AN UPPER BOUND FOR THE ABBES-SAITO FILTRATION OF FINITE FLAT GROUP SCHEMES AND APPLICATIONS

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Abstract. Let $\mathcal{O}_K$ be a complete discrete valuation ring of residue characteristic $p > 0$, and G be a finite flat group scheme over $\mathcal{O}_K$ of order a power of p. We prove in this paper that the Abbes-Saito filtration of G is bounded by a linear function of the degree of G. Assume $\mathcal{O}_K$ has generic characteristic 0 and the residue field of $\mathcal{O}_K$ is perfect. Fargues constructed the higher level canonical subgroups for a "near from being ordinary" Barsotti-Tate group $\mathcal{G}$ over $\mathcal{O}_K$. As an application of our bound, we prove that the canonical subgroup of $\mathcal{G}$ of level $n \geq 2$ constructed by Fargues appears in the Abbes-Saito filtration of the p^n-torsion subgroup of $\mathcal{G}$.

Let $\mathcal{O}_K$ be a complete discrete valuation ring with residue field k of characteristic $p > 0$ and fraction field K. We denote by v_π the valuation on K normalized by $v_\pi(K^\times) = \mathbf{Z}$. Let G be a finite and flat group scheme over $\mathcal{O}_K$ of order a power of p such that $G \otimes K$ is étale. We denote by $(G^a, a \in \mathbf{Q}_{\geq 0})$ the Abbes-Saito filtration of G. This is a decreasing and separated filtration of G by finite and flat closed subgroup schemes. We refer the readers to [AS02, AS03, AM04] for a full discussion, and to section 1 for a brief review of this filtration. Let ω_G be the module of invariant differentials of G. The generic étaleness of G implies that ω_G is a torsion $\mathcal{O}_K$-module of finite type. There exist thus nonzero elements $a_1, \cdots, a_d \in \mathcal{O}_K$ such that

$$\omega_G \simeq \bigoplus_{i=1}^{d} \mathcal{O}_K/(a_i).$$

We put $\deg(G) = \sum_{i=1}^d v_\pi(a_i)$, and call it the degree of G. The aim of this note is to prove the following

Theorem 0.1. *Let G be a finite and flat group scheme over $\mathcal{O}_K$ of order a power of p such that $G \otimes K$ is étale. Then we have $G^a = 0$ for $a > \frac{p}{p-1}\deg(G)$.*

Our bound is quite optimal when G is killed by p. Let $E_\delta = \mathrm{Spec}(\mathcal{O}_K[X]/(X^p - \delta X))$ be the group scheme of Tate-Oort over $\mathcal{O}_K$. We have $\deg(E_\delta) = v_\pi(\delta)$, and an easy computation by Newton polygons gives [Fa09, Lemme 5]

$$E_\delta^a = \begin{cases} E_\delta & \text{if } 0 \leq a \leq \frac{p}{p-1}\deg(E_\delta) \\ 0 & \text{if } a > \frac{p}{p-1}\deg(E_\delta). \end{cases}$$

However, our bound may be improved when G is not killed by p or G contains many identical copies of a closed subgroup. In [Hat06, Thm. 7], Hattori proves that if K has characteristic 0 and G is killed by p^n, then the Abbes-Saito filtration of G is bounded by that of the multiplicative group μ_{p^n}, i.e., we have $G^a = 0$ if $a > en + \frac{e}{p-1}$ where e is the absolute ramification index of K. Compared with Hattori's result, our bound has the advantage that it works in both characteristic 0 and characteristic p, and that it is good if $\deg(G)$ is small.

The basic idea to prove 0.1 is to approximate general power series over $\mathcal{O}_K$ by linear functions. First, we choose a "good" presentation of the algebra of G such that the defining equations of G involve only terms of total degree $m(p-1)+1$ with $m \in \mathbf{Z}_{\geq 0}$ (Prop. 1.6). The existence of such a presentation is a consequence of the classical theory on p-typical curves of formal groups. With this good presentation, we can prove that the neutral connected component of the a-tubular neighborhood of G is isomorphic to a closed rigid ball for $a > \frac{p}{p-1}\deg(G)$ (Lemma 1.9), and the only zero of the defining equations of G in the neutral component is the unit section.

The motivation of our theorem comes from the theory of canonical subgroups. We assume that K has characteristic 0, and the residue field k is perfect of characteristic $p \geq 3$. Let G be a Barsotti-Tate group of dimension $d \geq 1$ over $\mathcal{O}_K$. If G comes from an abelian scheme over $\mathcal{O}_K$, the canonical subgroup of level 1 of G was first constructed by Abbes and Mokrane in [AM04]. Then the author generalized their result to the Barsotti-Tate case [Ti06]. We actually proved that if a Barsotti-Tate group G over $\mathcal{O}_K$ is "near from being ordinary", a condition expressed explicitly as a bound on the Hodge height of G (cf. 2.1), then a certain piece of the Abbes-Saito filtration of $G[p]$ lifts the kernel of Frobenius of the special fiber of G [Ti06, Thm. 1.4]. Later on, Fargues [Fa09] gave another construction of the canonical subgroup of level 1 using Hodge-Tate maps, and his approach also allowed us to construct by induction the canonical subgroups of level $n \geq 2$, i.e., the canonical lifts of the kernel of n-th iteration of the Frobenius. He proved that the canonical subgroup of higher level appears in the Harder-Narasihman filtration of $G[p^n]$, which was introduced by him in [Fa07]. It is conjectured that the canonical subgroup of higher level also appears in the Abbes-Saito filtration of $G[p^n]$. In this paper, we prove this conjecture as a corollary of 0.1 (Thm. 2.5). Fargues's result on the degree of the quotient of $G[p^n]$ by its canonical subgroup of level n (see Thm. 2.4(i)) will play an essential role in our proof.

