Euler System of CM Points on Shimura Curves

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# Contents

1 Introduction 5
   1.1 The Main Results 5
   1.2 The BSD Conjecture 13

2 Galois Cohomology 15
   2.1 Local Cohomology 15
      2.1.1 Some General Lemmas 15
      2.1.2 Some Local Cohomology Computation 20
   2.2 Global Cohomology 24
      2.2.1 Explicit Description of Selmer Groups 25
      2.2.2 Global Duality 34

3 CM Points 37
   3.1 Euler System of CM Points 37
      3.1.1 CM Points on Shimura Curves 37
      3.1.2 Properties of the Euler System 39
   3.2 Derived Cohomology Classes 45
      3.2.1 Derivative Operators and Derived Classes 45
      3.2.2 Properties of the Derivative Classes 47
   3.3 Bounding the Selmer Group 49

4 Congruence 55
   4.1 Reduction at Bad Places 55
      4.1.1 Reduction of the Shimura Curve 55
      4.1.2 Reduction of the Jacobian 58
   4.2 Cohomology Classes from Congruences 61
      4.2.1 Congruence of Modular Forms 61
      4.2.2 The Non-triviality of the Cohomology Classes 68
   4.3 Bounding the Selmer Group 70
5 Application to Diopantine Equations 75
  5.1 Twisted Fermat Curves 75

6 Galois Images 79
  6.1 Introduction 79
  6.2 Mumford-Tate Group 80
    6.2.1 CM Cases 82
  6.3 Openness 84
    6.3.1 Local Openness 84
    6.3.2 Surjectivity on $\mod \ell$ Points 84
    6.3.3 Global Openness 86
    6.3.4 CM Case 87
  6.4 Geometric Simplicity 87
    6.4.1 Mod $\Lambda$ Images 88
    6.4.2 Geometric Simplicity 91
Chapter 1

Introduction

1.1 The Main Results

Let $A$ be an abelian variety defined over a number field $F$, $K$ a finite extension of $F$. The Birch and Swinnerton-Dyer (BSD, in short) conjecture for $(A, K)$ predicts that the rank of the Mordell-Weil group $A(K)$ is equal to the vanishing order at $s = 1$ of the L-function $L(s, A/K)$ of $A$ over $K$:

$$\text{rank}_{\mathbb{Z}} A(K) = \text{ord}_{s=1} L(s, A/K). \quad (1)$$

There is a refined version of the BSD conjecture on the leading coefficient of the Taylor expansion at $s = 1$ of the L-function in terms of arithmetic invariants of $A$ (see §1.2). In this book, we give some evidence to these conjectures.

Assume that $A$ is an abelian variety over a totally real field $F$, and is associated with an automorphic form $\phi$ for $GL_2, \mathbb{F}$ of weight $(2, \cdots, 2)$, conductor $N$, and with trivial central character, in the sense that

$$L_v(s, A) = \prod_{\sigma} L_v(s - \frac{1}{2}, \phi^\sigma),$$

for all finite places $v$ of $F$, where $\phi^\sigma$ are all distinct conjugations of $\phi$ under $\sigma \in \text{Aut}(\mathbb{C})$. Then $A$ has real multiplication over $F$ by the subring $\mathbb{Z}[\phi]$ of $\mathbb{C}$ generated over $\mathbb{Z}$ by Hecke eigenvalues of $\phi$. By an isogenous, we may assume $A$ has full real multiplication by $\mathcal{O}_\phi$, the integral closure of $\mathbb{Z}[\phi]$. Let $K/F$ be a totally imaginary quadratic extension with discriminant prime to $N$. Using the generalization of Gross-Zagier formula to totally real fields by the second author [24], we know that if $\text{ord}_{s=1/2} L(s, \phi_K) \leq 1$ then

$$\text{ord}_{s=1} L(s, A/K) = [\mathcal{O}_\phi : \mathbb{Z}] \cdot \text{ord}_{s=1/2} L(s, \phi_K). \quad (2)$$

Theorem 1.1.1. If $\text{ord}_{s=1/2} L(s, \phi_K) = 1$, then $\text{rank}_{\mathbb{Z}} A(K) = \text{ord}_{s=1} L(s, A/K)$ and the Shafarevich-Tate group $\text{III}(K, A)$ is finite.

If $\text{ord}_{s=1/2} L(s, \phi_K) = 0$ and $A$ does not have complex multiplication, then $A(K)$ is finite. If furthermore, $A$ is geometrically simple, then $\text{III}(K, A)$ is also finite.

With a non-vanishing result [10], we have