p-Adic Monodromy of the Universal Deformation of a HW-Cyclic Barsotti-Tate Group

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1. Introduction

1.1. A classical theorem of Igusa says that the monodromy representation associated with a versal family of ordinary elliptic curves in characteristic $p > 0$ is surjective [Igu, Ka2]. This important result has deep consequences in the theory of $p$-adic modular forms, and inspired various generalizations. Faltings and Chai [Ch2, FC] extended it to the universal family over the moduli space of higher dimensional principally polarized ordinary abelian varieties in characteristic $p$, and Ekedahl [Eke] generalized it to the jacobian of the universal $n$-pointed curve in characteristic $p$, equipped with a symplectic level structure. Recently, Chai and Oort [CO] proved the maximality of the $p$-adic monodromy over each “central leaf” in the moduli space of abelian varieties which is not contained in the supersingular locus. We refer to Deligne-Ribet [DR] and Hida [Hid] for other generalizations to some moduli spaces of PEL-type and their...
arithmetic applications. Though it has been formulated in a global setting, the proof of Igusa’s theorem is purely local, and it has got also local generalizations. Gross [Grol] generalized it to one-dimensional formal \( \mathcal{O} \)-modules over a complete discrete valuation ring of characteristic \( p \), where \( \mathcal{O} \) is the integral closure of \( \mathbb{Z}_p \) in a finite extension of \( \mathbb{Q}_p \). We refer to Chai [Ch2] and Achter-Norman [AN] for more results on local monodromy of Barsotti-Tate groups. Motivated by these results, it has been longly expected/conjectured that the monodromy of a \emph{versal} family of ordinary Barsotti-Tate groups in characteristic \( p > 0 \) is maximal. The aim of this paper is to prove the surjectivity of the monodromy representation associated with the universal deformation in characteristic \( p \) of a certain class of Barsotti-Tate groups.

1.2. To describe our main result, we introduce first the notion of HW-cyclic Barsotti-Tate groups. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \), and \( G \) be a Barsotti-Tate group over \( k \). We denote by \( G^\vee \) the Serre dual of \( G \), and by \( \text{Lie}(G^\vee) \) its Lie algebra. The Frobenius homomorphism of \( G \) (or dually the Verschiebung of \( G^\vee \)) induces a semi-linear endomorphism \( \varphi_G \) on \( \text{Lie}(G^\vee) \), called the Hasse-Witt map of \( G \) (2.6.1). We say that \( G \) is HW-cyclic, if \( c = \dim(G^\vee) \geq 1 \) and there is a \( v \in \text{Lie}(G^\vee) \) such that \( v, \varphi_G(v), \ldots, \varphi_G^{s/r}(v) \) form a basis of \( \text{Lie}(G^\vee) \) over \( k \) (4.1). We prove in 4.7 that \( G \) is HW-cyclic and non-ordinary if and only if the \( a \)-number of \( G \), defined previously by Oort, equals 1. Basic examples of HW-cyclic Barsotti-Tate groups are given as follows. Let \( r, s \) be relatively prime integers such that \( 0 \leq s \leq r \) and \( r \neq 0 \), \( \lambda = s/r \), \( G^\lambda \) be the Barsotti-Tate group over \( k \) whose (contravariant) Dieudonné module is generated by an element \( e \) over the non-commutative Dieudonné ring with the relation \( (F^{r-s} - V^s) \cdot e = 0 \) (4.10). It is easy to see that \( G^\lambda \) is HW-cyclic for any \( 0 < \lambda < 1 \). Any connected Barsotti-Tate group over \( k \) of dimension 1 and height \( h \) is isomorphic to \( G^{1/h} \) [Dem, Chap.IV §8].

Let \( G \) be a Barsotti-Tate group of dimension \( d \) and height \( c + d \) over \( k \); assume \( c \geq 1 \). We denote by \( S \) the “algebraic” local moduli of \( G \) in characteristic \( p \), and by \( G \) be the universal deformation of \( G \) over \( S \) (cf. 3.8). The scheme \( S \) is affine of ring \( R = k[[t_{ij} | 1 \leq i \leq c, 1 \leq j \leq d]] \), and the Barsotti-Tate group \( G \) is obtained by algebraizing the formal universal deformation of \( G \) over \( \text{Spf}(R) \) (3.7). Let \( U \) be the ordinary locus of \( G \) (i.e. the open subscheme of \( S \) parametrizing the ordinary fibers of \( G \)), and \( \eta \) a geometric point over the generic point of \( U \). For any integer \( n \geq 1 \), we denote by \( G(n) \) the kernel of the multiplication by \( p^n \) on \( G \), and by \[ T_p(G, \eta) = \varprojlim_n G(n)(\eta) \] the Tate module of \( G \) at \( \eta \). This is a free \( \mathbb{Z}_p \)-module of rank \( c \). We consider the monodromy representation attached to the étale part of \( G \) over \( U \)

\[ \rho_G : \pi_1(U, \eta) \to \text{Aut}_{\mathbb{Z}_p}(T_p(G, \eta)) \simeq \text{GL}_c(\mathbb{Z}_p). \]

The aim of this paper is to prove the following:
Theorem 1.3. If $G$ is connected and HW-cyclic, then the monodromy representation $\rho_G$ is surjective.

Igusa’s theorem mentioned above corresponds to Theorem 1.3 for $G = G^{1/2}$ (cf. 5.7). My interest in the $p$-adic monodromy problem started with the second part of my PhD thesis [Ti1], where I guessed 1.3 for $G = G^\lambda$ with $0 < \lambda < 1$ and proved it for $G^{1/3}$. After I posted the manuscript on ArXiv [Ti2], Strauch proved the one-dimensional case of 1.3 by using Drinfeld’s level structures [Str, Theorem 2.1]. Later on, Lau [Lau] proved 1.3 without the assumption that $G$ is HW-cyclic. By using the Newton stratification of the universal deformation space of $G$ due to Oort, Lau reduced the higher dimensional case to the one-dimensional case treated by Strauch. In fact, Strauch and Lau considered more generally the monodromy representation over each $p$-rank stratum of the universal deformation space. In this paper, we provide first a different proof of the one-dimensional case of 1.3. Our approach is purely characteristic $p$, while Strauch used Drinfeld’s level structure in characteristic 0. Then by following Lau’s strategy, we give a new (and easier) argument to reduce the general case of 1.3 to the one-dimensional case for HW-cyclic groups. The essential part of our argument is a versality criterion by Hasse-Witt maps of deformations of a connected one-dimensional Barsotti-Tate group (Prop. 4.11). This criterion can be considered as a generalization of another theorem of Igusa which claims that the Hasse invariant of a versal family of elliptic curves in characteristic $p$ has simple zeros. Compared with Strauch’s approach, our characteristic $p$ approach has the advantage of giving also results on the monodromy of Barsotti-Tate groups over a discrete valuation ring of characteristic $p$.

1.4. Let $A = k[[\pi]]$ be the ring of formal power series over $k$ in the variable $\pi$, $K$ its fraction field, and $v$ the valuation on $K$ normalized by $v(\pi) = 1$. We fix an algebraic closure $\overline{K}$ of $K$, and let $K^{\text{sep}}$ be the separable closure of $K$ contained in $\overline{K}$, $I$ be the Galois group of $K^{\text{sep}}$ over $K$, $I_p \subset I$ be the wild inertia subgroup, and $I_t = I/I_p$ the tame inertia group. For every integer $n \geq 1$, there is a canonical surjective character $\theta_{p^{n-1}} : I_t \to F_{p^n}^\times$ (5.2), where $F_{p^n}$ is the finite subfield of $k$ with $p^n$ elements.

We put $S = $ Spec$(A)$. Let $G$ be a Barsotti-Tate group over $S$, $G'$ be its Serre dual, Lie$(G')$ the Lie algebra of $G'$, and $\phi_G$ the Hasse-Witt map of $G$, i.e. the semi-linear endomorphism of Lie$(G')$ induced by the Frobenius of $G$. We define $h(G)$ to be the valuation of the determinant of a matrix of $\phi_G$, and call it the Hasse invariant of $G$ (5.4). We see easily that $h(G) = 0$ if and only if $G$ is ordinary over $S$, and $h(G) < \infty$ if and only if $G$ is generically ordinary. If $G$ is connected of height 2 and dimension 1, then $h(G) = 1$ is equivalent to that $G$ is versal (5.7).

Proposition 1.5. Let $S = $ Spec$(A)$ be as above, $G$ be a connected HW-cyclic Barsotti-Tate group with Hasse invariant $h(G) = 1$, and $G(1)$ the kernel of the multiplication by $p$ on $G$. Then the action of $I$ on $G(1)(\overline{K})$ is tame; moreover,
$G(1)(\overline{K})$ is an $\mathbb{F}_p'$-vector space of dimension 1 on which the induced action of $I_t$ is given by the surjective character $\theta_{p'-1}: I_t \to \mathbb{F}_p^\times$.

This proposition is an analog in characteristic $p$ of Serre’s result [Se3, Prop. 9] on the tameness of the monodromy associated with one-dimensional formal groups over a trait of mixed characteristic. We refer to 5.8 for the proof of this proposition and more results on the $p$-adic monodromy of HW-cyclic Barsotti-Tate groups over a trait in characteristic $p$.

1.6. This paper is organized as follows. In Section 2, we review some well known facts on ordinary Barsotti-Tate groups. Section 3 contains some preliminaries on the Dieudonné theory and the deformation theory of Barsotti-Tate groups. In Section 4, after establishing some basic properties of HW-cyclic groups and the coefficients of its Hasse-Witt matrix (Prop. 4.11), Section 5 is devoted to the study of the monodromy of a HW-cyclic Barsotti-Tate group over a complete trait of characteristic $p$. Section 6 is totally elementary, and contains a criterion (6.3) for the surjectivity of a homomorphism from a profinite group to $\text{GL}_n(\mathbb{Z}_p)$. Section 7 is the heart of this work, and it contains a proof of Theorem 1.3 in the one-dimensional case. Finally in Section 8, we follow Lau’s strategy and complete the proof of 1.3 by reducing the general case to the one-dimensional case treated in Section 7.

The proof in Section 7 of 1.3 in the one-dimensional case is based on an induction on the height $n + 1 \geq 2$ of $G$. The case $n = 1$ is just the classical Igusa’s theorem (5.7). For $n \geq 2$, by lemmas 6.3 and 6.5, it suffices to prove the following two statements: (a) the image of reduction modulo $p$ of $\rho_G$ contains a non-split Cartan subgroup; (b) under a suitable basis, the image of $\rho_G$ contains all matrix of the form \[
\begin{pmatrix}
B & b \\
0 & 1
\end{pmatrix}
\] with $B \in \text{GL}_{n-1}(\mathbb{Z}_p)$ and $b \in M_{(n-1) \times 1}(\mathbb{Z}_p)$.

The first statement follows easily from 1.5 by considering a certain base change of $G$ to a complete discrete valuation ring. To prove (b), we consider the formal completion $\text{Spec}(R')$ of the localization of the local moduli $S = \text{Spec}(R)$ of $G$ at the generic point of the locus where the universal deformation $G$ has $p$-rank $\leq 1$ (7.4). The ring $R'$ is a complete regular ring of dimension $n - 1$, and the Barsotti-Tate group $G' = G \otimes_R R'$ has a connected part of height $n$ and an étale part of height 1. Let $K_0$ be the residue field of $R'$, and $\overline{K}_0$ an algebraic closure of $K_0$. In order to apply the induction hypothesis, we consider the set of $k$-algebra homomorphisms $\sigma: R' \to \overline{R}' = \overline{K}_0[[t_1, \cdots, t_{n-1}]]$ lifting the natural inclusion $K_0 \to \overline{K}_0$. The key point is that, the natural map $\sigma \mapsto G'_{R',\sigma} = G' \otimes_{R',\sigma} \overline{R}'$ gives a bijection between the set of such $\sigma$’s and the set of deformations of $G_{K_0} = G' \otimes_{R',\sigma} \overline{R}'$; moreover, we can compute explicitly the Hasse-Witt map of the connected component $\mathcal{G}_{R',\sigma}^{\infty}$ of $G'_{R',\sigma}$ (Lemma 7.8). From the versality criterion for one-dimensional Barsotti-Tate groups in terms of the Hasse-Witt map established in Section 4 (Prop. 4.11), it follows immediately that there exists a $\sigma$ such that the Barsotti-Tate group $G'_{R',\sigma}^{\infty}$, which
is connected and one-dimensional of height $n$, is the universal deformation of its closed fiber. We fix such a $\sigma$. Then the set of all $\sigma'$ with $G_{R'}^{\sigma'} \simeq G_{R'}^{\sigma}$ as deformations of their common closed fiber is actually a group isomorphic to $\text{Ext}^1_{R'}(\mathbb{Q}_p/\mathbb{Z}_p,G_{R'}^{\sigma})$ (Prop. 3.10). Let $\sigma_1$ be the element corresponding to neutral element in $\text{Ext}^1_{R'}(\mathbb{Q}_p/\mathbb{Z}_p,G_{R'}^{\sigma})$. Applying the induction hypothesis to $G_{R',\sigma_1}$, we see that the monodromy group of $G_{R',\sigma_1}$, hence that of $G$, contains the subgroup $\left( \begin{array}{cc} \text{GL}_{n-1}(\mathbb{Z}_p) & 0 \\ 0 & 1 \end{array} \right)$ under a suitable basis of the Tate module (7.5.3). In order to conclude the proof, we need another $\sigma_2$ such that $G_{R',\sigma_2}$ has the same connected component as $G_{R',\sigma_1}$, and that the induced extension between the Tate module of the étale part of $G_{R',\sigma_2}$ and that of $G_{R',\sigma_2}$ is non-trivial after reduction modulo $p$ (see 7.5 and 7.5.4). To verify the existence of such a $\sigma_2$, we reduce the problem to a similar situation over a complete trait of characteristic $p$ (see 7.9), and we use a criterion of non-triviality of extensions by Hasse-Witt maps (5.12).

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1.8. Notations. Let $S$ be a scheme of characteristic $p > 0$. A BT-group over $S$ stands for a Barsotti-Tate group over $S$. Let $G$ be a commutative finite group scheme (resp. a BT-group) over $S$. We denote by $G^\vee$ its Cartier dual (resp. its Serre dual), by $\omega_G$ the sheaf of invariant differentials of $G$ over $S$, and by $\text{Lie}(G)$ the sheaf of Lie algebras of $G$. If $S = \text{Spec}(A)$ is affine and there is no risk of confusions, we also use $\omega_G$ and $\text{Lie}(G)$ to denote the corresponding $A$-modules of global sections. We put $G^{(p)}$ the pull-back of $G$ by the absolute Frobenius of $S$, $F_G : G \to G^{(p)}$ the Frobenius homomorphism and $V_G : G^{(p)} \to G$ the Verschiebung homomorphism. If $G$ is a BT-group and $n$ an integer $\geq 1$, we denote by $G(n)$ the kernel of the multiplication by $p^n$ on $G$; we have $G^\vee(n) = (G^\vee)(n)$ by definition. For an $\mathcal{O}_S$-module $M$, we denote by $M^{(p)} = \mathcal{O}_S \otimes_{F_S} M$ the scalar extension of $M$ by the absolute Frobenius of $\mathcal{O}_S$. If $\varphi : M \to N$ be a semi-linear homomorphism of $\mathcal{O}_S$-modules, we denote by $\tilde{\varphi} : M^{(p)} \to N$ the linearization of $\varphi$, i.e. we have $\tilde{\varphi}(\lambda \otimes x) = \lambda \cdot \varphi(x)$, where $\lambda$ (resp. $x$) is a local section of $\mathcal{O}_S$ (resp. of $M$).

Starting from Section 5, $k$ will denote an algebraically closed field of characteristic $p > 0$.

2. Review of ordinary Barsotti-Tate groups

In this section, $S$ denotes a scheme of characteristic $p > 0$. 
2.1. Let $G$ be a commutative group scheme, locally free of finite type over $S$. We have a canonical isomorphism of coherent $\mathcal{O}_S$-modules [III, 2.1]

\[(2.1.1)\quad \text{Lie}(G^\vee) \simeq \mathcal{H}om_{\mathcal{S}_{\text{fppf}}}(G, \mathcal{G}_a),\]

where $\mathcal{H}om_{\mathcal{S}_{\text{fppf}}}$ is the sheaf of homomorphisms in the category of abelian fppf-sheaves over $S$, and $\mathcal{G}_a$ is the additive group scheme. Since $\mathcal{G}_a^{(p)} \simeq \mathcal{G}_a$, the Frobenius homomorphism of $\mathcal{G}_a$ induces an endomorphism

\[(2.1.2)\quad \varphi_G : \text{Lie}(G^\vee) \to \text{Lie}(G^\vee),\]

semi-linear with respect to the absolute Frobenius map $F_S : \mathcal{O}_S \to \mathcal{O}_S$; we call it the Hasse-Witt map of $G$. By the functoriality of Frobenius, $\varphi_G$ is also the canonical map induced by the Frobenius of $G$, or dually by the Verschiebung of $G^\vee$.

2.2. By a commutative $p$-Lie algebra over $S$, we mean a pair $(L, \varphi)$, where $L$ is an $\mathcal{O}_S$-module locally free of finite type, and $\varphi : L \to L$ is a semi-linear endomorphism with respect to the absolute Frobenius $F_S : \mathcal{O}_S \to \mathcal{O}_S$. When there is no risk of confusions, we omit $\varphi$ from the notation. We denote by $p\text{-Lie}_S$ the category of commutative $p$-Lie algebras over $S$.

Let $(L, \varphi)$ be an object of $p\text{-Lie}_S$. We denote by

\[\mathcal{U}(L) = \text{Sym}(L) = \oplus_{n \geq 0} \text{Sym}^n(L),\]

the symmetric algebra of $L$ over $\mathcal{O}_S$. Let $\mathcal{I}_p(L)$ be the ideal sheaf of $\mathcal{U}(L)$ defined, for an open subset $V \subset S$, by

\[\Gamma(V, \mathcal{I}_p(L)) = \{x^\otimes_p - \varphi(x) : x \in \Gamma(V, \mathcal{U}(L))\},\]

where $x^\otimes_p = x \otimes x \otimes \cdots \otimes x \in \Gamma(V, \text{Sym}^p(L))$. We put $\mathcal{U}_p(L) = \mathcal{U}(L)/\mathcal{I}_p(L)$, and call it the $p$-enveloping algebra of $(L, \varphi)$. We endow $\mathcal{U}_p(L)$ with the structure of a Hopf-algebra with the comultiplication given by $\Delta(x) = 1 \otimes x + x \otimes 1$ and the counverse given by $\iota(x) = -x$.

Let $G$ be a commutative group scheme, locally free of finite type over $S$. We say that $G$ is of coheight one if the Verschiebung $V_G : G^{(p)} \to G$ is the zero homomorphism. We denote by $\mathfrak{g}V_S$ the category of such objects. For an object $G$ of $\mathfrak{g}V_S$, the Frobenius $F_G^{\vee}$ of $G^\vee$ is zero, so the Lie algebra $\text{Lie}(G^\vee)$ is locally free of finite type over $\mathcal{O}_S$ ([DG] VIIA Théo. 7.4(iii)). The Hasse-Witt map of $G$ (2.1.2) endows $\text{Lie}(G^\vee)$ with a commutative $p$-Lie algebra structure over $S$.

**Proposition 2.3** ([DG] VIIA, Théo. 7.2 et 7.4). The functor $\mathfrak{g}V_S \to p\text{-Lie}_S$ defined by $G \mapsto \text{Lie}(G^\vee)$ is an anti-equivalence of categories; a quasi-inverse is given by $(L, \varphi) \mapsto \text{Spec}(\mathcal{U}_p(L))$.

2.4. Assume $S = \text{Spec}(A)$ affine. Let $(L, \varphi)$ be an object of $p\text{-Lie}_S$ such that $L$ is free of rank $n$ over $\mathcal{O}_S$, $(e_1, \cdots, e_n)$ be a basis of $L$ over $\mathcal{O}_S$, $(h_{ij})_{1 \leq i, j \leq n}$ be the matrix of $\varphi$ under the basis $(e_1, \cdots, e_n)$, i.e. $\varphi(e_j) = \sum_{i=1}^n h_{ij} e_i$ for
1 ≤ j ≤ n. Then the group scheme attached to \((L, \varphi)\) is explicitly given by

\[
\text{Spec}(\mathcal{U}_p(L)) = \text{Spec}
\left( A[X_1, \ldots, X_n]/\left( X_j^j - \sum_{i=1}^n h_{ij} X_i \right)_{1 \leq i \leq n} \right),
\]

with the comultiplication \(\Delta(X_j) = 1 \otimes X_j + X_j \otimes 1\). By the Jacobian criterion of étaleness \([\text{EGA, IV}_0 \, 22.6.7]\), the finite group scheme \(\text{Spec}(\mathcal{U}_p(L))\) is étale over \(S\) if and only if the matrix \((h_{ij})_{1 \leq i \leq n} \leq n\) is invertible. This condition is equivalent to that the linearization of \(\varphi\) is an isomorphism.

**Corollary 2.5.** An object \(G\) of \(\mathcal{G}V_S\) is étale over \(S\), if and only if the linearization of its Hasse-Witt map \((2.1.2)\) is an isomorphism.

