EKOR STRATA FOR SHIMURA VARIETIES WITH PARAHORIC LEVEL STRUCTURE

XU SHEN, CHIA-FU YU, AND CHAO ZHANG

Abstract. In this paper we study the geometry of reduction modulo p of the Kisin-Pappas integral models for certain Shimura varieties of abelian type with parahoric level structure. We give some direct and geometric constructions for the EKOR strata on these Shimura varieties, using the theories of G-zips and mixed characteristic local $\mathcal{G}$-Shtukas. We establish several basic properties of these strata, including the smoothness, dimension formula, and closure relation. Moreover, we apply our results to the study of Newton strata and central leaves on these Shimura varieties.

Contents

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Introduction

In this paper, we study the geometry of reduction modulo p of the Kisin-Pappas integral models ([37]) for certain Shimura varieties of abelian type with parahoric level structure. Our goal is to develop a geometric approach to define and study the EKOR (*Ekedahl-Kottwitz-Oort-Rapoport*) stratifications on the reductions of these Shimura varieties, which were first introduced by He and Rapoport ([31]) in a different setting.

The reduction modulo p of Shimura varieties admits very rich geometric structures. We refer to the excellent survey papers [68, 69, 17] for some overview. In this paper, we focus on one important perspective of the mod p geometry of Shimura varieties with parahoric level structure. In [31], He and Rapoport have proposed a general guideline to study the geometry of reduction modulo p of Shimura varieties with parahoric level structure. In particular, they postulated *five basic axioms* ([31] section 3) on the integral models of Shimura varieties. Assuming the verification of these axioms, in [31] section 6, He and Rapoport introduced the EKOR stratification on the special fibers of these integral models, based on the works of Lusztig and He on G-stable piece decompositions ([51, 52, 24]). This new stratification has the following key features:

- for hyperspecial levels, by works of Viehmann ([80]), the EKOR stratification coincides with the EO (*Ekedahl-Oort*) stratification;
- for Iwahori levels, by works of Lusztig and He ([52, 24] and [31, Corollary 6.2]), it coincides with the KR (*Kottwitz-Rapoport*) stratification;
- for a general parahoric level, the EKOR stratification is a *refinement* of the KR stratification.

The finer structure of EKOR stratification makes it easier to be compared with other natural stratifications, e.g. the *Newton* stratification. In fact, under their axioms He-Rapoport showed that for a general parahoric level, each Newton stratum contains a certain EKOR stratum ([31] Theorem 6.18), while in general there is no KR stratum that is entirely contained in a given Newton stratum. One can expect that the geometry of EKOR strata will lead to interesting arithmetic applications, see [7, 83] for examples in the good reduction case.

In [31] section 7, the He-Rapoport axioms were verified for the Siegel modular varieties. For PEL-type Shimura varieties associated to unramified groups of type A and C and to odd ramified unitary groups, these axioms were verified by He-Zhou in [32]. For certain Shimura varieties of Hodge type associated to tamely ramified and residually split groups, in [95] Zhou has verified these axioms for the Kisin-Pappas integral models constructed in [37]. Hence in these cases we get the EKOR stratifications by [31] section 6. For a hyperspecial or an Iwahori level, by the above works of Viehmann or Lusztig and He, some basic geometric properties of EKOR strata, like smoothness and quasi-affineness, are known by the corresponding geometric properties of EO or KR strata. However, for a general parahoric level, even if one had verified the He-Rapoport axioms, the smoothness of EKOR strata was still unknown. In fact, the geometric meaning of EKOR types was quite mysterious, since the construction in [31] is purely group theoretic.

In this paper we would like to find a direct and geometric construction of these strata, for some concrete integral models. More precisely, similar to [95], we also work with the concrete integral models constructed by Kisin and Pappas in [37] for certain Shimura varieties of abelian type. The main results of Kisin and Pappas are that, under certain conditions, there exist some local model diagrams[1], which relate the integral models of Shimura varieties with the Pappas-Zhu local model schemes [65]. Roughly speaking, our construction of the EKOR stratification will be about certain refinement of the local model diagram in characteristic p. It can be viewed as a geometric realization of the group theoretical considerations in [31] section 6.

To state our main results, we need some notations. Let $p > 2$ be a fixed prime throughout the paper. Let (G, X) be a Shimura datum of abelian type, $\mathsf{K} = K_pK^p \subset G(\mathbb{A}_f)$ an open compact subgroup with $K^p \subset G(\mathbb{A}_f^p)$ sufficiently small and $K_p \subset G(\mathbb{Q}_p)$ a *parahoric* subgroup. We have the associated Shimura variety $\mathrm{Sh}_\mathsf{K} = \mathrm{Sh}_\mathsf{K}(G, X)$ over the reflex field E. Let $v|p$ be a place of E and $E = \mathsf{E}_v$. Let x be a point of the Bruhat-Tits building $\mathcal{B}(G, \mathbb{Q}_p)$, with the attached Bruhat-Tits stabilizer group scheme $\mathcal{G} = \mathcal{G}_x$ and its neutral connected component $\mathcal{G}^\circ = \mathcal{G}_x^\circ$, such that $K_p = \mathcal{G}^\circ(\mathbb{Z}_p)$. Assume that G splits over a tamely ramified extension of $\mathbb{Q}_p$ and $p \nmid |\pi_1(G^{\mathrm{der}})|$. We will consider the following cases:

- (G, X) is of Hodge type and $\mathcal{G} = \mathcal{G}^\circ$.
- (G, X) is of abelian type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$ (we refer to [4] Table 2.3.8 for the precise meaning of type $D^{\mathbb{H}}$).
- (G, X) is of abelian type such that G is unramified over $\mathbb{Q}_p$ and K_p is contained in some hyperspecial subgroup of $G(\mathbb{Q}_p)$.

Let $\mathscr{S}_\mathsf{K} = \mathscr{S}_\mathsf{K}(G, X)$ be the integral model over O_E of the Shimura variety Sh_K constructed by Kisin-Pappas in [37]. The above cases are exactly when we have the local model diagram by Theorem 4.2.7 (for the first case) and Theorem 4.6.23 (for the last two cases) of [37]. This is also why we restrict to Shimura data in the above cases (note that each case may overlap with the others, and the first case is used to deduce the other cases in [37] 4.6). We are interested in the geometry of the special fiber $\mathscr{S}_{\mathsf{K},0}$ over $k = \overline{\mathbb{F}}_p$. When K_p is hyperspecial (thus G is unramified), $\mathscr{S}_{\mathsf{K},0}$ is smooth. In the general case, by [37, Corollary 0.3], $\mathscr{S}_{\mathsf{K},0}$ is reduced, and the strict henselizations of the local rings on $\mathscr{S}_{\mathsf{K},0}$ have irreducible components which are normal and Cohen-Macaulay. We will simply write $\mathscr{S}_0$ for the special fiber when the level K is fixed.

We have in fact two approaches for the constructions of EKOR strata on $\mathscr{S}_0$:

- a local construction which uses the theory of G-zips due to Moonen-Wedhorn [55] and Pink-Wedhorn-Ziegler [66, 67]. Here "local" means that we first work on a fixed KR stratum; then we let the KR stratum vary;
- a global construction which uses the theory of local $\mathcal{G}$-Shtukas, generalizing some constructions of Xiao-Zhu in [84] (see also [94]) in the case of good reductions. Here "global" means that we work directly on the whole special fiber (up to perfection).

To explain the ideas, we assume that (G, X) is of Hodge type for simplicity. Let $G = G_{\mathbb{Q}_p}$ and $\{\mu\}$ be the attached Hodge cocharacter of G. Kottwitz and Rapoport defined a finite

[1]The existence of the local model diagram is in fact one of the He-Rapoport axioms, and of course, one hopes that eventually all the He-Rapoport axioms should be verified for the Kisin-Pappas integral models.

subset

$$\mathrm{Adm}(\{\mu\})$$

of the Iwahori Weyl group of G, *the* $\{\mu\}$*-admissible set*, cf. [44, 69]. Writing $K = K_p$, from $\mathrm{Adm}(\{\mu\})$ we get finite sets $\mathrm{Adm}(\{\mu\})_K$ and ${}^K\mathrm{Adm}(\{\mu\})$, which will parametrize the types of the KR stratification and EKOR stratification of level K respectively. We have the relations

$$\mathrm{Adm}(\{\mu\}) \supset {}^K\mathrm{Adm}(\{\mu\}) \twoheadrightarrow \mathrm{Adm}(\{\mu\})_K.$$

We have a partial order $\leq_K$ on $\mathrm{Adm}(\{\mu\})_K$, induced from the Bruhat order on the associated affine Weyl group. On the finite set ${}^K\mathrm{Adm}(\{\mu\})$, we have also a partial order $\leq_{K,\sigma}$, introduced by He in [22] section 4. See 1.2 for more details on these sets $\mathrm{Adm}(\{\mu\}), \mathrm{Adm}(\{\mu\})_K$ and ${}^K\mathrm{Adm}(\{\mu\})$.

The starting point of our local construction is the observation that the EKOR stratification (constructed in [31] section 6) is a refinement of the KR stratification, and for a KR type $w \in \mathrm{Adm}(\{\mu\})_K$, there is always an attached algebraic zip datum $\mathcal{Z}_w$ (see 1.3.6) in the sense of Pink-Wedhorn-Ziegler. Let $\mathcal{G}_0 = \mathcal{G} \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p$ and $\mathcal{G}_0^{\mathrm{rdt}}$ be its reductive quotient. Let M^{loc} be the special fiber over $k = \overline{\mathbb{F}}_p$ of the Pappas-Zhu local model scheme attached to the triple $(G, \{\mu\}, K)$ constructed in [65]. Then by construction $\mathcal{G}_0$ acts on M^{loc} and the underling topological space $|[\mathcal{G}_0\backslash M^{\mathrm{loc}}]|$ of the quotient stack $[\mathcal{G}_0\backslash M^{\mathrm{loc}}]$ is homeomorphic to $\mathrm{Adm}(\{\mu\})_K$ (for which the topology is defined by its partial order $\leq_K$). Under the above assumptions, the existence of the local model diagram gives us a morphism of algebraic stacks

$$\lambda_K : \mathscr{S}_0 \to [\mathcal{G}_0\backslash M^{\mathrm{loc}}],$$

such that the fibers are the KR strata of $\mathscr{S}_0$. We identify the finite Weyl group W_K attached to K with the Weyl group of $\mathcal{G}_0^{\mathrm{rdt}}$. There exists an explicit set J_w (see 1.3.6) of simple reflections in W_K defined from w and K. After choosing a suitable Siegel embedding, we can construct a $\mathcal{G}_0^{\mathrm{rdt}}$-zip of type J_w on the KR stratum $\mathscr{S}_0^w$ attached to w. Let $\mathcal{G}_0^{\mathrm{rdt}}$-$Zip_{J_w}$ be the algebraic stack over k of $\mathcal{G}_0^{\mathrm{rdt}}$-zips of type J_w. By [67] we have an isomorphism of algebraic stacks $\mathcal{G}_0^{\mathrm{rdt}}$-$Zip_{J_w} \simeq [E_{\mathcal{Z}_w}\backslash \mathcal{G}_0^{\mathrm{rdt}}]$, where $E_{\mathcal{Z}_w}$ is the zip group attached to $\mathcal{Z}_w$, see Definition 1.1.1. Thus we get a morphism of algebraic stacks

$$\zeta_w : \mathscr{S}_0^w \to [E_{\mathcal{Z}_w}\backslash \mathcal{G}_0^{\mathrm{rdt}}]$$

whose fibers are precisely the EKOR strata in $\mathscr{S}_0^w$.

Let $\pi_K : {}^K\mathrm{Adm}(\{\mu\}) \to \mathrm{Adm}(\{\mu\})_K$ be the natural surjection, then the local construction gives us a geometrization of $\pi_K^{-1}(w)$, as we have the following bijection

$$\pi_K^{-1}(w) \simeq {}^{J_w}W_K \simeq |[E_{\mathcal{Z}_w}\backslash \mathcal{G}_0^{\mathrm{rdt}}]|,$$

where ${}^{J_w}W_K \subset W_K$ is the set of minimal length representatives for $W_{J_w}\backslash W_K$. The strategy of the construction is similar to [92], but making more systematical use of the local model diagram. We refer to subsections 3.3 and 3.4 for details of the construction.

The main property of the morphism ζ_w is the following result.

Theorem A (Theorem 3.4.11)**.** *The morphism* ζ_w *is smooth.*

As a consequence, we can translate many geometric properties of the quotient stack $[E_{\mathcal{Z}_w}\backslash \mathcal{G}_0^{\mathrm{rdt}}]$ to those of $\mathscr{S}_0^w$. In particular, each EKOR stratum is a locally closed smooth subvariety of $\mathscr{S}_0^w$, and the closure of an EKOR stratum in $\mathscr{S}_0^w$ is a union of EKOR strata. Moreover, we can also describe the dimension of an EKOR stratum (if non-empty) once we know the dimension of the KR stratum containing it.

The disadvantage of the local construction is that the type J_w varies on different KR strata. The theory developed in [66, 67] is not enough to put these $\mathcal{G}_0^{\mathrm{rdt}}$-zips of different types uniformly together. In particular, when we want to show the closure relation of EKOR strata defined locally as above, we will meet a serious problem. To overcome these difficulties, in section 4 we adapt some ideas of Xiao and Zhu in [84]. One of the key observations is that by the works of Scholze ([76, Corollary 21.6.10]) and He-Pappas-Rapoport ([30, Theorem 2.15]), the (perfection of the) special fiber of the Pappas-Zhu local model admits an embedding into the Witt vector affine flag varieties $\mathrm{Gr}_{\mathcal{G}}$ ([93, 1]). So we introduce the notions of local $(\mathcal{G},\mu)$-Shtukas and their truncations in level 1, cf. Definition 4.1.3 and subsection 4.2, generalizing those in [84] in the unramified case. Roughly, a local $(\mathcal{G},\mu)$-Shtuka over a perfect ring R is a $\mathcal{G}$-torsor $\mathcal{E}$ over $W(R)$, together with an isomorphism $\beta: \sigma^*\mathcal{E}[\frac{1}{p}] \xrightarrow{\sim} \mathcal{E}[\frac{1}{p}]$ over $W(R)[\frac{1}{p}]$, such that the relative position between $\sigma^*\mathcal{E}$ and $\mathcal{E}$ at any geometric point of Spec R lies in $\mathrm{Adm}(\{\mu\})_K$. Here $\sigma: W(R)\to W(R)$ is the Frobenius. Let Aff_k^{pf} be the category of perfect k-algebras. We need to pass to the world of perfect algebraic geometry as in [93] and [84].

Consider the prestack $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}}$ of local $(\mathcal{G},\mu)$-Shtukas, which can be described as (cf. Lemma 4.1.4)

$$\mathrm{Sht}_{\mu,K}^{\mathrm{loc}} \simeq \Big[\frac{M^{\mathrm{loc},\infty}}{Ad_\sigma L^+\mathcal{G}}\Big],$$

where $M^{\mathrm{loc},\infty}\subset LG$ is the pre-image of $M^{\mathrm{loc}}\subset \mathrm{Gr}_{\mathcal{G}} = LG/L^+\mathcal{G}$, and the quotient means that we take the σ-conjugation action of $L^+\mathcal{G}$ on $M^{\mathrm{loc},\infty}$. Consider the reductive 1-truncation group $L^{1-\mathrm{rdt}}\mathcal{G}$, i.e. for any $R\in \mathrm{Aff}_k^{pf}$, $L^{1-\mathrm{rdt}}\mathcal{G}(R)=\mathcal{G}_0^{\mathrm{rdt}}(R)$. Let $L^+\mathcal{G}^{(1)-\mathrm{rdt}} := \ker(L^+\mathcal{G}\to L^{1-\mathrm{rdt}}\mathcal{G})$ and

$$M^{\mathrm{loc},(1)-\mathrm{rdt}}\subset LG/L^+\mathcal{G}^{(1)-\mathrm{rdt}}$$

be the image of $M^{\mathrm{loc},\infty}\subset LG$ under the projection $LG\to LG/L^+\mathcal{G}^{(1)-\mathrm{rdt}}$. Fix any integer $m\geq 2$, we have the algebraic stack of $(m,1)$-restricted local $(\mathcal{G},\mu)$-Shtukas

$$\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)} = \Big[\frac{M^{\mathrm{loc},(1)-\mathrm{rdt}}}{Ad_\sigma L^m\mathcal{G}}\Big].$$

Here $L^m\mathcal{G}$ is the m-truncation group, i.e. for any $R\in\mathrm{Aff}_k^{pf}$, $L^m\mathcal{G}(R)=\mathcal{G}(W_m(R))$. There are natural perfectly smooth morphisms

$$\mathrm{Sht}_{\mu,K}^{\mathrm{loc}}\to \mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)} \xrightarrow{\pi_K} [\mathcal{G}_0\backslash M^{\mathrm{loc}}].$$

Recall that we have a homeomorphism of topological spaces $\mathrm{Adm}(\{\mu\})_K\simeq |[\mathcal{G}_0\backslash M^{\mathrm{loc}}]|$. By the works of Lusztig and He, we have a homeomorphism of topological spaces (see Lemma 4.2.4)

$${}^K\mathrm{Adm}(\{\mu\})\simeq |\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)}|,$$

where the topology on ${}^K\mathrm{Adm}(\{\mu\})$ is defined by the partial order $\leq_{K,\sigma}$. By the works of Hamacher-Kim [20] and Pappas [59], we have a universal local $(\mathcal{G},\mu)$-Shtukas over $\mathscr{S}_0^{pf}$, the perfection of the special fiber of our Shimura variety. Taking its $(m,1)$-restriction, we get a morphism of algebraic stacks

$$\upsilon_K: \mathscr{S}_0^{pf}\to \mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)},$$

which lifts the morphism of algebraic stacks $\lambda_K:\mathscr{S}_0^{pf}\to[\mathcal{G}_0\backslash M^{\mathrm{loc}}]$ induced by the local model digram.

