HEEGNER POINTS OVER TOWERS OF KUMMER EXTENSIONS

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1. Introduction

Let E be an elliptic curve defined over a number field F. For each finite extension L of F, write $r_E(L)$ for the rank of the group $E(L)$ of L-rational points on E. A *p-adic Lie extension* of F is a Galois extension L_∞/F whose Galois group G is a p-adic Lie group (for example, the splitting field of any continuous representation of the absolute Galois group of F acting on a finite dimensional $\mathbb{Q}_p$-vector space). The present note is motivated by the following general problem:

Question 1.1. *To understand the variation of $r_E(L)$ as L ranges over all finite extensions of F contained in L_∞.*

This question dates back at least to the foundational article [Ma], which considers the case when $G = \mathbb{Z}_p$, and makes the first steps towards examining this problem by the methods of Iwasawa theory. As in classical descent theory, it is convenient to replace the Mordell-Weil group $E(L)$ by the p-power Selmer group of E over L, thus sidestepping the difficulties associated with the Shafarevich-Tate conjecture. This Selmer group is defined to be

$$\mathrm{Sel}_p(E/L) := \ker\left(H^1(L, E[p^\infty]) \longrightarrow \bigoplus_v H^1(L_v, E)[p^\infty]\right), \qquad (1.1)$$

where $E[p^\infty]$ denotes the Galois module of all p-power division points on E, and v runs over all places of L. The idea of Iwasawa theory is to exploit the structure of the Selmer group of E over L_∞ as a module for the Galois group G to show that the groups $\mathrm{Sel}_p(E/L)$ exhibit some coherence as L varies.

A rich, well-developed theory now paints a fairly precise picture when $F = \mathbb{Q}$ and G is either abelian or dihedral.

The last decade has seen the emergence of a program of non-abelian Iwasawa theory whose goal is to study Question 1.1 in settings which are further removed from the abelian setting. A prototypical example is the case where $L_\infty = \mathbb{Q}(A[p^\infty])$ is the field generated over $\mathbb{Q}$ by the coordinates of the p-power division points of an elliptic curve A over $\mathbb{Q}$. The article [Har] exhibits cases where $r_E(\mathbb{Q}(A[p^n]))$ is unbounded with n, but it is fair to say that the type of growth it could exhibit is at present only poorly understood.