**Proof.** The problem being local over \(S\), we may assume \(S\) affine and \(L = \text{Lie}(G^\vee)\) free over \(\mathcal{G}_S\). By Theorem 2.3, \(G\) is isomorphic to \(\text{Spec}(\mathcal{U}_p(L))\), and we conclude by the last remark of 2.4. □

2.6. Let \(G\) be a BT-group over \(S\) of height \(c + d\) and dimension \(d\). The Lie algebra \(\text{Lie}(G^\vee)\) is an \(\mathcal{G}_S\)-module locally free of rank \(c\), and canonically identified with \(\text{Lie}(G^\vee(1))\) \([\text{BBM}] \, 3.3.2\). We define the **Hasse-Witt map of \(G\)**

\[
(2.6.1) \quad \varphi_G : \text{Lie}(G^\vee) \to \text{Lie}(G^\vee)
\]

to be that of \(G(1)\) \((2.1.2)\).

2.7. Let \(k\) be a field of characteristic \(p > 0\), \(G\) be a BT-group over \(k\). Recall that we have a canonical exact sequence of BT-groups over \(k\)

\[
(2.7.1) \quad 0 \to G^o \to G \to G^\text{ét} \to 0
\]

with \(G^o\) connected and \(G^\text{ét}\) étale \([\text{Dem}] \, \text{Chap.II, §7}\). This induces an exact sequence of Lie algebras

\[
(2.7.2) \quad 0 \to \text{Lie}(G^\text{ét}^\vee) \to \text{Lie}(G^\vee) \to \text{Lie}(G^o^\vee) \to 0,
\]

compatible with Hasse-Witt maps.

**Proposition 2.8.** Let \(k\) be a field of characteristic \(p > 0\), \(G\) be a BT-group over \(k\). Then \(\text{Lie}(G^\text{ét}^\vee)\) is the unique maximal \(k\)-subspace \(V\) of \(\text{Lie}(G^\vee)\) with the following properties:

(a) \(V\) is stable under \(\varphi_G\);

(b) the restriction of \(\varphi_G\) to \(V\) is injective.

**Proof.** It is clear that \(\text{Lie}(G^\text{ét}^\vee)\) satisfies property (a). We note that the Verschiebung of \(G^\text{ét}(1)\) vanishes; so \(G^\text{ét}(1)\) is in the category \(\mathcal{G}V_{\text{Spec}(k)}\). Since \(k\) is a field, 2.5 implies that the restriction of \(\varphi_G\) to \(\text{Lie}(G^\text{ét}^\vee)\), which coincides with \(\varphi_G^n\), is injective. This proves that \(\text{Lie}(G^\text{ét}^\vee)\) verifies (b). Conversely, let \(V\) be an arbitrary \(k\)-subspace of \(\text{Lie}(G^\vee)\) with properties (a) and (b). We have to show that \(V \subset \text{Lie}(G^\text{ét}^\vee)\). Let \(\sigma\) be the Frobenius endomorphism of \(k\). If \(M\) is a \(k\)-vector space, for each integer \(n \geq 1\), we put \(M^{(p^n)} = k \otimes_{\sigma^n} M\), i.e. we have \(1 \otimes ax = \sigma^n(a) \otimes x\) in \(k \otimes_{\sigma^n} M\) for \(a \in k, x \in M\). Since \(\varphi_G|_V : V \to V\) is injective by assumption, the linearization \(\varphi_G^n|_{V^{(p^n)}} : V^{(p^n)} \to V\) of \(\varphi_G^n|_V\)
is injective (hence bijective) for any \( n \geq 1 \). We have \( V = \tilde{\varphi}^n_G(V(p^n)) \). Since \( G^\circ \) is connected, there is an integer \( n \geq 1 \) such that the \( n \)-th iterated Frobenius \( F^n_G(1) : G^\circ(1) \to G^\circ(1)(p^n) \) vanishes. Hence by definition, the linearized \( n \)-iterated Hasse-Witt map \( \tilde{\varphi}_G^n : \text{Lie}(G^\circ)(p^n)^{\vee} \to \text{Lie}(G^\circ)^{\vee} \) is zero. By the compatibility of Hasse-Witt maps, we have \( \tilde{\varphi}_G^n(V(p^n)) \subset \text{Lie}(G^\text{ét})^{\vee} \); in particular, we have \( V = \tilde{\varphi}_G^n(V(p^n)) \subset \text{Lie}(G^\text{ét})^{\vee} \). This completes the proof. □

**Corollary 2.9.** Let \( k \) be a field of characteristic \( p > 0 \), \( G \) be a BT-group over \( k \). Then \( G \) is connected if and only if \( \varphi_G \) is nilpotent.

**Proof.** In the proof of the proposition, we have seen that the Hasse-Witt map of the connected part of \( G \) is nilpotent. So the “only if” part is verified. Conversely, if \( \varphi_G \) is nilpotent, \( \text{Lie}(G^\text{ét})^{\vee} \) is zero by the proposition. Therefore \( G \) is connected. □

**Definition 2.10.** Let \( S \) be a scheme of characteristic \( p > 0 \), \( G \) be a BT-group over \( S \). We say that \( G \) is ordinary if there exists an exact sequence of BT-groups over \( S \)

\[
0 \to G^{\text{mult}} \to G \to G^\text{ét} \to 0,
\]

such that \( G^{\text{mult}} \) is multiplicative and \( G^\text{ét} \) is étale.

We note that when it exists, the exact sequence (2.10.1) is unique up to a unique isomorphism, because there is no non-trivial homomorphisms between a multiplicative BT-group and an étale one in characteristic \( p > 0 \). The property of being ordinary is clearly stable under arbitrary base change and Serre duality. If \( S \) is the spectrum of a field of characteristic \( p > 0 \), \( G \) is ordinary if and only if its connected part \( G^\circ \) is of multiplicative type.

**Proposition 2.11.** Let \( G \) be a BT-group over \( S \). The following conditions are equivalent:

(a) \( G \) is ordinary over \( S \).

(b) For every \( x \in S \), the fiber \( G_x = G \otimes_S \kappa(x) \) is ordinary over \( \kappa(x) \).

(c) The finite group scheme \( \text{Ker} V_G \) is étale over \( S \).

(c') The finite group scheme \( \text{Ker} F_G \) is of multiplicative type over \( S \).

(d) The linearization of the Hasse-Witt map \( \varphi_G \) is an isomorphism.

First, we prove the following lemmas.

**Lemma 2.12.** Let \( T \) be a scheme, \( H \) be a commutative group scheme locally free of finite type over \( T \). Then \( H \) is étale (resp. of multiplicative type) over \( T \) if and only if, for every \( x \in T \), the fiber \( H \otimes_T \kappa(x) \) is étale (resp. of multiplicative type) over \( \kappa(x) \).

**Proof.** We will consider only the étale case; the multiplicative case follows by duality. Since \( H \) is \( T \)-flat, it is étale over \( T \) if and only if it is unramified over \( T \). By [EGA, IV 17.4.2], this condition is equivalent to that \( H \otimes_T \kappa(x) \) is unramified over \( \kappa(x) \) for every point \( x \in T \). Hence the conclusion follows. □
Lemma 2.13. Let $G$ be a BT-group over $S$. Then $\text{Ker} V_G$ is an object of the category $\mathfrak{S} V_S$, i.e. it is locally free of finite type over $S$, and its Verschiebung is zero. Moreover, we have a canonical isomorphism $(\text{Ker} V_G)^\vee \simeq \text{Ker} F_{G^\vee}$, which induces an isomorphism of Lie algebras $\text{Lie}((\text{Ker} V_G)^\vee) \simeq \text{Lie}(\text{Ker} F_{G^\vee}) = \text{Lie}(G^\vee)$, and the Hasse-Witt map (2.1.2) of $\text{Ker} V_G$ is identified with $\varphi_G$ (2.6.1).

Proof. The group scheme $\text{Ker} V_G$ is locally free of finite type over $S$ ([III] 1.3(b)), and we have a commutative diagram

$$
\begin{array}{ccc}
(Ker V_G)(p) & \xrightarrow{V_{ker V_G}} & Ker V_G \\
\downarrow & & \downarrow \\
(G(p))(p) & \xrightarrow{V_{G(p)}} & G(p)
\end{array}
$$

By the functoriality of Verschiebung, we have $V_{G(p)} = (V_G)(p)$ and $\text{Ker} V_{G(p)} = (\text{Ker} V_G)(p)$. Hence the composition of the left vertical arrow with $V_{G(p)}$ vanishes, and the Verschiebung of $\text{Ker} V_G$ is zero.

By Cartier duality, we have $(\text{Ker} V_G)^\vee \simeq \text{Coker}(F_{G^\vee(1)})$. Moreover, the exact sequence

$$
\cdots \to G^\vee(1) \xrightarrow{F_{G^\vee(1)}} (G^\vee(1))(p) \xrightarrow{V_{G^\vee(1)}} G^\vee(1) \to \cdots,
$$

induces a canonical isomorphism

$$
(2.13.1) \quad \text{Coker}(F_{G^\vee(1)}) \xrightarrow{\sim} \text{Im}(V_{G^\vee(1)}) = \text{Ker} F_{G^\vee(1)} = \text{Ker} F_{G^\vee}.
$$

Hence, we deduce that

$$
(2.13.2) \quad (\text{Ker} V_G)^\vee \simeq \text{Coker}(F_{G^\vee(1)}) \xrightarrow{\sim} \text{Ker} F_{G^\vee} \hookrightarrow G^\vee(1).
$$

Since the natural injection $\text{Ker} F_{G^\vee} \hookrightarrow G^\vee(1)$ induces an isomorphism of Lie algebras, we get

$$
(2.13.3) \quad \text{Lie}((\text{Ker} V_G)^\vee) \simeq \text{Lie}(\text{Ker} F_{G^\vee}) = \text{Lie}(G^\vee(1)) = \text{Lie}(G^\vee).
$$

It remains to prove the compatibility of the Hasse-Witt maps with (2.13.3). We note that the dual of the morphism (2.13.2) is the canonical map $F : G(1) \to \text{Ker} V_G = \text{Im}(F_{G(1)})$ induced by $F_{G(1)}$. Hence by (2.1.1), the isomorphism (2.13.3) is identified with the functorial map

$$
\mathcal{H}om_{S_{fppf}}(\text{Ker} V_G, \mathbb{G}_a) \to \mathcal{H}om_{S_{fppf}}(G(1), \mathbb{G}_a)
$$

induced by $F$, and its compatibility with the Hasse-Witt maps follows easily from the definition (2.1.2). □

Proof of 2.11. (a)$\Rightarrow$(b). Indeed, the ordinarity of $G$ is stable by base change. (b)$\Rightarrow$(c). By Lemma 2.12, it suffices to verify that for every point $x \in S$, the fiber $(\text{Ker} V_G) \otimes_S \kappa(x) \simeq \text{Ker} V_{G_x}$ is étale over $\kappa(x)$. Since $G_x$ is assumed to be ordinary, its connected part $(G_x)^\vee$ is multiplicative. Hence, the Verschiebung of
$\left(G_x\right)^{\circ}$ is an isomorphism, and $\text{Ker} V_{G_x}$ is canonically isomorphic to $\text{Ker} V_{G_{\text{et}}'} \subset \left(G_x^{\text{et}}\right)^{(p)} \cong \left(G_x^{(p)}\right)^{\text{et}}$, so our assertion follows.

$(c) \Leftrightarrow (d)$. It follows immediately from Lemma 2.13 and Corollary 2.5.

$(c) \Leftrightarrow (c')$. By 2.12, we may assume that $S$ is the spectrum of a field. So the category of commutative finite group schemes over $S$ is abelian. We will just prove $(c) \Rightarrow (c')$; the converse can be proved by duality. We have a fundamental short exact sequence of finite group schemes

\[(2.13.4) \quad 0 \to \text{Ker} F_G \to G(1) \xrightarrow{F} \text{Ker} V_G \to 0,\]

where $F$ is induced by $F_{G(1)}$. That induces a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (\text{Ker} F_G)^{(p)} & \rightarrow & (G(1))^{(p)} & \xrightarrow{F^{(p)}} & (\text{Ker} V_G)^{(p)} & \rightarrow & 0 \\
& & \downarrow{V'} & & \downarrow{V_{G(1)}} & & \downarrow{V''} & & \\
0 & \rightarrow & \text{Ker} F_G & \rightarrow & G(1) & \xrightarrow{F} & \text{Ker} V_G & \rightarrow & 0
\end{array}
\]

where vertical arrows are the Verschiebung homomorphisms. We have seen that $V'' = 0$ (2.13). Therefore, by the snake lemma, we have a long exact sequence

\[(2.13.5) \quad 0 \to \text{Ker} V' \to \text{Ker} V_{G(1)} \xrightarrow{\alpha} (\text{Ker} V_G)^{(p)} \to \\
\quad \to \text{Coker} V' \to \text{Coker} V_{G(1)} \xrightarrow{\beta} \text{Ker} V_G \to 0,
\]

where the map $\alpha$ is the Frobenius of $\text{Ker} V_G$ and $\beta$ is the composed isomorphism $\text{Coker}(V_{G(1)}) \cong G(1)/\text{Ker} F_{G(1)} \xrightarrow{\sim} \text{Im}(F_{G(1)}) \cong \text{Ker} V_G$.

Then condition $(c)$ is equivalent to that $\alpha$ is an isomorphism; it implies that $\text{Ker} V' = \text{Coker} V' = 0$, i.e. the Verschiebung of $\text{Ker} F_G$ is an isomorphism, and hence $(c')$. $(c) \Rightarrow (a)$. For every integer $n > 0$, we denote by $F^n_G$ the composed homomorphism

\[G \xrightarrow{F_G} G^{(p)} \xrightarrow{F_G^{(p)}} \cdots \xrightarrow{F_G^{(p^{n-1})}} G^{(p^n)},\]

and by $V^n_G$ the composed homomorphism

\[G^{(p^n)} \xrightarrow{V_G^{(p^{n-1})}} G^{(p^{n-1})} \xrightarrow{V_G^{(p^{n-2})}} \cdots \xrightarrow{V_G^{(p^0)}} G;\]

$F^n_G$ and $V^n_G$ are isogenies of BT-groups. From the relation $V^n_G \circ F^n_G = p^n$, we deduce an exact sequence

\[(2.13.6) \quad 0 \to \text{Ker} F^n_G \to G(n) \xrightarrow{F^n} \text{Ker} V^n_G \to 0,
\]
where $F^n$ is induced by $F^n_G$. For $1 \leq j < n$, we have a commutative diagram

$$
\begin{array}{ccc}
G^{(p^n)} & \xrightarrow{V^{n-j}_{G^{(p^n)}}} & G^{(p^j)} \\
\downarrow V^n_G & & \downarrow V^j_G \\
G & & \\
\end{array}
$$

One notices that $\text{Ker} V^{n-j}_{G^{(p^n)}} = (\text{Ker} V^{n-j}_{G})(p^j)$ by the functoriality of Verschiebung. Since all maps in (2.13.7) are isogenies, we have an exact sequence

$$
0 \to (\text{Ker} V^{n-j}_{G})(p^j) \xrightarrow{i_{n-j,n}} \text{Ker} V^n_G \xrightarrow{\text{Id}} \text{Ker} V^n_G \to 0.
$$

Therefore, condition (c) implies by induction that $\text{Ker} V^n_G$ is an étale group scheme over $S$. Hence the $j$-th iteration of the Frobenius $\text{Ker} V^{n-j}_{G^{(p^n)}} \to (\text{Ker} V^{n-j}_{G})(p^j)$ is an isomorphism, and $\text{Ker} V^{n-j}_{G}$ is identified with a closed subgroup scheme of $\text{Ker} V^n_G$ by the composed map

$$
i_{n-j,n} : \text{Ker} V^{n-j}_{G} \xrightarrow{\sim} (\text{Ker} V^{n-j}_{G})(p^j) \xrightarrow{i_{n-j,n}} \text{Ker} V^n_G.
$$

We claim that the kernel of the multiplication by $p^{n-j}$ on $\text{Ker} V^n_G$ is $\text{Ker} V^n_G$. Indeed, from the relation $p^{n-j} \cdot \text{Id}_{G^{(p^n)}} = F^{n-j}_{G^{(p^n)}} \circ V^{n-j}_{G^{(p^n)}}$, we deduce a commutative diagram (without dotted arrows)

$$
\begin{array}{ccc}
\text{Ker} V^n_G & \xrightarrow{p_{n,j}} & G^{(p^n)} \\
\downarrow V^{n-j}_{G^{(p^n)}} & & \downarrow V^{n-j}_{G^{(p^j)}} \\
\text{Ker} V^j_G & & G^{(p^j)} \\
\downarrow i_{j,n} & & \downarrow p^{n-j} \\
\text{Ker} V^n_G & \xrightarrow{i_{n-j,n}} & G^{(p^n)}.
\end{array}
$$

It follows from (2.13.8) that the subgroup $\text{Ker} V^n_G$ of $G^{(p^n)}$ is sent by $V^{n-j}_{G^{(p^n)}}$ onto $\text{Ker} V^j_G$. Therefore diagram (2.13.9) remains commutative when completed by the dotted arrows, hence our claim. It follows from the claim that $(\text{Ker} V^n_G)_{n \geq 1}$ constitutes an étale BT-group over $S$, denoted by $G^{\text{et}}$. By duality, we have an exact sequence

$$
0 \to \text{Ker} F^j_G \to \text{Ker} F^n_G \to (\text{Ker} F^{n-j}_{G^{(p^n)}})(p^j) \to 0.
$$

Condition (c') implies by induction that $\text{Ker} F^n_G$ is of multiplicative type. Hence the $j$-th iteration of Verschiebung $(\text{Ker} F^{n-j}_{G^{(p^n)}})(p^j) \to \text{Ker} F^{n-j}_{G^{(p^n)}}$ is an isomorphism. We deduce from (2.13.10) that $(\text{Ker} F^n_G)_{n \geq 1}$ form a multiplicative BT-group over $S$ that we denote by $G^{\text{mult}}$. Then the exact sequences (2.13.6) give a decomposition of $G$ of the form (2.10.1).
Corollary 2.14. Let $G$ be a BT-group over $S$, and $S^{\text{ord}}$ be the locus in $S$ of the points $x \in S$ such that $G_x = G \otimes_{S} \kappa(x)$ is ordinary over $\kappa(x)$. Then $S^{\text{ord}}$ is open in $S$, and the canonical inclusion $S^{\text{ord}} \to S$ is affine.

The open subscheme $S^{\text{ord}}$ of $S$ is called the ordinary locus of $G$.

3. Preliminaries on Dieudonné Theory and Deformation Theory

3.1. We will use freely the conventions of 1.8. Let $S$ be a scheme of characteristic $p > 0$, $G$ be a Barsotti-Tate group over $S$, and $\mathbf{M}(G) = \mathbb{D}(G)_{(S,S)}$ be the coherent $\mathcal{O}_S$-module obtained by evaluating the (contravariant) Dieudonné crystal of $G$ at the trivial divided power immersion $S \hookrightarrow S$ [BBM, 3.3.6]. Recall that $\mathbf{M}(G)$ is an $\mathcal{O}_S$-module locally free of finite type satisfying the following properties:

(i) Let $F_M : \mathbf{M}(G)^{(p)} \to \mathbf{M}(G)$ and $V_M : \mathbf{M}(G) \to \mathbf{M}(G)^{(p)}$ be the $\mathcal{O}_S$-linear maps induced respectively by the Frobenius and the Verschiebung of $G$. We have the following exact sequence:

$$\cdots \to \mathbf{M}(G)^{(p)} \xrightarrow{F_M} \mathbf{M}(G) \xrightarrow{V_M} \mathbf{M}(G)^{(p)} \to \cdots.$$ 

(ii) There is a connection $\nabla : \mathbf{M}(G) \to \mathbf{M}(G) \otimes_{\mathcal{O}_S} \Omega^1_{S/F_p}$ for which $F_M$ and $V_M$ are horizontal morphisms.

(iii) We have two canonical filtrations on $\mathbf{M}(G)$ by $\mathcal{O}_S$-modules locally free of finite type:

\begin{align}
(3.1.1) & \quad 0 \to \omega_G \to \mathbf{M}(G) \to \text{Lie}(G^\vee) \to 0, \\
(3.1.2) & \quad 0 \to \text{Lie}(G^\vee)^{(p)} \xrightarrow{\omega_G^{(p)}} \mathbf{M}(G) \to \omega_G^{(p)} \to 0,
\end{align}

which is obtained by applying the Dieudonné functor to the exact sequence of finite group schemes $0 \to \text{Ker } F_G \to G(1) \to \text{Ker } V_G \to 0$ [BBM, 4.3.1, 4.3.6, 4.3.11]. Moreover, we have the following commutative diagram (cf. [Ka1, 2.3.2].
and 2.3.4])

\[(3.1.3)\]

\[
\begin{array}{ccc}
\text{0} & \text{0} & \text{0} \\
\downarrow & \downarrow & \downarrow \\
\omega_G^{(p)} & \omega_G & \omega_G^{(p)} \\
\downarrow & \downarrow & \downarrow \\
\text{M}(G)^{(p)} & \text{M}(G) & \text{M}(G)^{(p)} \\
\downarrow & \downarrow & \downarrow \\
\text{Lie}(G^\vee)^{(p)} & \text{Lie}(G^\vee) & \text{Lie}(G^\vee)^{(p)} \\
\downarrow & \downarrow & \downarrow \\
\text{0} & \text{0} & \text{0}
\end{array}
\]

where the columns are the Hodge filtrations and the anti-diagonal is the conjugate filtration. By functoriality, we see easily that \(\tilde{\phi}_G\) above is nothing but the linearization of the Hasse-Witt map \(\phi_G\) (2.6.1), and the morphism \(\psi^*_G : \text{Lie}(G)^{(p)} \to \text{Lie}(G)\), which is obtained by applying the functor \(\text{Hom}_{\mathcal{O}_S}(\_ , \mathcal{O}_S)\) to \(\psi_G\), is identified with the linearization \(\tilde{\phi}_G^\vee\) of \(\phi_G^\vee\).