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0.3. **Notation.** In this paper, $\mathcal{O}_K$ will denote a complete discrete valuation ring with residue field k of characteristic $p > 0$, and with fraction field K. Let π be a uniformizer of $\mathcal{O}_K$, and v_π be the valuation on K normalized by $v_\pi(\pi) = 1$. Let $\overline{K}$ be an algebraic closure of K, K^{sep} be the separable closure of K contained in $\overline{K}$, and $\mathcal{G}_K$ be the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. We denote still by v_π the unique extension of the valuation to $\overline{K}$.

1. Proof of Theorem 0.1

We recall first the definition of the filtration of Abbes-Saito for finite flat group schemes according to [AM04, AS03].

1.1. For a semi-local ring R, we denote by $\mathfrak{m}_R$ its Jacobson radical. An algebra R over $\mathcal{O}_K$ is called *formally of finite type*, if R is semi-local, complete with respect to the $\mathfrak{m}_R$-adic topology, Noetherian and $R/\mathfrak{m}_R$ is finite over k. We say an $\mathcal{O}_K$-algebra R formally of finite type is formally smooth, if each of the factors of R is formally smooth over $\mathcal{O}_K$.

Let $\mathbf{FEA}_{\mathcal{O}_K}$ be the category of finite, flat and generially étale $\mathcal{O}_K$-algebras, and $\mathbf{Set}_{\mathcal{G}_K}$ be the category of finite sets endowed with a discrete action of the Galois group $\mathcal{G}_K$. We have the fiber functor

$$\mathscr{F} : \mathbf{FEA}_{\mathcal{O}_K} \to \mathbf{Set}_{\mathcal{G}_K},$$

which associates with an object A of $\mathbf{FEA}_{\mathcal{O}_K}$ the set $\mathrm{Spec}(A)(\overline{K})$ equipped with the natural action of $\mathcal{G}_K$. We define a filtration on the functor $\mathscr{F}$ as follows. For each object A in $\mathbf{FEA}_{\mathcal{O}_K}$, we choose a presentation

$$0 \to I \to \mathscr{A} \to A \to 0, \tag{1.1.1}$$

where $\mathscr{A}$ is an $\mathcal{O}_K$-algebra formally of finite type and formally smooth. For any $a = \frac{m}{n} \in \mathbf{Q}_{>0}$ with m prime to n, we define $\mathscr{A}^a$ to be the π-adic completion of the subring $\mathscr{A}[I^n/\pi^m] \subset \mathscr{A} \otimes_{\mathcal{O}_K} K$ generated over $\mathscr{A}$ by all the f/π^m with $f \in I^n$. The $\mathcal{O}_K$-algebra $\mathscr{A}^a$ is topologically of finite type, and the tensor product $\mathscr{A}^a \otimes_{\mathcal{O}_K} K$ is an affinoid algebra over K [AS03, Lemma 1.4]. We put $X^a = \mathrm{Sp}(\mathscr{A}^a \otimes_{\mathcal{O}_K} K)$, which is a smooth affinoid variety over K [AS03, Lemma 1.7]. We call it the *a-th tubular neighborhood of* $\mathrm{Spec}(A)$ *with respect to the presentation* (1.1.1). The $\mathcal{G}_K$-set of the geometric connected components of X^a, denoted by $\pi_0(X^a(A)_{\overline{K}})$, depends only on the $\mathcal{O}_K$-algebra A and the rational number a, but not on the choice of the presentation [AS03, Lemma 1.9.2]. For rational numbers $b > a > 0$, we have natural inclusions of affinoid varieties $\mathrm{Sp}(A \otimes_{\mathcal{O}_K} K) \hookrightarrow X^b \hookrightarrow X^a$, which induce natural morphisms $\mathrm{Spec}(A)(\overline{K}) \to \pi_0(X^b(A)_{\overline{K}}) \to \pi_0(X^a(A)_{\overline{K}})$. For a morphism $A \to B$ in $\mathbf{FEA}_{\mathcal{O}_K}$, we can choose properly presentations of A and B so that we have a functorial map $\pi_0(X^a(B)_{\overline{K}}) \to \pi_0(X^a(A)_{\overline{K}})$. Hence we get, for any $a \in \mathbf{Q}_{>0}$, a (contravariant) functor

$$\mathscr{F}^a : \mathbf{FEA}_{\mathcal{O}_K} \to \mathbf{Set}_{\mathcal{G}_K}$$

given by $A \mapsto \pi_0(X^a(A)_{\overline{K}})$. We have natural morphisms of functors $\phi_a : \mathscr{F} \to \mathscr{F}^a$, and $\phi_{a,b} : \mathscr{F}^b \to \mathscr{F}^a$ for rational numbers $b > a > 0$ with $\phi_a = \phi_{b,a} \circ \phi_b$. For any A in $\mathbf{FEA}_{\mathcal{O}_K}$, we have $\mathscr{F}(A) \xrightarrow{\sim} \varprojlim_{a \in \mathbf{Q}_{>0}} \mathscr{F}^a(A)$ [AS02, 6.4]; if A is a complete intersection over $\mathcal{O}_K$, the map $\mathscr{F}(A) \to \mathscr{F}^a(A)$ is surjective for any a [AS02, 6.2].