Theorem B (Theorem 4.4.3)**.** *The morphism υ_K is perfectly smooth.*

The relation between the local and global constructions is as follows. Recall that we have $\pi_K:\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)}\to[\mathcal{G}_0\backslash M^{\mathrm{loc}}]$. For any $w\in|[\mathcal{G}_0\backslash M^{\mathrm{loc}}]|$, there is a natural perfectly smooth morphism

$$\pi_K^{-1}(w)\to [E_{\mathcal{Z}_w}\backslash \mathcal{G}_0^{\mathrm{rdt}}]^{pf}$$

which induces a homeomorphism of underlying topological spaces, see Proposition 4.2.6. Here $[E_{\mathcal{Z}_w}\backslash\mathcal{G}_0^{\mathrm{rdt}}]^{pf}$ is the perfection of the algebraic stack $[E_{\mathcal{Z}_w}\backslash\mathcal{G}_0^{\mathrm{rdt}}]$. The morphism

υ_K thus interpolates the morphisms ζ_w^{pf} when $w \in \mathrm{Adm}(\{\mu\})_K$ varies.

The fibers of υ_K give the EKOR strata on $\mathscr{S}_0^{pf}$. Since the perfection does not change underlying topological spaces, we get the closure relation of EKOR strata on $\mathscr{S}_0$.

We collect some of the main results for EKOR strata as follows.

Theorem C. (1) (Theorem 3.4.12, Theorem 5.4.5 (1), Corollary 4.4.4) *We have the EKOR stratification*

$$\mathscr{S}_0 = \coprod_{x \in {}^K\mathrm{Adm}(\{\mu\})} \mathscr{S}_0^x.$$

For each $x \in {}^K\mathrm{Adm}(\{\mu\})$, the corresponding EKOR stratum $\mathscr{S}_0^x$ is a locally closed subvariety which is non-empty, smooth, equi-dimensional of dimension $\ell(x)$. Moreover, we have the closure relation

$$\overline{\mathscr{S}_0^x} = \coprod_{x' \leq_{K,\sigma} x} \mathscr{S}_0^{x'}.$$

(2) (Theorem 3.5.9, Theorem 5.4.5 (2)) *If K is Iwahori, then any KR stratum is quasi-affine. In general, if the He-Rapoport axiom 4 (c) holds, any EKOR stratum is quasi-affine. In particular, if $G_{\mathbb{Q}_p}$ is residually split, then any EKOR stratum is quasi-affine.*

In (1), the non-emptiness is deduced by results of several other authors, see for example [38, 91, 95]. We refer to Corollary 3.5.3 for more details. The smoothness and dimension formula are conjectures of He and Rapoport in [31], based on their axioms together with some different observations (see also our discussions after Theorem D). It is an interesting question to investigate the singularities of the closure of an EKOR stratum. If the level K is Iwahori, then it is known that each closure of KR stratum is normal and Cohen-Macaulay, see [37, Corollary 4.2.12] and [65, Theorem 1.2]. We don't know whether this is true for a general parahoric level.

Quasi-affineness of EKOR strata is also conjectured by He and Rapoport in [31]. Statement (2) says that this conjecture holds as long as the He-Rapoport axiom 4 (c) holds. It is reduced to quasi-affineness of Iwahori KR strata using Theorem D. Noting that axiom 4 (c) has been verified by Zhou if $G_{\mathbb{Q}_p}$ is residually split [95], EKOR strata are quasi-affine in this case. Quasi-affineness of Iwahori KR strata in the Siegel case was known by the work of Görtz-Yu [16].

It is somehow surprising that quasi-affineness of Iwahori KR strata is proved using techniques to study EKOR strata: KR strata are defined using the local model diagram, so it sounds reasonable to expect that one could study Iwahori KR strata simply using the local model diagram. On the contrary, our proof of the quasi-affineness of Iwahori KR strata does use techniques from the study of EKOR strata, and in particular it involves the morphism ζ_w which contains the geometric information that can not be seen from the local model diagram. We refer to the proof of Theorem 3.5.9 for more details.

We can define various ordinary loci and superspecial loci using EKOR strata. More precisely,

(1) we can do this for each KR stratum. Namely, for $w \in \mathrm{Adm}(\{\mu\})_K$, in $\mathscr{S}_0^w$,
   (a) there is a unique EKOR stratum, namely $\mathscr{S}_0^{{}^K w_K}$ with ${}^K w_K$ as in 1.2.6. This stratum is open dense in $\mathscr{S}_0^w$ by the smoothness of the map ζ_w, and is called the *w-ordinary locus*;
   (b) there is a unique EKOR stratum, namely $\mathscr{S}_0^{x_w}$. Here x_w is as in 1.2.8. This stratum is closed in $\mathscr{S}_0^w$, and is called the *w-superspecial locus*;

(2) we can also do this globally in $\mathscr{S}_0$. Namely,

(a) the *ordinary locus* $\mathscr{S}_0^{\mathrm{ord}}$ is the union of EKOR strata $\mathscr{S}_0^x$ with $\ell(x) = \dim(\mathscr{S}_0)$. It is clear by definition that

$$\mathscr{S}_0^{\mathrm{ord}} = \coprod_{\substack{w \in \mathrm{Adm}(\{\mu\})_K, \\ \ell({}^K w_K) = \dim(\mathscr{S}_0)}} \mathscr{S}_0^{{}^K w_K},$$

and it is an open dense smooth subscheme in $\mathscr{S}_0$ (note that the density follows from the smoothness of the maps ζ_w for all $w \in \mathrm{Adm}(\{\mu\})_K$). By a result of He-Nie in [29] (see the following Proposition 1.2.4) we can rewrite the above disjoint union as

$$\mathscr{S}_0^{\mathrm{ord}} = \coprod_{\mu' \in W_0(\underline{\mu}), t^{\mu'} \in {}^K\widetilde{W}} \mathscr{S}_0^{t^{\mu'}}.$$

We refer to Proposition 1.2.4 for the precise meaning of the notation $W_0(\underline{\mu})$,

(b) the *superspecial locus* is the EKOR strata $\mathscr{S}_0^{\tau}$, where τ is as in 1.2.5. $\mathscr{S}_0^{\tau}$ is the unique closed EKOR stratum in $\mathscr{S}_0$, and it is of dimension 0. Moreover, $\mathscr{S}_0^{\tau}$ is a central leaf (see [34] for some discussions on central leaves in the reduction of Hodge-type Shimura varieties with parahoric level structure).

Let $I \subset K$ be an Iwahori subgroup. Fix the prime to p level K^p and we simply write $\mathscr{S}_I = \mathscr{S}_{IK^p}$ and $\mathscr{S}_K = \mathscr{S}_{KK^p}$. Then by [95] section 7, we have a morphism between special fibers

$$\pi_{I,K} : \mathscr{S}_{I,0} \to \mathscr{S}_{K,0}.$$

Theorem D (Theorem 3.5.5, Theorem 5.4.5 (3))**.** *For $x \in {}^K\mathrm{Adm}(\{\mu\})$ viewed as an element of ${}^I\mathrm{Adm}(\{\mu\}) = \mathrm{Adm}(\{\mu\})$, the morphism*

$$\pi_{I,K}^x : \mathscr{S}_{I,0}^x \to \mathscr{S}_{K,0}^x$$

induced by $\pi_{I,K}$ is finite étale. If in addition the He-Rapoport axiom 4 (c) is satisfied, $\pi_{I,K}^x$ is a finite étale covering.

The étaleness of $\pi_{I,K}^x$ is also a conjecture of He and Rapoport in [31]. They prove that $\pi_{I,K}^x$ is finite and surjective assuming their five axioms, and deduce the dimension formula from it. If $\pi_{I,K}^x$ is, in addition, étale, then $\pi_{I,K}^x : \mathscr{S}_{I,0}^x \to \mathscr{S}_{K,0}^x$ becomes a finite étale covering, and hence the smoothness of $\mathscr{S}_{K,0}^x$ follows directly from that of $\mathscr{S}_{I,0}^x$. We only need to prove the étaleness of $\pi_{I,K}^x$, as the finiteness is actually a consequence of the axioms (without axiom 4 (c)) together with the étaleness of $\pi_{I,K}^x$.

We can apply the results of EKOR strata to study Newton strata and central leaves for reductions of Shimura varieties with parahoric level. By [70], using the universal F-isocrystal with G-structure in the Hodge type case, we can define the Newton stratification on $\mathscr{S}_0$. Then we can extend to the case of abelian type as in [78]. Similarly we can define central leaves in this setting. In [31] subsection 6.6, He and Rapoport proved that each Newton stratum contains a certain EKOR stratum in their setting, in particular for Shimura varieties with integral models satisfying their axioms. In [12] Görtz-He-Nie introduced the notion of fully Hodge-Newton decomposable pairs $(G, \{\mu\})$, and they made a deep study for these pairs. In particular, under the assumption that the He-Rapoport axioms were verified, they proved that for Hodge-Newton decomposable Shimura varieties, each Newton stratum is a union of certain EKOR strata. The methods to prove the above mentioned results in [31] and [12] are group theoretic. With our geometric constructions of EKOR strata at hand, these results of [31] and [12] become unconditional for the Kisin-Pappas integral models.

Theorem E. (1) (Corollary 3.4.14, Corollary 5.5.6) *Each Newton stratum contains an EKOR stratum $\mathscr{S}_0^x$ such that x is σ-straight. Moreover, $\mathscr{S}_0^x$ is a central leaf* *of dimension $\langle \rho, \nu(b) \rangle$. Here ρ is the half sum of positive roots, and $\nu(b)$ is the Newton cocharacter of the Newton stratum.*

(2) (Theorem 5.5.8) *Assume that the attached pair $(G, \{\mu\})$ is fully Hodge-Newton decomposable. Then*
   (a) *each Newton stratum of $\mathscr{S}_0$ is a union of EKOR strata;*
   (b) *each EKOR stratum in a non-basic Newton stratum is an adjoint central leaf, and it is open and closed in the Newton stratum, in particular, non-basic Newton strata are smooth;*
   (c) *the basic Newton stratum is a union of certain Deligne-Lusztig varieties.*

We refer the reader to subsections 3.6 and 4.3 for geometric constructions of EKOR strata for affine Deligen-Lusztig varieties, and to Propositions 3.6.3 and 3.6.4 for the relation between local and global EKOR strata.

In section 6 we discuss the case of Siegel modular varieties and study the example of Siegel threefolds in details. This is also an example of fully Hodge-Newton decomposable Shimura varieties.

We briefly describe the structure of this article. In the first section we recollect some facts about G-zips, the Iwahori Weyl group and some related group theoretic sets, which will be used later. In section 2, we review some constructions of the Pappas-Zhu local models and the Kisin-Pappas integral models of Shimura varieties of abelian type, which are the objects to be studied in this paper. In section 3, we construct and study the EKOR stratification for Shimura varieties of Hodge type by a local method. More precisely, we construct a $\mathcal{G}_0^{\mathrm{rdt}}$-zip and thus an EO stratification on each KR stratum. In section 4, we give some global constructions of the EKOR strata by adapting and generalizing some ideas of Xiao-Zhu in [84]. More precisely, we will introduce the notions of local $(\mathcal{G}, \mu)$-Shtukas and their truncations in level 1. We study their moduli and apply them to our Shimura varieties. In section 5, we extend our constructions to the abelian type case. We also apply the results of EKOR strata to the study of Newton strata and central leaves for these Shimura varieties. In section 6, we discuss the Siegel case and investigate the example of GSp_4 in details. Finally, in the appendix (Proposition A.3.5) we verify He-Rapoport's axiom 4 (c) for Shimura varieties of PEL-type, which improves some of our main results (Theorem C (2) and Theorem D) for the PEL-type case.

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1. Recollections of some group theoretic results

In this section, we review and collect some known facts about G-zips, the Iwahori Weyl group and some related group theoretic sets, which will be used later. Fix a prime p.

1.1. **Algebraic zip data.** For a linear algebraic group H over a field, we denote its unipotent radical by R_uH. For an element $h \in H$, let $\overline{h}$ be its image in H/R_uH.

Definition 1.1.1. ([66, Definition 1.1]) An algebraic zip datum is a tuple $\mathcal{Z} = (G, P, Q, \sigma')$ consisting of a (connected) reductive group $G/\mathbb{F}_p$, together with parabolic subgroups P and Q defined over a finite extension $k/\mathbb{F}_p$, and an isogeny $\sigma' : P/R_uP \to Q/R_uQ$. The group

$$E_{\mathcal{Z}} := \{(p, q) \in P \times Q \mid \sigma'(\overline{p}) = \overline{q}\}$$

is called the *zip group* attached to $\mathcal{Z}$; it acts on G_k through the map $((p, q), g) \mapsto pgq^{-1}$.

By abuse of notation, we still denote by $E_{\mathcal{Z}}$ the base change to $\overline{k}$ of the zip group attached to $\mathcal{Z}$. We are interested in the decomposition of $G_{\overline{k}}$ into $E_{\mathcal{Z}}$-orbits. To describe it, we fix a Borel subgroup B of $G_{\overline{k}}$, a maximal torus $T \subset B$ and an element $g \in G(\overline{k})$ such that $B \subset Q_{\overline{k}}$, ${}^gB \subset P_{\overline{k}}$, $\sigma'(\overline{{}^gB}) = \overline{B}$ and $\sigma'(\overline{{}^gT}) = \overline{T}$. Here $\overline{B}$ is the image of B in $Q_{\overline{k}}/R_uQ_{\overline{k}}$, and similarly for the other objects. Let W be the Weyl group of $G_{\overline{k}}$ with respect to T and $\mathtt{S} \subset W$ be the set of simple reflections corresponding to B. Let $J \subset \mathtt{S}$ be the type of P, W_J be the subgroup of W generated by J, and ${}^JW$ be the set of minimal length representatives for $W_J\backslash W$. For $w \in {}^JW \subset W = N_G(T)/T$, let $\dot{w} \in N_G(T)$ be a representative which maps to w. We set

$$G^w := E_{\mathcal{Z}} \cdot gB\dot{w}B.$$

Theorem 1.1.2. ([66, Theorem 1.3, Proposition 7.1, Proposition 7.3]) *The subsets G^w for $w \in {}^JW$ form a pairwise disjoint decomposition of $G_{\overline{k}}$ into locally closed smooth subvarieties. The dimension of G^w is* $\dim P + \ell(w)$. *If the differential of σ' at 1 vanishes, G^w is a single orbit of $E_{\mathcal{Z}}$.*

Remark 1.1.3. Using the choice of (B, T, g), we can identify P/R_uP (resp. Q/R_uQ) with a Levi subgroup L_P (resp. L_Q) of P (resp. Q), and view σ' as an isogeny $L_P \to L_Q$. We can rewrite $E_{\mathcal{Z}}$ as

$$E_{\mathcal{Z}} = \{(u_1 l,\ u_2\sigma'(l)) \mid u_1 \in R_uP,\ u_2 \in R_uQ,\ l \in L_P\}.$$

Remark 1.1.4. The closure of G^w is a union of $G^{w'}$s as described in [66, Theorem 1.4]. In particular, there is a unique open dense stratum (given by the unique maximal element in ${}^JW$), called the ordinary locus; and there is a unique closed stratum (given by $\mathrm{id} \in {}^JW$), called the superspecial locus.