The formation of these structures on \(\text{M}(G)\) commutes with arbitrary base changes of \(S\). In the sequel, we will use \((\text{M}(G), F_M, \nabla)\) to emphasize these structures on \(\text{M}(G)\).

3.2. In the reminder of this section, \(k\) will denote an algebraically closed field of characteristic \(p > 0\). Let \(S\) be a scheme formally smooth over \(k\) such that \(\Omega^1_{S/F} = \Omega^1_{S/k}\) is an \(\mathcal{O}_S\)-module locally free of finite type, e.g. \(S = \text{Spec}(A)\) with \(A\) a formally smooth \(k\)-algebra with a finite \(p\)-basis over \(k\). Let \(G\) be a BT-group over \(S\). We put \(K_S\) to be the composed morphism

\[(3.2.1)\]

\[
\text{KS} : \omega_G \to \text{M}(G) \xrightarrow{\nabla} \text{M}(G) \otimes_{\mathcal{O}_S} \Omega^1_{S/k} \xrightarrow{pr} \text{Lie}(G^\vee) \otimes_{\mathcal{O}_S} \Omega^1_{S/k}
\]

which is \(\mathcal{O}_S\)-linear. We put \(\mathcal{I}_{S/k} = \text{Hom}_{\mathcal{O}_S}(\Omega^1_{S/k}, \mathcal{O}_S)\), and define the Kodaira-Spencer map of \(G\)

\[(3.2.2)\]

\[
\text{Kod} : \mathcal{I}_{S/k} \to \text{Hom}_{\mathcal{O}_S}(\omega_G, \text{Lie}(G^\vee))
\]

to be the morphism induced by \(\text{KS}\). We say that \(G\) is versal if \(\text{Kod}\) is surjective.

3.3. Let \(r\) be an integer \(\geq 1\), \(R = k[[t_1, \cdots, t_r]]\), \(m\) be the maximal ideal of \(R\). We put \(\mathcal{S} = \text{Spf}(R)\), \(S = \text{Spec}(R)\), and for each integer \(n \geq 0\), \(S_n = \text{Spec}(R/m^{n+1})\). By a BT-group \(\mathcal{G}\) over the formal scheme \(\mathcal{S}\), we mean a sequence of BT-groups \((G_n)_{n \geq 0}\) over \((S_n)_{n \geq 0}\) equipped with isomorphisms \(G_{n+1} \times_{S_{n+1}} S_n \simeq G_n\).
According to [deJ, 2.4.4], the functor \( G \mapsto (G \times SS_n)_{n \geq 0} \) induces an equivalence of categories between the category of BT-groups over \( S \) and the category of BT-groups over \( \mathcal{S} \). For a BT-group \( \mathcal{G} \) over \( \mathcal{S} \), the corresponding BT-group \( G \) over \( S \) is called the algebraization of \( \mathcal{G} \). We say that \( \mathcal{G} \) is versal over \( \mathcal{S} \), if its algebraization \( G \) is versal over \( S \). Since \( S \) is local, by Nakayama’s Lemma, \( \mathcal{G} \) or \( G \) is versal if and only if the reduction of Kod modulo the maximal ideal

\[
\text{Kod}_0 : \mathcal{G}_{/k} \otimes \mathcal{G}_S k \longrightarrow \text{Hom}_k(\omega_{G_0}, \text{Lie}(G'_0))
\]

is surjective.

3.4. We recall briefly the deformation theory of a BT-group. Let \( \mathfrak{A} \mathfrak{L}_k \) be the category of local artinian \( k \)-algebras with residue field \( k \). We notice that all morphisms of \( \mathfrak{A} \mathfrak{L}_k \) are local. A morphism \( \mathcal{A}' \to \mathcal{A} \) in \( \mathfrak{A} \mathfrak{L}_k \) is called a small extension, if it is surjective and its kernel \( I \) satisfies \( I \cdot \mathcal{A}' = 0 \), where \( \mathcal{A}' \) is the maximal ideal of \( \mathcal{A}' \).

Let \( G_0 \) be a BT-group over \( k \), and \( A \) an object of \( \mathfrak{A} \mathfrak{L}_k \). A deformation of \( G_0 \) over \( A \) is a pair \((G, \phi)\), where \( G \) is a BT-group over \( \text{Spec}(A) \) and \( \phi \) is an isomorphism \( \phi : G \otimes_A k \iso G_0 \). When there is no risk of confusions, we will denote a deformation \((G, \phi)\) simply by \( G \). Two deformations \((G, \phi)\) and \((G', \phi')\) over \( A \) are isomorphic if there exists an isomorphism of BT-groups \( \psi : G \iso G' \) over \( A \) such that \( \phi = \phi' \circ (\psi \otimes_A k) \). Let’s denote by \( \mathcal{D} \) the functor which associates with each object \( A \) of \( \mathfrak{A} \mathfrak{L}_k \) the set of isomorphism classes of deformations of \( G_0 \) over \( A \). If \( f : A \to B \) is a morphism of \( \mathfrak{A} \mathfrak{L}_k \), then the map \( \mathcal{D}(f) : \mathcal{D}(A) \to \mathcal{D}(B) \) is given by extension of scalars. We call \( \mathcal{D} \) the deformation functor of \( G_0 \) over \( \mathfrak{A} \mathfrak{L}_k \).

**Proposition 3.5** ([III], 4.8). Let \( G_0 \) be a BT-group over \( k \) of dimension \( d \) and height \( c + d \), \( \mathcal{D} \) be the deformation functor of \( G_0 \) over \( \mathfrak{A} \mathfrak{L}_k \).

(i) Let \( \mathcal{A}' \to \mathcal{A} \) be a small extension in \( \mathfrak{A} \mathfrak{L}_k \) with ideal \( I \), \( x = (G, \phi) \) be an element in \( \mathcal{D}(A) \), \( \mathcal{D}_x(A') \) be the subset of \( \mathcal{D}(A') \) with image \( x \) in \( \mathcal{D}(A) \). Then the set \( \mathcal{D}_x(A') \) is a nonempty homogenous space under the group \( \text{Hom}_k(\omega_{G_0}, \text{Lie}(G'_0)) \otimes_k I \).

(ii) The functor \( \mathcal{D} \) is pro-representable by a formally smooth formal scheme \( \mathcal{S} \) over \( k \) of relative dimension \( cd \), i.e. \( \mathcal{S} = \text{Spf}(R) \) with \( R \simeq k[[u_{ij}, \leq 1 \leq c, 1 \leq j \leq d]] \), and there exists a unique deformation \((\mathcal{G}, \psi)\) of \( G_0 \) over \( \mathcal{S} \) such that, for any object \( A \) of \( \mathfrak{A} \mathfrak{L}_k \) and any deformation \((G, \phi)\) of \( G_0 \) over \( A \), there is a unique homomorphism of local \( k \)-algebras \( \varphi : R \to A \) with \( (G, \phi) = \Psi(\varphi)(\mathcal{G}, \psi) \).

(iii) Let \( \mathcal{T}_{\mathcal{S}/k}(0) = \mathcal{T}_{\mathcal{S}/k} \otimes_{\mathcal{S}/k} k \) be the tangent space of \( \mathcal{S} \) at its unique closed point,

\[
\text{Kod}_0 : \mathcal{T}_{\mathcal{S}/k}(0) \longrightarrow \text{Hom}_k(\omega_{G_0}, \text{Lie}(G'_0))
\]

be the Kodaira-Spencer map of \( \mathcal{G} \) evaluated at the closed point of \( \mathcal{S} \). Then \( \text{Kod}_0 \) is bijective, and it can be described as follows. For an element \( f \in \mathcal{T}_{\mathcal{S}/k}(0) \), i.e. a homomorphism of local \( k \)-algebras \( f : R \to k[\epsilon]/\epsilon^2 \), \( \text{Kod}_0(\epsilon^2) = \epsilon^2 \)

\[
[\mathcal{G} \otimes_R (k[\epsilon]/\epsilon^2)] - [G_0 \otimes_k (k[\epsilon]/\epsilon^2)],
\]

which is a well-defined element in \( \text{Hom}_k(\omega_{G_0}, \text{Lie}(G'_0)) \) by (i).
Remark 3.6. Let $(e_j)_{1 \leq j \leq d}$ be a basis of $\omega_{G_0}$, $(f_i)_{1 \leq i \leq e}$ be a basis of $\text{Lie}(G_0^\text{et})$.
In view of 3.5(iii), we can choose a system of parameters $(t_{ij})_{1 \leq i \leq e, 1 \leq j \leq d}$ of $\mathcal{S}$ such that

$$\text{Kod}_0\left( \frac{\partial}{\partial t_{ij}} \right) = e_j^* \otimes f_i,$$

where $(e_j^*)_{1 \leq j \leq d}$ is the dual basis of $(e_j)_{1 \leq j \leq d}$. Moreover, if $m$ is the maximal ideal of $R$, the parameters $t_{ij}$ are determined uniquely modulo $m^2$.

Corollary 3.7 (Algebraization of the Universal Deformation). The assumptions being those of (3.5), we put moreover $\mathbf{S} = \text{Spec}(R)$ and $\mathbf{G}$ the algebraization of the universal formal deformation $\mathcal{G}$. Then the BT-group $\mathbf{G}$ is versal over $\mathbf{S}$, and satisfies the following universal property: Let $A$ be a noetherian complete local $k$-algebra with residue field $k$, $G$ be a BT-group over $A$ endowed with an isomorphism $G \otimes_A k \simeq G_0$. Then there exists a unique continuous homomorphism of local $k$-algebras $\varphi : R \to A$ such that $G \simeq G \otimes_R A$.

Proof. By the last remark of 3.3, $\mathbf{G}$ is clearly versal. It remains to prove that it satisfies the universal property in the corollary. Let $G$ be a deformation of $G_0$ over a noetherian complete local $k$-algebra $A$ with residue field $k$. We denote by $m_A$ the maximal ideal of $A$, and put $A_n = A/m_A^{n+1}$ for each integer $n \geq 0$. Then by 3.5(b), there exists a unique local homomorphism $\varphi_n : R \to A_n$ such that $G \otimes A_n \simeq G \otimes_R A_n$. The $\varphi_n$'s form a projective system $(\varphi_n)_{n \geq 0}$, whose projective limit $\varphi : R \to A$ answers the question. □

Definition 3.8. The notations are those of (3.7). We call $\mathbf{S}$ the local moduli in characteristic $p$ of $G_0$, and $\mathbf{G}$ the universal deformation of $G_0$ in characteristic $p$.

If there is no confusions, we will omit “in characteristic $p$” for short.

9. Let $G$ be a BT-group over $k$, $G^\circ$ be its connected part, and $G^\text{et}$ be its étale part. Let $r$ be the height of $G^\text{et}$. Then we have $G^\text{et} \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r$, since $k$ is algebraically closed. Let $\mathcal{D}_G$ (resp. $\mathcal{D}_{G^\circ}$) be the deformation functor of $G$ (resp. $G^\circ$) over $\mathfrak{A}L_k$. If $A$ is an object in $\mathfrak{A}L_k$ and $\mathcal{G}$ is a deformation of $G$ (resp. $G^\circ$) over $A$, we denote by $[\mathcal{G}]$ its isomorphism class in $\mathcal{D}_G(A)$ (resp. in $\mathcal{D}_{G^\circ}(A)$).

Proposition 3.10. The assumptions are as above, let $\Theta : \mathcal{D}_G \to \mathcal{D}_{G^\circ}$ be the morphism of functors that maps a deformation of $G$ to its connected component.
(i) The morphism $\Theta$ is formally smooth of relative dimension $r$.
(ii) Let $A$ be an object of $\mathfrak{A}L_k$, and $\mathcal{G}$ be a deformation of $G^\circ$ over $A$. Then the subset $\Theta^{-1}_A([\mathcal{G}])$ of $\mathcal{D}_G(A)$ is canonically identified with $\text{Ext}^1_A(\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{G}^\circ)^\vee$, where $\text{Ext}^1_A$ means the group of extensions in the category of abelian fpf-sheaves on $\text{Spec}(A)$.

Proof. (i) Since $\mathcal{D}_G$ and $\mathcal{D}_{G^\circ}$ are both pro-representable by a noetherian local complete $k$-algebra and formally smooth over $k$ (3.5), by a formal completion version of [EGA, IV 17.11.1(d)], we only need to check that the tangent map $\Theta_{k[e]/e^2} : \mathcal{D}_G(k[e]/e^2) \to \mathcal{D}_{G^\circ}(k[e]/e^2)$
is surjective with kernel of dimension \( r \) over \( k \). By 3.5(iii), \( \mathcal{D}_G(k[e]/e^2) \) (resp. \( \mathcal{D}_G(k[e]/e^2) \)) is isomorphic to \( \text{Hom}_k(\omega_G, \text{Lie}(G^\vee)) \) (resp. \( \text{Hom}_k(\omega_G, \text{Lie}(G^\vee)) \)) by the Kodaira-Spencer morphism. In view of the canonical isomorphism \( \omega_G \cong \omega_{G^r} \), \( \Theta_{k[e]/e^2} \) corresponds to the map
\[
\Theta'_{k[e]/e^2} : \text{Hom}_k(\omega_G, \text{Lie}(G^\vee)) \to \text{Hom}_k(\omega_G, \text{Lie}(G^\vee))
\]
induced by the canonical surjection \( \text{Lie}(G^\vee) \to \text{Lie}(G^\vee) \). It is clear that \( \Theta'_{k[e]/e^2} \) is surjective of kernel \( \text{Hom}_k(\omega_G, \text{Lie}(G^\vee)) \), which has dimension \( r \) over \( k \).

(ii) Since \( G^{et} \) is isomorphic to \( (\mathbb{Q}_p/\mathbb{Z}_p)^r \), every element in \( \text{Ext}_A^1(\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{G}^\circ) \) defines clearly an element of \( \mathcal{D}_G(A) \) with image \( \mathcal{G}^\circ \) in \( \mathcal{D}_G(A) \). Conversely, for any \( \mathcal{G} \in \mathcal{D}_G(A) \) with connected component isomorphic to \( \mathcal{G}^\circ \), the isomorphism \( G^{et} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^r \) lifts uniquely to an isomorphism \( \mathcal{G}^{et} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^r \) because \( A \) is henselian. The canonical exact sequence \( 0 \to \mathcal{G}^\circ \to \mathcal{G} \to \mathcal{G}^{et} \to 0 \) shows that \( \mathcal{G} \) comes from an element of \( \text{Ext}_A^1(\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{G}^\circ)^r \).

\[\Box\]

4. HW-cyclic Barsotti-Tate Groups

**Definition 4.1.** Let \( S \) be a scheme of characteristic \( p > 0 \), \( G \) be a BT-group over \( S \) such that \( c = \text{dim}(G^\vee) \) is constant. We say that \( G \) is HW-cyclic, if \( c \geq 1 \) and there exists an element \( v \in \Gamma(S, \text{Lie}(G^\vee)) \) such that
\[
v, \varphi_G(v), \ldots, \varphi_G^{c-1}(v)
\]
generate \( \text{Lie}(G^\vee) \) as an \( \mathcal{O}_S \)-module, where \( \varphi_G \) is the Hasse-Witt map (2.6.1) of \( G \).

**Remark 4.2.** It is clear that a BT-group \( G \) over \( S \) is HW-cyclic, if and only if \( \text{Lie}(G^\vee) \) is free over \( \mathcal{O}_S \) and there exists a basis of \( \text{Lie}(G^\vee) \) over \( \mathcal{O}_S \) under which \( \varphi_G \) is expressed by a matrix of the form
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & -a_1 \\
1 & 0 & \cdots & 0 & -a_2 \\
0 & 1 & \cdots & 0 & -a_3 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & -a_c
\end{pmatrix},
\]
where \( a_i \in \Gamma(S, \mathcal{O}_S) \) for \( 1 \leq i \leq c \).

**Lemma 4.3.** Let \( R \) be a local ring of characteristic \( p > 0 \), \( k \) be its residue field.

(i) A BT-group \( G \) over \( R \) is HW-cyclic if and only if so is \( G \otimes k \).

(ii) Let \( 0 \to G' \to G \to G'' \to 0 \) be an exact sequence of BT-groups over \( R \). If \( G \) is HW-cyclic, then so is \( G' \). In particular, if \( R \) is henselian, the connected part of a HW-cyclic BT-group over \( R \) is HW-cyclic.

**Proof.** (i) The property of being HW-cyclic is clearly stable under arbitrary base changes, so the “only if” part is clear. Assume that \( G_0 = G \otimes k \) is HW-cyclic. Let \( \overline{v} \) be an element of \( \text{Lie}(G_0^\vee) = \text{Lie}(G^\vee) \otimes k \) such that
Then by Nakayama’s lemma, $(v, \varphi_G(v), \cdots, \varphi_G^{-1}(v))$ is a basis of $\text{Lie}(G^\vee)$. Let $v$ be any lift of $\overline{v}$ in $\text{Lie}(G^\vee)$. Then by Nakayama’s lemma, $(v, \varphi_G(v), \cdots, \varphi_G^{-1}(v))$ is a basis of $\text{Lie}(G^\vee)$.

(ii) By statement (i), we may assume $R = k$. The exact sequence of BT-groups induces an exact sequence of Lie algebras

$$0 \to \text{Lie}(G^{\vee}) \to \text{Lie}(G^\vee) \to \text{Lie}(G^\vee) \to 0,$$

and the Hasse-Witt map $\varphi_G$ is induced by $\varphi_G$ by functoriality. Assume that $G$ is HW-cyclic and $G^\vee$ has dimension $c$. Let $u$ be an element of $\text{Lie}(G^\vee)$ such that

$$u, \varphi_G(u), \cdots, \varphi_G^{-1}(u)$$

form a basis of $\text{Lie}(G^\vee)$ over $k$. We denote by $u'$ the image of $u$ in $\text{Lie}(G^\vee)$. Let $r \leq c$ be the maximal integer such that the vectors

$$u', \varphi_G(u'), \cdots, \varphi_G^{-1}(u')$$

are linearly independent over $k$. It is easy to see that they form a basis of the $k$-vector space $\text{Lie}(G^\vee)$. Hence $G'$ is HW-cyclic.

**Lemma 4.4.** Let $S = \text{Spec}(R)$ be an affine scheme of characteristic $p > 0$, $G$ be a HW-cyclic BT-group over $R$ with $c = \dim(G^\vee)$ constant, and

$$\begin{pmatrix}
0 & 0 & \cdots & 0 & -a_1 \\
1 & 0 & \cdots & 0 & -a_2 \\
0 & 1 & \cdots & 0 & -a_3 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & -a_c
\end{pmatrix} \in M_{c \times c}(R),$$

be a matrix of $\varphi_G$. Put $a_{c+1} = 1$, and $P(X) = \sum_{i=0}^{c} a_{i+1}X^p \in R[X]$. (i) Let $V_G : G^{(p)} \to G$ be the Verschiebung homomorphism of $G$. Then $\text{Ker} V_G$ is isomorphic to the group scheme $\text{Spec}(R[X]/P(X))$ with comultiplication given by $X \mapsto 1 \otimes X + X \otimes 1$.

(ii) Let $x \in S$, and $G_x$ be the fibre of $G$ at $x$. Put

$$(4.4.1) \quad i_0(x) = \min_{0 \leq i \leq c} \{i; a_{i+1}(x) \neq 0\},$$

where $a_i(x)$ denotes the image of $a_i$ in the residue field of $x$. Then the étale part of $G_x$ has height $c - i_0(x)$, and the connected part of $G_x$ has height $d + i_0(x)$. In particular, $G_x$ is connected if and only if $a_i(x) = 0$ for $1 \leq i \leq c$.

**Proof.** (i) By 2.3 and 2.13, $\text{Ker} V_G$ is isomorphic to the group scheme

$$\text{Spec} \left( R[X_1, \ldots, X_c]/(X_1^p - X_2, \cdots, X_{c-1}^p - X_c, X_p^c - X_1, X_1 + \cdots + a_c X_c) \right)$$

with comultiplication $\Delta(X_i) = 1 \otimes X_i + X_i \otimes 1$ for $1 \leq i \leq c$. By sending $(X_1, X_2, \cdots, X_c) \to (X, X^p, \cdots, X^{p^{c-1}})$, we see that the above group scheme is isomorphic to $\text{Spec}(R[X]/P(X))$ with comultiplication $\Delta(X) = 1 \otimes X + X \otimes 1$. 