1.2. Let $G = \mathrm{Spec}(A)$ be a finite and flat group scheme over $\mathcal{O}_K$ such that $G \otimes K$ is étale over K, and $a \in \mathbf{Q}_{>0}$. The group structure of G induces a group structure on $\mathscr{F}^a(A)$, and the natural map $G(\overline{K}) = \mathscr{F}(A) \to \mathscr{F}^a(A)$ is a homomorphism of groups. Hence the kernel $G^a(\overline{K})$ of $G(\overline{K}) \to \mathscr{F}^a(A)$ is a $\mathcal{G}_K$-invariant subgroup of $G(\overline{K})$, and it defines a closed subgroup scheme G^a_K of the generic fiber $G \otimes K$. The scheme theoretic closure of G^a_K in G, denoted by G^a, is a closed subgroup of G finite and flat over $\mathcal{O}_K$. Putting $G^0 = G$, we get a decreasing and separated filtration $(G^a, a \in \mathbf{Q}_{\geq 0})$ of G by finite and flat closed subgroup schemes. We call it *Abbes-Saito filtration* of G. For any real number $a \geq 0$, we put $G^{a+} = \cup_{b \in \mathbf{Q}_{>a}} G^a$.

Assume G is connected, i.e., the ring A is local. Let

$$0 \to I \to \mathcal{O}_K[[X_1, \cdots, X_d]] \to A \to 0 \tag{1.2.1}$$

be a presentation of A by the ring of formal power series such that the unit section of G corresponds to the point $(X_1, \cdots, X_d) = (0, \cdots, 0)$. Since A is a relative complete intersection over $\mathcal{O}_K$, I is generated by d elements $f_1, \cdots, f_d$. For $a \in \mathbf{Q}_{>0}$, the $\overline{K}$-valued points of the a-th tubular neighborhood of G are given by

$$X^a(\overline{K}) = \big\{(x_1, \cdots, x_d) \in \mathfrak{m}_{\overline{K}}^d \mid v_\pi(f_i(x_1, \cdots, x_d)) \geq a \text{ for } 1 \leq i \leq d\big\}, \tag{1.2.2}$$

where $\mathfrak{m}_{\overline{K}}$ is the maximal ideal of $\mathcal{O}_{\overline{K}}$. The subset $G(\overline{K}) \subset X^a(\overline{K})$ corresponds to the zeros of the f_i's. Let X_0^a be the connected component of X^a containing 0. Then the subgroup $G^a(\overline{K})$ is the intersection of $X_0^a(\overline{K})$ with $G(\overline{K})$.

The basic properties of Abbes-Saito filtration that we need are summarized as follows.

Proposition 1.3 ([AM04] 2.3.2, 2.3.5)**.** *Let G and H be finite and flat group schemes, generically étale over $\mathcal{O}_K$, $f : G \to H$ be a homomorphism of group schemes.*

(i) *G^{0+} is the connected component of G, and we have $(G^{0+})^a = G^a$ for any $a \in \mathbf{Q}_{>0}$.*

(ii) *For $a \in \mathbf{Q}_{>0}$, f induces a canonical homomorphism $f^a : G^a \to H^a$. If f is flat and surjective, then $f^a(\overline{K}) : G^a(\overline{K}) \to H^a(\overline{K})$ is surjective.*

Now we return to the proof of Theorem 0.1.

Lemma 1.4. *Let R be a $\mathbf{Z}_p$-algebra, $\mathscr{X}$ be a formal group of dimension d over R such that* $\mathrm{Lie}(\mathscr{X})$ *is a free R-module of rank d. Then*

(i) *the ring $\mathbf{Z}_p$ acts naturally on $\mathscr{X}$, and its image in* $\mathrm{End}_R(\mathscr{X})$ *lies in the center of* $\mathrm{End}_R(\mathscr{X})$*;*

(ii) *there exist parameters $(X_1, \cdots, X_d)$ of $\mathscr{X}$, such that we have $[\zeta](X_1, \cdots, X_d) = (\zeta X_1, \cdots \zeta X_d)$ for any $(p-1)$-th root of unity $\zeta \in \mathbf{Z}_p$.*