We will need to construct morphisms from a scheme S over k to the quotient stack $[E_{\mathcal{Z}}\backslash G_k]$. We refer the reader to [48] for some basics about algebraic stacks. A morphism

$$f : S \to [E_{\mathcal{Z}}\backslash G_k]$$

is, by definition, an $E_{\mathcal{Z}}$-torsor $\mathcal{E}$ over S together with an $E_{\mathcal{Z}}$-equivariant morphism $f^{\#} : \mathcal{E} \to G_k$. However, it is not always obvious how to construct $(\mathcal{E}, f^{\#})$ directly from certain structures (e.g. Dieudonné modules) on S. The following notation will be used in this paper.

Definition 1.1.5. ([67, Definition 3.1]) A G-zip of type J over S is a tuple (I, I_P, I_Q, ι'), where

- I is a right G-torsor over S,
- $I_P \subset I$ is a right P-torsor over S,
- $I_Q \subset I$ is a right Q-torsor over S, and

- $\iota' : I_P/R_uP \to I_Q/R_uQ$ is a P/R_uP-equivariant morphism over S, i.e. we have $\iota'(xp) = \iota'(x)\sigma'(p)$, for all $x \in I_P/R_uP$ and $p \in P/R_uP$.

Let $G\text{-}Zip_J$ be the stack of G-zips of type J over k. By [66, Theorem 12.7] and [67, Theorem 1.5], we have an isomorphism of algebraic stacks

$$G\text{-}Zip_J \simeq [E_{\mathcal{Z}}\backslash G_k],$$

and they are smooth algebraic stacks of dimension 0 over k. Thus to give a morphism

$$S \to [E_{\mathcal{Z}}\backslash G_k]$$

is equivalent to give a G-zip of type J over S. We will only explain how to construct a morphism $S \to [E_{\mathcal{Z}}\backslash G_k]$ from a G-zip (I, I_P, I_Q, ι'). Let $\mathcal{E}$ be the cartesian product

$$\begin{array}{ccccc} \mathcal{E} & \longrightarrow & & & I_Q \\ \downarrow & & & & \downarrow \\ I_P & \longrightarrow & I_P/R_uP & \xrightarrow{\iota'} & I_Q/R_uQ. \end{array}$$

It is an $E_{\mathcal{Z}}$-torsor over S. There is a morphism $f^\# : \mathcal{E} \to G_k$ as follows. For $t = (t_1, t_2) \in \mathcal{E}(S)$ with $t_1 \in I_P(S)$ and $t_2 \in I_Q(S)$ such that $\iota'(t_1R_uP) = t_2R_uQ$, there is a unique $g \in G(S)$ such that $t_1g = t_2$. We set $f^\#(t) := g$. One checks easily that $f^\#$ is $E_{\mathcal{Z}}$-equivariant. We will write $f^\#$ as the composition

$$\mathcal{E} \xrightarrow{(t_1,t_2)\mapsto(t_1,t_2)} I \times I \xrightarrow{(f_1,f_2)\mapsto d(f_1,f_2)} G_k\ . \tag{1.1.6}$$

Here $d(f_1, f_2)$ is the unique element in G_k which takes f_1 to f_2.

Remark 1.1.7. Let σ' be of the form $i \circ \sigma$, where $\sigma : P/R_uP \to P^{(p)}/R_uP^{(p)}$ is the relative Frobenius, and $i : P^{(p)}/R_uP^{(p)} \to Q/R_uQ$ is an isomorphism of group varieties. It is clear that the differential of σ' at 1 vanishes in this case, and hence Theorem 1.1.2 applies. Moreover, a G-zip (I, I_P, I_Q, ι') is equivalent to the tuple (I, I_P, I_Q, ι), where ι is the induced $P^{(p)}/R_uP^{(p)}$-equivariant isomorphism

$$(I_P/R_uP)^{(p)} \to I_Q/R_uQ.$$

Let ω_0 be the element of maximal length in W, and $\sigma : W \to W$ be the Frobenius map. Set $K := {}^{\omega_0}\sigma(J)$. Here we write ${}^gJ$ for gJg^{-1}. Let

$$x \in {}^KW^{\sigma(J)}$$

be the element of minimal length in $W_K\omega_0W_{\sigma(J)}$. Then x is the unique element of maximal length in ${}^KW^{\sigma(J)}$ (see [81] 5.2). There is a partial order $\preceq$ on ${}^JW$, defined by $w' \preceq w$ if and only if there exists $y \in W_J$, such that

$$yw'x\sigma(y^{-1})x^{-1} \leq w$$

(see [81, Definition 5.8]). Here $\leq$ is the Bruhat order (see A.2 of [81] for the definition). As usual, the partial order $\preceq$ makes ${}^JW$ into a topological space (see [67, Proposition 2.1] for example). On the other hand, for an algebraic stack X, we have its underlying topological space $|X|$ (see [48] section 5). By [67, Proposition 2.2], the above Theorem 1.1.2 actually tells us that we have a homeomorphism of topological spaces

$$|[E_{\mathcal{Z}}\backslash G_{\overline{k}}]| \simeq {}^JW.$$

1.2. **The Iwahori Weyl groups.** Our notations in this subsection will be different from those in the previous one, namely, G will be a reductive group over $\mathbb{Q}_p$, and $\mathcal{G}$ will be a parahoric group scheme over $\mathbb{Z}_p$. Let $K = \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ be the associated parahoric subgroup. We also set $k := \overline{\mathbb{F}}_p$, $\breve{\mathbb{Z}}_p := W(k)$, $\breve{\mathbb{Q}}_p := \breve{\mathbb{Z}}_p[\frac{1}{p}]$, $\breve{K} := \mathcal{G}(\breve{\mathbb{Z}}_p)$, and $\breve{G} := G(\breve{\mathbb{Q}}_p)$. Let $\sigma : \breve{\mathbb{Z}}_p \to \breve{\mathbb{Z}}_p$ be the Frobenius map, which induces Frobenius maps on $\breve{\mathbb{Q}}_p, \breve{K}, \breve{G}$ and related objects.

Let $\mathcal{B}$ be the Bruhat-Tits building of $G_{\breve{\mathbb{Q}}_p}$ and let $T \subset G$ be a maximal torus which after extension of scalars is contained in a Borel subgroup B of $G_{\breve{\mathbb{Q}}_p}$. The split part of T defines an apartment of $\mathcal{B}$. Let $\mathfrak{a}$ be an alcove of $\mathcal{B}$ inside this apartment. Let $I \subset \mathcal{G}(\mathbb{Z}_p)$ be an Iwahori subgroup, such that $\breve{I}$ is the Iwahori subgroup of $G(\breve{\mathbb{Q}}_p)$ fixing the alcove $\mathfrak{a}$. Let $N(T)$ be the normalizer of T. The *Iwahori Weyl group* is given by

$$\widetilde{W} = N(T)(\breve{\mathbb{Q}}_p)/(T(\breve{\mathbb{Q}}_p) \cap \breve{I}),$$

and the relative Weyl group of $G_{\breve{\mathbb{Q}}_p}$ is given by

$$W_0 = N(T)(\breve{\mathbb{Q}}_p)/T(\breve{\mathbb{Q}}_p).$$

Choosing a special vertex in the alcove corresponding to $\breve{I}$ which will be fixed once and for all, we have

$$\widetilde{W} \cong X_*(T)_{\Gamma_0} \rtimes W_0$$

and

$$\widetilde{W} \cong W_a \rtimes \pi_1(G)_{\Gamma_0},$$

where $\Gamma_0 = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\breve{\mathbb{Q}}_p)$ and W_a is the affine Weyl group, which is a Coxeter group. One can then use the Bruhat partial order (resp. length function) on W_a to define a partial order (resp. length function) on $\widetilde{W}$. To be more precise, for $w = (w_a, t) \in \widetilde{W}$ with $w_a \in W_a$ and $t \in \pi_1(G)_{\Gamma_0}$, we set $\ell(w) = \ell(w_a)$, and $w = (w_a, t) \leq w' = (w'_a, t')$ if an only if $w_a \leq w'_a$ and $t = t'$.

We recall the definition of $\{\mu\}$-admissible sets. For $\mu' \in X_*(T)_{\Gamma_0}$, we write $t^{\mu'}$ when viewed as an element in $\widetilde{W}$. Let $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ be a cocharacter and $\{\mu\}$ be the associated conjugacy class. The attached $\{\mu\}$-*admissible set* is the following finite subset of $\widetilde{W}$ introduced by Kottwitz and Rapoport (cf. [44, 69, 64]):

$$\mathrm{Adm}(\{\mu\}) = \{w \in \widetilde{W} \mid w \leq t^{x(\underline{\mu})} \text{ for some } x \in W_0\}. \tag{1.2.1}$$

Here $\underline{\mu}$ is the image in $X_*(T)_{\Gamma_0}$ of a dominant representative in the conjugacy class $\{\mu\}$. We will be interested in some related sets taking into account the information of the parahoric subgroup K. Let

$$W_K = (N(T)(\breve{\mathbb{Q}}_p) \cap \breve{K})/(T(\breve{\mathbb{Q}}_p) \cap \breve{I}),$$

then by [19, Proposition 12] we have an isomorphism

$$W_K \simeq W_{\mathcal{G}_0^{\mathrm{rdt}}},$$

where $\mathcal{G}_0 = \mathcal{G} \otimes_{\mathbb{Z}_p} k$, $\mathcal{G}_0^{\mathrm{rdt}}$ is its reductive quotient, and $W_{\mathcal{G}_0^{\mathrm{rdt}}}$ is the associated Weyl group. Let ${}^K\widetilde{W} \subset \widetilde{W}$ be the subset of minimal length representatives for $W_K \backslash \widetilde{W}$. We set[2]

- $\mathrm{Adm}(\{\mu\})^K = W_K \mathrm{Adm}(\{\mu\}) W_K$, a subset of $\widetilde{W}$;
- $\mathrm{Adm}(\{\mu\})_K = W_K \backslash \mathrm{Adm}(\{\mu\})^K / W_K$, a subset of $W_K \backslash \widetilde{W} / W_K$;
- ${}^K\mathrm{Adm}(\{\mu\}) = \mathrm{Adm}(\{\mu\})^K \cap {}^K\widetilde{W}$, a subset of ${}^K\widetilde{W}$.

[2] We learned the notation ${}^K\mathrm{Adm}(\{\mu\})$ from [13].

The natural map ${}^K\widetilde{W} \to W_K\backslash\widetilde{W}/W_K$ induces a surjection

$$ {}^K\mathrm{Adm}(\{\mu\}) \twoheadrightarrow \mathrm{Adm}(\{\mu\})_K. $$

We have the following important result due to He ([27]) and Haines-He ([18]):

Theorem 1.2.2. ([27, Theorem 6.1], [18, Proposition 5.1]) *We have*

$$ {}^K\mathrm{Adm}(\{\mu\}) = \mathrm{Adm}(\{\mu\}) \cap {}^K\widetilde{W}. $$

1.2.3. *The partial order $\leq_{K,\sigma}$ on ${}^K\mathrm{Adm}(\{\mu\})$.* There is a partial order $\leq_{K,\sigma}$ on ${}^K\widetilde{W}$. It is defined by

$$ x_1 \leq_{K,\sigma} x_2 \quad \text{if there exists } y \in W_K \text{ such that} \quad yx_1\sigma(y)^{-1} \leq x_2. $$

Here $\leq$ is the Bruhat order. By [22, 4.7], $\leq_{K,\sigma}$ is indeed a partial order. By [31, Remark 6.14], $\leq_{K,\sigma}$ is finer than the Bruhat order in general. We will then restrict the partial order $\leq_{K,\sigma}$ on ${}^K\widetilde{W}$ to ${}^K\mathrm{Adm}(\{\mu\})$.

Proposition 1.2.4. ([29, Proposition 2.1]) *The maximal elements in ${}^K\mathrm{Adm}(\{\mu\})$ with respect to the partial order $\leq_{K,\sigma}$ are $t^{\mu'}$, where μ' runs over elements in the W_0-orbit of $\underline{\mu}$ with $t^{\mu'} \in {}^K\widetilde{W}$.*

1.2.5. *The element τ.* There is a distinguished element $\tau_{\{\mu\}} \in \widetilde{W}$ attached to the conjugacy class $\{\mu\}$. In terms of the isomorphism $\widetilde{W} \cong W_a \rtimes \pi_1(G)_{\Gamma_0}$, it corresponds to the element

$$ (\mathrm{id}, \mu^{\#}) \in W_a \rtimes \pi_1(G)_{\Gamma_0}, $$

where $\mu^{\#}$ is the common image of $\mu \in \{\mu\}$ in $\pi_1(G)_{\Gamma_0}$. It is then clear that $\tau_{\{\mu\}}$ is of length 0, and lies in ${}^K\mathrm{Adm}(\{\mu\})$. It is the minimal element in ${}^K\mathrm{Adm}(\{\mu\})$ with respect to the above partial order $\leq_{K,\sigma}$.

The conjugacy class $\{\mu\}$ will be fixed in our practical applications, and hence we will simply write τ for $\tau_{\{\mu\}}$.

1.2.6. ${}^K\widetilde{W}^K, {}_K\widetilde{W}^K$ *and* ${}^K\widetilde{W}_K$. Let ${}^K\widetilde{W}^K \subset \widetilde{W}$ be the subset of minimal length representatives for $W_K\backslash\widetilde{W}/W_K$. Since each double coset $W_K w W_K$ admits a unique element of minimal length, we will identify $\mathrm{Adm}(\{\mu\})_K$ as a subset of ${}^K\widetilde{W}^K$, which is again denoted by $\mathrm{Adm}(\{\mu\})_K$. For $w \in \mathrm{Adm}(\{\mu\})_K$, by $\ell(w)$ we mean that we view $w \in {}^K\widetilde{W}^K$ and $\ell(w)$ is its length. Sometimes we will also write $x_w \in {}^K\widetilde{W}^K$ for the corresponding $w \in \mathrm{Adm}(\{\mu\})_K$ to distinguish the notation.

A certain subset ${}_K\widetilde{W}^K \subset \widetilde{W}$ is introduced in [74] to describe the dimension and closure of Schubert cells. For $w \in \widetilde{W}$, let w^K be the unique element of minimal length in wW_K. Then by [74, Lemma 1.6], there is a unique element ${}_Kw^K$ of maximal length in $\{(vw)^K \mid v \in W_K\}$. We set

$$ {}_K\widetilde{W}^K := \{{}_Kw^K \mid w \in \widetilde{W}\}. $$

It seems that a slightly different subset ${}^K\widetilde{W}_K \subset \widetilde{W}$ is more convenient for our purpose. For $w \in \widetilde{W}$, let ${}^Kw$ be the unique element of minimal length in $W_K w$, there is a unique element ${}^Kw_K$ of maximal length in $\{{}^K(wv) \mid v \in W_K\}$. We set

$$ {}^K\widetilde{W}_K := \{{}^Kw_K \mid w \in \widetilde{W}\}. $$

The natural projection $\widetilde{W} \to W_K\backslash\widetilde{W}/W_K$ induces bijections

$$ {}_K\widetilde{W}^K \xrightarrow{\sim} W_K\backslash\widetilde{W}/W_K \xleftarrow{\sim} {}^K\widetilde{W}_K. $$

The following statement should be well known to experts. It follows essentially from a result of Howlett (see e.g. [6, Theorem 4.18]).