(ii) By base change, we may assume that $S = x = \text{Spec}(k)$ and hence $G = G_x$. Let $G(1)$ be the kernel of the multiplication by $p$ on $G$. Then we have an exact sequence

$$0 \to \text{Ker} F_G \to G(1) \to \text{Ker} V_G \to 0.$$  

Since $\text{Ker} F_G$ is an infinitesimal group scheme over $k$, we have $G(1)(\bar{k}) = (\text{Ker} V_G)(\bar{k})$, where $\bar{k}$ is an algebraic closure of $k$. By the definition of $i_0(x)$, we have $P(X) = Q(X_{p^{i_0(x)}})$, where $Q(X)$ is an additive separable polynomial in $k[X]$ with $\deg(Q) = p^{c-i_0(x)}$. Hence the roots of $P(X)$ in $\bar{k}$ form an $\mathbb{F}_p$-vector space of dimension $c - i_0(x)$. By (i), $(\text{Ker} V_G)(\bar{k})$ can be identified with the additive group consisting of the roots of $P(X)$ in $\bar{k}$. Therefore, the étale part of $G$ has height $c - i_0(x)$, and the connected part of $G$ has height $d + i_0(x)$. □

4.5. Let $k$ be a perfect field of characteristic $p > 0$, and $\alpha_p = \text{Spec}(k[X]/X^p)$ be the finite group scheme over $k$ with comultiplication map $\Delta(X) = 1 \otimes X + X \otimes 1$. Let $G$ be a BT-group over $k$. Following Oort, we call

$$a(G) = \dim_k \text{Hom}_{\text{fppf}}(\alpha_p, G)$$

the $a$-number of $G$, where $\text{Hom}_{\text{fppf}}$ means the homomorphisms in the category of abelian fppf-sheaves over $k$. Since the Frobenius of $\alpha_p$ vanishes, any morphism of $\alpha_p$ in $G$ factorize through $\text{Ker}(F_G)$. Therefore we have

$$\text{Hom}_{\text{fppf}}(\alpha_p, G) = \text{Hom}_{\text{fppf}}(\alpha_p, \text{Ker}(F_G))$$

$$= \text{Hom}_{\text{fppf}}(\alpha_p, \text{Ker}(F_G)^{\vee}, \alpha_p)$$

$$= \text{Hom}_{\text{fppf}}(\text{Lie}(\alpha_p), \text{Ker}(F_G))$$,

where $\text{Hom}_{\text{fppf}}(\alpha_p, G) = \text{Hom}_{\text{fppf}}(\alpha_p, \text{Ker}(F_G))$ denotes the homomorphisms in the category of commutative group schemes over $k$, and the last equality uses Proposition 2.3. Since we have a canonical isomorphism $\text{Lie}(\text{Ker}(F_G)) \simeq \text{Lie}(G)$ and $\text{Lie}(\alpha_p)$ has dimension one over $k$ with $\varphi_{\alpha_p} = 0$, we get

$$a(G) = \dim_k \{ x \in \text{Lie}(G) | \varphi_{G^\vee}(x) = 0 \} = \dim_k \text{Ker}(\varphi_{G^\vee}).$$

Due to the perfectness of $k$, we have also $a(G) = \dim_k \text{Ker}(\varphi_{G^{\vee}})$, where $\varphi_{G^{\vee}}$ is the linearization of $\varphi_{G^\vee}$. By Proposition 2.11, we see that $a(G) = 0$ if and only if $G$ is ordinary.

**Lemma 4.6.** Let $G$ be a BT-group over $k$, and $G^\vee$ its Serre dual. Then we have $a(G) = a(G^\vee)$.

**Proof.** Let $\psi_G : \omega_G \to \omega_G^{(p)}$ be the $k$-linear map induced by the Verschiebung of $G$. Then $\psi_G^{\vee}$, the morphism obtained by applying the functor $\text{Hom}_k(\_, k)$ to $\psi_G$, is identified with $\varphi_{G^{\vee}}$. By (4.5.1) and the exactitude of the functor $\text{Hom}_k(\_, k)$, we have $a(G) = \dim_k \text{Ker}(\psi_G) = \dim_k \text{Coker}(\psi_G)$. Using the additivity of $\dim_k$, we get finally $a(G) = \dim_k \text{Ker}(\psi_G)$. By considering the commutative diagram (3.1.3), we have

$$a(G) = \dim_k \left( \omega_G \cap \psi_G(\text{Lie}(G^{\vee})^{(p)}) \right).$$
On the other hand, it follows also from (3.1.3) that
\[ a(G^\vee) = \dim_k \ker(\tilde{\varphi}_G) = \dim_k \left( \phi_G(\text{Lie}(G^\vee)^{(p)}) \cap \omega_G \right). \]

The lemma now follows immediately. \(\square\)

**Proposition 4.7.** Let \(k\) be a perfect field of characteristic \(p > 0\), \(G\) a BT-group over \(k\). Consider the following conditions:

(i) \(G\) is HW-cyclic and non-ordinary;
(ii) the connected part \(G^o\) of \(G\) is HW-cyclic and not of multiplicative type;
(iii) \(a(G^\vee) = a(G) = 1\).

We have (i) \(\Rightarrow\) (ii) \(\Leftrightarrow\) (iii). If \(k\) is algebraically closed, we have moreover (ii) \(\Rightarrow\) (i).

**Remark 4.8.** In [Oo1, Lemma 2.2], Oort proved the following assertion, which is a generalization of (iii) \(\Rightarrow\) (ii): Let \(k\) be an algebraically closed field of characteristic \(p > 0\), and \(G\) be a connected BT-group with \(a(G) = 1\). Then there exists a basis of the Dieudonné module \(M\) of \(G\) over \(W(k)\), such that the action of Frobenius on \(M\) is given by a display-matrix of “normal form” in the sense of [Oo1, 2.1].

**Proof.** (i) \(\Rightarrow\) (ii) follows from 4.3(ii).
(ii) \(\Rightarrow\) (iii). First, we note that \(a(G) = a(G^o)\), so we may assume \(G\) connected. Since \(G\) is not of multiplicative type, we have \(c = \dim(G^\vee) \geq 1\). By Lemma 4.4(ii), there exists a basis of \(\text{Lie}(G^\vee)\) over \(k\) under which \(\varphi_G\) is expressed by
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix} \in M_{c \times c}(k).
\]

According to (4.5.1), \(a(G^\vee)\) equals to \(\dim_k \ker(\varphi_G)\), i.e. the \(k\)-dimension of the solutions of the equation system in \((x_1, \cdots, x_c)\)
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x_1^p \\
x_2^p \\
\vdots \\
x_c^p
\end{pmatrix} = 0
\]

The solutions \((x_1, \cdots, x_c)\) form clearly a vector space over \(k\) of dimension 1, i.e. we have \(a(G^\vee) = 1\).

(iii) \(\Rightarrow\) (ii). Let \(G^\text{ét}\) be the étale part of \(G\). Since \(k\) is perfect, the exact sequence (2.7.1) splits [Dem, Chap. II §7]; so we have \(G \simeq G^o \times G^\text{ét}\). We put \(M = \text{Lie}(G^\vee)\), \(M_1 = \text{Lie}(G^o^\vee)\) and \(M_2 = \text{Lie}(G^\text{ét}^\vee)\) for short. By 2.8 and 2.9, we have a decomposition \(M = M_1 \oplus M_2\), such that \(M_1, M_2\) are stable under \(\varphi_G\), and the action of \(\varphi_G\) is nilpotent on \(M_1\) and bijective on \(M_2\). We note
We consider the Dieudonné module of multiplicative type, hence a is not of multiplicative type, hence that a = 300. Yichao Tian

Since \( \dim \ker(G) = 1 \), we prove that \( G \) is HW-cyclic. Let \( \lambda \) be determined later. Then we have \( u = \lambda_1 e_1 + \cdots + \lambda_m e_m \), where \( \lambda_i(1 \leq i \leq m) \) are some elements in \( k \) to be determined later. Then we have

\[
\begin{pmatrix}
\varphi_G^n(u)
\vdots
\varphi_G^{n+m-1}(u)
\end{pmatrix} =
\begin{pmatrix}
\lambda_1^n & \cdots & \lambda_m^n \\
\vdots & \ddots & \vdots \\
\lambda_1^{n+m-1} & \cdots & \lambda_m^{n+m-1}
\end{pmatrix}
\begin{pmatrix}
e_1 \\
\vdots \\
e_m
\end{pmatrix}.
\]

Let \( L(\lambda_1, \cdots, \lambda_m) \in k[\lambda_1, \cdots, \lambda_m] \) be the determinant polynomial of the matrix on the right side. An elementary computation shows that the polynomial \( L(\lambda_1, \cdots, \lambda_m) \) is not null. We can choose \( \lambda_1, \cdots, \lambda_m \in k \) such that \( L(\lambda_1, \cdots, \lambda_m) \neq 0 \) because \( k \) is algebraically closed. So \( \varphi_G^n(u), \cdots, \varphi_G^{n+m-1}(u) \) form a basis of \( M_2 \) over \( k \). Since

\[
\varphi_G^i(u) \equiv \varphi_G^i(v) \mod M_2 \quad \text{for} \quad 0 \leq i \leq n,
\]

by the choice of \( u \), we see that \( \{u, \varphi_G(u), \cdots, \varphi_G^{n+m-1}(u)\} \) form a basis of \( M = \text{Lie}(G^\vee) \) over \( k \).

By combining 4.6 and 4.7, we obtain the following

**Corollary 4.9.** Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Then a BT-group over \( k \) is HW-cyclic if and only if so is its Serre dual.

---

**4.10. Examples.** Let \( k \) be a perfect field, \( W(k) \) be the ring of Witt vectors with coefficients in \( k \), and \( \sigma \) be the Frobenius automorphism of \( W(k) \). Let \( s, r \) be relatively prime integers such that \( 0 \leq s \leq r \) and \( r \neq 0 \); put \( \lambda = \frac{r}{s} \). We consider the Dieudonné module \( M^\lambda \simeq W(k)[F, V]/(F^{r-s} - V^s) \), where \( W(k)[F, V] \) is the non-commutative ring with relations \( FV = VF = p, Fa = \sigma(a)F \) and \( V\sigma(a) = aV \) for all \( a \in W(k) \). We note that \( M^\lambda \) is free of rank
where over $\text{theory}$, $M_r^{\overline{\Gamma}}(\text{BT-group of slope } \lambda)$ be a matrix of $(i)$ be a lift of $(ii)$ Assume that $k$ algebraically closed. Then by the Dieudonné-Manin’s classification of isocrystals [Dem, Chap.IV §4], any BT-group over $k$ is isogenous to a finite product of $G^\lambda$’s; moreover, any connected one-dimensional BT-group over $k$ of height $r$ is necessarily isomorphic to $G^{1/r}$ [Dem, Chap.IV §8], hence in particular HW-cyclic.

**Proposition 4.11.** Let $k$ be an algebraically closed field of characteristic $p > 0$, $R$ be a noetherian complete regular local $k$-algebra with residue field $k$, and $S = \text{Spec}(R)$. Let $G$ be a connected HW-cyclic BT-group over $R$ of dimension $d \geq 1$ and height $c + d$,

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & -a_1 \\
1 & 0 & \cdots & 0 & -a_2 \\
0 & 1 & \cdots & 0 & -a_3 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & -a_c
\end{pmatrix} \in M_{c \times c}(R)
$$

be a matrix of $\varphi_G$.

(i) If $G$ is versal over $S$, then $\{a_1, \cdots, a_c\}$ is a subset of a regular system of parameters of $R$.

(ii) Assume that $d = 1$. The converse of (i) is also true, i.e. if $\{a_1, \cdots, a_c\}$ is a subset of a regular system of parameters of $R$ then $G$ is versal over $S$. Furthermore, $G$ is the universal deformation of its special fiber if and only if $\{a_1, \cdots, a_c\}$ is a system of regular parameters of $R$.

**Proof.** Let $(M(G), F_M, \nabla)$ be the finite free $\mathcal{O}_S$-module equipped with a semi-linear endomorphism $F_M$ and a connection $\nabla : M(G) \rightarrow M(G) \otimes_{\mathcal{O}_S} \Omega^1_S/k$, obtained by evaluating the Dieudonné crystal of $G$ at the trivial immersion $S \hookrightarrow S$ (cf. 3.1). Recall that we have a commutative diagram

$$
\begin{array}{ccc}
M(G)^{(p)} & \xrightarrow{F_M} & M(G) \\
pr \downarrow & & \downarrow pr \\
\text{Lie}(G'^{(p)}) & \xrightarrow{\varphi_{G'}} & \text{ Lie}(G')
\end{array}
$$

where $\phi_G$ is universally injective (3.1.3). Let $\{v_1, \cdots, v_c\}$ be a basis of $\text{Lie}(G')$ over $\mathcal{O}_S$ under which $\varphi_G$ is expressed by $\mathfrak{h}$, i.e. we have $\varphi_G^{-1}(v_i) = v_i$ for $1 \leq i \leq c$ and $\varphi_G(v_i) = -\sum_{i=1}^c a_i v_i$. Let $f_i$ be a lift of $v_i$ to $\Gamma(S, M(G))$, and put $f_{i+1} = \phi_G(v_i^{(p)})$ for $1 \leq i \leq c - 1$, where $v_i^{(p)} = 1 \otimes v_i \in \Gamma(S, \text{Lie}(G'^{(p)}))$. The image of $f_i$ in $\Gamma(S, \text{Lie}(G'^{(p)}))$ is thus $v_i$ for $1 \leq i \leq c$ by
We have
\[ e_1 = \phi_G(v_c^{(p)}) + a_1 f_1 + \cdots + a_c f_c \in \Gamma(S, M(G)). \]

The image of \( e_1 \) in \( \Gamma(S, \text{Lie}(G'^v)) \) is \( \varphi_G(v_c) + \sum_{i=1}^c a_i v_i = 0 \); so we have \( e_1 \in \Gamma(S, \omega_G) \). By (4.11.2), we notice that \( a_1, \ldots, a_c \) belong to the maximal ideal \( \mathfrak{m}_R \) of \( R \), as \( G \) is connected. Hence, we have \( \overline{e_1} = \phi_G(v_c^{(p)}) \), where for a \( R \)-module \( M \) and \( x \in M \), we denote by \( \varphi \) the canonical image of \( x \) in \( M \otimes k \). Since \( \varphi_G \) commutes with base change and is universally injective, we get \( \overline{e_1} = \phi_G(v_c^{(p)}) = \phi_G \otimes_k (v_c^{(p)}) \neq 0 \). Therefore, we can choose \( e_2, \ldots, e_d \in \Gamma(S, \omega_G) \) such that \( (e_1, \ldots, e_d) \) becomes a basis of \( \omega_G \) over \( \mathcal{O}_S \), so \( (e_1, \ldots, e_d, f_1, \ldots, f_c) \) is a basis of \( M(G) \). Since \( F_M \) is horizontal for the connection \( \nabla \) (cf. 3.1(ii)), we have
\[ \nabla(\phi_G(v_c^{(p)})) = \nabla(F_M(f_c^{(p)})) = 0. \]

In view of (4.11.2), we get
\[ \nabla(e_1) = \sum_{i=1}^c f_i \otimes da_i + \sum_{i=1}^c a_i \nabla(f_i) \]
\[ = \sum_{i=1}^c f_i \otimes da_i \mod \mathfrak{m}_R. \]

Let \( \text{Kod}_0 \) and \( \text{Kod}_0 \) be respectively the reductions modulo \( \mathfrak{m}_R \) of (3.2.1) and (3.2.2). Since \( (\overline{v}_i)_{1 \leq i \leq c} \) is a base of \( \text{Lie}(G'^v) \otimes k \), we can write
\[ \text{Kod}_0(e_j) = \sum_{i=1}^d \overline{v}_i \otimes \theta_{i,j} \quad \text{for} \ 1 \leq j \leq d, \]
where \( \theta_{i,j} \in \Omega_{S/k} \otimes k \). From (4.11.3), we deduce that \( \theta_{i,1} = da_i \). By the definition of \( \text{Kod}_0 \), we have
\[ \text{Kod}_0(\partial) = \sum_{j=1}^d \sum_{i=1}^c < \partial, \theta_{i,j} > \overline{v}_j^* \otimes \overline{v}_i \]
where \( \partial \in \mathcal{I}_{S/k} \otimes k, < \bullet, \bullet > \) is the canonical pairing between \( \mathcal{I}_{S/k} \otimes k \) and \( \Omega_{S/k}^1 \otimes k \), and \( (\overline{v}_i^*)_{1 \leq i \leq d} \) denotes the dual basis of \( (\overline{v}_i)_{1 \leq i \leq d} \). Now assume that \( G \) is versal over \( S \), \( i.e. \) \( \text{Kod}_0 \) is surjective by definition (3.2). In particular, there are \( \partial_1, \ldots, \partial_c \in \mathcal{I}_{S/k} \otimes k \) such that \( \text{Kod}_0(\partial_i) = \overline{v}_i^* \otimes v_i \) for \( 1 \leq i \leq c \), \( i.e. \) we have
\[ < \partial, da_j > = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for } 1 \leq i, j \leq c, \]
and
\[ < \partial, \theta_{j,\ell} > = 0 \quad \text{for } 1 \leq i, j \leq c, 2 \leq \ell \leq d. \]

From (4.11.5), we see easily that \( da_1, \ldots, da_c \) are linearly independent in \( \Omega_{S/k}^1 \otimes k \approx \mathfrak{m}_R/\mathfrak{m}_R^2 \); therefore, \( (a_1, \ldots, a_c) \) is a part of a regular system of parameters of \( R \). Statement (i) is proved.
For statement (ii), we assume $d = 1$ and that $(a_1, \ldots, a_c)$ is a part of a regular system of parameters of $R$. Then the formula (4.11.4) is simplified as
\[
\text{Kod}_0(\partial) = \sum_{i=1}^{c} <\partial, da_i > \pi^* \otimes \pi_i.
\]
Since $da_1, \ldots, da_c$ are linearly independent in $\Omega^1_{S/k} \otimes k$, there exist $\partial_1, \ldots, \partial_c \in \mathcal{I}_{S/k} \otimes k$ such that (4.11.5) holds, i.e. $(\pi^* \otimes \pi_i)_{1 \leq i \leq c}$ are in the image of Kod$_0$. But the elements $(\pi^* \otimes \pi_i)_{1 \leq i \leq c}$ form already a basis of $\mathcal{H}om_{\mathcal{O}_S}(\omega_G, \text{Lie}(G^0)) \otimes k$. So Kod$_0$ is surjective, and hence $G$ is versal over $S$ by Nakayama’s lemma.

Let $G_0$ be the special fiber of $G$. It remains to prove that when $d = 1$, $G$ is the universal deformation of $G_0$ if and only if $\dim(S) = c$ and $G$ is versal over $S$.

Let $S$ be the local moduli in characteristic $p$ of $G_0$. By the universal property of $G$ (3.7), there exists a unique morphism $f : S \to S$ such that $G \simeq G \times_S S$.

Since $S$ and $S$ are local complete regular schemes over $k$ with residue field $k$ of the same dimension, $f$ is an isomorphism if and only if the tangent map of $f$ at the closed point of $S$, denoted by $T_f$, is an isomorphism. By the functoriality of Kodaira-Spencer maps (3.2.2), we have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{I}_{S/k} \otimes \mathcal{O}_S & \xrightarrow{\text{Kod}_0^k} & \text{Hom}_k(\omega_{G_0}, \text{Lie}(G_0)) \\
\downarrow T_f & & \downarrow \\
\mathcal{I}_{S/k} \otimes \mathcal{O}_S & \xrightarrow{\text{Kod}_0^S} & \text{Hom}_k(\omega_{G_0}, \text{Lie}(G_0))
\end{array}
\]
where horizontal arrows are the Kodaira-Spencer maps evaluated at the closed points (3.3.1). Since Kod$_0^k$ and Kod$_0^S$ are isomorphisms according to the first part of this proposition, we deduce that so is $T_f$. This completes the proof. □

5. **Monodromy of a HW-cyclic BT-group over a Complete Trait of Characteristic $p > 0$**

5.1. Let $k$ be an algebraically closed field of characteristic $p > 0$, $A$ be a complete discrete valuation ring of characteristic $p$, with residue field $k$ and fraction field $K$. We put $S = \text{Spec}(A)$, and denote by $s$ its closed point, by $\eta$ its generic point. Let $\overline{K}$ be an algebraic closure of $K$, $K^{\text{sep}}$ be the maximal separable extension of $K$ contained in $\overline{K}$, $K^t$ be the maximal tamely ramified extension of $K$ contained in $K^{\text{sep}}$. We put $I = \text{Gal}(K^{\text{sep}}/K)$, $I_p = \text{Gal}(K^{\text{sep}}/K^t)$ and $I_t = I/I_p = \text{Gal}(K^t/K)$.

Let $\pi$ be a uniformizer of $A$; so we have $A \simeq k[[\pi]]$. Let $v$ be the valuation on $K$ normalized by $v(\pi) = 1$; we denote also by $v$ the unique extension of $v$ to $\overline{K}$. For every $\alpha \in \mathbb{Q}$, we denote by $m_\alpha$ (resp. by $m_\alpha^+$) the set of elements $x \in K^{\text{sep}}$ such that $v(x) \geq \alpha$ (resp. $v(x) > \alpha$). We put
\[
(5.1.1) \quad V_\alpha = m_\alpha/m_\alpha^+,
\]
which is a $k$-vector space of dimension 1 equipped with a continuous action of the Galois group $I$.
5.2. First, we recall some properties of the inertia groups $I_p$ and $I_t$ [Se1, Chap. IV]. The subgroup $I_p$, called the wild inertia subgroup, is the unique maximal pro-$p$-group contained in $I$ and hence normal in $I$. The quotient $I_t = I/I_p$ is a commutative profinite group, called the tame inertia group. We have a canonical isomorphism
\[
\theta : I_t \overset{\sim}{\rightarrow} \lim_{(d,p)=1} \mu_d,
\]
where the projective system is taken over positive integers prime to $p$, $\mu_d$ is the group of $d$-th roots of unity in $k$, and the transition maps $\mu_m \to \mu_d$ are given by $\zeta \mapsto \zeta^{m/d}$, whenever $d$ divides $m$. We denote by $\theta_d : I_t \to \mu_d$ the projection induced by (5.2.1). Let $q$ be a power of $p$, $\mathbb{F}_q$ be the finite subfield of $k$ with $q$ elements. Then $\mu_{q-1} = \mathbb{F}_q^\times$, and we can write $\theta_{q-1} : I_t \to \mathbb{F}_q^\times$. The character $\theta_q$ is characterized by the following property.