Proof. This is actually a classical result on formal groups. In the terminology of [Haz78], the formal group $\mathscr{X}$ comes from the base change of $\mathscr{X}^{\mathrm{univ}}$ defined by the d-dimensional universal p-typical formal group law (denoted by $F_V(X, Y)$ in [Haz78, 15.2.8]) over $\mathbf{Z}_p[V] = \mathbf{Z}_p[V_i(j,k); i \in \mathbf{Z}_{\geq 0}, j, k = 1, \cdots, d]$, where the $V_i(j,k)$'s are free variables. So we are reduced to proving the Lemma for $\mathscr{X}^{\mathrm{univ}}$. If X and Y are short for the column vectors $(X_1, \cdots, X_d)$ and $(Y_1, \cdots, Y_d)$ respectively, the formal group law on $\mathscr{X}^{\mathrm{univ}}$ is determined by

$$F_V(X, Y) = f_V^{-1}(f_V(X) + f_V(Y)), \quad \text{with } f_V(X) = \sum_{i=0}^{\infty} a_i(V) X^{p^i},$$

where $a_i(V)$'s are certain $d \times d$ matrices with coefficients in $\mathbf{Q}_p[V]$ with $a_1(V)$ invertible, X^{p^i} is short for $(X_1^{p^i}, \cdots, X_d^{p^i})$, and f_V^{-1} is the unique d-tuple of power series in $(X_1, \cdots, X_d)$ with coefficients in $\mathbf{Q}_p[V]$ such that $f_V^{-1} \circ f_V = 1$ [Haz78, 10.4]. We note that $F_V(X, Y)$ is a d-tuple of power series with coefficient in $\mathbf{Z}_p[V]$, although $f_V(X)$ has coefficients in $\mathbf{Q}_p[V]$

[Haz78, 10.2(i)]. Via approximation by integers, we see easily that the multiplication by an element $\xi \in \mathbf{Z}_p$ can be well defined as $[\xi](X) = f_V^{-1}(\xi f_V(X))$. This proves (i). Statement (ii) is an immediate consequence of the fact that $f_V(X)$ involves just p-powers of X. $\square$

Remark 1.5. The referee gives the following alternative proof of this Lemma via the Cartier theory of formal groups. Let $\mathscr{X}$ be the formal group over R as in the Lemma. We denote by $\mathscr{X}(R[[T]])$ the group of $R[[T]]$-valued points of $\mathscr{X}$ whose reduction modulo T is the neutral element $0 \in \mathscr{X}(R)$. A formal group law over $\mathscr{X}$ is a datum $(\mathscr{X}; \gamma_1, \cdots, \gamma_d)$, where $\gamma_1, \cdots, \gamma_d \in \mathscr{X}(R[[T]])$ are such that their image in $\mathscr{X}(R[T]/T^2)$ forms a basis of $\mathrm{Lie}(\mathscr{X})$. In particular, $(\gamma_i)_{1\leq i\leq d}$ establish an isomorphism of formal schemes over R $\mathscr{X} \simeq \mathrm{Spf}(R[[X_1, \cdots, X_d]])$. Recall that $\mathscr{X}(R[[T]])$ is the Cartier module associated with $\mathscr{X}$ over the big Cartier ring (denoted by $\mathrm{Cart}(R)$ in [Ch94, 2.3]). Since R is a $\mathbf{Z}_p$-algebra, the Cartier theory [Ch94, 4.3, 4.4] implies that there exists a p-typical formal group law $(\mathscr{X}; \gamma_1, \cdots, \gamma_d)$ over $\mathscr{X}$, i.e. we have $\epsilon_p \cdot \gamma_i = 0$, where

$$\epsilon_p = \prod_{\substack{\ell \text{ prime} \\ (\ell,p)=1}} (1 - \frac{1}{\ell} V_\ell F_\ell)$$

is Cartier's idempotent in $\mathrm{Cart}(R)$ (see [Ch94, 4.1]). Let $\Delta : \mathbf{Z}_p = W(\mathbf{F}_p) \to W(\mathbf{Z}_p)$ be the Cartier homomorphism given by $(x_0, x_1, \ldots) \mapsto ([x_0], [x_1], \ldots)$, where $x_n \in \mathbf{F}_p$ and $[x_n]$ denotes its Teichmüller lift. Then we get a natural map $u : \mathbf{Z}_p \xrightarrow{\Delta} W(\mathbf{Z}_p) \to W(R)$. For a $(p-1)$-th root of unity $\zeta \in \mathbf{Z}_p$, we have $u(\zeta) = [\zeta] \in W(R)$. Note that for any $a \in R$ and $1 \leq i \leq d$, the p-typical curve $[a] \cdot \gamma_i$ is the image of γ_i under the map $\mathscr{X}(R[[T]]) \to \mathscr{X}(R[[T]])$ induced by $T \mapsto aT$. Applying this fact to $u(\zeta) \cdot \gamma_i = [\zeta] \cdot \gamma_i$, one obtains the Lemma immediately.

Proposition 1.6. *Let $G = \mathrm{Spec}(A)$ be a connected finite and flat group scheme over $\mathcal{O}_K$ of order a power of p. Then there exists a presentation of A of type* (1.2.1) *such that the defining equations f_i for $1 \leq i \leq d$ have the form*

$$f_i(X_1, \cdots, X_d) = \sum_{|n|\geq 1}^{\infty} a_{i,\underline{n}} X^{\underline{n}} \qquad \text{with } a_{i,\underline{n}} = 0 \ \text{ if } (p-1) \nmid (|\underline{n}| - 1),$$

where $\underline{n} = (n_1, \cdots, n_d) \in (\mathbf{Z}_{\geq 0})^d$ are multi-indexes, $|\underline{n}| = \sum_{j=1}^d n_j$, and $X^{\underline{n}}$ is short for $\prod_{j=1}^d X_j^{n_j}$.