Lemma 1.2.7. *For $w \in \widetilde{W}$, we have $\ell({}_Kw^K) = \ell({}^Kw_K)$.*

Proof. To simplify notations, we assume that w is the unique element of minimal length in $W_K w W_K$, i.e. $w \in {}^K\widetilde{W}^K$. Let $W_J = w^{-1}W_K w \cap W_K$, and ${}^J W_K \subset W_K$ be the set of minimal length representatives of $W_J \backslash W_K$. For $v \in W_K$, we have a unique decomposition $v = v_J \cdot {}^J v$ with $v_J \in W_J$ and ${}^J v \in {}^J W_K$, and hence

$$wv = wv_J \cdot {}^J v = (wv_J w^{-1}) \cdot w \cdot {}^J v.$$

For $x \in {}^J W_K$, we have, by [6, Theorem 4.18] (and its proof), that $wx \in {}^K\widetilde{W}$ and $\ell(wx) = \ell(w) + \ell(x)$. In particular, noting that $wv_J w^{-1} \in W_K$, we have ${}^K(wv) = w \cdot {}^J v$, and hence ${}^K w_K = wx_0$ with $\ell({}^K w_K) = \ell(w) + \ell(x_0)$. Here x_0 is the unique element of maximal length in ${}^J W_K$.

There is a similar description for ${}_K w^K$. Namely, let $W_I = wW_K w^{-1} \cap W_K$, and $W_K^I \subset W_K$ be the set of minimal length representatives of W_K/W_I, we have ${}_K w^K = y_0 w$ and $\ell({}_K w^K) = \ell(w) + \ell(y_0)$, where y_0 is the maximal element in W_K^I. One sees immediately that $\ell({}_K w^K) = \ell({}^K w_K)$, as $\ell(x_0) = \ell(y_0)$. □

1.2.8. By definition we have inclusion

$${}^K\widetilde{W}_K \subset {}^K\widetilde{W}.$$

For any $w \in \mathrm{Adm}(\{\mu\})_K$, we get a unique representative of w in ${}^K\widetilde{W}_K$, which we denote by ${}^K w_K$. Then

$${}^K w_K \in \mathrm{Adm}(\{\mu\})^K \cap {}^K\widetilde{W} = {}^K\mathrm{Adm}(\{\mu\}).$$

In this way we get a section $w \mapsto {}^K w_K$ of the surjection ${}^K\mathrm{Adm}(\{\mu\}) \twoheadrightarrow \mathrm{Adm}(\{\mu\})_K$. On the other hand, we have also the inclusion

$${}^K\widetilde{W}^K \subset {}^K\widetilde{W}.$$

Recall that for $w \in \mathrm{Adm}(\{\mu\})_K$, we get a unique representative $x_w \in {}^K\widetilde{W}^K \subset {}^K\widetilde{W}$. Then

$$x_w \in \mathrm{Adm}(\{\mu\})^K \cap {}^K\widetilde{W} = {}^K\mathrm{Adm}(\{\mu\}).$$

In this way we get another section $w \mapsto x_w$ of the surjection ${}^K\mathrm{Adm}(\{\mu\}) \twoheadrightarrow \mathrm{Adm}(\{\mu\})_K$. We note that ${}^K w_K$ and x_w are the maximal and minimal elements respectively in $W_K w W_K \cap {}^K\widetilde{W}$, the fiber at w of the surjection ${}^K\mathrm{Adm}(\{\mu\}) \twoheadrightarrow \mathrm{Adm}(\{\mu\})_K$.

1.2.9. *The Newton map.* We identify $X_*(T)_{\Gamma_0,\mathbb{Q}} = X_*(T)_{\mathbb{Q}}^{\Gamma_0}$. Let $X_*(T)^+_{\Gamma_0,\mathbb{Q}}$ be the set of dominant elements in $X_*(T)_{\Gamma_0,\mathbb{Q}}$ defined by the positive relative roots of $G_{\breve{\mathbb{Q}}_p}$ corresponding to B (cf. [26] 1.7). The action of σ on $X_*(T)_{\Gamma_0,\mathbb{Q}}/W_0$ transfers to an action on $X_*(T)^+_{\Gamma_0,\mathbb{Q}}$ (the so called L-action), and hence gives a subset $(X_*(T)^+_{\Gamma_0,\mathbb{Q}})^{\langle\sigma\rangle}$.

There is a map (cf. [26])

$$\nu : \widetilde{W} \to (X_*(T)^+_{\Gamma_0,\mathbb{Q}})^{\langle\sigma\rangle}$$

as follows. For $w \in \widetilde{W}$, there exists $n \in \mathbb{N}$ such that σ^n acts trivially on $\widetilde{W}$ and that

$$\lambda = w\sigma(w) \cdots \sigma^{n-1}(w) \in X_*(T)_{\Gamma_0}.$$

The element

$$\frac{1}{n}\lambda \in X_*(T)_{\Gamma_0,\mathbb{Q}}$$

is independent of the choice of n. The map $\nu : \widetilde{W} \to (X_*(T)^+_{\Gamma_0,\mathbb{Q}})^{\langle\sigma\rangle}$ is then given by setting $\nu(w)$ to be the unique dominant element in the W_0-orbit of $\frac{1}{n}\lambda$. It is called the Newton map. By construction, it is constant on each σ-conjugacy class of $\widetilde{W}$.

1.2.10. *σ-straight elements.* Let us recall definition and basic properties of σ-straight elements in $\widetilde{W}$. An element $w \in \widetilde{W}$ is called σ-straight if

$$\ell(w\sigma(w)\sigma^2(w)\cdots\sigma^{m-1}(w)) = m\ell(w)$$

for all $m \in \mathbb{N}$. By [25, Lemma 1.1], it is equivalent to

$$\ell(w) = \langle \nu(w), 2\rho \rangle,$$

where $\nu(w)$ is as in 1.2.9, and ρ is the half sum of positive roots.

We denote by

$$\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}} \subset \mathrm{Adm}(\{\mu\})$$

the subset of σ-straight elements. We also set

$${}^K\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}} := \mathrm{Adm}(\{\mu\})_{\sigma\text{-str}} \cap {}^K\widetilde{W} \subset {}^K\mathrm{Adm}(\{\mu\}),$$

where the last inclusion is by Theorem 1.2.2. One checks easily by definition that the element τ is σ-straight, so $\tau \in {}^K\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}}$.

1.3. $C(\mathcal{G}, \{\mu\})$ **and related quotients.** We continue the notations in the last subsection.

1.3.1. *The set $B(G, \{\mu\})$.* Let $B(G)$ be the set of σ-conjugacy classes in $\breve{G}$. Kottwitz constructed two maps in [41], namely, the Newton map

$$\nu : B(G) \to (X_*(T)^+_{\Gamma_0,\mathbb{Q}})^{\langle\sigma\rangle}$$

and the Kottwitz map

$$\kappa : B(G) \to \pi_1(G)_\Gamma.$$

Here $\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. An element $[b] \in B(G)$ is uniquely determined by its images $\nu([b])$ and $\kappa([b])$. The relation between the Newton maps on $B(G)$ and $\widetilde{W}$ respectively is as follows. A σ-conjugacy class of $\widetilde{W}$ is called straight if it contains a σ-straight element. Let $B(\widetilde{W})_{\sigma\text{-str}}$ be the set of straight σ-conjugacy classes of $\widetilde{W}$. By [26, Theorem 3.3], the map

$$\Psi : B(\widetilde{W})_{\sigma\text{-str}} \to B(G)$$

induced by the inclusion $N(T)(\breve{\mathbb{Q}}_p) \subset \breve{G}$ is bijective, and compatible with the Newton maps on both sides.

There is a partial order $\le$ on $X_*(T)_\mathbb{Q}$ defined as follows. For $v_1, v_2 \in X_*(T)_\mathbb{Q}$, $v_1 \le v_2$ if and only if $v_2 - v_1$ is a non-negative $\mathbb{Q}$-sum of positive relative coroots. One can the define a partial order on $B(G)$ as follows:

$$[b_1] \le [b_2] \text{ if and only if } \kappa([b_1]) = \kappa([b_2]) \text{ and } \nu([b_1]) \le \nu([b_2]).$$

One then define

$$B(G, \{\mu\}) := \{[b] \in B(G) \mid \kappa([b]) = \mu^\natural, \nu([b]) \le \overline{\mu}\}.$$

Here $\mu^\natural$ is the common image of $\mu \in \{\mu\}$ in $\pi_1(G)_\Gamma$, and $\overline{\mu}$ is the Galois average of a dominant representative of the image of an element of $\{\mu\}$ in $X_*(T)_{\Gamma_0,\mathbb{Q}}$ with respect to the L-action of σ on $X_*(T)^+_{\Gamma_0,\mathbb{Q}}$. By [27, Proposition 4.1], the above Ψ maps the image of $\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}}$ in $B(\widetilde{W})_{\sigma\text{-str}}$ bijectively to $B(G, \{\mu\})$.

1.3.2. *The set* $C(\mathcal{G},\{\mu\})$. We consider the (right) action of $\breve{K}\times\breve{K}$ on $\breve{G}$ given by $(g,(k_1,k_2))\mapsto k_1^{-1}gk_2$. Let $\breve{K}_\sigma\subset\breve{K}\times\breve{K}$ be the graph of the Frobenius map and set $C(\mathcal{G})=\breve{G}/\breve{K}_\sigma$. We have a natural surjection $C(\mathcal{G})\to B(G)$. The subset

$$\breve{K}\mathrm{Adm}(\{\mu\})\breve{K}\subset\breve{G}$$

is stable with respect to the action of $\breve{K}_\sigma$. We denote by

$$C(\mathcal{G},\{\mu\})=\breve{K}\mathrm{Adm}(\{\mu\})\breve{K}/\breve{K}_\sigma$$

the set of its orbits. Then $C(\mathcal{G},\{\mu\})\subset C(\mathcal{G})$.

Let $\breve{K}_1$ be the pro-unipotent radical of $\breve{K}$, then $\breve{K}_\sigma(\breve{K}_1\times\breve{K}_1)\subset\breve{K}\times\breve{K}$ is a subgroup. We define

$$\mathrm{EKOR}(K,\{\mu\}):=\breve{K}\mathrm{Adm}(\{\mu\})\breve{K}/\breve{K}_\sigma(\breve{K}_1\times\breve{K}_1),\quad \mathrm{KR}(K,\{\mu\}):=\breve{K}\mathrm{Adm}(\{\mu\})\breve{K}/\breve{K}\times\breve{K}.$$

We then have natural maps

$$B(G,\{\mu\})\longleftarrow C(\mathcal{G},\{\mu\})\longrightarrow \mathrm{EKOR}(K,\{\mu\})\longrightarrow \mathrm{KR}(K,\{\mu\}), \tag{1.3.3}$$

where all the arrows are surjections.

By the Bruhat-Tits decomposition, we see that the natural inclusion $N(T)(\breve{\mathbb{Q}}_p)\to\breve{G}$ induces a bijection

$$\mathrm{Adm}(\{\mu\})_K\xrightarrow{\sim}\mathrm{KR}(K,\{\mu\}).$$

To identify ${}^K\mathrm{Adm}(\{\mu\})$ and $\mathrm{EKOR}(K,\{\mu\})$, we need the following results due to Lusztig and He.

Theorem 1.3.4. ([31, Theorem 6.1]) *Notations as above, we have*

(1) *for any* $x\in{}^K\widetilde{W}$, $\breve{K}_\sigma(\breve{K}_1x\breve{K}_1)=\breve{K}_\sigma(\breve{I}x\breve{I})$;
(2) $\breve{G}=\coprod_{x\in{}^K\widetilde{W}}\breve{K}_\sigma(\breve{K}_1x\breve{K}_1)=\coprod_{x\in{}^K\widetilde{W}}\breve{K}_\sigma(\breve{I}x\breve{I})$.

In particular, the inclusion $N(T)(\breve{\mathbb{Q}}_p)\to\breve{G}$ *induces a bijection*

$${}^K\mathrm{Adm}(\{\mu\})\xrightarrow{\sim}\mathrm{EKOR}(K,\{\mu\}).$$

We will identify ${}^K\mathrm{Adm}(\{\mu\})$ with $\mathrm{EKOR}(K,\{\mu\})$ and $\mathrm{Adm}(\{\mu\})_K$ with $\mathrm{KR}(K,\{\mu\})$ from now on.

Following [10, Definition 13.1.1], an element $w\in\widetilde{W}$ is said to be *fundamental* if $\breve{I}\dot{w}\breve{I}$ consists of a single $\breve{I}$-σ-conjugacy class. To explain relations between fundamental elements, σ-straight elements and $C(\mathcal{G},\{\mu\})$ (resp. $B(G,\{\mu\})$), we need the following results. Recall the set

$${}^K\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}}={}^K\mathrm{Adm}(\{\mu\})\cap\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}}.$$

The Ψ above induces a map

$${}^K\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}}\longrightarrow B(G,\{\mu\}).$$

Theorem 1.3.5. ([26, Proposition 4.5], [31, Theorem 6.17]) *Notations as above, we have*

(1) *σ-straight elements are fundamental,*
(2) *the map* ${}^K\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}}\longrightarrow B(G,\{\mu\})$ *is surjective.*

If $x\in{}^K\widetilde{W}$ is σ-straight, then we have $\breve{K}_\sigma(\breve{K}_1x\breve{K}_1)=\breve{K}_\sigma(\breve{I}x\breve{I})$ by Theorem 1.3.4 (1), and $\breve{I}x\breve{I}=\breve{K}_\sigma\cdot x$ by Theorem 1.3.5 (1). In particular, we have

$$\breve{K}_\sigma(\breve{K}_1x\breve{K}_1)=\breve{K}_\sigma\cdot x.$$

This means that the natural surjection

$$C(\mathcal{G},\{\mu\})\twoheadrightarrow{}^K\mathrm{Adm}(\{\mu\})$$

admits a natural section (explicitly given as above) when restricting to the subset

$${}^K\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}}\subset{}^K\mathrm{Adm}(\{\mu\}).$$

Combined with Theorem 1.3.5 (2), we have the following commutative diagram:

$$\begin{array}{ccccc} & & & & {}^K\mathrm{Adm}(\{\mu\}) \\ & \nearrow (\hookrightarrow) & & \nearrow\!\!\!\nearrow & \\ {}^K\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}} & \hookrightarrow & C(\mathcal{G},\{\mu\}) & & \\ & \searrow & & \searrow\!\!\!\searrow & \\ & & & & B(G,\{\mu\}). \end{array}$$

1.3.6. Now we make the link between the theory of algebraic zip data in subsection 1.1, the theory of Iwahori Weyl groups 1.2 and $C(\mathcal{G},\{\mu\})$. We need some constructions which are slightly modified version of those in the proof of [31, Theorem 6.1]. Let $\widetilde{\mathbb{S}}$ be the set of simple reflections in $\widetilde{W}$ and $J_K \subset \widetilde{\mathbb{S}}$ be the set of simple reflections in W_K. We have $\sigma(J_K) = J_K$ by construction. Let $w \in {}^K\widetilde{W}^K$, i.e., w is of shortest length in $W_K w W_K$. We set $\sigma' = \sigma \circ \mathrm{Ad}(w)$.

The map $\breve{K} \to \breve{K}w\breve{K}$, $k \mapsto wk$ induces a bijection

$$\breve{K}/(\breve{K} \cap w^{-1}\breve{K}w)_{\sigma'} \xrightarrow{\sim} \breve{K}w\breve{K}/\breve{K}_\sigma.$$

We remind the readers that, by our conventions at the beginning of 1.3, the $\breve{K}_\sigma$-action is given by

$$g \cdot k = k^{-1} g \sigma(k),$$

where $g \in \breve{K}w\breve{K}$ and $k \in \breve{K}$. The $(\breve{K} \cap w\breve{K}w^{-1})_{\sigma'}$-action is defined in the same way.

