dots$$

for $a_i\in\mathbb{I}[[\Gamma_{\mathscr{K}}^+]]$, then some a_i must be in $(\mathbb{I}[[\Gamma_{\mathscr{K}}^+]])^\times$. This implies (easy exercise) that $\mathscr{A}_{\mathbf{h}}$ is outside any height one prime P of $\mathbb{I}[[\Gamma_{\mathscr{K}}]]$ containing $\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma}$ (since $\mathscr{A}_{\mathbf{g}}$ belongs to $\mathbb{I}[[\Gamma_{\mathscr{K}}^+]]$, one may assume $P=P_+\mathbb{I}[[\Gamma_{\mathscr{K}}^+]]$ for some height one prime P_+ of $\mathbb{I}[[\Gamma_{\mathscr{K}}^+]]$. By Vatsal's result, $\mathrm{ord}_{P_+}(\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma})=0$.)

We are ready to prove the result promised in the previous section: $\mathrm{ord}_P(\mathscr{E}_{\mathscr{D}})\geq\mathrm{ord}_P(\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma}.\mathscr{L}_1^{\Sigma})$ for any height one prime P (Here $\mathscr{L}_1^{\Sigma}$ is the p-adic L-function for the trivial character, which is co-prime to $(\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma})$ by the work of Vatsal. In [11] Skinner-Urban actually worked in a more general setting by allowing non-trivial characters). First recall that all constant terms of the Klingen Eisenstein family are divisible by $\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma}\mathscr{L}_1^{\Sigma}$. By the fundamental exact sequence one can find some family $\mathbf{F}$ of forms on $GU(2,2)$ such that $\mathbf{E}_{\mathscr{D}}-(\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma})(\mathscr{L}_1^{\Sigma})\mathbf{F}:=\mathbf{H}$ is a cuspidal family. Now we prove the desired inequality. Suppose $r=\mathrm{ord}_P(\mathscr{L}_{\mathbf{f},\mathscr{K}}^{\Sigma})\geq 1$. By construction there is a Fourier coefficient of the above constructed cuspidal family $\mathbf{H}$ outside P. Denote it as $c(\beta,x;\mathbf{H})$ where $\beta\in S_2(\mathbb{Q})$ and $x\in GU(2,2)(\mathbb{A}_{\mathbb{Q}})$. We define a map:

$$\mu:=h_{\mathscr{D}}\to\Lambda_P/P^r\Lambda_P$$

by: $\mu(h)=c(\beta,x;h\mathbf{H})/c(\beta,x;\mathbf{H})$. This is $\Lambda_{\mathscr{D}}$-linear and surjective. Moreover,

$$c(\beta,x;h\mathbf{H})\equiv c(\beta,x;h\mathbf{E}_{\mathscr{D}})\equiv\lambda_{\mathscr{D}}(h)c(\beta,x;\mathbf{E}_{\mathscr{D}})\equiv\lambda_{\mathscr{D}}(h)c(\beta,x,\mathbf{H})(\mathrm{mod}\ P^r).$$

Thus $I_{\mathscr{D}}\subseteq\ker\mu$. So we have a surjection $\mu: h_{\mathscr{D}}/I_{\mathscr{D}}\twoheadrightarrow\Lambda_P/P^r\Lambda_P$. But the right hand side is $\Lambda_{\mathscr{D}}/\mathscr{E}_{\mathscr{D}}$. This gives the inequality.

10 Generalizations of the Skinner-Urban Work

We have seen that the key ingredient of this work is a study of the p-adic properties of the Fourier coefficients of the Klingen Eisenstein series. To generalize this argument to more general unitary groups we need some non-vanishing modulo p results for special values of L-functions, which so far is only available for forms on unitary groups of rank at most 2. We are able to study the Klingen Eisenstein series for $U(1,1)\hookrightarrow U(2,2)$ and $U(2,0)\hookrightarrow U(3,1)$, proving the corresponding main conjectures for two different Rankin Selberg p-adic L-functions. Here we only mention the following by product ([13]):

Theorem 5. *Let F be a totally real field in which p splits completely. Let f be a Hilbert modular form over F with trivial character and parallel weight 2. Let ρ_f be the p-adic representation of G_F associated to f. Suppose:*

1 f has good ordinary reduction at all primes dividing p;

2 $\bar{\rho}_f$ *is absolutely irreducible.*
3 *If* $[F:\mathbb{Q}]$ *is even and the global sign of* f *if* -1*, then the automorphic representation of* f *is not principal series in at least one finite place.*

If the central value $L(f,1)=0$*, then* $H^1_f(F,\rho_f)$*is infinite.*

In the case when the sign of $L(f,s)$ is -1 this is an early result of Nekovar [7] and Zhang [16]. The cases when this sign is $+1$ is new. Note that even in the case when $F=\mathbb{Q}$ our result is slightly stronger than the one in [11]. The reason is that by working with general totally real fields we can use a base change trick to remove some of the technical local conditions [11].

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