Proof. By a theorem of Raynaud [BBM82, 3.1.1], there is a projective abelian variety V over $\mathcal{O}_K$, and an embedding of group schemes $j : G \hookrightarrow V$. Let V' be the quotient of V by G. Let $\mathscr{X}$, $\mathscr{Y}$ be respectively the formal completion of V and V' along their unit sections. They are formal groups over $\mathcal{O}_K$. Since G is connected, it's identified with the kernel of the natural isogeny $\phi : \mathscr{X} \to \mathscr{Y}$. Let $(X_1, \cdots, X_d)$ (*resp.* $(Y_1, \cdots, Y_d)$) be parameters of $\mathscr{X}$ (*resp.* $\mathscr{Y}$) satisfying the preceding lemma. The isogeny ϕ is thus given by

$$(X_1, \cdots, X_d) \mapsto (f_1(X_1, \cdots, X_d), \cdots, f_d(X_1, \cdots, X_d)),$$

where $f_i = \sum_{|\underline{n}|\geq 1} a_{i,\underline{n}} X^{\underline{n}} \in \mathcal{O}_K[[X_1, \cdots, X_d]]$. Since for any $(p-1)$-th root of unity $\zeta \in \mathbf{Z}_p$ we have $f_i(\zeta X_1, \cdots, \zeta X_d) = \zeta f_i(X_1, \cdots, X_d)$, it's easy to see that $a_{i,\underline{n}} = 0$ if $(p-1) \nmid (|\underline{n}| - 1)$.

$\square$

Remark 1.7. As pointed out by the referee, we can avoid using Raynaud's deep theorem to realize G as the kernel of an isogeny of formal groups over $\mathcal{O}_K$. In fact, by the biduality formula $G \simeq (G^D)^D$, where G^D denotes the Cartier dual of G, we have a canonical closed embedding $u : G \hookrightarrow U = \mathrm{Res}_{G^D/S}(\mathbf{G}_m)$ of group schemes over $S = \mathrm{Spec}(\mathcal{O}_K)$. Here, "$\mathrm{Res}_{G^D/S}$" means Weil's restriction of scalars, so U is an affine smooth group scheme over S. Since the quotient of an affine scheme by a finite flat group scheme is always representable by a scheme [Ra67], we can consider the quotient $U' = U/G$ and the formal groups $\mathscr{X}, \mathscr{Y}$ associated with U and U', so that G is the kernel of the natural isogeny $\phi : \mathscr{X} \to \mathscr{Y}$.

1.8. Proof of Theorem 0.1. Let $H = G^{0+}$ be the connected component of G. By 1.3(i), we have $G^a = H^a$ for $a \in \mathbf{Q}_{>0}$. The exact sequence of finite flat group schemes $0 \to H \to G \to G/H \to 0$ induces a long exact sequence of finite $\mathcal{O}_K$-modules

$$0 \to H^{-1}(\ell_{G/H}) \to H^{-1}(\ell_G) \to H^{-1}(\ell_H) \to \omega_{G/H} \to \omega_G \to \omega_H \to 0,$$

where ℓ_G means the co-Lie complex of G [BBM82, 3.2.9]. Since the generic fiber of G/H is étale, it's easy to see that $H^{-1}(\ell_H) = 0$. It follows thus that $0 \to \omega_{G/H} \to \omega_G \to \omega_H \to 0$ is exact. Since G/H is étale, we have $\omega_{G/H} = 0$ and hence $\deg(G) = \deg(H)$. Up to replacing G by H, we may assume that $G = \mathrm{Spec}(A)$ is connected.

We choose a presentation of A as in Prop. 1.6 so that we have an isomorphism of $\mathcal{O}_K$-algebras

$$A \simeq \mathcal{O}_K[[X_1, \cdots, X_d]]/(f_1, \cdots, f_d)$$

where

$$f_i(X_1, \cdots, X_d) = \sum_{j=1}^{d} a_{i,j} X_j + \sum_{|\underline{n}|\geq p} a_{i,\underline{n}} X^{\underline{n}}.$$

As A is finite as an $\mathcal{O}_K$-module, we have

$$\Omega^1_{A/\mathcal{O}_K} = \widehat{\Omega}^1_{A/\mathcal{O}_K} \simeq \Big(\bigoplus_{i=1}^{d} A\, dX_i\Big)/(df_1, \cdots, df_d).$$

Since $\omega_G \simeq e^*(\Omega^1_{A/\mathcal{O}_K})$, where e is the unit section of G, we get

$$\omega_G \simeq \big(\oplus_{i=1}^{d} \mathcal{O}_K dX_i\big)/\Big(\sum_{1\leq j\leq d} a_{i,j} dX_j\Big)_{1\leq i\leq d}.$$

In particular, if U denotes the matrix $(a_{i,j})_{1\leq i,j\leq d}$, then we have $\deg(G) = v_\pi(\det(U))$.