Let $\mathcal{G}_0 = \mathcal{G} \otimes \overline{\mathbb{F}}_p$ and $\mathcal{G}_0^{\mathrm{rdt}}$ be the maximal reductive quotient of $\mathcal{G}_0 \otimes \overline{\mathbb{F}}_p$. Then $\mathcal{G}_0^{\mathrm{rdt}}$ is a reductive group defined over $\mathbb{F}_p$. Let $\overline{B}$ be the image in $\mathcal{G}_0^{\mathrm{rdt}}$ of $\breve{I}$ and $\overline{T}$ be a maximal torus of $\overline{B}$. Let

$$J_w = J_K \cap \mathrm{Ad}(w^{-1})(J_K),$$

which is a subset of simple reflections in the Weyl group of $\mathcal{G}_0^{\mathrm{rdt}}$ (with respect to $(\overline{B},\overline{T})$). Let $\overline{L}_{J_w} \subset \mathcal{G}_0^{\mathrm{rdt}}$ (resp. $\overline{P}_{J_w} \subset \mathcal{G}_0^{\mathrm{rdt}}$) be the standard Levi subgroup (resp. parabolic subgroup) of type J_w, and $\overline{L}_{\sigma'(J_w)} \subset \mathcal{G}_0^{\mathrm{rdt}}$ (resp. $\overline{P}_{\sigma'(J_w)} \subset \mathcal{G}_0^{\mathrm{rdt}}$) be the standard Levi subgroup (resp. parabolic subgroup) of type $\sigma'(J_w)$. Then we have a natural isogeny $\overline{L}_{J_w} \to \overline{L}_{\sigma'(J_w)}$ which is again denoted by σ'. The tuple

$$\mathcal{Z}_w := (\mathcal{G}_0^{\mathrm{rdt}}, \overline{P}_{J_w}, \overline{P}_{\sigma'(J_w)}, \sigma' : \overline{L}_{J_w} \to \overline{L}_{\sigma'(J_w)})$$

is an algebraic zip datum. Let

$$E_{\mathcal{Z}_w} = (\overline{L}_{J_w})_{\sigma'}(U_{J_w} \times U_{\sigma'(J_w)}),$$

where U_{J_w} (resp. $U_{\sigma'(J_w)}$) is the unipotent radical of $\overline{P}_{J_w}$ (resp. $\overline{P}_{\sigma'(J_w)}$). It has a left action on $\mathcal{G}_0^{\mathrm{rdt}}$. Moreover, we have a natural map

$$f_w : \breve{K}w\breve{K}/\breve{K}_\sigma \to E_{\mathcal{Z}_w}\backslash \mathcal{G}_0^{\mathrm{rdt}}, \qquad k_1 w k_2 \mapsto \overline{\sigma(k_2)k_1} \tag{1.3.7}$$

which induces a bijection

$$\breve{K}w\breve{K}/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1) \xrightarrow{\sim} E_{\mathcal{Z}_w}\backslash \mathcal{G}_0^{\mathrm{rdt}}.$$

The induced map will also be denoted by f_w. Let $w \in \mathrm{Adm}(\{\mu\})_K$ and view it as an element of ${}^K\widetilde{W}^K$. Then $\breve{K}w\breve{K}/\breve{K}_\sigma(\breve{K}_1 \times \breve{K}_1)$ is identical to the fiber of the surjection

$${}^K\mathrm{Adm}(\{\mu\}) = \mathrm{EKOR}(K,\{\mu\}) \twoheadrightarrow \mathrm{KR}(K,\{\mu\}) = \mathrm{Adm}(\{\mu\})_K$$

at w. On the other hand, noting that $\sigma' = \sigma \circ \mathrm{Ad}(w) = \mathrm{Ad}(\sigma(w)) \circ \sigma$, by Theorem 1.1.2 and Remark 1.1.7, the underlying space of $E_{\mathcal{Z}_w}\backslash \mathcal{G}_0^{\mathrm{rdt}}$ is the finite set

$${}^{J_w}W_K.$$

One also sees easily that it is precisely the set ${}^J W_K$ in the proof of Lemma 1.2.7. Note that

$$w({}^{J_w} W_K) = W_K w W_K \cap {}^K\widetilde{W}$$

by [51] 2.1 (b) (see also the proof of [31, Theorem 6.1]).

2. Local models and Shimura varieties

In this section, we review the Pappas-Zhu local models [65] and the Kisin-Pappas integral models [37] for certain Shimura varieties of abelian type. We refer to [71, 8, 9, 60, 61, 63, 64] for more information on the theory of local models.

2.1. **Local models.**

2.1.1. *Loop groups and affine flag varieties.* Let $k = \overline{k}$ be an algebraically closed field. Let $\mathcal{G}$ be an affine group scheme of finite type over $k[[t]]$. We set $L\mathcal{G}$ and $L^+\mathcal{G}$ to be the group functors on the category of k-algebras given by

$$LG(R) := \mathcal{G}(R((t))) = G(R((t))) \quad \text{and} \quad L^+\mathcal{G}(R) := \mathcal{G}(R[[t]])$$

respectively. Then $L^+\mathcal{G}$ is represented by an affine group scheme over k, and LG is represented by an ind-affine ind-group scheme over k. Moreover, if $\mathcal{G}$ is smooth, then $L^+\mathcal{G}$ is reduced.

The notations in this part are different from other sections. We will switch back to our original notations from 2.1.7. Let $G/k((t))$ be a reductive group, and $\mathcal{G}/k[[t]]$ be the parahoric model of G corresponding to a facet $\mathfrak{a}$ in the enlarged Bruhat-Tits building of G. The *affine flag variety* $\mathrm{Gr}_{\mathcal{G}}$ is the fpqc-sheaf associated to the functor on the category of k-algebras given by

$$R \mapsto LG(R)/L^+\mathcal{G}(R).$$

It has a distinguished base point e_0 given by the identity section of G.

Remark 2.1.2. If $\mathcal{G}$ is an affine group scheme of finite type over $W(k)$, we can also define, by abuse of notations, $L^+\mathcal{G}$ to be the group functor on the category of k-algebras given by $L^+\mathcal{G}(R) := \mathcal{G}(W(R))$. Like in the equi-characteristic case, it is represented by an affine group scheme over k which is reduced if $\mathcal{G}$ is smooth. Moreover, the functor on the category of perfect k-algebras given by $LG(R) := \mathcal{G}(W(R)[1/p]) = G(W(R)[1/p])$ is represented by an ind perfect scheme. We can consider the Witt vector affine flag variety $\mathrm{Gr}_{\mathcal{G}} = \mathrm{Gr}_{\mathcal{G}}^W$ which is given by

$$\mathrm{Gr}_{\mathcal{G}} = LG/L^+\mathcal{G}$$

on the category of perfect k-algebras. By [93] and [1], this is an ind-proper perfect scheme over k. We will work with these affine flag varieties in section 4.

2.1.3. *Affine Schubert cells and affine Schubert varieties.* As in 1.2, we have the Iwahori Weyl group $\widetilde{W}$ with Bruhat order $\leq$, the finite Coxeter group $W_{\mathfrak{a}}$ (denoted by W_K with $K = \mathcal{G}(k[[t]])$ in 1.2 in the mixed characteristic setting) isomorphic to the Weyl group of $\mathcal{G}_0^{\mathrm{rdt}}$, the reductive quotient of $\mathcal{G}_0 = \mathcal{G} \otimes k$, and the subset ${}_{\mathfrak{a}}\widetilde{W}^{\mathfrak{a}} \subset \widetilde{W}$ in bijection with $W_{\mathfrak{a}}\backslash\widetilde{W}/W_{\mathfrak{a}}$. For $w \in \widetilde{W}$, the orbit map

$$L^+\mathcal{G} \to \mathrm{Gr}_{\mathcal{G}}, \quad g \mapsto g \cdot \dot{w}e_0$$

factors through $\mathcal{G}_0$. The $\mathcal{G}_0$-orbit of $\dot{w}e_0$, denoted by $\mathrm{Gr}_w = \mathrm{Gr}_{\mathcal{G},w}$, is a connected smooth and locally closed subscheme of $\mathrm{Gr}_{\mathcal{G}}$. Let Gr_w^- be its closure, which is a reduced closed subscheme of $\mathrm{Gr}_{\mathcal{G}}$. The variety Gr_w (resp. Gr_w^-) is called the *affine Schubert cell* (resp. affine Schubert variety) attached to w. By the Bruhat-Tits decomposition, we have a set-theoretically disjoint union of locally closed subsets

$$\mathrm{Gr}_{\mathcal{G}} = \coprod_{v \in W_{\mathfrak{a}}\backslash\widetilde{W}/W_{\mathfrak{a}}} \mathrm{Gr}_v = \coprod_{w \in {}_{\mathfrak{a}}\widetilde{W}^{\mathfrak{a}}} \mathrm{Gr}_w.$$

Theorem 2.1.4. ([75, Proposition 2.2]) *For $w \in {}_{\mathfrak{a}}\widetilde{W}^{\mathfrak{a}}$, we have*

(1) $\dim \mathrm{Gr}_w = \ell(w)$;
(2) Gr_w^- *is proper, and* $\mathrm{Gr}_w^- = \coprod_{v \in {}_{\mathfrak{a}}\widetilde{W}^{\mathfrak{a}}, v \leq w} \mathrm{Gr}_v$.

Remark 2.1.5. As in 1.2, we also have ${}^{\mathfrak{a}}\widetilde{W}_{\mathfrak{a}} \subset \widetilde{W}$, and by Lemma 1.2.7, $\dim \mathrm{Gr}_w = \ell({}^{\mathfrak{a}}w_{\mathfrak{a}})$ for all $w \in {}_{\mathfrak{a}}\widetilde{W}^{\mathfrak{a}}$.

We consider the $\mathcal{G}_0$-action on Gr_w.

Lemma 2.1.6. *For $x \in \mathrm{Gr}_w(k)$, its stabilizer $\mathcal{G}_{0,x}$ is smooth.*

Proof. The $\mathcal{G}_0$-action on Gr_w is transitive, so we can take $x = we_0$. The $\mathcal{G}_0$-action on Gr_w is induced by the $L^+\mathcal{G}$-action, and the stabilizer of x with respect to this $L^+\mathcal{G}$-action is

$$L^+\mathcal{G}_w := L^+\mathcal{G} \cap (\dot{w} \cdot L^+\mathcal{G} \cdot \dot{w}^{-1}).$$

Let $\mathfrak{b} = \mathfrak{a} \cup w\mathfrak{a}$, where $w\mathfrak{a}$ is the w-translation of $\mathfrak{a}$, and $\mathcal{G}_{\mathfrak{b}}$ be the Bruhat-Tits group scheme attached to $\mathfrak{b}$. Then $L^+\mathcal{G}_w = L^+(\mathcal{G}_{\mathfrak{b}})$, and the homomorphism $\mathcal{G}_{\mathfrak{b}} \to \mathcal{G}$ induces a commutative diagram

$$\begin{array}{ccc} L^+(\mathcal{G}_{\mathfrak{b}}) & \longrightarrow & L^+\mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{G}_{\mathfrak{b},0} & \longrightarrow & \mathcal{G}_0. \end{array}$$

In particular, $\mathcal{G}_{0,x}$ is a quotient of $\mathcal{G}_{\mathfrak{b},0}$, and hence smooth. $\square$

2.1.7. *Constructions and properties of local models.* We refer the readers to [64] for a general exposition on the theory of local models. Here we will briefly recall the construction of the Pappas-Zhu local models in [65]. For a reductive group G over $\mathbb{Q}_p$ and a prime number $p > 2$, we denote by $\mathcal{B}(G, \mathbb{Q}_p)$ the (extended) Bruhat-Tits building of G. For $x \in \mathcal{B}(G, \mathbb{Q}_p)$, we write $\mathcal{G}$ for the parahoric group scheme attached to x, i.e. the connected stabilizer of x. It is then a linear algebraic groups over $\mathbb{Z}_p$ with generic fiber G. We will assume from now on that

(2.1.8) $\qquad G$ splits over a tamely ramified extension and $p \nmid |\pi_1(G^{\mathrm{der}})|$.

In [65, §3], there is a construction of a smooth affine group scheme $\underline{\mathcal{G}}$ over $\mathbb{Z}_p[u]$ which specializes to the parahoric group scheme $\mathcal{G}$ via the base change $\mathbb{Z}_p[u] \to \mathbb{Z}_p$ given by $u \to p$ (see [65, §4]), and such that $\underline{G} := \underline{\mathcal{G}}|_{\mathbb{Z}_p[u,u^{-1}]}$ is reductive. There is a corresponding ind-projective ind-scheme (the global affine Grassmannian)

$$\mathrm{Gr}_{\underline{\mathcal{G}},\mathbb{A}^1} \to \mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}_p[u]$$

(see [65, §6]). The base change $\mathrm{Gr}_{\underline{\mathcal{G}},\mathbb{A}^1} \times_{\mathbb{A}^1} \operatorname{Spec} \mathbb{Q}_p$ given by $u \to p$ can be identified with the affine Grassmannian Gr_G of G over $\mathbb{Q}_p$. (Recall that Gr_G represents the fpqc sheaf associated to the quotient $R \mapsto G(R((t)))/G(R[[t]])$; the identification is via $t = u - p$.)

Let $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \longrightarrow G_{\overline{\mathbb{Q}}_p}$ be a minuscule cocharacter, and $E/\mathbb{Q}_p$ be the local reflex field, i.e. the field of definition of the conjugacy class $\{\mu\}$. The cocharacter μ defines a projective homogeneous space $G_{\overline{\mathbb{Q}}_p}/P_{\mu^{-1}}$. Here, P_ν denotes the parabolic subgroup that corresponds to the cocharacter ν, i.e. the parabolic subgroup whose Lie algebra is the subset $\mathrm{Lie}(G_{\overline{\mathbb{Q}}_p})$ consisting of elements of non-negative weights with respect to ν. Since the conjugacy class $\{\mu\}$ is defined over E, we can see that this homogeneous space has a canonical model X_μ defined over E. Since μ is minuscule, the corresponding Schubert cell $\mathrm{Gr}_{G_{\overline{\mathbb{Q}}_p},\mu}$, whose set of $\overline{\mathbb{Q}}_p$-points is by definition

$$G(\overline{\mathbb{Q}}_p[[t]])\mu(t)G(\overline{\mathbb{Q}}_p[[t]])/G(\overline{\mathbb{Q}}_p[[t]]),$$

is closed in the affine Grassmannian $\mathrm{Gr}_{G_{\overline{\mathbb{Q}}_p}}$. Moreover, $\mathrm{Gr}_{G_{\overline{\mathbb{Q}}_p},\mu}$ is defined over E and can be G_E-equivariantly identified with X_μ.

The *Pappas-Zhu local model* $\mathrm{M}^{\mathrm{loc}}_{G,\mathcal{G},\{\mu\}}$ is the Zariski closure of $X_\mu \subset \mathrm{Gr}_{G,E}$ in

$$\mathrm{Gr}_{\underline{\mathcal{G}},\mathbb{A}^1} \times_{\mathbb{A}^1} \operatorname{Spec} O_E,$$

where the base change $\mathbb{Z}_p[u] \to O_E$ is given by $u \mapsto p$. We will simply write $\mathrm{M}^{\mathrm{loc}}_G$ for $\mathrm{M}^{\mathrm{loc}}_{G,\mathcal{G},\{\mu\}}$ when $\mathcal{G}$ and $\{\mu\}$ are fixed. By its construction, $\mathrm{M}^{\mathrm{loc}}_G$ is a projective flat scheme over O_E which admits an action of the group scheme $\mathcal{G}_{O_E}$.

Theorem 2.1.9. ([65, Theorem 9.1], [37, Corollary 2.1.3]) *Under the assumptions in (2.1.8), the scheme* $\mathrm{M}^{\mathrm{loc}}_G$ *is normal. The geometric special fiber of* $\mathrm{M}^{\mathrm{loc}}_G$ *is reduced and admits a stratification with locally closed smooth strata; the closure of each stratum is normal and Cohen-Macaulay. Under the above assumptions, the base change* $\mathrm{M}^{\mathrm{loc}}_G \otimes_{O_E} O_L$ *is normal, for every finite extension* L/E.

Remark 2.1.10. In the PEL type case (A) and (C), Görtz proved in [8, 9] that if the group G is unramified over $\mathbb{Q}_p$, then the naive local models defined by Rapoport-Zink in [71] are flat over O_E, thus they coincide with the Pappas-Zhu local models above. In particular in these cases there exists a moduli interpretation for $\mathrm{M}^{\mathrm{loc}}_G$.

We will need the geometry of the geometric special fiber of $\mathrm{M}^{\mathrm{loc}}_G$, denoted by M_0 for simplicity. Let $\underline{G}$ and $\underline{\mathcal{G}}$ be as at the beginning of 2.1.7. We set $G' = \underline{G} \times k((u))$, and $\mathcal{G}' = \underline{\mathcal{G}} \times k[[u]]$. Here both base-changes are induced by reduction mod p. Noting that $\mathcal{G}'_0 \cong \mathcal{G}_{0,k}$, we view W_K as the Weyl group of $\mathcal{G}'_0$. By [37, §9.2.2], the construction of $\underline{G}$ induces an isomorphism between the Iwahori Weyl groups $\widetilde{W}$ of $G_{\breve{\mathbb{Q}}_p}$ and $\widetilde{W}'$ of G', and hence we will view $\mathrm{Adm}(\{\mu\}) \subset \widetilde{W}$ as an admissible subset of $\widetilde{W}'$.