**Proposition 5.3** ([Se3] Prop.7). Let $a, d$ be relatively prime positive integers with $d$ prime to $p$. Then the natural action of $I_p$ on the $k$-vector space $V_a/d$ (5.1.1) is trivial, and the induced action of $I_t$ on $V_a/d$ is given by the character $(\theta_d)^a : I_t \to \mu_d$. In particular, if $q$ is a power of $p$, the action of $I_t$ on $V_1/(q-1)$ is given by the character $\theta_{q-1} : I_t \to \mathbb{F}_q^\times$ and any $I$-equivariant $\mathbb{F}_p$-subspace of $V_1/(q-1)$ is an $\mathbb{F}_q$-vector space.

5.4. Let $G$ be a BT-group over $S$. We define $h(G)$ to be the valuation of the determinant of a matrix of $\varphi_G$ if $\dim(G^{\vee}) \geq 1$, and $h(G) = 0$ if $\dim(G^{\vee}) = 0$. We call $h(G)$ the Hasse invariant of $G$.

(a) $h(G)$ does not depend on the choice of the matrix representing $\varphi_G$. Indeed, let $c$ be the rank of $\text{Lie}(G^{\vee})$ over $A$, $\mathfrak{g} \in \text{Mat}_{c\times c}(A)$ be a matrix of $\varphi_G$. Any other matrix representing $\varphi_G$ can be written in the form $U^{-1} \cdot A \cdot U^{\{p\}}$, where $U \in \text{GL}_c(A)$, $U^{-1}$ is the inverse of $U$, and $U^{\{p\}}$ is the matrix obtained by applying the Frobenius map of $A$ to the coefficients of $U$.

(b) By 2.11, the generic fiber $G_n$ is ordinary if and only if $h(G) < \infty$; $G$ is ordinary over $T$ if and only $h(G) = 0$.

(c) Let $0 \to G' \to G \to G'' \to 0$ be a short exact sequence of BT-groups over $T$, then we have $h(G) = h(G') + h(G'')$. Indeed, the exact sequence of BT-groups induces a short exact sequence of Lie algebras (cf. [BBM] 3.3.2)
\[
0 \to \text{Lie}(G^{''\vee}) \to \text{Lie}(G^{\vee}) \to \text{Lie}(G^{''\vee}) \to 0,
\]
from which our assertion follows easily.

**Proposition 5.5.** Let $G$ be a BT-group over $S$. Then we have $h(G) = h(G^{\vee})$.

**Proof.** The proof is very similar to that of Lemma 4.6. First, we have
\[
h(G) = \text{leng}(\text{Lie}(G^{\vee})/(\varphi_G(\text{Lie}(G^{\vee})^{\{p\}})),
\]
where $\varphi_G$ is the linearization of $\varphi_G$, and “leng” means the length of a finite $A$-module (note that this formulae holds even if $\dim(G^{\vee}) = 0$). By the commutative diagram (3.1.3), we have
\[
h(G) = \text{leng} M(G)/(\phi_G(\text{Lie}(G^{\vee})^{\{p\}}) + \omega_G).
\]
On the other hand, by applying the functor $\text{Hom}_A(\_ , A)$ to the $A$-linear map $\tilde{\varphi}_{G^\vee} : \text{Lie}(G)^{(p)} \to \text{Lie}(G)$, we obtain a map $\psi_G : \omega_G \to \omega_G^{(p)}$. If $U$ is a matrix of $\tilde{\varphi}_{G^\vee}$, then the transpose of $U$, denoted by $U^t$, is a matrix of $\psi_G$. So we have

$$h(G^\vee) = v(\det(U)) = v(\det(U^t)) = \text{leng}(\omega_G^{(p)} / \psi_G(\omega_G)).$$

By diagram 3.1.3, we get

$$h(G^\vee) = \text{leng} M(G) / (\phi_G(\text{Lie}(G^\vee)^{(p)}) + \omega_G) = h(G).$$

5.6. Let $G$ be a BT-group over $S$, $c = \dim(G^\vee)$. We put

$$T_p(G) = \lim_{\longleftarrow} G(n)(K)$$

the Tate module of $G$, where $G(n)$ is the kernel of $p^n : G \to G$. It is a free $\mathbb{Z}_p$-module of rank $\leq c$, and the equality holds if and only if the generic fiber $G_\eta$ is ordinary. The Galois group $I$ acts continuously on $T_p(G)$. We are interested in the image of the monodromy representation

$$(5.6.2)\quad \rho : I = \text{Gal}(K^{\text{sep}}/K) \to \text{Aut}_{\mathbb{Z}_p}(T_p(G)).$$

We denote by

$$(5.6.3)\quad \overline{\rho} : I = \text{Gal}(K^{\text{sep}}/K) \to \text{Aut}_{\mathbb{F}_p}(G(1)(\overline{K}))$$

its reduction mod $p$.

**Theorem 5.7 (Reformulation of Igusa’s theorem).** Let $G$ be a connected BT-group over $S$ of height 2 and dimension 1. Then $G$ is versal (3.2) if and only if $h(G) = 1$; moreover, if this condition is satisfied, the monodromy representation $\rho : I \to \text{Aut}_{\mathbb{Z}_p}(T_p(G)) \simeq \mathbb{Z}_p^\times$ is surjective.

**Proof.** Since $\text{Lie}(G^\vee)$ is a $\mathcal{O}_S$-module free of rank 1, the condition that $h(G) = 1$ is equivalent to that any matrix of $\varphi_G$ is represented by a uniformizer of $A$. Hence the first part of this theorem follows from Proposition 4.11(ii).

We follow [Ka2, Thm 4.3] to prove the surjectivity of $\rho$ under the assumption that $h(G) = 1$. For each integer $n \geq 1$, let

$$\rho_n : I \to \text{Aut}_{\mathbb{Z}/p^n\mathbb{Z}}(G(n)(\overline{K})) \simeq (\mathbb{Z}/p^n\mathbb{Z})^\times$$

be the reduction mod $p^n$ of $\rho$, $K_n$ be the subfield of $K^{\text{sep}}$ fixed by the kernel of $\rho_n$. Then $\rho_n$ induces an injective homomorphism $\text{Gal}(K_n/K) \to (\mathbb{Z}/p^n\mathbb{Z})^\times$. By taking projective limits, we are reduced to proving the surjectivity of $\rho_n$ for every $n \geq 1$. It suffices to verify that

$$|\text{Im}(\rho_n)| = [K_n : K] \geq p^{n-1}(p - 1)$$

(then the equality holds automatically).
We regard $G$ as a formal group over $S$. Then by [Ka2, 3.6], there exists a parameter $X$ of the formal group $G$ normalized by the condition that $[\xi](X) = \xi(X)$ for all $(p - 1)$-th root of unity $\xi \in \mathbb{Z}_p$. For such a parameter, we have

$$[p](X) = a_1X^p + aX^{p^2} + \sum_{m \geq 2} c_mX^{p^m + (p - 1)} \in A[[X]],$$

where we have $v(a) = h(G) = 1$ by [Ka2, 3.6.1 and 3.6.5], and $v(\alpha) = 0$, as $G$ is of height 2. For each integer $i \geq 0$, we put

$$V^{(p^i)}(X) = a_1^{p^i}X + a^{p^i}X^p + \sum_{m \geq 2} c_m^{p^i}X^{p^m + (p - 1)} \in A[[X]].$$

then we have $[p^n](X) = V^{(p^{n-1})} \circ V^{(p^{n-2})} \circ \cdots \circ V(X^{p^n})$. Hence each point of $G(n)(\overline{K})$ is given by a sequence $y_1, \ldots, y_n \in K^{sep}$ (or simply an element $y_n \in K^{sep}$) satisfying the equations

\[
\begin{align*}
V(y_1) &= a_1y_1 + \alpha y_1^p + \cdots = 0; \\
V^{(p)}(y_2) &= a_1^{p}y_2 + \alpha^{p}y_1^{p} + \cdots = y_1; \\
& \vdots \\
V^{(p^{n-1})}(y_n) &= a_1^{p^{n-1}}y_n + \alpha^{p^{n-1}}y_1^{p^{n-1}} + \cdots = y_{n-1}.
\end{align*}
\]

Let $y_n \in K^{sep}$ be such that $y_1 \neq 0$. By considering the Newton polygons of the equations above, we verify that

$$v(y_i) = \frac{1}{p^{i-1}(p-1)} \quad \text{for } 1 \leq i \leq n.$$

In particular, the ramification index $e(K_n/K)$ is at least $p^{n-1}(p - 1)$. By the definition of $K_n$, the Galois group $\text{Gal}(K^{sep}/K_n)$ must fix $y_n \in K^{sep}$, i.e. $K_n$ is an extension of $K(y_n)$. Therefore, we have $[K_n : K] \geq [K(y_n) : K] \geq e(K(y_n)/K) \geq p^{n-1}(p - 1)$.

**Proposition 5.8.** Let $G$ be a HW-cyclic BT-group over $S$ of height $c + d$ and dimension $d$ such that $G \otimes K$ is ordinary,

$$\mathfrak{g} = \begin{pmatrix}
0 & 0 & \cdots & 0 & -a_1 \\
1 & 0 & \cdots & 0 & -a_2 \\
0 & 1 & \cdots & 0 & -a_3 \\
& & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -a_c
\end{pmatrix}$$

be a matrix of $\varphi_G$. Put $q = p^c$, $a_{c+1} = 1$, and $P(X) = \sum_{i=0}^{c} a_{i+1}X^p^i \in A[X]$.

(i) Assume that $G$ is connected and the Hasse invariant $h(G) = 1$. Then the representation $\overline{p}$ (5.6.3) is tame, $G(1)(\overline{K})$ is endowed with the structure of an $\mathbb{F}_q$-vector space of dimension 1, and the induced action of $I_t$ is given by the character $\theta_{q-1} : I_t \to \mathbb{F}_q^\times$.

(ii) Assume that $c > 1$, $v(a_i) \geq 2$ for $1 \leq i \leq c - 1$ and $v(a_c) = 1$. Then the order of $\text{Im}(\overline{p})$ is divisible by $p^{c-1}(p - 1)$. 

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*Yichao Tian*

**306**

*Documenta Mathematica* 14 (2009) 281–324
(iii) Put \( i_0 = \min_{0 \leq i \leq c} \{ i; \nu(a_{i+1}) = 0 \} \). Assume that there exists \( \alpha \in k \) such that \( \nu(P(\alpha)) = 1 \). Then we have \( i_0 \leq c - 1 \) and the order of \( \text{Im}(\overline{\mathfrak{p}}) \) is divisible by \( p^{i_0} \).

**Proof.** Since \( G \) is generically ordinary, we have \( a_1 \neq 0 \) by 2.11(d). Hence \( P(X) \in K[X] \) is a separable polynomial. By 4.4, \( G(1)(\overline{K}) \simeq (\text{Ker} V_G)(K^{\text{sep}}) \) is identified with the additive group consisting of the roots of \( P(X) \) in \( K^{\text{sep}} \).

(i) By definition of the Hasse invariant, we have \( \nu(a_1) = h(G) = 1 \). By 4.4(ii), the assumption that \( G \) is connected is equivalent to saying \( \nu(a_i) \geq 1 \) for \( 1 \leq i \leq c \). From the Newton polygon of \( P(X) \), we deduce that all the non-zero roots of \( P(X) \) in \( K^{\text{sep}} \) have the same valuation \( 1/(q-1) \). We denote by

\[
\psi : G(1)(\overline{K}) \rightarrow V_{1/(q-1)}
\]

the map which sends each root \( x \in K^{\text{sep}} \) of \( P(X) \) to the class of \( x \) in \( V_{1/(q-1)} = m_1/(q-1)/m_1/(q-1) \) (5.1.1). We remark that \( G(1)(\overline{K}) \) is an \( \mathbb{F}_q \)-vector space of dimension \( c \). Hence \( G(1)(\overline{K}) \) is automatically of dimension 1 over \( \mathbb{F}_q \) once we know it is an \( \mathbb{F}_q \)-vector space. By 5.3, it suffices to show that \( \psi \) is an injective \( I \)-equivariant homomorphism of groups. By 4.4(i), \( \psi \) is obviously an \( I \)-equivariant homomorphism of groups. Let \( x_0 \) be a root of \( P(X) \), and put \( Q(y) = P(x_0y) \). Then the polynomial \( Q(y) \) has the form \( Q(y) = x_0^q Q_1(y) \), where

\[
Q_1(y) = y^q + b_1 y^{q-1} + \cdots + b_2 y^2 + b_1 y
\]

with \( b_i = a_i/x_0^{(q-p^{i-1})} \in K^{\text{sep}} \). We have \( \nu(b_i) > 0 \) for \( 2 \leq i \leq c \) and \( \nu(b_1) = 0 \).

Let \( \overline{b}_1 \) be the class of \( b_1 \) in the residue field \( k = \mathfrak{m}_0/\mathfrak{m}_0^2 \). Then the images of the roots of \( P(X) \) in \( V_{1/(q-1)} \) are \( x_0 \overline{b}_1^{1/(q-1)} \zeta \), where \( \zeta \) runs over the finite field \( \mathbb{F}_q \). Therefore, \( \psi \) is injective.

(ii) By computing the slopes of the Newton polygon of \( P(X) \), we see that \( P(X) \) has \( p^{c-1}(p-1) \) roots of valuation \( 1/(p^c - p^{c-1}) \). Let \( L \) be the sub-extension of \( K^{\text{sep}} \) obtained by adding to \( K \) all the roots of \( P(x) \). Then the ramification index \( e(L/K) \) is divisible by \( p^{c-1}(p-1) \). Let \( \overline{L} \) be the sub-extension of \( K^{\text{sep}} \) fixed by the kernel of \( \overline{\mathfrak{p}} \) (5.6.3). The Galois group \( \text{Gal}(K^{\text{sep}}/\overline{L}) \) fixes the roots of \( P(x) \) by definition. Hence we have \( L \subset \overline{L} \), and \( |\text{Im}(\overline{\mathfrak{p}})| = [\overline{L} : K] \) is divisible by \( [L : K] \); in particular, it is divisible by \( p^{c-1}(p-1) \).

(iii) Note that the relation \( i_0 \leq c - 1 \) is equivalent to saying that \( G \) is not connected by 4.4(ii). Assume conversely \( i_0 = c, i.e. \ G \) is connected. Then we would have

\[
P(X) \equiv X^q \mod (\pi A[X]).
\]

But \( \nu(P(\alpha)) = 1 \) implies that \( \alpha^{p^c} \in \pi A \), i.e. \( \alpha = 0 \); hence we would have \( P(\alpha) = 0 \), which contradicts the condition \( \nu(P(\alpha)) = 1 \).

We put \( Q(X) = P(X + \alpha) = P(X) + P(\alpha) \). As \( \nu(P(\alpha)) = 1 \), then \( (0, 1) \) and \( (p^{i_0}, 0) \) are the first two break points of the Newton polygon of \( Q(X) \). Hence there exists \( p^{i_0} \) roots of \( Q(X) \) of valuation \( 1/p^{i_0} \). Let \( L \) be the subextension of \( K \) in \( K^{\text{sep}} \) generated by the roots of \( P(X) \). The ramification index \( e(L/K) \) is divisible by \( p^{i_0} \). As in the proof of (ii), if \( \overline{L} \) is the subextension of \( K^{\text{sep}} \),
fixed by the kernel of \( \overline{\rho} \), then it is an extension of \( L \). Therefore, we have \( |\text{Im}(\overline{\rho})| = |\overline{L} : K| \) is divisible by \( |L : K| \), and in particular, divisible by \( p^6 \). □

5.9. Let \( G \) be a BT-group over \( S \) with connected part \( G^c \), and étale part \( G^\text{ét} \) of height \( r \). We have a canonical exact sequence of \( I \)-modules
\[
0 \to G^c(1)(\overline{K}) \to G(1)(\overline{K}) \to G^\text{ét}(1)(\overline{K}) \to 0
\]
giving rise to a class \( \overline{C} \in \text{Ext}^1_{\mathbb{F}_p[I]}(G^\text{ét}(1)(\overline{K}), G^c(1)(\overline{K})) \), which vanishes if and only if (5.9.1) splits. Since \( I \) acts trivially on \( G^\text{ét}(1)(\overline{K}) \), we have an isomorphism of \( I \)-modules \( G^\text{ét}(1)(\overline{K}) \cong \mathbb{F}_p^r \). Recall that for any \( \mathbb{F}_p[I] \)-module \( M \), we have a canonical isomorphism (Se1 Chap.VII, §2)
\[
\text{Ext}^1_{\mathbb{F}_p[I]}(\mathbb{F}_p, M) \cong H^1(I, M).
\]
Hence we deduce that
\[
\overline{C} \in \text{Ext}^1_{\mathbb{F}_p[I]}(G^\text{ét}(1)(\overline{K}), G^c(1)(\overline{K})) \cong H^1(I, G^c(1)(\overline{K}))^r.
\]

**Proposition 5.10.** Let \( G \) be a HW-cyclic BT-group over \( S \) such that \( h(G) = 1 \), \( \overline{\rho} \) (5.6.3) be the representation of \( I \) on \( G(1)(\overline{K}) \). Then the cohomology class \( \overline{C} \) does not vanish if and only if the order of the group \( \text{Im}(\overline{\rho}) \) is divisible by \( p \).

First, we prove the following result on cohomology of groups.

**Lemma 5.11.** Let \( F \) be a field, \( \Gamma \) be a commutative group, and \( \chi : \Gamma \to F^\times \) be a non-trivial character of \( \Gamma \). We denote by \( F(\chi) \) an \( F \)-vector space of dimension 1 endowed with an action of \( \Gamma \) given by \( \chi \). Then we have \( H^1(\Gamma, F(\chi)) = 0 \).

**Proof.** Let \( C \) be a 1-cocycle of \( \Gamma \) with values in \( F(\chi) \). We prove that \( C \) is a 1-coboundary. For any \( g, h \in \Gamma \), we have
\[
C(gh) = C(g) + \chi(g)C(h),
\]
\[
C(hg) = C(h) + \chi(h)C(g).
\]
Since \( \Gamma \) is commutative, it follows from the relation \( C(gh) = C(hg) \) that
\[
(\chi(g) - 1)C(h) = (\chi(h) - 1)C(g).
\]
If \( \chi(g) \neq 1 \) and \( \chi(h) \neq 1 \), then
\[
\frac{1}{\chi(g) - 1}C(g) = \frac{1}{\chi(h) - 1}C(h).
\]
Therefore, there exists \( x \in F(\chi) \) such that \( C(g) = (\chi(g) - 1)x \) for all \( g \in \Gamma \) with \( \chi(g) \neq 1 \). If \( \chi(g) = 1 \), we have also \( C(g) = 0 = (\chi(g) - 1)x \) by (5.11.1). This shows that \( C \) is a 1-coboundary. □

**Proof of 5.10.** By 4.3(ii) and 5.4(c), the connected part \( G^c \) of \( G \) is HW-cyclic with \( h(G^c) = h(G) = 1 \). Assume that \( T_p(G^c) \) has rank \( \ell \) over \( \mathbb{Z}_p \), and \( T_p(G^\text{ét}) \) has rank \( r \). Then by 5.8(a), \( G^c(1)(\overline{K}) \) is an \( \mathbb{F}_q \)-vector space of dimension 1 with \( q = p^\ell \), and the action of \( I \) on \( G^c(1)(\overline{K}) \) factors through the character \( \overline{\chi} : I \to I, \theta \mapsto \theta^{q-1} \mathbb{F}_q^\times \). We write \( G^c(1)(\overline{K}) = \mathbb{F}_q(\overline{\chi}) \) for short. If the cohomology class \( \overline{C} \) is zero, then the exact sequence (5.9.1) splits, i.e. we have an isomorphism.
of Galois modules $G(1)(\overline{K}) \simeq \mathbb{F}_q(\chi) \oplus \mathbb{F}_p^r$. It is clear that the group $\text{Im}(\overline{\rho})$ has order $q - 1$.

Conversely, if the cohomology class $\overline{c}$ is not zero, we will show that there exists an element in $\text{Im}(\overline{\rho})$ of order $p$. We choose a basis adapted to the exact sequence (5.9.1) such that the action of $g \in I$ is given by

\[
\overline{\rho}(g) = \begin{pmatrix} \overline{\chi}(g) & \overline{C}(g) \\ 0 & 1 \end{pmatrix},
\]

where $1_r$ is the unit matrix of type $(r)$ with coefficients in $\mathbb{F}_p$, and the map $g \mapsto \overline{C}(g)$ gives rise to a 1-cocycle representing the cohomology class $\overline{c}$. Let $I_1$ be the kernel of $\overline{\chi} : I \rightarrow \mathbb{F}_q^\times$, $\Gamma$ be the quotient $I/I_1$, so $\overline{\chi}$ induces an isomorphism $\overline{\chi} : \Gamma \xrightarrow{\sim} \mathbb{F}_q^\times$. We have an exact sequence

\[
0 \rightarrow H^1(\Gamma, \mathbb{F}_q(\overline{\chi}))^r \xrightarrow{\text{Inf}} H^1(I, \mathbb{F}_q(\overline{\chi}))^r \xrightarrow{\text{Res}} H^1(I_1, \mathbb{F}_q(\overline{\chi}))^r,
\]

where “Inf” and “Res” are respectively the inflation and restriction homomorphisms in group cohomology. Since $H^1(\Gamma, \mathbb{F}_q(\overline{\chi}))^r = 0$ by 5.11, the restriction of the cohomology class $\overline{c}$ to $H^1(I_1, \mathbb{F}_q(\overline{\chi}))^r$ is non-zero. Hence there exists $h \in I_1$ such that $\overline{C}(h) \neq 0$. As we have $\overline{\chi}(h) = 1$, then

\[
\overline{\rho}(h)^p = \begin{pmatrix} 1_t & p\overline{C}(h) \\ 0 & 1_r \end{pmatrix} = 1_t + r.
\]

Thus the order of $\overline{\rho}(h)$ is $p$. \hfill \Box

**Corollary 5.12.** Let $G$ be a HW-cyclic BT-group over $S$,

\[
b = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_1 \\ 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 1 & \cdots & 0 & -a_3 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -a_c \end{pmatrix}
\]

be a matrix of $\varphi_G$, $P(X) = X^r + a_1X^r - 1 + \cdots + a_1X \in A[X]$. If $h(G) = 1$ and if there exists $\alpha \in k \subset A$ such that $\nu(P(\alpha)) = 1$, then the cohomology class (5.9.2) is not zero, i.e. the extension of $I$-modules (5.9.1) does not split.