For any rational number λ, we denote by $\mathbf{D}^d(0, |\pi|^\lambda)$ (*resp.* $\mathring{\mathbb{D}}^d(0, |\pi|^\lambda)$) the rigid analytic closed (*resp.* open) disk of dimension d over K consisting of points $(x_1, \cdots, x_d)$ with $v_\pi(x_i) \geq \lambda$ (*resp.* $v_\pi(x_i) > \lambda$) for $1 \leq i \leq d$; we put $\mathbf{D}^d(0,1) = \mathbf{D}^d(0, |\pi|^0)$ and $\mathring{\mathbb{D}}^d(0,1) = \mathring{\mathbb{D}}^d(0, |\pi|^0)$. Let $a > \frac{p}{p-1}\deg(G)$ be a rational number, X^a be the a-th tubular neighborhood of G with respect to the chosen presentation. By (1.2.2), we have a cartesian diagram of rigid analytic spaces

(1.8.1)
$$\begin{tikzcd} X^a \arrow[r, hook] \arrow[d, "\mathbf{f}"] & \mathring{\mathbb{D}}^d(0,1) \arrow[d, "\mathbf{f}=(f_1,\cdots,f_d)"] \\ \mathbf{D}^d(0,|\pi|^a) \arrow[r, hook] & \mathring{\mathbb{D}}^d(0,1), \end{tikzcd}$$

where horizontal arrows are inclusions, and $\mathbf{f}(y_1,\cdots,y_d)=(f_1(y_1,\cdots,y_d),\cdots,f_d(y_1,\cdots,y_d))$. Let X^a_0 be the connected component of X^a containing 0. By the discussion below (1.2.2), we just need to prove that 0 is the only zero of the f_i's contained in X^a_0.

Let $V=(b_{i,j})_{1\leq i,j\leq d}$ be the unique $d\times d$ matrix with coefficients in $\mathcal{O}_K$ such that $UV=VU=\det(U)I_d$, where I_d is the $d\times d$ identity matrix. If $\mathbf{A}^d_K$ denotes the d-dimensional rigid affine space over K, then V defines an isomorphism of rigid spaces

$$\mathbf{g}:\mathbf{A}^d_K\to\mathbf{A}^d_K;\qquad (x_1,\cdots,x_d)\mapsto(\sum_{j=1}^d b_{1,j}x_j,\cdots,\sum_{j=1}^d b_{d,j}x_j).$$

It's clear that $\mathbf{g}(\mathring{\mathbb{D}}^d(0,1))\subset\mathring{\mathbb{D}}^d(0,1)$, so that $\mathbf{f}$ is defined on $\mathbf{g}(\mathring{\mathbb{D}}^d(0,1))$. The composite morphism $\mathbf{f}\circ\mathbf{g}:\mathring{\mathbb{D}}^d(0,1)\to\mathring{\mathbb{D}}^d(0,1)$ is given by

(1.8.2)
$$(x_1,\cdots,x_d)\mapsto(\det(U)x_1+R_1,\cdots,\det(U)x_d+R_d),$$

where $R_i=\sum_{|\underline{n}|\geq p}a_{i,\underline{n}}\prod_{j=1}^d(\sum_{k=1}^d b_{j,k}x_k)^{n_j}$ involves only terms of order $\geq p$ for $1\leq i\leq d$. For $1\leq i\leq d$, we have basic estimations

(1.8.3)
$$v_\pi(\det(U)x_i)=\deg(G)+v_\pi(x_i)\quad\text{and}\quad v_\pi(R_i)\geq p\min_{1\leq j\leq d}\{v_\pi(x_j)\}.$$

Lemma 1.9. *For any rational number $a>\frac{p}{p-1}\deg(G)$, the map $\mathbf{g}$ induces an isomorphism of affinoid rigid spaces*
$$\mathbf{g}:\mathbf{D}^d(0,|\pi|^{a-\deg(G)})\xrightarrow{\sim}X^a_0.$$

Assuming this Lemma for a moment, we can complete the proof of 0.1 as follows. Consider the composite

$$\mathbf{h}=\mathbf{f}\circ\mathbf{g}|_{\mathbf{D}^d(0,|\pi|^{a-\deg(G)})}:\mathbf{D}^d(0,|\pi|^{a-\deg(G)})\xrightarrow{\sim}X^a_0\hookrightarrow X^a\xrightarrow{\mathbf{f}}\mathbf{D}^d(0,|\pi|^a).$$

In order to complete the proof of 0.1, we just need to prove that the inverse image $\mathbf{h}^{-1}(0)=\{0\}$. Let $(x_1,\cdots,x_d)$ be a point of $\mathbf{D}^d(0,|\pi|^{a-\deg(G)})$, and $(z_1,\cdots,z_d)=\mathbf{h}(x_1,\cdots,x_d)$. We may assume $v_\pi(x_1)=\min_{1\leq i\leq d}\{v_\pi(x_i)\}$. We have $v_\pi(x_1)\geq a-\deg(G)>\frac{1}{p-1}\deg(G)$ by the assumption on a. It follows thus from (1.8.3) that

$$v_\pi(R_1)\geq pv_\pi(x_1)>\deg(G)+v_\pi(x_1)=v_\pi(\det(U)x_1).$$

Hence, we deduce from (1.8.2) that $v_\pi(z_1)=\deg(G)+v_\pi(x_1)$. In particular, $z_1=0$ if and only if $x_1=0$. Therefore, we have $\mathbf{h}^{-1}(0)=\{0\}$. This achieves the proof of Theorem 0.1.

Proof of 1.9. Let ϵ be any rational number with $0 < \epsilon < \frac{p-1}{p}a - \deg(G)$. We will prove that

$$\mathbf{D}^d(0, |\pi|^{a-\deg(G)}) = \mathbf{D}^d(0, |\pi|^{a-\deg(G)-\epsilon}) \cap \mathbf{g}^{-1}(X^a).$$

This will imply that $\mathbf{D}^d(0, |\pi|^{a-\deg(G)})$ is a connected component of $\mathbf{g}^{-1}(X^a)$. Since $\mathbf{g} : \mathbf{A}^d_K \to \mathbf{A}^d_K$ is an isomorphism, the lemma will follow immediately.