We set

$$\mathcal{S}_{\mathcal{G}',\mu} = \bigcup_{w \in \mathrm{Adm}(\{\mu\})_K} \mathrm{Gr}^-_{\mathcal{G}',w}$$

with the reduced-induced scheme structure. It is a closed subvariety in $\mathrm{Gr}_{\mathcal{G}'}$. By [65, Theorem 9.3],

$$\mathcal{S}_{\mathcal{G}',\mu} = \mathrm{M}_0$$

as closed subschemes of $\mathrm{Gr}_{\mathcal{G}'}$. In view of this identification, we write $\mathrm{M}_0^w := \mathrm{Gr}_{\mathcal{G}',w}$ and $\mathrm{M}_0^{w,-} := \mathrm{Gr}^-_{\mathcal{G}',w}$ for $w \in \mathrm{Adm}(\{\mu\})_K$. In particular, combined with Theorem 2.1.4, Remark 2.1.5 and Lemma 2.1.6, we have the following.

Corollary 2.1.11. *There is a set-theoretically disjoint union of locally closed subsets*

$$\mathrm{M}_0 = \coprod_{w \in \mathrm{Adm}(\{\mu\})_K} \mathrm{M}_0^w.$$

Here $\mathrm{Adm}(\{\mu\})_K$ *is as in (1.2.1). Moreover,*

(1) $\mathrm{M}_0^{w,-} = \coprod_{v \in \mathrm{Adm}(\{\mu\})_K, v \leq w} \mathrm{M}_0^v$;
(2) *each* M_0^w *consists of a single* $\mathcal{G}_0$*-orbit, and the stabilizer of each closed point is smooth;*
(3) $\dim \mathrm{M}_0^w = \ell({}^K w_K)$;

Remark 2.1.12. In [30] 2.6, there is a modified version of the Pappas-Zhu local models for which we can remove the condition $p \nmid |\pi_1(G^{\mathrm{der}})|$. Moreover, in the abelian type case, by [30, Theorem 2.15], the associated small v-sheaves (in the sense of Scholze) satisfy Scholze's conjecture in [76] (Conjecture 21.4.1). Therefore, up to a mild modification as in [30] there exist embeddings of (the perfection of) the special fibers M_0 of the above local models into the Witt vector affine flag varieties $\mathrm{Gr}^W_{\mathcal{G}}$, cf. the above Remark 2.1.2. We will come back to this point of view in section 4.

2.2. **Integral models for Shimura varieties of Hodge type.** Let (G, X) be a Shimura datum of Hodge type and $p > 2$. For $x \in \mathcal{B}(G, \mathbb{Q}_p)$ (the extended Bruhat-Tits building of $G_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$), we write $\mathcal{G} = \mathcal{G}_x$ for the stabilizer of x and $\mathcal{G}^\circ$ for its connected component of the identity. Then $\mathcal{G}^\circ$ is the parahoric group scheme attached to x, which is an integral model of G over $\mathbb{Z}_p$. We will assume

$$G_{\mathbb{Q}_p} \text{ splits over a tamely ramified extension, } p \nmid |\pi_1(G^{\mathrm{der}}_{\mathbb{Q}_p})| \text{ and } \mathcal{G} = \mathcal{G}^\circ. \tag{2.2.1}$$

Let $\mathsf{E} = E(G, X)$ be the reflex field of (G, X), and v be a place of E over p that will be fixed once and for all. Let $E = \mathsf{E}_v$ and O_E be its ring of integers. The residue field of O_E will be denoted by κ as usual. Fixing $K_p := \mathcal{G}(\mathbb{Z}_p)$, for $K^p \subset G(\mathbb{A}_f^p)$ small enough, we set $\mathsf{K} := K_p K^p$, and we are interested in certain integral models of $\mathrm{Sh}_{\mathsf{K}}(G, X)_E$.

By [37, 4.1.5, 4.1.6], there is a symplectic embedding

$$i : (G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^{\pm})$$

such that there is a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ such that $V_{\mathbb{Z}} \subset V_{\mathbb{Z}}^\vee$, and that the base-change to $\mathbb{Z}_p$ of $\widetilde{\mathcal{G}}$, the Zariski closure of G in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$, is $\mathcal{G}$. For here and after, we simply write $\mathrm{GSp} = \mathrm{GSp}(V, \psi)$ and $V_R = V_{\mathbb{Z}} \otimes R$ for an algebra R. Let $g = \frac{1}{2}\mathrm{dim} V$, $H = H_p H^p$ where H_p is the subgroup of $\mathrm{GSp}(\mathbb{Q}_p)$ leaving $V_{\mathbb{Z}_p}$ stable, and H^p is a compact open subgroup of $\mathrm{GSp}(\mathbb{A}_f^p)$ containing K^p, leaving $V_{\widehat{\mathbb{Z}}^p}$ stable and small enough. Let $\mathscr{S}_H(\mathrm{GSp}, S^{\pm})$ be the moduli scheme over $\mathbb{Z}_{(p)}$ of isomorphism classes of certain triples $(\mathcal{A}, \lambda, \varepsilon^p)$ consisting of a g-dimensional abelian scheme $\mathcal{A}$ equipped with a polarization $\lambda : \mathcal{A} \to \mathcal{A}^t$ of degree $|V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}|$ and a level structure ε^p; for more details see [35] 2.3.3. The above symplectic embedding induces an embedding

$$i : \mathrm{Sh}_{\mathsf{K}}(G, X)_E \hookrightarrow \mathscr{S}_H(\mathrm{GSp}, S^{\pm})_{O_E},$$

and we write $\mathscr{S}_K^-(G, X)$ for the closure of $\mathrm{Sh}_{\mathsf{K}}(G, X)_E$ and

$$\mathscr{S}_{\mathsf{K}}(G, X)$$

for the normalization of $\mathscr{S}_{\mathsf{K}}^-(G, X)$. In particular we have morphisms

$$\mathscr{S}_{\mathsf{K}}(G, X) \to \mathscr{S}_{\mathsf{K}}^-(G, X) \subset \mathscr{S}_H(\mathrm{GSp}, S^{\pm})_{O_E}.$$

In particular we get a "universal" abelian scheme $\mathcal{A} \to \mathscr{S}_{\mathsf{K}}(G, X)$ by pulling back the universal abelian scheme over $\mathscr{S}_H(\mathrm{GSp}, S^{\pm})_{O_E}$ under the composition of the above morphisms.

2.2.2. *An alternative construction of local models.* We will follow [37] 4.1.5. Notations and assumptions as before, the Shimura datum (G, X) gives a $G(\overline{\mathbb{Q}}_p)$-conjugacy class $\{\mu\}$ of cocharacters. Its induced $\mathrm{GL}(V_{\mathbb{Z}_p})(\overline{\mathbb{Q}}_p)$-conjugacy class contains a cocharacter μ' defined over $\mathbb{Z}_p$. Let P (resp. P') be the parabolic subgroup of $G_{\overline{\mathbb{Q}}_p}$ (resp. $\mathrm{GL}(V_{\mathbb{Z}_p})$) with non-negative weights with respect to μ' (resp. μ). We denote by $\mathrm{M}^{\mathrm{loc}}_{G,X}$ the closure of the G-orbit of y in $\mathrm{GL}(V_{\mathbb{Z}_p})/P'$, where y is the point corresponding to P. Then $\mathrm{M}^{\mathrm{loc}}_{G,X}$ is defined over O_E and equipped with an action of $\mathcal{G}$. By [37, Corollary 2.3.16], it is the Pappas-Zhu local model attached to $(G_{\mathbb{Q}_p}, \{\mu\}, \mathcal{G})$.

To explain properties of $\mathscr{S}_{\mathsf{K}}(G, X)$ and its relations with $\mathrm{M}^{\mathrm{loc}}_{G,X}$, we need the following data. By [35, Lemma 1.3.2], there is a tensor $s \in V_{\mathbb{Z}_{(p)}}^{\otimes}$ defining $\mathcal{G}$. Consider

$$\mathcal{V} = \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathscr{S}_{\mathsf{K}}(G, X)), \quad \mathcal{V}_E = \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathrm{Sh}_{\mathsf{K}}(G, X)_E).$$

Let $\mathcal{V}^1 \subset \mathcal{V}$ be the Hodge filtration, and $s_{\mathrm{dR},E} \in \mathcal{V}_E^{\otimes}$ be the section induced by s.

Theorem 2.2.3. *Notations an assumptions as above, we have the following.*

(1) $\mathrm{M}^{\mathrm{loc}}_{G,X}$ *is normal with reduced geometric special fiber. Moreover,* $\mathrm{M}^{\mathrm{loc}}_{G,X}$ *depends only on the pair* $(G_{\mathbb{Q}_p}, \{\mu\})$.

(2) ([37, Proposition 4.2.6]) *The tensor* $s_{\mathrm{dR},E}$ *extends to a tensor* $s_{\mathrm{dR}} \in \mathcal{V}^{\otimes}$. *The* $\mathscr{S}_{\mathsf{K}}(G,X)$*-scheme* $\pi : \widetilde{\mathscr{S}_{\mathsf{K}}}(G,X) \to \mathscr{S}_{\mathsf{K}}(G,X)$ *which classifies isomorphisms* $f : V_{\mathbb{Z}_p}^{\vee} \to \mathcal{V}$ *mapping* s *to* s_{dR} *is a* $\mathcal{G}$*-torsor.*

(3) ([37, Theorem 4.2.7]) *The morphism* $q : \widetilde{\mathscr{S}_{\mathsf{K}}}(G,X) \to \mathrm{GL}(V_{\mathbb{Z}_p})/P'$, $f \mapsto f^{-1}(\mathcal{V}^1)$ *factors through* $\mathrm{M}^{\mathrm{loc}}_{G,X}$. *Moreover, it is smooth. In particular we get the following local model diagram*

$$\begin{array}{ccccc} & & \widetilde{\mathscr{S}_{\mathsf{K}}}(G,X) & & \\ & \swarrow^{\pi} & & \searrow^{q} & \\ \mathscr{S}_{\mathsf{K}}(G,X) & & & & \mathrm{M}^{\mathrm{loc}}_{G,X}. \end{array}$$

Remark 2.2.4. For $(V_{\mathbb{Z}}, \psi)$ as above, we set $V'_{\mathbb{Z}} := (V_{\mathbb{Z}} \oplus V_{\mathbb{Z}}^{\vee})^4$ and $V' := V'_{\mathbb{Z}} \otimes \mathbb{Q}$. By Zarhin's trick, there is a perfect alternating form ψ' on $V'_{\mathbb{Z}}$, such that the (faithful) representation of G on V' factors through $\mathrm{GSp}(V', \psi')$, and hence induces a closed immersion $\mathcal{G} \to \mathrm{GSp}(V'_{\mathbb{Z}_p}, \psi')$. One can then use this $(V'_{\mathbb{Z}}, \psi')$ as $(V_{\mathbb{Z}}, \psi)$, and the integral model thus obtained also satisfies all the properties in the above theorem.

2.2.5. Now we consider the general case that $\mathcal{G}^{\circ} \subset \mathcal{G}$ not necessarily equal. Let $K_p^{\circ} = \mathcal{G}^{\circ}(\mathbb{Z}_p) \subset K_p = \mathcal{G}(\mathbb{Z}_p)$ be the associated open compact subgroups of $G(\mathbb{Q}_p)$. Let $K^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small open compact subgroup. Set $\mathsf{K} = K_p K^p$, and $\mathsf{K}^{\circ} = K_p^{\circ} K^p$. We get the natural projection

$$\mathrm{Sh}_{\mathsf{K}^{\circ}}(G,X) \to \mathrm{Sh}_{\mathsf{K}}(G,X).$$

Denote by $\mathscr{S}_{\mathsf{K}^{\circ}}(G,X)$ the normalization of $\mathscr{S}_{\mathsf{K}}(G,X)$ in $\mathrm{Sh}_{\mathsf{K}^{\circ}}(G,X)$. Since K^p is sufficiently small, by [37, Proposition 4.3.7], the covering

$$\mathscr{S}_{\mathsf{K}^{\circ}}(G,X) \to \mathscr{S}_{\mathsf{K}}(G,X)$$

is étale, and it splits over an unramified extension of O_E. Let $\widetilde{\mathscr{S}_{\mathsf{K}^{\circ}}}(G,X) \to \mathscr{S}_{\mathsf{K}^{\circ}}(G,X)$ be the pullback of the $\mathcal{G}$-torsor over $\mathscr{S}_{\mathsf{K}}(G,X)$. By abuse of notation we still denote this morphism by π, and the composition $\widetilde{\mathscr{S}_{\mathsf{K}^{\circ}}}(G,X) \to \widetilde{\mathscr{S}_{\mathsf{K}}}(G,X) \to \mathrm{M}^{\mathrm{loc}}_{G,X}$ by q. Thus we get a diagram

$$\begin{array}{ccccc} & & \widetilde{\mathscr{S}_{\mathsf{K}^{\circ}}}(G,X) & & \\ & \swarrow^{\pi} & & \searrow^{q} & \\ \mathscr{S}_{\mathsf{K}^{\circ}}(G,X) & & & & \mathrm{M}^{\mathrm{loc}}_{G,X}, \end{array}$$

where π is a $\mathcal{G}$-torsor, and q is $\mathcal{G}$-equivariant and smooth of relative dimension $\dim G$. As conjectured in [37] 4.3.10, this $\mathcal{G}$-torsor $\widetilde{\mathscr{S}_{\mathsf{K}^{\circ}}}(G,X) \to \mathscr{S}_{\mathsf{K}^{\circ}}(G,X)$ should have a reduction to a $\mathcal{G}^{\circ}$-torsor.

2.3. **Deformations of p-divisible groups with crystalline tensors.** To understand the local geometric structures of $\mathscr{S}_{\mathsf{K}} = \mathscr{S}_{\mathsf{K}}(G,X)$ or $\mathrm{M}^{\mathrm{loc}}_{G,X}$, we need to study the deformation theory of p-divisible groups with crystalline tensors. In this subsection we will mainly follow [37] section 3. As previously we assume $p > 2$. Let k be either a finite extension of $\mathbb{F}_p$ or $\overline{\mathbb{F}}_p$.

Let R be a complete local ring with residue field k and maximal ideal $\mathfrak{m}$. We set

$$\mathbb{W}(R) := W(k) \oplus \mathbb{W}(\mathfrak{m}) \subset W(R),$$

where $\mathbb{W}(\mathfrak{m})$ consists of Witt vectors $(w_i)_{i \geq 1}$ such that $w_i \in \mathfrak{m}$ and $(w_i)_{i \geq 1}$ goes to 0 $\mathfrak{m}$-adically. Let $I_R := \ker(\mathbb{W}(R) \to R)$. We denote by σ the Frobenius endomorphism on $\mathbb{W}(R)$, and $\sigma_1 : I_R \to \mathbb{W}(R)$ the inverse of the Verschiebung v.

Definition 2.3.1. ([96, Definition 1]) A Dieudonné display over R is a tuple $(M, M_1, \varphi, \varphi_1)$, where

- M is a finite free $\mathbb{W}(R)$-module;
- $M_1 \subset M$ is a $\mathbb{W}(R)$-submodule such that $I_R M \subset M_1 \subset M$ and M/M_1 is a projective R-module;
- $\varphi : M \to M$ is a σ-linear map;
- $\varphi_1 : M_1 \to M$ is a σ-linear map whose image generates M as a $\mathbb{W}(R)$-module, and which satisfies $\varphi_1(v(w)m) = w\varphi(m)$, for all $w \in \mathbb{W}(R)$ and $m \in M$.

It is constructed in [96] a functor from the category of p-divisible groups over R to the category of Dieudonné displays over R. Moreover, by the main results there, this functor induces an equivalence of categories.

We are particularly interested in cases when $W(R)$, and hence $\mathbb{W}(R)$, is p-torsion free. This holds when R is p-torsion free, or $pR = 0$ and R is reduced. In this case, one can recover $(M, M_1, \varphi, \varphi_1)$ from (M, M_1, φ_1) by taking $\varphi(m) := \varphi_1(v(1)m)$. We will also call the tuple (M, M_1, φ_1) satisfying related conditions in Definition 2.3.1 a Dieudonné display.

Let $\widetilde{M}_1$ be the image of the homomorphism

$$\sigma^*(i) : \sigma^* M_1 := \mathbb{W}(R) \otimes_{\sigma, \mathbb{W}(R)} M_1 \to \sigma^* M = \mathbb{W}(R) \otimes_{\sigma, \mathbb{W}(R)} M$$

induced by the inclusion $i : M_1 \to M$.