**Proof.** Since $\nu(a_1) = h(G) = 1$, the integer $i_0$ defined in 5.8(iii) is at least 1. Then the corollary follows from 5.8(iii) and 5.10. \hfill \Box

### 6. Lemmas in Group Theory

In this section, we fix a prime number $p \geq 2$ and an integer $n \geq 1$.

#### 6.1. Recall that the general linear group $\text{GL}_n(\mathbb{Z}_p)$ admits a natural exhaustive decreasing filtration by normal subgroups

\[
\text{GL}_n(\mathbb{Z}_p) \supset 1 + p\mathbb{M}_n(\mathbb{Z}_p) \supset \cdots \supset 1 + p^n\mathbb{M}_n(\mathbb{Z}_p) \supset \cdots
\]

where $\mathbb{M}_n(\mathbb{Z}_p)$ denotes the ring of matrix of type $(n, n)$ with coefficients in $\mathbb{Z}_p$. We endow $\text{GL}_n(\mathbb{Z}_p)$ with the topology for which $(1 + p^n\mathbb{M}_n(\mathbb{Z}_p))_{m \geq 1}$ form a
6.2. Let $\mathfrak{g}$ be a profinite group, $\rho : \mathfrak{g} \to \text{GL}_n(\mathbb{Z}_p)$ be a continuous homomorphism of topological groups. By taking inverse images, we obtain a decreasing filtration $(F^m\mathfrak{g}, m \in \mathbb{Z}_{\geq 0})$ on $\mathfrak{g}$ by open normal subgroups:

$$F^0\mathfrak{g} = \mathfrak{g}, \quad \text{and} \quad F^m\mathfrak{g} = \rho^{-1}(1 + p^m\text{M}_n(\mathbb{Z}_p)) \text{ for } m \geq 1.$$ 

Furthermore, the homomorphism $\rho$ induces a sequence of injective homomorphisms of finite groups

(6.2.1) $\rho_0 : F^0\mathfrak{g}/F^1\mathfrak{g} \to \text{GL}_n(\mathbb{F}_p)$

(6.2.2) $\rho_m : F^m\mathfrak{g}/F^{m+1}\mathfrak{g} \to \text{M}_n(\mathbb{F}_p)$, for $m \geq 1$.

**Lemma 6.3.** The homomorphism $\rho$ is surjective if and only if the following conditions are satisfied:

(i) The homomorphism $\rho_0$ is surjective.

(ii) For every integer $m \geq 1$, the subgroup $\text{Im}(\rho_m)$ of $\text{M}_n(\mathbb{F}_p)$ contains an element of the form

$$\begin{pmatrix}
  x & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0
\end{pmatrix}$$

with $x \neq 0$; or equivalently, there exists, for every $m \geq 1$, an element $g_m \in \mathfrak{g}$ such that $\rho(g_m)$ is of the form

$$\begin{pmatrix}
  1 + p^m a_{1,1} & p^{m+1} a_{1,2} & \cdots & p^{m+1} a_{1,n} \\
  p^{m+1} a_{2,1} & 1 + p^{m+1} a_{2,2} & \cdots & p^{m+1} a_{2,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  p^{m+1} a_{n,1} & p^{m+1} a_{n,2} & \cdots & 1 + p^{m+1} a_{n,n}
\end{pmatrix},$$

where $a_{i,j} \in \mathbb{Z}_p$ for $1 \leq i, j \leq n$ and $a_{1,1}$ is not divisible by $p$.

**Proof.** We notice first that $\rho$ is surjective if and only if $\rho_m$ is surjective for every $m \geq 0$, because $\mathfrak{g}$ is complete and $\text{GL}_n(\mathbb{Z}_p)$ is separated [Bou, Chap. III §2 n°8 Cor.2 au Théo. 1]. The surjectivity of $\rho_0$ is condition (i). Condition (ii) is clearly necessary. We prove that it implies the surjectivity of $\rho_m$ for all $m \geq 1$, under the assumption of (i). First, we remark that under condition (i), if $A$ lies in $\text{Im}(\rho_m)$, then for any $U \in \text{GL}_n(\mathbb{F}_p)$ the conjugate matrix $U \cdot A \cdot U^{-1}$ lies also in $\text{Im}(\rho_m)$. In fact, let $A$ be a lift of $A$ in $\text{M}_n(\mathbb{Z}_p)$ and $U \in \text{GL}_n(\mathbb{Z}_p)$ a lift of $U$. By assumption, there exist $g, h \in \mathfrak{g}$ such that

$$\rho(g) \equiv 1 + p^m \overline{A} \mod (1 + p^{m+1}\text{M}_n(\mathbb{Z}_p)) \quad \text{and} \quad \rho(h) \equiv \overline{U} \mod (1 + p\text{M}_n(\mathbb{Z}_p)).$$

Therefore, we have $\rho(gh^{-1}) \equiv (1 + p^m \overline{U} \cdot \overline{A} \cdot \overline{U}^{-1}) \mod (1 + p^{m+1}\text{M}_n(\mathbb{Z}_p))$. Hence $gh^{-1} \in F^m\mathfrak{g}$ and $\rho_m(gh^{-1}) = U \cdot A \cdot U^{-1}$.

For $1 \leq i, j \leq n$, let $E_{i,j} \in \text{M}_n(\mathbb{F}_p)$ be the matrix whose $(i,j)$-th entry is 0 and the other entries are 0. The matrices $E_{i,j}(1 \leq i, j \leq n)$ form clearly
a basis of $M_n(\mathbb{F}_p)$ over $\mathbb{F}_p$. To prove the surjectivity of $\rho_m$, we only need to verify that $E_{i,j} \in \text{Im}(\rho_m)$ for $1 \leq i, j \leq n$, because $\text{Im}(\rho_m)$ is an $\mathbb{F}_p$-subspace of $M_n(\mathbb{F}_p)$. By assumption, we have $E_{1,1} \in \text{Im}(\rho_m)$. For $2 \leq i \leq n$, we put $U_i = E_{1,i} - E_{i,1} + \sum_{j \neq 1, i} E_{j,j}$. Then we have $U_i \in \text{GL}_n(\mathbb{Z}_p)$ and $U_i \cdot E_{1,1} \cdot U_i^{-1} = E_{i,i} \in \text{Im}(\rho_m)$. For $1 \leq i < j \leq n$, we put $U_{i,j} = I + E_{i,j}$ where $I$ is the unit matrix. Then we have $U_{i,j}, E_{i,j} \cdot U_{i,j}^{-1} = E_{i,i} + E_{i,j} \in \text{Im}(\rho_m)$, and hence $E_{i,j} \in \text{Im}(\rho_m)$. This completes the proof. 

\[ \square \]

Remark 6.4. By using the arguments in [Se2, Chap. IV 3.4 Lemma 3], we can prove the following stronger form of Lemma 6.3: If $p = 2$, condition (i) and (ii) for $m = 1, 2$ are sufficient to guarantee the surjectivity of $\rho$; if $p \geq 3$, then (i) and (ii) just for $m = 1$ suffice already.

A subgroup $C$ of $\text{GL}_n(\mathbb{F}_p)$ is called a non-split Cartan subgroup, if the subset $C \cup \{0\}$ of the matrix algebra $M_n(\mathbb{F}_p)$ is a field isomorphic to $\mathbb{F}_{p^n}$; such a group is cyclic of order $p^n - 1$.

Lemma 6.5. Assume that $n \geq 2$. We denote by $H$ the subgroup of $\text{GL}_n(\mathbb{F}_p)$ consisting of all the elements of the form $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$, where $A \in \text{GL}_{n-1}(\mathbb{F}_p)$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}$ with $b_i \in \mathbb{F}_p (1 \leq i \leq n - 1)$. Let $G$ be a subgroup of $\text{GL}_n(\mathbb{F}_p)$. Then $G = \text{GL}_n(\mathbb{F}_p)$ if and only if $G$ contains $H$ and a non-split Cartan subgroup of $\text{GL}_n(\mathbb{F}_p)$.

Proof. The “only if” part is clear. For the “if” part, let $C$ be a non-split Cartan subgroup contained in $G$. For a finite group $\Lambda$, we denote by $|\Lambda|$ its order. An easy computation shows that $|\text{GL}_n(\mathbb{F}_p)| = |H| \cdot |C|$. So we just need to prove that $U \cap C = \{1\}$; since then we will have $|\text{GL}_n(\mathbb{F}_p)| = |G|$, hence $G = \text{GL}_n(\mathbb{F}_p)$. Let $g \in H \cap C$, and $P(T) \in \mathbb{F}_p[T]$ be its characteristic polynomial. We fix an isomorphism $C \cong \mathbb{F}_{p^n}^\times$, and let $\zeta \in \mathbb{F}_{p^n}^\times$ be the element corresponding to $g$. We have $P(T) = \prod_{\sigma \in \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)} (T - \sigma(\zeta))$ in $\mathbb{F}_{p^n}[T]$. On the other hand, the fact that $g \in H$ implies that $(T - 1)$ divides $P(T)$. Therefore, we get $\zeta = 1$, i.e. $g = 1$. 

\[ \square \]

Remark 6.6. E. Lau point out the following strengthened version of 6.5: When $n \geq 3$, a subgroup $G \subset \text{GL}_n(\mathbb{F}_p)$ coincides with $\text{GL}_n(\mathbb{F}_p)$ if and only if $G$ contains a non-split Cartan subgroup and the subgroup $\begin{pmatrix} \text{GL}_{n-1}(\mathbb{F}_p) & 0 \\ 0 & 1 \end{pmatrix}$. This can be used to simplify the induction process in the proof of Theorem 7.3 when $n \geq 3$. 

Documenta Mathematica 14 (2009) 281–324
7. Proof of Theorem 1.3 in the One-dimensional Case

7.1. We start with a general remark on the monodromy of BT-groups. Let $X$ be a scheme, $G$ be an ordinary BT-group over a scheme $X$, $G^{\text{ét}}$ be its étale part (2.10.1). If $\eta$ is a geometric point of $X$, we denote by $T_p(G, \eta) = \lim_{n \to \infty} G(n)(\eta) = \lim_{n \to \infty} G^{\text{ét}}(n)(\eta)$ the Tate module of $G$ at $\eta$, and by $\rho(G)$ the monodromy representation of $\pi_1(X, \eta)$ on $T_p(G, \eta)$. Let $f : Y \to X$ be a morphism of schemes, $\xi$ be a geometric point of $Y$, $G_Y = G \times_X Y$. Then by the functoriality, we have a commutative diagram

$$
\begin{array}{ccc}
\pi_1(Y, \xi) & \xrightarrow{\pi_1(f)} & \pi_1(X, f(\xi)) \\
\rho(G_Y) \downarrow & & \downarrow \rho(G) \\
\text{Aut}_{Z_p}(T_p(G_Y, \xi)) & \xrightarrow{\gamma} & \text{Aut}_{Z_p}(T_p(G, f(\xi)))
\end{array}
$$

In particular, the monodromy of $G_Y$ is a subgroup of the monodromy of $G$. In the sequel, diagram (7.1.1) will be refereed as the functoriality of monodromy for the BT-group $G$ and the morphism $f$.

7.2. Let $k$ be an algebraically closed field of characteristic $p > 0$, $G$ be the unique connected BT-group over $k$ of dimension 1 and height $n + 1 \geq 2$ (4.10). We denote by $S$ the algebraic local moduli of $G$ in characteristic $p$, by $G$ the universal deformation of $G$ over $S$, and by $U$ the ordinary locus of $G$ over $S$ (3.8). Recall that $S$ is affine of ring $R \simeq k[[t_1, \cdots, t_n]]$ (3.7), and that $G$ and $G$ are HW-cyclic (cf. 4.3(i) and 4.10). Let $\eta$ be a geometric point of $U$ over its generic point. We put $T_p(G, \eta) = \lim_{m \in \mathbb{Z}_{\geq 1}} G(m)(\eta)$ to be the Tate module of $G$ at the point $\eta$. This is a free $\mathbb{Z}_p$-module of rank $n$. We have the monodromy representation $\rho_n : \pi_1(U, \eta) \to \text{Aut}_{\mathbb{Z}_p}(T_p(G, \eta)) \simeq \text{GL}_n(\mathbb{Z}_p)$.

The following is the one-dimensional case of Theorem 1.3.

**Theorem 7.3.** Under the above assumptions, the homomorphism $\rho_n$ is surjective for $n \geq 1$.

7.4. First, we assume $n \geq 2$. By Proposition 4.11(ii), we may assume that

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & -t_1 \\
1 & 0 & \cdots & 0 & -t_2 \\
0 & 1 & \cdots & 0 & -t_3 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -t_n
\end{pmatrix}
$$

(7.4.1)
We fix an imbedding \( \tilde{\varphi}_G \). Let \( \mathfrak{p} \) be the prime ideal of \( R \) generated by \( t_1, \cdots, t_{n-1} \). Then the closed subscheme of \( S \) defined by \( \mathfrak{p} \) is just the locus where the \( p \)-rank of \( G \) is \( \leq 1 \) by 4.4(ii). Let \( K_0 \simeq k((t_n)) \) be the fraction field of \( R/\mathfrak{p}, \ R' = \tilde{R}_p \) be the completion of the localization of \( R \) at \( \mathfrak{p} \), and \( \mathcal{G}_{R'} = G \otimes_R R' \). Since the natural map \( R \to R' \) is injective, for any \( a \in R \), we will denote also by \( a \) its image in \( R' \). Since the Hasse-Witt map commutes with base change, the image of \( \mathfrak{h} \) in \( M_{n \times n}(R') \), denoted also by \( \mathfrak{h} \), is a matrix of \( \varphi_{G_0} \). We see easily that the \( \acute{e}tale \) part of \( \mathcal{G}_{R'} \) has height 1 and its connected part \( \mathcal{G}^0_{R'} \) has height \( n \). We have an exact sequence of BT-groups over \( R' \)

\[(7.4.2) \quad 0 \to \mathcal{G}^0_{R'} \to \mathcal{G}_{R'} \to \mathcal{G}^\acute{e}t_{R'} \to 0.\]

We fix an imbedding \( i : K_0 \to \tilde{K}_0 \) of \( K_0 \) into an algebraically closed field. Put \( \mathcal{G}^*_R = \mathcal{G}^*_R \otimes \tilde{K}_0 \) for \( * = 0, \acute{e}t, \circ \). We have \( \mathcal{G}^\acute{e}t_{\tilde{K}_0} \simeq Q_p/\mathbb{Z}_p \), and \( \mathcal{G}^0_{\tilde{K}_0} \) is the unique connected-dimensional BT-group over \( \tilde{K}_0 \) of height \( n \) (cf. 4.10). We put \( \tilde{R}' = \tilde{K}_0[[x_1, \cdots, x_{n-1}]] \), and

\[(7.4.3) \quad \Sigma = \{ \text{ring homomorphisms } \sigma : R' \to \tilde{R}' \text{ lifting } R' \to K_0 \xrightarrow{i} \tilde{K}_0 \}\]

Let \( \sigma \in \Sigma \). We deduce from (7.4.2) by base change an exact sequence of \( \text{BT-groups over } \tilde{R}' \)

\[(7.4.4) \quad 0 \to \mathcal{G}^0_{R', \sigma} \to \mathcal{G}_{R', \sigma} \to \mathcal{G}^\acute{e}t_{R', \sigma} \to 0,\]

where we have put \( \mathcal{G}^*_R = \mathcal{G}^*_R \otimes \tilde{R}' \) for \( * = 0, \acute{e}t, \circ \). Due to the henselian property of \( \tilde{R}' \), the isomorphism \( \mathcal{G}^\acute{e}t_{\tilde{K}_0} \simeq Q_p/\mathbb{Z}_p \) lifts uniquely to an isomorphism \( \mathcal{G}^\acute{e}t_{R', \sigma} \simeq Q_p/\mathbb{Z}_p \). Assume that \( \mathcal{G}^0_{R', \sigma} \) is generically ordinary over \( \tilde{S} = \text{Spec}(\tilde{R}') \). Let \( \tilde{U}'_{\sigma} \subset \tilde{S}' \) be its ordinary locus, and \( \tilde{\varphi} \) be a geometric point over the generic point of \( \tilde{U}'_{\sigma} \). The exact sequence (7.4.4) induces an exact sequence of Tate modules

\[(7.4.5) \quad 0 \to T_p(\mathcal{G}^0_{R', \sigma}, \tilde{\varphi}) \to T_p(\mathcal{G}_{R', \sigma}, \tilde{\varphi}) \to T_p(\mathcal{G}^\acute{e}t_{R', \sigma}, \tilde{\varphi}) \to 0\]

compatible with the actions of \( \pi_1(\tilde{U}'_{\sigma}, \tilde{\varphi}) \). Since we have \( T_p(\mathcal{G}^0_{R', \sigma}, \tilde{\varphi}) \simeq T_p(Q_p/\mathbb{Z}_p, \tilde{\varphi}) = \mathbb{Z}_p \), this determines a cohomology class

\[(7.4.6) \quad C_{\sigma} \in \text{Ext}^1_{\mathbb{Z}_p[\pi_1(\tilde{U}'_{\sigma}, \tilde{\varphi})]}(\mathbb{Z}_p, T_p(\mathcal{G}^0_{R', \sigma}, \tilde{\varphi})) \simeq H^1(\pi_1(\tilde{U}'_{\sigma}, \tilde{\varphi}), T_p(\mathcal{G}^0_{R', \sigma}, \tilde{\varphi})).\]

We consider also the “mod-p version” of (7.4.5)

\[0 \to \mathcal{G}^0_{R', \sigma}(1)(\tilde{\varphi}) \to \mathcal{G}_{R', \sigma}(1)(\tilde{\varphi}) \to \mathbb{F}_p \to 0,\]

which determines a cohomology class

\[(7.4.7) \quad \overline{C}_{\sigma} \in \text{Ext}^1_{\mathbb{F}_p[\pi_1(\tilde{U}'_{\sigma}, \tilde{\varphi})]}(\mathbb{F}_p, \mathcal{G}^0_{R', \sigma}(1)(\tilde{\varphi})) \simeq H^1(\pi_1(\tilde{U}'_{\sigma}, \tilde{\varphi}), \mathcal{G}^0_{R', \sigma}(1)(\tilde{\varphi})).\]

It is clear that \( \overline{C}_{\sigma} \) is the image of \( C_{\sigma} \) by the canonical reduction map

\[H^1(\pi_1(\tilde{U}'_{\sigma}, \tilde{\varphi}), T_p(\mathcal{G}^0_{R', \sigma}, \tilde{\varphi})) \to H^1(\pi_1(\tilde{U}'_{\sigma}, \tilde{\varphi}), \mathcal{G}^0_{R', \sigma}(1)(\tilde{\varphi})).\]
LEMMA 7.5. Under the above assumptions, there exist \( \sigma_1, \sigma_2 \in \Sigma \) satisfying the following properties:

(i) We have \( \mathcal{G}^0_{R'}, \sigma_1 = \mathcal{G}^0_{R'}, \sigma_2 \), and it is the universal deformation of \( \mathcal{G}^0_{K_0} \).

(ii) We have \( C_{\sigma_1} = 0 \) and \( \overline{C}_{\sigma_2} \neq 0 \).

Before proving this lemma, we prove first Theorem 7.3.