We prove first the inclusion "$\subset$". It suffices to show $\mathbf{g}(\mathbf{D}^d(0, |\pi|^{a-\deg(G)})) \subset X^a$. Let $(x_1, \cdots, x_d)$ be a point of $\mathbf{D}^d(0, |\pi|^{a-\deg(G)})$. By (1.8.1), we have to check that $(z_1, \cdots, z_d) = \mathbf{f}(\mathbf{g}(x_1, \cdots, x_d))$ lies in $\mathbf{D}^d(0, |\pi|^a)$. We get from (1.8.3) that $v_\pi(\det(U)x_i) = \deg(G) + v_\pi(x_i) \geq a$ and $v_\pi(R_i) \geq p(a - \deg(G))$. As $a > \frac{p}{p-1}\deg(G)$, we have $v_\pi(R_i) > a$. It follows from (1.8.2) that

$$v_\pi(z_i) \geq \min\{v_\pi(\det(U)x_i), v_\pi(R_i)\} \geq a.$$

This proves $(z_1, \cdots, z_d)$ is contained in $\mathbf{D}^d(0, |\pi|^a)$, hence we have $\mathbf{g}(\mathbf{D}^d(0, |\pi|^{a-\deg(G)})) \subset X^a$.

To prove the inclusion "$\supset$", we just need to verify that every point in $\mathbf{D}^d(0, |\pi|^{a-\deg(G)-\epsilon})$ but outside $\mathbf{D}^d(0, |\pi|^{a-\deg(G)})$ does not lie in $\mathbf{g}^{-1}(X^a)$. Let $(x_1, \cdots, x_d)$ be such a point. We may assume that

$$(1.9.1) \quad a - \deg(G) - \epsilon \leq v_\pi(x_1) < a - \deg(G) \quad \text{and} \quad v_\pi(x_i) \geq a - \deg(G) - \epsilon \ \text{ for } 2 \leq i \leq d.$$

Let $(z_1, \cdots, z_d) = (\det(U)x_1 + R_d, \cdots, \det(U)x_d + R_d)$ be the image of $(x_1, \cdots, x_d)$ under the composite $\mathbf{f} \circ \mathbf{g}$. According to (1.8.1), the proof will be completed if we can prove that $(z_1, \cdots, z_d)$ is not in $\mathbf{D}^d(0, |\pi|^a)$. From (1.8.3) and (1.9.1), we get $v_\pi(\det(U)x_1) = \deg(G) + v_\pi(x_1) < a$ and $v_\pi(R_1) \geq p(a - \deg(G) - \epsilon)$. Thanks to the assumption on ϵ, we have $p(a - \deg(G) - \epsilon) > a$, so $v_\pi(z_1) = v_\pi(\det(U)x_1) < a$. This shows that $(z_1, \cdots, z_d)$ is not in $\mathbf{g}^{-1}(X^a)$, hence the proof of the lemma is complete.

$\square$

2. Applications to Canonical subgroups

In this section, we suppose the fraction field K has characteristic 0 and the residue field k is perfect of characteristic $p \geq 3$. Let e be the absolute ramification index of $\mathcal{O}_K$. For any rational number $\epsilon > 0$, we denote by $\mathcal{O}_{K,\epsilon}$ the quotient of $\mathcal{O}_K$ by the ideal consisting of elements with p-adic valuation greater or equal to ϵ.

2.1. First we recall some results on the canonical subgroups according to [AM04], [Ti06] and [Fa09]. Let $v_p : \mathcal{O}_K/p \to [0,1]$ be the truncated p-adic valuation (with $v_p(0) = 1$). Let G be a truncated Barsotti-Tate group of level $n \geq 1$ non-étale over $\mathcal{O}_K$, $G_1 = G \otimes_{\mathcal{O}_K} (\mathcal{O}_K/p)$. The Lie algebra of G_1, denoted by $\mathrm{Lie}(G_1)$ is a finite free $\mathcal{O}_K/p$-module. The Verschiebung homomorphism $V_{G_1} : G_1^{(p)} \to G_1$ induces a semi-linear endomorphism φ_{G_1} of $\mathrm{Lie}(G_1)$. We choose a basis of $\mathrm{Lie}(G_1)$ over $\mathcal{O}_K/p$, and let U be the matrix of φ under this basis. We define the Hodge height of G, denoted by $h(G)$, to be the truncated p-adic valuation of $\det(U)$. We note that the definition of $h(G)$ does not depend on the choice of U. The Hodge height of G is an analog of the Hasse invariant in mixed characteristic, and we have $h(G) = 0$ if and only if G is ordinary.

Theorem 2.2 ([Fa09] Théo. 4)**.** *Let G be a truncated Barsotti-Tate group of level 1 over $\mathcal{O}_K$ of dimension $d \geq 1$ and height h. Assume $h(G) < \frac{1}{2}$ if $p \geq 5$ and $h(G) < 1/3$ if $p = 3$.*

(i) *For any rational number $\frac{ep}{p-1}h(G) < a \leq \frac{ep}{p-1}(1 - h(G))$, the finite flat subgroup G^a of G given by the Abbes-Saito filtration has rank p^d.*

(ii) *Let C be the subgroup $G^{\frac{ep}{p-1}(1-h(G))}$ of G. We have* $\deg(G/C) = eh(G)$.