Lemma 2.3.2. ([37, Lemma 3.1.5]) *Suppose that $W(R)$ is p-torsion free. Then*

(1) *for a Dieudonné display (M, M_1, φ_1) over R, the linearization of φ_1 factors as a composition $\sigma^* M_1 \to \widetilde{M}_1 \xrightarrow{\Psi} M$ with Ψ an isomorphism;*
(2) *given an isomorphism $\Psi : \widetilde{M}_1 \to M$, there is a unique Dieudonné display (M, M_1, φ_1) over R which produces (M, M_1, Ψ) via the construction in* (1).

We will also call a triple (M, M_1, Ψ) as above a Dieudonné display.

2.3.3. *Versal deformations.* Let $\mathscr{G}_0$ be a p-divisible group over k, and (N, N_1, ϕ, ϕ_1) be the attached Dieudonné display. By Lemma 2.3.2 (and using similar notations), this is given by an isomorphism $\Psi_0 : \widetilde{N}_1 \xrightarrow{\sim} N$. We will describe a certain versal deformation of $\mathscr{G}_0$ constructed in [37] using Dieudonné displays. The Hodge filtration on $N \otimes k$ gives a parabolic subgroup $P_0 \subset \mathrm{GL}(N \otimes k)$. We will fix a parabolic subgroup $P \subset \mathrm{GL}(N)$ lifting P_0. Let $\mathrm{M}^{\mathrm{loc}} = \mathrm{GL}(N)/P$, and

$$\widehat{\mathrm{M}}^{\mathrm{loc}} = \mathrm{Spf}\, R$$

be the completion of $\mathrm{M}^{\mathrm{loc}}$ along the image of identity in $\mathrm{GL}(N \otimes k)$. In particular, R is a power series ring over $W(k)$.

Let $M = N \otimes_{W(k)} \mathbb{W}(R)$ and $\overline{M}_1 \subset M/I_R M$ be the filtration corresponding to the parabolic

$$\mathfrak{g} P \mathfrak{g}^{-1} \subset \mathrm{GL}(N_R),$$

where $\mathfrak{g} \in (\mathrm{GL}(N)/P)(R)$ is the universal point. Let M_1 be the preimage of $\overline{M}_1$ in M, $\widetilde{M}_1$ be as in Lemma 2.3.2, and $\Psi : \widetilde{M}_1 \xrightarrow{\sim} M$ be an isomorphism which reduces to Ψ_0 modulo $\mathbb{W}(\mathfrak{m})$. The triple

$$(M, M_1, \Psi)$$

gives a Dieudonné display, and hence a p-divisible group $\mathscr{G}$ over R which deforms $\mathscr{G}_0$.

Let $\mathfrak{a}_R = \mathfrak{m}^2 + pR \subset R$. By [37, Lemma 3.1.9], there is a natural identification

$$\widetilde{M}_1 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{a}_R) \xrightarrow{\cong} \widetilde{N}_1 \otimes_{W(k)} \mathbb{W}(R/\mathfrak{a}_R) \tag{2.3.4}$$

making the following diagram commutative.

$$\begin{array}{ccc} \widetilde{M}_1 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{a}_R) & \longrightarrow & \sigma^*\big(M \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{a}_R)\big) \\ \downarrow \cong & & \| \\ \widetilde{N}_1 \otimes_{W(k)} \mathbb{W}(R/\mathfrak{a}_R) & \longrightarrow & \sigma^*(N) \otimes_{W(k)} \mathbb{W}(R/\mathfrak{a}_R). \end{array}$$

As in [37, 3.1.11], Ψ is said to be *constant modulo* $\mathfrak{a}_R$ if the composition

$$\widetilde{N}_1 \otimes_{W(k)} \mathbb{W}(R/\mathfrak{a}_R) \cong \widetilde{M}_1 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{a}_R) \xrightarrow{\Psi} M \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{a}_R) \cong N \otimes_{W(k)} \mathbb{W}(R/\mathfrak{a}_R)$$

is equal to $\Phi_0 \otimes 1$. A useful class of versal deformations is as follows.

Lemma 2.3.5. ([37, Lemma 3.1.12]) *If Ψ is constant modulo $\mathfrak{a}_R$ then the deformation $\mathscr{G}$ is versal.*

2.3.6. *Deformations with crystalline tensors.* Let $N \otimes k \supset \mathrm{Fil}^1(N \otimes k)$ be the Hodge filtration. By [67, 4C, 4G], one has a natural descending filtration on $N^{\otimes} \otimes k$, and in particular a subspace $\mathrm{Fil}^0 \subset N^{\otimes} \otimes k$. Now we assume that there is a tensor $s_{\mathrm{cris}} \in N^{\otimes}$ which is ϕ-invariant and its image in $N^{\otimes} \otimes k$ lies in Fil^0. A deformation theory is developed in [37, 3.2] in a more general setting, but to apply to current problems, we simply assume that there is an isomorphism

$$f : V_{\mathbb{Z}_p} \otimes W(k) \to N$$

mapping s to s_{cris}. Here $V_{\mathbb{Z}_p}$ and s are as in Theorem 2.2.3. We will fix the isomorphism f in discussions here. In particular, we have

$$\mathrm{M}_G^{\mathrm{loc}} \subset (\mathrm{GL}(N)/P)_{O_E}$$

which is normal with reduced fibers.

Let Spf R_G be the completion of $\mathrm{M}_G^{\mathrm{loc}}$ along the image of identity in $\mathrm{GL}(N \otimes k)$. Then R_G is a quotient of $R_E := R \otimes_{W(k)} O_E$. We set

$$M_{R_E} := M \otimes_{\mathbb{W}(R)} \mathbb{W}(R_E), \quad M_{R_G} := M \otimes_{\mathbb{W}(R)} \mathbb{W}(R_G),$$

and $\widetilde{M}_{R_G,1} \subset M_{R_G}$ be constructed the same way as $\widetilde{M}_1$. Note that R_G is p-torsion free, since $\mathrm{M}_G^{\mathrm{loc}}$ is normal with reduced fibers and Spf R_G is the completion of $\mathrm{M}_G^{\mathrm{loc}}$ along a closed point. In particular, $W(R_G)$ is p-torsion free. Then we have, by [37, 3.1.6], that

$$\widetilde{M}_{R_G,1} = \widetilde{M}_1 \otimes_{\mathbb{W}(R)} \mathbb{W}(R_G).$$

Lemma 2.3.7. [37, Corollary 3.2.11] *The tensor s also lies in $\widetilde{M}_{R_G,1}^{\otimes}$. Moreover,*

$$\mathcal{T} := \mathrm{Isom}_{\mathbb{W}(R_G)}\big((\widetilde{M}_{R_G,1}, s), (M_{R_G}, s)\big)$$

is a trivial $\mathcal{G}$-torsor.

Let $\mathfrak{a}_E := \mathfrak{m}_{R_E}^2 + \pi R_E$, where $\pi \in O_E$ is a uniformizer, then $R_E/\mathfrak{a}_E \cong R/\mathfrak{a}_R$. By [37, 3.2.12], there is an isomorphism $\Psi : \widetilde{M}_{R_E,1} \to M_{R_E}$ which is constant modulo $\mathfrak{a}_E$ and such that

$$\Psi_{R_G} := \Psi \otimes_{\mathbb{W}(R_E)} \mathbb{W}(R_G) : \widetilde{M}_{R_G,1} \to M_{R_G}$$

respects the tensor s.

As in the previous case, $(M_{R_E}, M_{R_E,1}, \Psi)$ gives a Dieudonné display, and hence a p-divisible group $\mathscr{G}$ over R_E deforming $\mathscr{G}_0$. Moreover, by the previous lemma, any deformations thus obtained are isomorphic, and hence the deformation $\mathscr{G}/R_E$ is versal. For a finite extension L/E, an O_L-point of Spf R_E factors through the subspace Spf R_G if the deformation $\mathscr{G}_{O_L}$ is compatible with the tensor s. We refer to [37, Proposition 3.2.17] for the precise statement. If $\mathscr{G}_0$ is the p-divisible group attached to a point $x \in \mathscr{S}_{\mathsf{K}}(k)$, the p-divisible group $\mathcal{A}_{\widehat{O}_x}[p^\infty]$ is a deformation of $\mathscr{G}_0$, and hence, by versality of $\mathscr{G}$, is given by a morphism $\mathrm{Spf}\, \widehat{O}_x \to \mathrm{Spf}\, R_E$. By [37, Proposition 4.2.2], any such morphism induces an isomorphism

$$\mathrm{Spf}\, \widehat{O}_x \xrightarrow{\sim} \mathrm{Spf}\, R_G.$$

2.3.8. *Some variations.* Recall that we defined M to be $N \otimes \mathbb{W}(R)$, and we used this in a direct way almost everywhere in previous descriptions. In particular, the Dieudonné display $(M_{R_E}, M_{R_E,1}, \Psi)$, and hence

$$(M_{R_G}, M_{R_G,1}, \Psi_{R_G})$$

which determines the p-divisible group $\mathcal{A}_{\widehat{O}_x}[p^\infty]$, is based on this identification. However, it is sometimes more convenient to use it in a less direct way.

We start with the universal element of $(\mathrm{GL}(V_{\mathbb{Z}_p})/P)(R)$ restricted to $\mathrm{Spf}\, R_G$. It lifts to an element in $\mathfrak{g} \in \mathrm{GL}(V_{\mathbb{Z}_p})(R_G)$ such that $\mathfrak{g} \equiv \mathrm{id} \mod \mathfrak{m}_{R_E}$, and then to an element $\widetilde{\mathfrak{g}} \in \mathrm{GL}(V_{\mathbb{Z}_p})(\mathbb{W}(R_G))$. Let $N^1 \subset N$ be the filtration corresponding to P. Identifying $N \otimes \mathbb{W}(R_G)$ and M_{R_G} as before, then

$$\mathfrak{g} \cdot (N^1 \otimes R_G) \subset N \otimes R_G$$

is the Hodge filtration of $M_{R_G} \otimes R_G$. Let $(N \otimes \mathbb{W}(R_G))_1$ be the preimage of $N^1 \otimes R_G$ in $N \otimes \mathbb{W}(R_G)$, then

$$N_1 \otimes \mathbb{W}(R_G) \subset (N \otimes \mathbb{W}(R_G))_1$$

and

$$\widetilde{\mathfrak{g}} \cdot (N \otimes \mathbb{W}(R_G))_1 = M_{R_G,1}.$$

We have a natural identification

$$\widetilde{N}_1 \otimes \mathbb{W}(R_G) \cong (N \widetilde{\otimes \mathbb{W}(R_G)})_1,$$

and $\sigma(\widetilde{\mathfrak{g}})$ induces an isomorphism from $\widetilde{N}_1 \otimes \mathbb{W}(R_G)$ to $\widetilde{M}_{R_G,1}$, denoted by $\sigma(\widetilde{\mathfrak{g}})_\flat$. Now the Dieudonné display $(M_{R_G}, M_{R_G,1}, \Psi_{R_G})$ becomes

$$\big(N \otimes \mathbb{W}(R_G), (N \otimes \mathbb{W}(R_G))_1, \Psi'_{R_G}\big), \quad \text{where} \quad \Psi'_{R_G} := \widetilde{\mathfrak{g}}^{-1} \circ \Psi_{R_G} \circ \sigma(\widetilde{\mathfrak{g}})_\flat. \tag{2.3.9}$$

Remark 2.3.10. We would like to describe Ψ'_{R_G} restricted to some interesting sub-formal-schemes of $\mathrm{Spf}\, R_G$.

(1) One can not assume the lifting $\mathfrak{g}$ (and hence $\widetilde{\mathfrak{g}}$) to be an element in $\mathcal{G}$ in general, as we don't have smooth coverings $\mathcal{G}_{O_E} \to \mathrm{M}^{\mathrm{loc}}_G$. For a sub-formal-scheme $\mathrm{Spf}\, A \subset \mathrm{Spf}\, R_G$ such that $\mathbb{W}(A)$ is p-torsion free, we can describe the restriction of $(M_{R_G}, M_{R_G,1}, \Psi_{R_G})$ to A in the same way as in 2.3.8, and get a triple $\big(N \otimes \mathbb{W}(A), (N \otimes \mathbb{W}(A))_1, \Psi'_A\big)$. Here $(N \otimes \mathbb{W}(A))_1$ is the preimage of $N^1 \otimes A$ in $N \otimes \mathbb{W}(A)$, and

$$\Psi'_A = \widetilde{\mathfrak{g}_A}^{-1} \circ \Psi_{R_G} \circ \sigma(\widetilde{\mathfrak{g}_A})_\flat$$

with $\mathfrak{g}_A \in \mathrm{GL}(V_{\mathbb{Z}_p})(A)$ a lifting of the tautological morphism $\mathrm{Spf}\, A \to \mathrm{M}^{\mathrm{loc}}_G$, and $\widetilde{\mathfrak{g}_A} \in \mathrm{GL}(V_{\mathbb{Z}_p})(\mathbb{W}(A))$ a lifting of $\mathfrak{g}_A$. If $\mathrm{Spf}\, A$ is contained in the KR stratum of $\mathrm{Spf}\, R_G$ containing the closed point, i.e. if the morphism $\mathrm{Spf}\, A \subset \mathrm{Spf}\, R_G \to \mathrm{M}^{\mathrm{loc}}_G$ factors through $\mathrm{M}^w_0 \subset \mathrm{M}^{\mathrm{loc}}_G$, then we can choose $\mathfrak{g}_A \in \mathcal{G}(A)$ and $\widetilde{\mathfrak{g}_A} \in \mathcal{G}(\mathbb{W}(A))$ for Ψ_A.

(2) Let $\mathrm{Spf}\, A \subset \mathrm{Spf}\, R_G$ be a sub-formal-scheme such that $\mathbb{W}(A)$ is p-torsion free. Noting that

$$\sigma(\widetilde{\mathfrak{g}_A})_\flat = \mathrm{id} \in \mathrm{GL}(V_{\mathbb{Z}_p})(\mathbb{W}(A/\mathfrak{a}_E A))$$

by the canonical identification (2.3.4), we have

$$\Psi'_A = \widetilde{\mathfrak{g}_A}^{-1} \circ (\Psi_0 \otimes 1) \in \mathrm{GL}(V_{\mathbb{Z}_p})(\mathbb{W}(A/\mathfrak{a}_E A))$$

as Ψ is assumed to be constant mod $\mathfrak{a}_E$. If (A, m_A) is the complete local ring at x of the KR stratum containing it, then over A, we can choose $\widetilde{\mathfrak{g}_A} \in \mathcal{G}(\mathbb{W}(A))$ and find that

$$\Psi'_A = \widetilde{\mathfrak{g}_A}^{-1} \circ (\Psi_0 \otimes 1) \in \mathcal{G}(\mathbb{W}(A/m_A^2)).$$

2.3.11. *The global crystalline tensor and related torsors.* We recall the global crystalline tensor constructed in [20]. For a ring R, one can define displays over R as in Definition 2.3.1, but working with $W(R)$ and setting $I_R = \ker(W(R) \to R)$. If R is p-adically complete, Lau [46, Proposition 2.1] constructed a natural functor $\mathcal{M}$ from the category of p-divisible groups over R to that of displays over R. If R is, in addition, a complete local ring, then we have a canonical isomorphism

$$\mathcal{M} \cong \mathbb{M} \otimes_{\mathbb{W}(R)} W(R).$$

Here $\mathbb{M}$ is the Dieudonné display attached to a p-divisible group X, and $\mathcal{M} = \mathcal{M}(X)$ is the attached display. By descent theory of displays in [97, §1.3], one can define displays over p-adic formal schemes. In particular, we have a natural functor form the category of p-divisible groups over a p-adic formal scheme $\mathcal{S}$ to that of displays over $\mathcal{S}$.

Let $\mathscr{S}_{\mathsf{K}}^{\wedge}$ be the p-adic completion of the integral model $\mathscr{S}_{\mathsf{K}}/O_{\breve{E}}$, and $(\underline{M}, \underline{M}_1, \underline{\Psi})$ be the display attached to the p-divisible group $\mathcal{A}[p^\infty] \mid_{\mathscr{S}_{\mathsf{K}}^{\wedge}}$. Here

$$\underline{\Psi} : \widetilde{\underline{M}_1} \to \underline{M}$$

is an isomorphism constructed in a similar way as in Lemma 2.3.2.