PROOF OF 7.3. First, we notice that the monodromy of a BT-group is independent of the base point. So we can change \( \eta \) to any geometric point of \( U \) when discussing the monodromy of \( G \). We make an induction on the codimension \( n = \dim(G') \). The case of \( n = 1 \) is proved in Theorem 5.7. Assume that \( n \geq 2 \) and the theorem is proved for \( n - 1 \). We denote by

\[ \overline{\rho}_n : \pi_1(U, \eta) \to \text{Aut}_{\mathbb{F}_p}(G(1)(\eta)) \simeq \text{GL}_n(\mathbb{F}_p) \]

the reduction of \( \rho_n \) modulo by \( p \). By Lemma 6.3 and 6.5, to prove the surjectivity of \( \rho_n \), we only need to verify the following conditions:

(a) \( \text{Im}(\overline{\rho}_n) \) contains a non-split Cartan subgroup of \( \text{GL}_n(\mathbb{F}_p) \);

(b) \( \text{Im}(\rho_n) \) contains the subgroup \( H \subset \text{GL}_n(\mathbb{Z}_p) \) consisting of all the elements of the form \( \begin{pmatrix} B & b \\ 0 & 1 \end{pmatrix} \in \text{GL}_n(\mathbb{Z}_p) \), with \( B \in \text{GL}_{n-1}(\mathbb{Z}_p) \) and \( b \in M_{(n-1) \times 1}(\mathbb{Z}_p) \);

For condition (a), let \( A = k[[\pi]] \), \( T = \text{Spec}(A) \), \( \xi \) be its generic point, \( \overline{\xi} \) be a geometric point over \( \xi \), and \( I = \text{Gal}(\overline{\xi}/\xi) \) be the absolute Galois group over \( \xi \). We keep the notations of 7.4. Let \( f^* : R \to A \) be the homomorphism of \( k \)-algebras such that \( f^*(t_1) = \pi \) and \( f^*(t_i) = 0 \) for \( 2 \leq i \leq n \). We denote by \( f : T \to S \) the corresponding morphism of schemes, and put \( G_T = G \times_S T \). By the functoriality of Hasse-Witt maps,

\[ \varphi_T : \pi_1(U, \eta) \to \text{Aut}_{\mathbb{F}_p}(G_T(1)(\overline{\xi})) \]

is a matrix of \( \varphi_{G_T} \). By definition 5.4, the Hasse invariant of \( G_T \) is \( h(G) = 1 \). Hence \( G_T \) is generically ordinary; so \( f(\xi) \in U \). Let

\[ \overline{\rho}_T : I = \text{Gal}(\overline{\xi}/\xi) \to \text{Aut}_{\mathbb{F}_p}(G_T(1)(\overline{\xi})) \]

be the mod-\( p \) monodromy representation attached to \( G_T \). Proposition 5.8(i) implies that \( \text{Im}(\overline{\rho}_T) \) is a non-split Cartan subgroup of \( \text{GL}_n(\mathbb{F}_p) \). On the other hand, by the functoriality of monodromy, we get \( \text{Im}(\overline{\rho}_T) \subset \text{Im}(\overline{\rho}_n) \). This verifies condition (a).

To check condition (b), we consider the constructions in 7.4. Let \( S' = \text{Spec}(R') \), \( f : S' \to S \) be the morphism of schemes corresponding to the natural ring homomorphism \( R \to R' \); \( U' \) be the ordinary locus of \( \mathcal{G}_{R'} \); and \( \overline{\xi} \) be a geometric point of \( U' \). From (7.4.2), we deduce an exact sequence of Tate modules

\[ (7.5.1) \quad 0 \to T_p(\mathcal{G}^0_{R'}, \overline{\xi}) \to T_p(\mathcal{G}_{R'}, \overline{\xi}) \to T_p(\mathcal{G}^{et}_{R'}, \overline{\xi}) \to 0. \]
Let $\rho_{\mathcal{G}} : \pi_1(U', \overline{\xi}) \rightarrow \text{Aut}_{\mathbb{Z}_p}(T_p(\mathcal{G}_{R'}, \overline{\xi})) \simeq \text{GL}_n(\mathbb{Z}_p)$ be the monodromy representation of $\mathcal{G}_{R'}$. Under any basis of $T_p(\mathcal{G}_{R'}, \overline{\xi})$ adapted to (7.5.1), the action of $\pi_1(U', \overline{\xi})$ on $T_p(\mathcal{G}_{R'}, \overline{\xi})$ is given by

$$\rho_{\mathcal{G}_{R'}} : g \in \pi_1(U', \overline{\xi}) \mapsto \begin{pmatrix} \rho_{\mathcal{G}_{R'}}^\sigma(g) & 0 \\ 0 & \rho_{\mathcal{G}_{R'}}^\sigma(g) \end{pmatrix},$$

where $g \mapsto \rho_{\mathcal{G}_{R'}}^\sigma(g) \in \text{GL}_{n-1}(\mathbb{Z}_p)$ (resp. $g \mapsto \rho_{\mathcal{G}_{R'}}^{\sigma_1}(g) \in \mathbb{Z}_p^\times$) gives the action of $\pi_1(U', \overline{\xi})$ on $T_p(\mathcal{G}_{R'}, \overline{\xi})$ (resp. on $T_p(\mathcal{G}_{R'}^{\text{rig}}, \overline{\xi})$). Note that $f(U') \subset U$. So by the functoriality of monodromy, we get $\text{Im}(\rho_{\mathcal{G}}) \subset \text{Im}(\rho_n)$. To complete the proof of Theorem 7.3, it suffices to check condition (b) with $\rho_n$ replaced by $\rho_{\mathcal{G}_{R'}}$ under the induction hypothesis that 7.3 is valide for $n-1$. Let $\sigma_1, \sigma_2 : \mathcal{G} \rightarrow \mathcal{G}$ be the homomorphisms given by 7.5. For $i = 1, 2$, we denote by $f_i : \mathcal{S}' \rightarrow \text{Spec}(R') \rightarrow S' = \text{Spec}(R')$ the morphism of schemes corresponding to $\sigma_i$, and put $\mathcal{G}_i = \mathcal{G}_{R', \sigma_i} = \mathcal{G}_{R'} \circ \sigma_i, R'$ to simply the notations. By condition 7.5(i), we can denote by $\mathcal{G}$ the common connected component of $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $\overline{U}' \subset \mathcal{S}'$ be the ordinary locus of $\mathcal{G}$. Then we have $f_i(\overline{U}') \subset U'$ for $i = 1, 2$. Let $\pi$ be a geometric point over the generic point of $\overline{U}'$. We have an exact sequence of Tate modules

$$0 \rightarrow T_p(\mathcal{G}^\circ, \overline{x}) \rightarrow T_p(\mathcal{G}_i, \overline{x}) \rightarrow T_p(\mathcal{G}_{p/R}/\mathbb{Z}_p, \overline{x}) \rightarrow 0$$

compatible with the actions of $\pi_1(\overline{U}', \overline{x})$. We denote by

$$\rho_{\mathcal{G}_i} : \pi_1(U', \overline{x}) \rightarrow \text{Aut}_{\mathbb{Z}_p}(T_p(\mathcal{G}_i, \overline{x})) \simeq \text{GL}_n(\mathbb{Z}_p)$$

the monodromy representation of $\mathcal{G}_i$. In a basis adapted to (7.5.2), the action of $\pi_1(U', \overline{x})$ on $T_p(\mathcal{G}_i, \overline{x})$ is given by

$$\rho_{\mathcal{G}_i} : g \mapsto \begin{pmatrix} \rho_{\mathcal{G}^\circ}^\sigma(g) & C_{\sigma}(g) \\ 0 & 1 \end{pmatrix},$$

where $\rho_{\mathcal{G}^\circ} : \pi_1(U', \overline{x}) \rightarrow \text{GL}_{n-1}(\mathbb{Z}_p)$ is the monodromy representation of $\mathcal{G}^\circ$, and the cohomology class in $H^1(\pi_1(\overline{U}', \overline{x}), T_p(\mathcal{G}))$ given by $g \mapsto C_{\sigma}(g)$ is nothing but the class defined in (7.4.6). By 7.5(i) and the induction hypothesis, $C_{\sigma}(g)$ is surjective. Since the cohomology class $C_{\sigma_1} = 0$ by 7.5(ii), we may assume $C_{\sigma_1}(g) = 0$ for all $g \in \pi_1(U', \overline{x})$. Therefore $\text{Im}(\rho_{\mathcal{G}_i})$ contains all the matrix of the form $\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$ with $B \in \text{GL}_{n-1}(\mathbb{Z}_p)$. By the functoriality of monodromy, $\text{Im}(\rho_{\mathcal{G}_{R'}})$ contains $\text{Im}(\rho_{\mathcal{G}_i})$. Hence we have

$$\begin{pmatrix} \text{GL}_{n-1}(\mathbb{Z}_p) & 0 \\ 0 & 1 \end{pmatrix} \subset \text{Im}(\rho_{\mathcal{G}_i}) \subset \text{Im}(\rho_{\mathcal{G}_{R'}}).$$

On the other hand, since the cohomology class $\overline{C}_{\sigma_2} \neq 0$, there exists a $g \in \pi_1(U', \overline{x})$ such that $b_2 = \overline{C}_{\sigma_2}(g) \neq 0$. Hence the matrix $\rho_{\mathcal{G}_2}(g)$ has the form $\begin{pmatrix} B_2 & b_2 \\ 0 & 1 \end{pmatrix}$ such that $B_2 \in \text{GL}_{n-1}(\mathbb{Z}_p)$ and the image of $b_2 \in M_{1 \times n-1}(\mathbb{Z}_p)$
in $M_{1 \times n-1}(F_p)$ is non-zero. By the functoriality of monodromy, we have
$\text{Im}(\rho_{\mathfrak{g}_u}) \subset \text{Im}(\rho_{\mathfrak{g}_{ul}})$; in particular, we have\[
\begin{pmatrix}
B_2 & b_2 \\
0 & 1
\end{pmatrix} \in \text{Im}(\rho_{\mathfrak{g}_{ul}}).
\]
In view of (7.5.3), we get\[
(\text{7.5.4}) \quad \begin{pmatrix}
\text{GL}_{n-1}(\mathbb{Z}_p) & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
B_2 & b_2 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\text{GL}_{n-1}(\mathbb{Z}_p) & 0 \\
0 & 1
\end{pmatrix} \subset \text{Im}(\rho_{\mathfrak{g}_{ul}}).
\]
But the subset of $\text{GL}_n(\mathbb{Z}_p)$ on the left hand side is just the subgroup $H$ described in condition (b). Therefore, condition (b) is verified for $\rho_{\mathfrak{g}_{ul}}$, and the proof of 7.3 is complete.

The rest of this section is dedicated to the proof of Lemma 7.5.

**Lemma 7.6.** Let $k$ be an algebraically closed field of characteristic $p > 0$, $A$ be a noetherian henselian local $k$-algebra with residue field $k$, $G$ be a BT-group over $A$, and $G^{\acute{e}t}$ be its étale part. Put\[
\text{Lie}(G^\vee)^{\varphi=1} = \{x \in \text{Lie}(G^\vee) \text{ such that } \varphi_G(x) = x\}.
\]
Then $\text{Lie}(G^\vee)^{\varphi=1}$ is an $\mathbb{F}_p$-vector space of dimension equal to the rank of $\text{Lie}(G^{\acute{e}t, \vee})$, and the $A$-submodule $\text{Lie}(G^{\acute{e}t, \vee})$ of $\text{Lie}(G^\vee)$ is generated by $\text{Lie}(G^\vee)^{\varphi=1}$.

**Proof.** Let $r$ be the rank of $\text{Lie}(G^{\acute{e}t, \vee})$, $G^\circ$ be the connected part of $G$, and $s$ be the height of $\text{Lie}(G^{\circ, \vee})$. We have an exact sequence of $A$-modules\[
0 \rightarrow \text{Lie}(G^{\acute{e}t, \vee}) \rightarrow \text{Lie}(G^\vee) \rightarrow \text{Lie}(G^{\circ, \vee}) \rightarrow 0,
\]
compatible with Hasse-Witt maps. We choose a basis of $\text{Lie}(G^\vee)$ adapted to this exact sequence, so that $\varphi_G$ is expressed by a matrix of the form\[
\begin{pmatrix}
U & W \\
0 & V
\end{pmatrix}
\]
with $U \in \text{M}_{r \times r}(A)$, $V \in \text{M}_{s \times s}(A)$, and $W \in \text{M}_{r \times s}(A)$. An element of $\text{Lie}(G^\vee)^{\varphi=1}$ is given by a vector $\begin{pmatrix} x \\ y \end{pmatrix}$, where $x = \begin{pmatrix} x_1 \\
\vdots \\
x_r \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\
\vdots \\
y_s \end{pmatrix}$ with $x_i, y_j \in A$, satisfying\[
(\text{7.6.1}) \quad \begin{pmatrix}
U & W \\
0 & V
\end{pmatrix} \begin{pmatrix} x^{(p)} \\ y^{(p)} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases}
U \cdot x^{(p)} + W \cdot y^{(p)} = x \\
V \cdot y^{(p)} = y.
\end{cases}
\]
where $x^{(p)}$ (resp. $y^{(p)}$) is the vector obtained by applying $a \mapsto a^p$ to each $x_i (1 \leq i \leq r)$ (resp. $y_j (1 \leq j \leq s)$). By 2.9, the Hasse-Witt map of the special fiber of $G^\circ$ is nilpotent. So there exists an integer $N \geq 1$ such that $\varphi_G^{N}(\text{Lie}(G^{\circ, \vee})) \subset \mathfrak{m}_A \cdot \text{Lie}(G^{\circ, \vee})$, i.e. we have $V \cdot y^{(p)} \cdot \cdots \cdot V^{(p^{N-1})} \equiv 0 \pmod{\mathfrak{m}_A}$. From the equation $V \cdot y^{(p)} = y$, we deduce that\[
y = V \cdot y^{(p)} \cdot \cdots \cdot V^{(p^{N-1})} \cdot y^{(p^N)} \equiv 0 \pmod{\mathfrak{m}_A}.
\]
But this implies that \( y^{(p^N)} \equiv 0 \pmod{m_A^{p^N}} \). Hence we get \( y = V \cdot y^{(p)} \equiv 0 \pmod{m_A^{p+1}} \). Repeating this argument, we get finally \( y \equiv 0 \pmod{m_A} \) for all integers \( \ell \geq 1 \), so \( y = 0 \). This implies that \( \ker(G^{\nu}) \subset \ker(G^{\nu+1}) \), and the equation \((7.6.1)\) is simplified as \( U \cdot x^{(p)} = x \). Since the linearization of \( \varphi_{G^\nu} \) is bijective by \( 2.11 \), we have \( U \in \GL_r(A) \). Let \( U \) be the image of \( U \) in \( \GL_r(k) \), and \( \text{Sol} \) be the solutions of the equation \( U \cdot x^{(p)} = x \). As \( k \) is algebraically closed, \( \text{Sol} \) is an \( \mathbb{F}_p \)-space of dimension \( r \), and \( \ker(G^{\nu}) \otimes k \) is generated by \( \text{Sol} \) (cf. [Ka2, Prop. 4.1]). By the henselian property of \( A \), every elements in \( \text{Sol} \) lifts uniquely to a solution of \( U \cdot x^{(p)} = x \), i.e. the reduction map \( \ker(G^{\nu}) \to \text{Sol} \) is bijective. By Nakayama’s lemma, \( \ker(G^{\nu}) \) generates the \( A \)-module \( \ker(G^{\nu}) \).

\[ \square \]

7.7. We keep the notations of 7.4. Let \( \text{Comp}_{\mathcal{R}_0} \) be the category of noetherian complete local \( \mathcal{R}_0 \)-algebras with residue field \( \mathcal{K}_0 \), \( D_{\mathcal{R}_0} \) (resp. \( D_{\mathcal{R}_0}^\vee \)) be the functor which associates to every object \( A \) of \( \text{Comp}_{\mathcal{R}_0} \) the set of isomorphism classes of deformations of \( \mathcal{G}_{\mathcal{R}_0} \) (resp. \( \mathcal{G}_{\mathcal{R}_0}^\vee \)). If \( A \) is an object in \( \text{Comp}_{\mathcal{R}_0} \) and \( G \) is a deformation of \( \mathcal{G}_{\mathcal{R}_0} \) (resp. \( \mathcal{G}_{\mathcal{R}_0}^\vee \)) over \( A \), we denote by \( [G] \) its isomorphic class in \( D_{\mathcal{R}_0}^\vee(A) \) (resp. in \( D_{\mathcal{R}_0} \)).

**Lemma 7.8.** Let \( \Sigma \) be the set defined in \((7.4.3)\).

(i) The morphism of sets \( \Phi : \Sigma \to D_{\mathcal{R}_0}(\mathcal{R}) \) given by \( \sigma \mapsto [\mathcal{G}_{\mathcal{R}_0}] \) is bijective.

(ii) Let \( \sigma \in \Sigma \). Then there exists a basis of \( \ker(\mathcal{G}_{\mathcal{R}_0}) \) such that \( \mathcal{G}_{\mathcal{R}_0} \) is represented by a matrix of the form

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & a_1 \\
1 & 0 & \cdots & 0 & a_2 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & a_{n-1}
\end{pmatrix}
\]

with \( a_i \equiv \alpha \cdot \sigma(t_i) \pmod{m_{\mathcal{R}_0}^i} \) for \( 1 \leq i \leq n-1 \), where \( \alpha \in \mathcal{R}_0^\times \) and \( m_{\mathcal{R}_0}^i \) is the maximal ideal of \( \mathcal{R}_0 \). In particular, \( \mathcal{G}_{\mathcal{R}_0} \) is the universal deformation of \( \mathcal{G}_{\mathcal{R}_0} \), if and only if \( \{ \sigma(t_1), \ldots, \sigma(t_{n-1}) \} \) is a system of regular parameters of \( \mathcal{R}_0 \).

**Proof.** (i) We begin with a remark on the Kodaira-Spencer map of \( \mathcal{G}_{\mathcal{R}_0} \). Let \( \mathcal{S}_{/k} = \mathcal{H}om_{\mathcal{O}_S}(\Omega_{\mathcal{S}_{/k}}^1, \mathcal{O}_S) \) be the tangent sheaf of \( S \). Since \( G \) is universal, the Kodaira-Spencer map \((3.2.2)\)

\[
\text{Kod} : \mathcal{S}_{/k} \sim \mathcal{H}om_{\mathcal{O}_S}(\omega_G, \ker(G^{\nu}))
\]

is an isomorphism. By functoriality, this induces an isomorphism of \( R' \)-modules

\[
\text{Kod}_{R'} : T_{R'/k} \sim \text{Hom}_{\mathcal{O}_S}(\omega_{\mathcal{G}}^{\nu}, \ker(\mathcal{G}_{\mathcal{R}_0}^{\nu})),
\]

where \( T_{R'/k} = \text{Hom}_{\mathcal{R}'}(\Omega_{\mathcal{S}_{/k}}^1, \mathcal{R}') = \Gamma(S, \mathcal{S}_{/k}) \otimes_R \mathcal{R}' \).

For each integer \( \nu \geq 0 \), we put \( \mathcal{R}' = \mathcal{R}/m_{\mathcal{R}_0}^{\nu+1} \), \( \Sigma_{\nu} \) to be the set of liftings of \( \mathcal{R} \to K_0 \to \mathcal{K}_0 \to R \to \mathcal{R}' \), and \( \Phi_{\nu} : \Sigma_{\nu} \to D_{\mathcal{R}_0}(\mathcal{R}') \) to be the morphism of

\[ \text{Documenta Mathematica 14 (2009) 281–324} \]
sets $\sigma_\nu \mapsto [\mathcal{G}_{R'} \otimes_{\sigma_\nu} \tilde{R}'_\nu]$. We prove by induction on $\nu$ that $\Phi_\nu$ is bijective for all $\nu \geq 0$. This will complete the proof of (i). For $\nu = 0$, the claim holds trivially. Assume that it holds for $\nu - 1$ with $\nu \geq 1$. We have a commutative diagram

$$
\begin{array}{ccc}
\Sigma_\nu & \xrightarrow{\Phi_\nu} & D_{\mathcal{G}_{R_0'}}(\tilde{R}'_\nu) \\
\downarrow & & \downarrow \\
\Sigma_{\nu-1} & \xrightarrow{\Phi_{\nu-1}} & D_{\mathcal{G}_{R_0'}}(\tilde{R}'_{\nu-1})
\end{array}
$$

where the vertical arrows are the canonical reductions, and the lower arrow is an isomorphism by induction hypothesis. Let $\tau$ be an arbitrary element of $\Sigma_{\nu-1}$. We denote by $\Sigma_{\nu,\tau} \subset \Sigma_\nu$ the preimage of $\tau$, and by $D_{\Phi_{\nu-1}(\tau)}(\tilde{R}'_{\nu-1}) \subset D_{\mathcal{G}_{R_0'}}(\tilde{R}'_{\nu-1})$ the preimage of $\Phi_{\nu-1}(\tau)$. It suffices to prove that $\Phi_\nu$ induces a bijection between $\Sigma_{\nu,\tau}$ and $D_{\Phi_{\nu-1}(\tau)}(\tilde{R}'_{\nu})$. Let $I_\nu = m_{\tilde{R}'_\nu}^\nu/m_{\tilde{R}'_\nu}^{\nu+1}$ be the ideal of the reduction map $\tilde{R}'_\nu \rightarrow \tilde{R}'_{\nu-1}$. By [EGA, 0IV 21.2.5 and 21.9.4], we have $\Omega_{R'/k}^1 \simeq \tilde{\Omega}_{R'/k}^1$, and they are free over $A$ of rank $n$. By [EGA, 0IV 20.1.3], $\Sigma_{\nu,\tau}$ is a (nonempty) homogenous space under the group

$$\text{Hom}_{K_0}(\tilde{\Omega}_{R'/k}^1 \otimes_{R'} K_0, I_\nu) = T_{R'/k} \otimes_{R'} I_\nu.$$ 

On the other hand, according to 3.5(i), $D_{\Phi_{\nu-1}(\tau)}(\tilde{R}'_{\nu})$ is a homogenous space under the group

$$\text{Hom}_{K_0}(\mathcal{G}_{R_0'}, \text{Lie}(\mathcal{G}_{R_0'}')) \otimes_{K_0} I_\nu = \text{Hom}_{R'}(\mathcal{G}_{R'}, \text{Lie}(\mathcal{G}_{R'}')) \otimes_{R'} I_\nu.$$ 

Moreover, it is easy to check that the morphism of sets $\Phi_\nu : \Sigma_{\nu,\tau} \rightarrow D_{\Phi_{\nu-1}(\tau)}(\tilde{R}'_{\nu})$ is compatible with the homomorphism of groups

$$\text{Kod}_{R'} \otimes_{R'} \text{Id} : T_{R'/k} \otimes_{R'} I_\nu \rightarrow \text{Hom}_{R'}(\mathcal{G}_{R'}, \text{Lie}(\mathcal{G}_{R'}')) \otimes_{R'} I_\nu,$$

where $\text{Kod}_{R'}$ is the Kodaira-Spencer map (7.8.2) associated to $\mathcal{G}_{R'}$. The bijectivity of $\Phi_\nu$ now follows from the fact that $\text{Kod}_{R'}$ is an isomorphism.