(iii) *The subgroup $C \otimes \mathcal{O}_{K,1-h(G)}$ coincides with the kernel of the Frobenius homomorphism of $G \otimes \mathcal{O}_{K,1-h(G)}$. Moreover, for any rational number ϵ with $\frac{h(G)}{p-1} < \epsilon \leq 1 - h(G)$, if H is a finite and flat closed subgroup of G such that $H \otimes \mathcal{O}_{K,\epsilon}$ coincides with the kernel of Frobenius of $G \otimes \mathcal{O}_{K,\epsilon}$, then we have $H = C$.*

The subgroup C in this theorem, when it exists, is called the *canonical subgroup (of level 1)* of G.

Remark 2.3. (i) The conventions here are slightly different from those in [Fa09]. The Hodge height is called Hasse invariant in *loc. cit.*, while we choose to follow the terminologies in [AM04] and [Ti06]. Our index of Abbes-Saito filtration and the degree of G are e times those in [Fa09].

(ii) Statement (iii) of the theorem is not explicitly stated in [Fa09, Théo. 4], but it's an easy consequence of *loc. cit.* Prop. 11.

For the canonical subgroups of higher level, we have

Theorem 2.4 ([Fa09] Théo. 6)**.** *Let G be a truncated Barsotti-Tate group of level n over $\mathcal{O}_K$ of dimension $d \geq 1$ and height h. Assume $h(G) < \frac{1}{3^n}$ if $p = 3$ and $h(G) < \frac{1}{2p^{n-1}}$ if $p \geq 5$.*

(i) *There exists a unique closed subgroup of G that is finite and flat over $\mathcal{O}_K$ and satisfies*

- *$C_n(\overline{K})$ is free of rank d over $\mathbf{Z}/p^n\mathbf{Z}$.*
- *For each integer i with $1 \leq i \leq n$, let C_i be the scheme theoretic closure of $C_n(\overline{K})[p^i]$ in G. Then the subgroup $C_i \otimes \mathcal{O}_{K,1-p^{i-1}h(G)}$ coincides with the kernel of the i-th iterated Frobenius of $G \otimes \mathcal{O}_{K,1-p^{i-1}h(G)}$.*

(ii) *We have* $\deg(G/C_n) = \frac{e(p^n-1)}{p-1}h(G)$.

The subgroup C_n in the theorem above is called the canonical subgroup of level n of G. Fargues actually proves that C_n is a certain piece of the Harder-Narasimhan filtration of G. The aim of this section is to show that C_n appears also in the Abbes-Saito filtration.

Theorem 2.5. *Let G be a truncated Barsotti-Tate group of level n over $\mathcal{O}_K$ satisfying the assumptions in 2.4, and C_n be its canonical subgroup of level n. Then for any rational number a satisfying $\frac{ep(p^n-1)}{(p-1)^2}h(G) < a \leq \frac{ep}{p-1}(1 - h(G))$, we have $G^a = C_n$.*

Proof. We proceed by induction on n. If $n = 1$, the theorem is 2.2(i). We suppose $n \geq 2$ and the theorem is valid for truncated Barsotti-Tate groups of level $n - 1$. For each integer i with $1 \leq i \leq n$, let G_i denote the scheme theoretic closure of $G(\overline{K})[p^i]$ in G, and C_i the scheme theoretic closure of $C_n(\overline{K})[p^i]$ in C_n. By Theorem 2.4(i), it's clear that C_i is the canonical subgroup of level i of G_i. Let a be a rational number with

$\frac{ep(p^n-1)}{(p-1)^2}h(G) < a \leq \frac{ep}{p-1}(1-h(G))$. By the induction hypothesis and the functoriality of Abbes-Saito filtration 1.3(ii), we have $C_{n-1}(\overline{K}) = G^a_{n-1}(\overline{K}) \subset G^a(\overline{K})$, and the image of $G^a(\overline{K})$ in $G_1(\overline{K})$ is exactly $C_1(\overline{K}) = G^a_1(\overline{K})$. Note that we have a commutative diagram of exact sequences of groups

$$\begin{tikzcd}
0 \arrow[r] & C_{n-1}(\overline{K}) \arrow[r] \arrow[d, hook] & C_n(\overline{K}) \arrow[r] \arrow[d, hook] & C_1(\overline{K}) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & G_{n-1}(\overline{K}) \arrow[r] & G(\overline{K}) \arrow[r, "\times p^{n-1}"] & G_1(\overline{K}) \arrow[r] & 0,
\end{tikzcd}$$

where vertical arrows are natural inclusions. So we have $C_n(\overline{K}) \subset G^a(\overline{K})$. On the other hand, Theorems 0.1 and 2.4(ii) imply that $(G/C_n)^a(\overline{K}) = 0$ as $a > \frac{ep(p^n-1)}{(p-1)^2}h(G) = \frac{p}{p-1}\deg(G/C_n)$. Therefore, we get $G^a(\overline{K}) \subset C_n(\overline{K})$ by 1.3(ii). This completes the proof. □

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