(ii) The second part of the statement follows immediately from 4.11. It remains to compute the Hasse-Witt map of $\mathcal{G}_{R',\sigma}^\vee$. We determine first the submodule $\text{Lie}(\mathcal{G}_{R',\sigma}^\vee)$ of $\text{Lie}(\mathcal{G}_{R',\sigma})$. We choose a basis of $\text{Lie}(G^\vee)$ over $\mathcal{O}_S$ such that $\phi_G$ is expressed by the matrix $\mathfrak{h}$ (7.4.1). As $\mathfrak{g}_{R',\sigma}$ derives from $G$ by base change $R \rightarrow R' \rightarrow \tilde{R}'$, there exists a basis $(e_1, \ldots, e_n)$ of $\text{Lie}(\mathcal{G}_{R',\sigma}^\vee)$ such that $\phi_{\mathfrak{g}_{R',\sigma}}$ is expressed by

$$\mathfrak{h} = \begin{pmatrix}
0 & 0 & \cdots & 0 & -\sigma(t_1) \\
1 & 0 & \cdots & 0 & -\sigma(t_2) \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & -\sigma(t_n)
\end{pmatrix}.
$$

By Lemma 7.6, $\text{Lie}(\mathcal{G}_{R',\sigma}^\vee)$ is generated by $\text{Lie}(\mathcal{G}_{R',\sigma}^\vee)^\mathfrak{h} = 1$. If $\sum_{i=1}^n x_i e_n \in \text{Lie}(\mathcal{G}_{R',\sigma}^\vee)^\mathfrak{h} = 1$ with $x_i \in \tilde{R}'$ for $1 \leq i \leq n$, then $(x_i)_{1 \leq i \leq n}$ must satisfy the
equation $\Phi : \Sigma \rightarrow D$.

\[
\begin{pmatrix}
    x_1^0 \\
    \vdots \\
    x_n^0
\end{pmatrix} =
\begin{pmatrix}
    x_1 \\
    \vdots \\
    x_n
\end{pmatrix} ;
\]

or equivalently,

\[
\begin{pmatrix}
    x_1 = -\sigma(t_1)x_n^0 \\
    x_2 = -\sigma(t_2)x_n^0 - \sigma(t_1)x_n^0 \\
    \vdots \\
    x_{n-1} = -\sigma(t_{n-1})x_n^0 - \cdots - \sigma(t_1)x_n^0 \\
    x_n = -\sigma(t_n)x_n^0 - \sigma(t_1)x_n^0 + \sigma(t_1)x_n^0 + x_n = 0.
\end{pmatrix}
\]

(7.8.3)

We note that $\sigma(t_i) \in m_{\tilde{R}}^i$ for $1 \leq i \leq n - 1$ and $\sigma(t_n) \in \tilde{R}^\times$ with image $i(t_n) \in K_0$, where $i : K_0 \rightarrow K_0$ is the fixed imbedding. By Hensel’s lemma, every solution in $K_0$ of the equation $i(t_n)x_n + x_n = 0$ lifts uniquely to a solution of (7.8.3). As Lie($\mathscr{G}_{R, \sigma}^{\hat{}}$) has rank 1, by Lemma 7.6, these are all the solutions of (7.8.3). Let $(\lambda_1, \ldots, \lambda_n)$ be a non-zero solution of (7.8.3). We have

\[
\lambda_n \in \tilde{R}^\times \text{ and } \lambda_i \equiv -\lambda_i^p(t_i) \pmod{m_2^2}.
\]

(7.8.4)

We put $v = \lambda_1 e_1 + \cdots + \lambda_n e_n$; so $v$ is a basis of Lie($\mathscr{G}_{R, \sigma}^{\hat{}}$) by 7.6. For $1 \leq i \leq n$, let $f_i$ be the image of $e_i$ in Lie($\mathscr{G}_{R, \sigma}^{\hat{}}$). Then $f_1, \ldots, f_n$ clearly generate Lie($\mathscr{G}_{R, \sigma}^{\hat{}}$). By the explicit description above of Lie($\mathscr{G}_{R, \sigma}^{\hat{}}$), we have $f_n = -\lambda_n^{-1}(\lambda_1 f_1 + \cdots + \lambda_{n-1} f_{n-1})$. Hence $f_1, \ldots, f_{n-1}$ form a basis of Lie($\mathscr{G}_{R, \sigma}^{\hat{}}$). By the functoriality of Hasse-Witt maps, we have $\varphi_{\mathscr{G}_{R, \sigma}^{\hat{}}} (f_i) = f_{i+1}$ for $1 \leq i \leq n - 1$, or equivalently,

\[
\varphi_{\mathscr{G}_{R, \sigma}^{\hat{}}} (f_1, \ldots, f_{n-1}) = (f_1, \ldots, f_{n-1}) \cdot
\begin{pmatrix}
    0 & 0 & \cdots & 0 & -\lambda_n^{-1} \lambda_1 \\
    1 & 0 & \cdots & 0 & -\lambda_n^{-1} \lambda_2 \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    0 & 0 & \cdots & 1 & -\lambda_n^{-1} \lambda_{n-1}
\end{pmatrix}.
\]

In view of (7.8.4), we see that the above matrix has the form of (7.8.1) by setting $\alpha = \lambda_n^{-1} \in \tilde{R}^\times$. The second part of statement (ii) follows immediately from Proposition 4.11(ii) and the description above of $\varphi_{\mathscr{G}_{R, \sigma}^{\hat{}}}$.

Now we can turn to the proof of 7.5.

7.9. PROOF OF LEMMA 7.5. First, suppose that we have found a $\sigma_2 \in \Sigma$ such that $\sigma_2 = 0$ and $\mathscr{G}_{R, \sigma_2}$ is the universal deformation of $\mathscr{G}_{R_0}$. Since $\Phi : \Sigma \rightarrow D_{\mathscr{G}_{R_0}}(\tilde{R})$ is bijective by 7.8(i), there exists a $\sigma_1 \in \Sigma$ corresponding to the deformation $[\mathscr{G}_{R, \sigma_2} \oplus \mathbb{Q}_p / \mathbb{Z}_p] \in D_{\mathscr{G}_{R_0}}(\tilde{R})$. It is clear that $\mathscr{G}_{R, \sigma_1} \simeq \mathscr{G}_{R, \sigma_2}$. Besides, the exact sequence (7.4.5) for $\sigma_1$ splits; so we have $C_{\sigma_1} = 0$. It remains to prove the existence of $\sigma_2$. We note first that $K_0$ can be canonically imbedded into $\tilde{R}$, since it is perfect. Since $R'$ is formally smooth over $k$ and

Documenta Mathematica 14 (2009) 281–324
Let \( \sigma \) be the choice of \( G \) corresponding to \( \xi \) over \( \text{residue field} \). Let \( \eta = \mathcal{G}_\sigma \) verify that \( \eta \), and \( \xi \) imply that \( \mathcal{G}_\sigma \) is the universal deformation of \( \mathcal{G}_{\mathcal{K}_0} \). It remains to verify that \( \mathcal{C}_\sigma \neq 0 \).

Let \( A = \mathcal{K}_0[[\pi]] \) be a complete discrete valuation ring of characteristic \( p \) with residue field \( \mathcal{K}_0 \), \( T = \text{Spec}(A) \), \( \xi \) be the generic point of \( T \), \( \mathcal{E} \) be a geometric over \( \xi \), and \( I = \text{Gal}(\mathcal{E}/\xi) \) the Galois group. We define a homomorphism of \( \mathcal{K}_0 \)-algebras \( f^* : \mathcal{R} \rightarrow A \) by putting \( f^*(\mathcal{G}) = f_\mathcal{G} \) and \( f^*(\mathcal{G}) = 0 \) for \( 2 \leq i \leq n - 1 \). This is possible, since \( (\mathcal{G}(t_1), \cdots, \mathcal{G}(t_{n-1})) \) is a system of regular parameters of \( \mathcal{R} \). Let \( f : T \rightarrow \mathcal{S}^f \) be the homomorphism of schemes corresponding to \( f^* \), and \( \mathcal{G}_T = \mathcal{G}_{\mathcal{R}^f} \times_{\mathcal{S}^f} T \). By the functoriality of Hasse-Witt maps,

\[
\mathcal{G}_T = \begin{pmatrix}
0 & 0 & \cdots & 0 & -\pi \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -f^*(\mathcal{G}(\mathcal{G}}(t_n))
\end{pmatrix} \in M_{n \times n}(\mathcal{R})
\]

is a matrix of \( \varphi_{\mathcal{G}_T} \). By definition (5.4), the Hasse invariant of \( \mathcal{G}_T \) is \( h(\mathcal{G}_T) = 1 \). In particular, \( \mathcal{G}_T \) is generically ordinary. Let \( \mathcal{U}_T^f \subset \mathcal{S}^f \) be the ordinary locus of \( \mathcal{G}_{\mathcal{R}^f \sigma} \). We have \( f(\xi) \in \mathcal{U}_T^f \). By the functoriality of fundamental groups, \( f \) induces a homomorphism of groups

\[
\pi_1(f) : I = \text{Gal}(\mathcal{E}/\xi) \rightarrow \pi_1(\mathcal{U}_T^f, f(\xi)) \simeq \pi_1(\mathcal{U}_T^f, \mathfrak{R}).
\]

Let \( \mathcal{G}_T \) be the connected part of \( \mathcal{G}_T \), and \( \mathcal{G}_T^{et} \) be the étale part of \( \mathcal{G}_T \). Then \( \mathcal{G}_T^{et} \simeq \mathbb{Q}_p/\mathbb{Z}_p \). We have an exact sequence of \( \mathbb{F}_p[I] \)-modules

\[
0 \rightarrow \mathcal{G}_T^{et}(1)(\xi) \rightarrow \mathcal{G}_T(1)(\xi) \rightarrow \mathcal{G}_T^{et}(1)(\xi) \rightarrow 0,
\]

which determines a cohomology class \( \mathcal{C}_T \in H^1(I, \mathcal{G}_T^{et}(1)(\xi)) \). We notice that \( \mathcal{G}_T(1)(\xi) \) is isomorphic to \( \mathcal{G}_{\mathcal{R}^f(\sigma)}(1)(\mathfrak{R}) \) as an abelian group, and the action of \( I \) on \( \mathcal{G}_T(1)(\xi) \) is induced by the action of \( \pi_1(\mathcal{U}_T^f, \mathfrak{R}) \) on \( \mathcal{G}_{\mathcal{R}^f(\sigma)}(1)(\mathfrak{R}) \). Therefore, \( \mathcal{C}_T \) is the image of \( \mathcal{C}_\sigma \) by the functorial map

\[
H^1(\pi_1(\mathcal{U}_T^f, \mathfrak{R}), \mathcal{G}_{\mathcal{R}^f(\sigma)}(1)(\mathfrak{R})) \rightarrow H^1(I, \mathcal{G}_T^{et}(1)(\xi)).
\]

To verify that \( \mathcal{C}_\sigma \neq 0 \), it suffices to check that \( \mathcal{C}_T \neq 0 \). We consider the polynomial \( P(X) = X^{p^n} + f^*(\mathcal{G}(t_n))X^{p^{n-1}} + \pi X \in A[X] \). According to 5.12, it suffices to find an \( \alpha \in \mathcal{K}_0 \subset A \) such that \( P(\alpha) \) is a uniformizer of \( A \). But by the choice of \( \sigma \), we have \( \sigma(t_n) \in \mathcal{K}_0 \) and \( \sigma(t_n) \neq 0 \); so \( f^*(\mathcal{G}(t_n)) \neq 0 \) lies in \( \mathcal{K}_0 \). Let \( \alpha \) be a \( p^{n-1}(p - 1) \)-th root of \( -f^*(\mathcal{G}(t_n)) \) in \( \mathcal{K}_0 \). Then we have \( \alpha \in \mathcal{K}_0^{\times} \), and \( P(\alpha) = \alpha \pi \) is a uniformizer of \( A \). This completes the proof of 7.5.
8. End of the Proof of Theorem 1.3

In this section, \( k \) denotes an algebraically closed field of characteristic \( p > 0 \).

8.1. First, we recall some preliminaries on Newton stratification due to F. Oort. Let \( G \) be an arbitrary BT-group over \( k \), \( S \) be the local moduli of \( G \) in characteristic \( p \), and \( \mathbf{G} \) be the universal deformation of \( G \) over \( S \) (3.8). Put \( d = \dim(G) \) and \( c = \dim(G') \). We denote by \( N(G) \) the Newton polygon of \( G \), which has endpoints \((0, 0)\) and \((c + d, d)\). Here we use the normalization of Newton polygons such that slope \( 0 \) corresponds to étale BT-groups and slope \( 1 \) corresponds to groups of multiplicative type.

Let \( NP(c + d, d) \) be the set of Newton polygons with endpoints \((0, 0)\) and \((c + d, d)\) and slopes in \((0, 1)\). For \( \alpha, \beta \in NP(c + d, d) \), we say that \( \alpha \preceq \beta \) if no point of \( \alpha \) lies below \( \beta \); then \( \preceq \) is a partial order on \( NP(c + d, d) \).

For each \( \beta \in NP(c + d, d) \), we denote by \( V_\beta \) the subset of \( S \) consisting of points \( x \) with \( N(G_x) \preceq \beta \), and by \( V^\circ_\beta \) the subset of \( S \) consisting of points \( x \) with \( N(G_x) = \beta \). By Grothendieck-Katz’s specialization theorem of Newton polygons, \( V_\beta \) is closed in \( S \), and \( V^\circ_\beta \) is open (maybe empty) in \( V_\beta \). We put \( \diamond(\beta) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq y < d, y < x < c + d, (x, y) \text{ lies on or above the polygon } \beta \} \), and \( \dim(\beta) = \#(\diamond(\beta)) \).

**Theorem 8.2** ([Oo2] Theorem 2.11). Under the above assumptions, for each \( \beta \in NP(c + d, d) \), the subset \( V^\circ_\beta \) is non-empty if and only if \( N(G) \preceq \beta \). In that case, \( V_\beta \) is the closure of \( V^\circ_\beta \) and all irreducible components of \( V_\beta \) have dimension \( \dim(\beta) \).

8.3. Let \( G \) be a connected and HW-cyclic BT-group over \( k \) of dimension \( d = \dim(G) \geq 2 \). Let \( \beta \in NP(c + d, d) \) be the Newton polygon given by the following slope sequence:

\[
\beta = \left( \frac{1}{c + 1}, \ldots, \frac{1}{c + 1}, \frac{1}{d - 1}, \ldots, \frac{1}{d - 1} \right).
\]

We have \( N(G) \preceq \beta \) since \( G \) is supposed to be connected. By Oort’s Theorem 8.2, \( V_\beta \) is an equal dimensional closed subset of the local moduli \( S \) of dimension \( c(d - 1) \). We endow \( V_\beta \) with the structure of a reduced closed subscheme of \( S \).

**Lemma 8.4.** Under the above assumptions, let \( R \) be the ring of \( S \), and

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & -a_1 \\
1 & 0 & \cdots & 0 & -a_2 \\
0 & 1 & \cdots & 0 & -a_3 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -a_c
\end{pmatrix} \in M_{c \times c}(R)
\]

be a matrix of the Hasse-Witt map \( \varphi_G \). Then the closed reduced subscheme \( V_\beta \) of \( S \) is defined by the prime ideal \( (a_1, \ldots, a_c) \). In particular, \( V_\beta \) is irreducible.
Let $P$ be the ideal of $R$ defining $V_\beta$. Let $x$ be an arbitrary point of $V_\beta$, we denote by $p_x$ the prime ideal of $R$ corresponding to $x$. Since the Newton polygon of the fibre $G_x$ lies above $\beta$, $G_x$ is connected. By Lemma 4.4, we have $a_i \in p_x$ for $1 \leq i \leq c$. Since $V_\beta$ is reduced, we have $a_i \in I$. Let $\mathfrak{P} = (a_1, \cdots, a_c)$, and $V(\mathfrak{P})$ the closed subscheme of $S$ defined by $\mathfrak{P}$. Then $V(\mathfrak{P})$ is an integral scheme of dimension $c(d-1)$ and $V_\beta \subset V(\mathfrak{P})$. Since Theorem 8.2 implies that $\dim V_\beta = c(d-1)$, we have necessarily $V_\beta = V(\mathfrak{P})$. 

We keep the assumptions above. Let $(t_{i,j})_{1 \leq i \leq c, 1 \leq j \leq d}$ be a regular system of parameters of $R$ such that $t_{i,d} = a_i$ for all $1 \leq i \leq c$. Let $x$ be the generic point of the Newton strata $V_\beta$, $k' = \kappa(x)$, and $R' = \mathcal{O}_{S,x}$. Since $R$ is noetherian and integral, the canonical ring homomorphism $R \to \mathcal{O}_{S,x} \to R'$ is injective. The image in $R'$ of an element $a \in R$ will be denoted also by $a$. By choosing a $k$-section $k' \to R'$ of the canonical projection $R' \to k'$, we get a (non-canonical) isomorphism of $k$-algebras $R' \simeq k'[t_{1,d}, \cdots, t_{c,d}]$. Let $k''$ be an algebraic closure of $k'$, and $R'' = k''[t_{1,d}, \cdots, t_{c,d}]$. Then we have a natural injective homomorphism of $k$-algebras $R' \to R''$ mapping $t_{i,d}$ to $t_{i,d}$ for $1 \leq i \leq c$.

Let $S'' = \text{Spec}(R'')$, $x$ be its closed point. By the construction of $S''$, we have a morphism of $k$-schemes

\begin{equation}
(8.4.1)
f : S'' \to S
\end{equation}

sending $x$ to $x$. We put $\mathcal{G} = G \times_S S''$. By the choice of the Newton polygon $\beta$, the closed fibre $\mathcal{G}_x$ has a BT-subgroup $\mathcal{H}$ of multiplicative type of height $d-1$. Since $S''$ is henselian, $\mathcal{H}_x$ lifts uniquely to a BT-subgroup $\mathcal{H}$ of $\mathcal{G}$. We put $\mathcal{G}'' = \mathcal{G}/\mathcal{H}$. It is a connected BT-group over $S''$ of dimension 1 and height $c+1$.

**Lemma 8.5.** Under the above assumptions, $\mathcal{G}''$ is the universal deformation in equal characteristic of its special fiber.

This lemma is a particular case of [Lau, Lemma 3.1]. Here, we use 4.11(ii) to give a simpler proof.

**Proof.** We have an exact sequence of BT-groups over $S''$

$$0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{G}'' \to 0,$$

which induces an exact sequence of Lie algebras $0 \to \text{Lie}(\mathcal{G}'') \to \text{Lie}(\mathcal{G}) \to \text{Lie}(\mathcal{H}) \to 0$ compatible with Hasse-Witt maps. Since $\mathcal{H}$ is of multiplicative type, we get $\text{Lie}(\mathcal{H}) = 0$ and an isomorphism of Lie algebras $\text{Lie}(\mathcal{G}'') \simeq \text{Lie}(\mathcal{G})$. By the choice of the regular system $(t_{i,j})_{1 \leq i \leq c, 1 \leq j \leq d}$, there is a basis $(v_1, \cdots, v_c)$ of $\text{Lie}(\mathcal{G}'')$ such that $\varphi_{\mathcal{G}''}$ is given by the matrix

$$h = \begin{pmatrix}
0 & 0 & \cdots & 0 & -t_{1,d} \\
1 & 0 & \cdots & 0 & -t_{2,d} \\
0 & 1 & \cdots & 0 & -t_{3,d} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & -t_{c,d}
\end{pmatrix}.$$
Now the lemma results from Proposition 4.11(ii).

8.6. Proof of Theorem 1.3. The one-dimensional case is treated in 7.3. If \( \dim(G) \geq 2 \), we apply the preceding discussion to obtain the morphism \( f: S'' \to S \) and the BT-groups \( \mathcal{G} = G \times_S S'' \) and \( \mathcal{G}'' \), which is the quotient of \( \mathcal{G} \) by the maximal subgroup of \( \mathcal{G} \) of multiplicative type. Let \( U'' \) be the common ordinary locus of \( \mathcal{G} \) and \( \mathcal{G}'' \) over \( S'' \), and \( \bar{\xi} \) be a geometric point of \( U'' \). Then \( f \) maps \( U'' \) into the ordinary locus \( U \) of \( G \). We denote by

\[
\rho_{\mathcal{G}} : \pi_1(U'', \bar{\xi}) \to \text{Aut}_{\mathbb{Z}_p}(T_p(\mathcal{G}, \bar{\xi}))
\]

the monodromy representation associated to \( \mathcal{G} \), and the same notation for \( \rho_{\mathcal{G}''} \). By the functoriality of monodromy, we have \( \text{Im}(\rho_{\mathcal{G}}) \subset \text{Im}(\rho_{\mathcal{G}'}) \). On the other hand, the canonical map \( \mathcal{G} \to \mathcal{G}'' \) induces an isomorphism of Tate modules \( T_p(\mathcal{G}, \eta) \cong T_p(\mathcal{G}'', \eta) \) compatible with the action of \( \pi_1(U'', \eta) \). Therefore, the group \( \text{Im}(\rho_{\mathcal{G}}) \) is identified with \( \text{Im}(\rho_{\mathcal{G}'}) \). Since \( \mathcal{G}'' \) is one-dimensional, we conclude the proof by Lemma 8.5 and Theorem 7.3.

References


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