Proposition 2.3.12. ([20, Proposition 3.3.1]) *There is a tensor $\underline{s}_{\mathrm{cris}} \in \underline{M}^{\otimes}$ which is also in $\underline{M}_1^{\otimes}$, such that $\underline{s}_{\mathrm{cris}}$ is $\underline{\Psi}$-invariant, and that for each $z \in \mathscr{S}_{\mathsf{K}}(k)$, its restriction to* $\mathrm{Spf}\, R_G$ *is* s_{cris}.

Let R be p-adic ring, $I_R = \mathrm{Im}(V)$ as above, then by [97, Proposition 3], $W(R)$ is p-adic and I_R-adic. So an element $a \in W(R)$ is a unit if and only if its image in R is a unit. Now for $\mathfrak{n} \in \mathrm{Max}\, W(R)$, its image in R is a proper ideal, and hence a maximal ideal. In particular, the map $\mathrm{Max}\, R \to \mathrm{Max}\, W(R), \mathfrak{m} \mapsto \widetilde{\mathfrak{m}}$, where $\widetilde{\mathfrak{m}}$ is the pre-image in $W(R)$ of $\mathfrak{m}$, is a bijection.

Lemma 2.3.13. *Let R be a p-adic Noetherian ring which is reduced and X be a scheme over $W(R)$ of finite presentation. Then X is flat over $W(R)$ if $X \otimes_{W(R)} W(R_{\mathfrak{m}}^{\wedge})$ is flat over $W(R_{\mathfrak{m}}^{\wedge})$ for all $\mathfrak{m} \in \mathrm{Max}\, R$, where $R_{\mathfrak{m}}^{\wedge}$ is the $\mathfrak{m}$-adic completion of $R_{\mathfrak{m}}$.*

Proof. Let us denote by A^h the henselization of a ring A. It suffices to show that $X \otimes_{W(R)} W(R)_{\widetilde{\mathfrak{m}}}^h$ is flat over $W(R)_{\widetilde{\mathfrak{m}}}^h$ for all $\mathfrak{m} \in \mathrm{Max}\, R$. Let $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$ be the minimal primes of R, then we have $R \hookrightarrow \prod_i R_i$ and hence $W(R) \hookrightarrow \prod_i W(R_i)$, where $R_i := R/\mathfrak{p}_i$. Let $\mathfrak{m}_i \in \mathrm{Max}\, R_i$ be the image of $\mathfrak{m}$ in R_i, we have

$$W(R)_{\widetilde{\mathfrak{m}}} \hookrightarrow \prod_i W(R_i)_{\widetilde{\mathfrak{m}}_i}.$$

For each i, we have a natural homomorphism $f_i : W(R_i)_{\widetilde{\mathfrak{m}}_i} \to W(R_{i,\mathfrak{m}_i}^{\wedge})$ by the universality of localizations. We claim that f_i is injective. The lemma follows from this injectivity: we will then have injections

$$W(R)_{\widetilde{\mathfrak{m}}}^h \hookrightarrow \prod_i W(R_i)_{\widetilde{\mathfrak{m}}_i}^h \hookrightarrow \prod_i W(R_{i,\mathfrak{m}_i}^{\wedge}).$$

By [73, Theorem 4.2.8 (a)], flatness over $W(R)_{\widetilde{\mathfrak{m}}}^h$ reduces to that over $\prod_i W(R_{i,\mathfrak{m}_i}^{\wedge})$ which is clear by our assumption.

Now we prove the injectivity of f_i. To simplify notations, we remove the subscript i. Noting that R is an integral domain, we see that R is either an $\mathbb{F}_p$-algebra or p-torsion free.

(1) If R is an $\mathbb{F}_p$-algebra, then $W(R)$ and $W(R^\wedge_{\mathfrak{m}})$ are integral domains with $W(R) \subset W(R^\wedge_{\mathfrak{m}})$. The induced homomorphism $W(R)_{\widetilde{\mathfrak{m}}} \subset W(R^\wedge_{\mathfrak{m}})$ is clearly injective.
(2) If R is p-torsion free, then we have a commutative diagram of injective homomorphisms

$$\begin{array}{ccc} W(R) & \longrightarrow & \prod_{i=0}^{\infty} R \\ \downarrow & & \downarrow \\ W(R^\wedge_{\mathfrak{m}}) & \longrightarrow & \prod_{i=0}^{\infty} R^\wedge_{\mathfrak{m}}. \end{array}$$

Here the horizontal maps are given by the Witt polynomials. Elements in $W(R) - \widetilde{\mathfrak{m}}$ are of the form $r = (r_0, r_1, \cdots)$ with $r_0 \in R - \mathfrak{m}$. Its image in $\prod_{i=0}^{\infty} R$ is of the form

$$(r_0, r_0^p + pr_1', \cdots, r_0^{p^n} + pr_n', \cdots).$$

Noting that $p \in \mathfrak{m}$, r is not a zero-divisor in $\prod_{i=0}^{\infty} R$, and hence $W(R)_{\widetilde{\mathfrak{m}}} \subset W(R^\wedge_{\mathfrak{m}})$. □

Let $\mathscr{S}_{\mathsf{K}}/O_{\breve{E}}$, and $(\underline{M}, \underline{M}_1, \underline{\Psi})$ be as before. We will simply work with its base-change to R, where $\operatorname{Spf} R \subset \mathscr{S}^\wedge_{\mathsf{K}}$ is open affine.

Corollary 2.3.14. *Notations as above, we have*

(1) *the scheme* $\widetilde{\mathscr{S}}^\wedge_{\mathsf{K},\mathrm{cris}} := \mathbf{Isom}_{W(R)}\big((V^\vee_{\mathbb{Z}_p}, s) \otimes W(R), (\underline{M}, \underline{s}_{\mathrm{cris}})\big)$ *is a* $\mathcal{G}$*-torsor over* $W(R)$*;*
(2) *the scheme* $\widetilde{\mathscr{S}}'^\wedge_{\mathsf{K},\mathrm{cris}} := \mathbf{Isom}_{W(R)}\big((V^\vee_{\mathbb{Z}_p}, s) \otimes W(R), (\widetilde{\underline{M}}_1, \underline{s}_{\mathrm{cris}})\big)$ *is a* $\mathcal{G}$*-torsor over* $W(R)$*.*

Proof. Let $R^\wedge$ be the complete local ring at a closed point of $\operatorname{Spf} R$. Then $\widetilde{\mathscr{S}}^\wedge_{\mathsf{K},\mathrm{cris}} \otimes W(R^\wedge)$ is a $\mathcal{G}$-torsor by Proposition 2.3.12, and hence statement (1) follows from Lemma 2.3.13. By Lemma 2.3.7, $\widetilde{\mathscr{S}}'^\wedge_{\mathsf{K},\mathrm{cris}} \otimes W(R^\wedge)$ is a $\mathcal{G}$-torsor, and hence statement (2) follows the same way. □

2.4. **Integral models for Shimura varieties of abelian type.** Recall that a Shimura datum (G, X) is said to be of abelian type, if there is a Shimura datum of Hodge type (G_1, X_1) and a central isogeny $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ which induces an isomorphism of adjoint Shimura data $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \xrightarrow{\sim} (G^{\mathrm{ad}}, X^{\mathrm{ad}})$.

Let (G, X) be a Shimura datum of abelian type and fix a choice of a Hodge type datum (G_1, X_1) as above. As before $p > 2$.

2.4.1. Let $x \in \mathcal{B}(G, \mathbb{Q}_p)$ and $x_1 \in \mathcal{B}(G_1, \mathbb{Q}_p)$ such that $x^{\mathrm{ad}} = x_1^{\mathrm{ad}} \in \mathcal{B}(G^{\mathrm{ad}}, \mathbb{Q}_p)$. We denote the model of G (resp. G_1) over $\mathbb{Z}_p$ defined as the stabilizer of x (resp. x_1) by $\mathcal{G}$ (resp. $\mathcal{G}_1$), with connected model $\mathcal{G}^\circ$ (resp. $\mathcal{G}_1^\circ$). We have group schemes $G_{\mathbb{Z}_{(p)}}$ (resp. $G_{1,\mathbb{Z}_{(p)}}$) and $G^\circ_{\mathbb{Z}_{(p)}}$ (resp. $G^\circ_{1,\mathbb{Z}_{(p)}}$) over $\mathbb{Z}_{(p)}$ corresponding to $\mathcal{G}$ (resp. $\mathcal{G}_1$) and $\mathcal{G}^\circ$ (resp. $\mathcal{G}_1^\circ$). Write $K_p = \mathcal{G}(\mathbb{Z}_p)$, $K_p^\circ = \mathcal{G}^\circ(\mathbb{Z}_p)$, $K_{1,p} = \mathcal{G}_1(\mathbb{Z}_p)$, and $K_{1,p}^\circ = \mathcal{G}_1^\circ(\mathbb{Z}_p)$.

Let $G^{\mathrm{ad}\circ}_{\mathbb{Z}_{(p)}}$ be the parahoric model of $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ over $\mathbb{Z}_{(p)}$ defined by $x^{\mathrm{ad}} \in \mathcal{B}(G^{\mathrm{ad}}, \mathbb{Q}_p)$. Similarly we have $G^{\mathrm{ad}\circ}_{1,\mathbb{Z}_{(p)}}$. Then

$$G^{\mathrm{ad}\circ}_{\mathbb{Z}_{(p)}} = G^{\mathrm{ad}\circ}_{1,\mathbb{Z}_{(p)}}$$

as $G^{\mathrm{ad}}_{\mathbb{Q}_p} = G^{\mathrm{ad}}_{1,\mathbb{Q}_p}$ and $x^{\mathrm{ad}} = x_1^{\mathrm{ad}}$ by assumption. Set $G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}} = G_{\mathbb{Z}_{(p)}}/Z_{\mathbb{Z}_{(p)}}$, where $Z_{\mathbb{Z}_{(p)}}$ is the closure of the center Z_G of G in $G_{\mathbb{Z}_{(p)}}$. Similarly, we have $G^{\mathrm{ad}}_{1,\mathbb{Z}_{(p)}} = G_{1,\mathbb{Z}_{(p)}}/Z_{1,\mathbb{Z}_{(p)}}$. In general, $G^{\mathrm{ad}\circ}_{\mathbb{Z}_{(p)}}$ is not equal to the neutral component $(G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}})^\circ$ of $G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$. However, suppose that either the center Z_G is connected or that $Z_{G^{\mathrm{der}}}$ has rank prime to p, then by [37, Lemma 4.6.2 (2)], $G^{\mathrm{ad}\circ}_{\mathbb{Z}_{(p)}} = (G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}})^\circ$. In what follows, we will assume that either Z_{G_1} is connected or that $Z_{G_1^{\mathrm{der}}}$ has rank prime to p. Then by the above discussion, we have

$$G^{\mathrm{ad}\circ}_{\mathbb{Z}_{(p)}} = G^{\mathrm{ad}\circ}_{1,\mathbb{Z}_{(p)}} = (G^{\mathrm{ad}}_{1,\mathbb{Z}_{(p)}})^\circ,$$

which is the neutral component of $G^{\mathrm{ad}}_{1,\mathbb{Z}_{(p)}}$.

2.4.2. Recall ([4]) that the group

$$\mathscr{A}(G) = G(\mathbb{A}_f)/Z_G(\mathbb{Q})^- *_{G(\mathbb{Q})_+/Z_G(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^+$$

acts on $\mathrm{Sh}(G,X) = \varprojlim_K \mathrm{Sh}_K(G,X)$. Set

$$\mathscr{A}(G)^\circ = G(\mathbb{Q})_+^-/Z_G(\mathbb{Q})^- *_{G(\mathbb{Q})_+/Z_G(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^+,$$

which depends only on G^{der} and not on G. Similarly for G_1 we have $\mathscr{A}(G_1)$ and $\mathscr{A}(G_1)^\circ$.

Using the integral models $G^\circ = G^\circ_{\mathbb{Z}_{(p)}}$ and $G^{\mathrm{ad}\circ} = G^{\mathrm{ad}\circ}_{\mathbb{Z}_{(p)}}$ we introduce

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}}) = G(\mathbb{A}_f^p)/Z_G(\mathbb{Z}_{(p)})^- *_{G^\circ(\mathbb{Z}_{(p)})_+/Z_G(\mathbb{Z}_{(p)})} G^{\mathrm{ad}\circ}(\mathbb{Z}_{(p)})^+$$

and

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ = G(\mathbb{Z}_{(p)})_+^-/Z_G(\mathbb{Z}_{(p)})^- *_{G^\circ(\mathbb{Z}_{(p)})_+/Z_G(\mathbb{Z}_{(p)})} G^{\mathrm{ad}\circ}(\mathbb{Z}_{(p)})^+.$$

Similarly for G_1 we have $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})$ and $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$.

2.4.3. Let $\mathrm{Sh}_{K_p^\circ}(G,X) = \varprojlim_{K^p} \mathrm{Sh}_{K_p^\circ K^p}(G,X)$ be the limit with the $G(\mathbb{A}_f^p)$-action. We want to construct an integral model

$$\mathscr{S}_{K_p^\circ}(G,X)$$

of $\mathrm{Sh}_{K_p^\circ}(G,X)$ together with the $G(\mathbb{A}_f^p)$-action, such that for any $K^p \subset G(\mathbb{A}_f^p)$,

$$\mathscr{S}_{\mathsf{K}^\circ}(G,X) := \mathscr{S}_{K_p^\circ}(G,X)/K^p$$

defines an integral model of $\mathrm{Sh}_{\mathsf{K}^\circ}(G,X)$ with $\mathsf{K}^\circ = K_p^\circ K^p$.

Consider the Hodge type datum (G_1,X_1) and the associated integral models $\mathscr{S}_{K_{1,p}^\circ K_1^p}(G_1,X_1)$. Set

$$\mathscr{S}_{K_{1,p}^\circ}(G_1,X_1) = \varprojlim_{K_1^p} \mathscr{S}_{K_{1,p}^\circ K_1^p}(G_1,X_1).$$

This is an integral model of $\mathrm{Sh}_{K_{1,p}^\circ}(G_1,X_1) = \varprojlim_{K_1^p} \mathrm{Sh}_{K_{1,p}^\circ K_1^p}(G_1,X_1)$ by subsection 2.2.

Let $X_1^+ \subset X_1$ be a connected component. For $\mathsf{K}_1 = K_{1,p}^\circ K_1^p$, let

$$\mathrm{Sh}_{\mathsf{K}_1^\circ}(G_1,X_1)^+ \subset \mathrm{Sh}_{\mathsf{K}_1^\circ}(G_1,X_1)$$

be the geometrically connected component which is the image of $X_1^+ \times 1$. Then the scheme $\mathrm{Sh}_{K_1^\circ}(G_1,X_1)^+ = \varprojlim_{K_1^p} \mathrm{Sh}_{K_{1,p}^\circ K_1^p}(G_1,X_1)^+$ is defined over E_1^p, where E_1 is the reflex field of (G_1,X_1), and E_1^p is the maximal extension of E_1 which is unramified at p. Let $O_{(p)}$ be the localization at (p) of the ring of integers of E_1^p. We write

$$\mathscr{S}_{\mathsf{K}_1^\circ}(G_1,X_1)^+$$

for the closure of $\mathrm{Sh}_{\mathsf{K}_1^\circ}(G_1,X_1)^+$ in $\mathscr{S}_{\mathsf{K}_1^\circ}(G_1,X_1) \otimes O_{(p)}$, and set

$$\mathscr{S}_{K_{1,p}^\circ}(G_1,X_1)^+ := \varprojlim_{K_1^p} \mathscr{S}_{\mathsf{K}_1^\circ}(G_1,X_1)^+,$$

which is an integral model of $\mathrm{Sh}_{K_1^\circ}(G_1,X_1)^+$ over $O_{(p)}$. The $G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\mathrm{Sh}_{K_{1,p}^\circ}(G_1,X_1)^+$ extends to $\mathscr{S}_{K_{1,p}^\circ}(G_1,X_1)^+$, which (after converting to a right action) induces an action of $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$ on $\mathscr{S}_{K_{1,p}}(G_1,X_1)^+$.

By [37, Lemma 4.6.10] we have an injection

$$\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ \backslash \mathscr{A}(G_{\mathbb{Z}_{(p)}}) \hookrightarrow \mathscr{A}(G_1)^\circ \backslash \mathscr{A}(G)/K_p^\circ.$$

Let $J \subset G(\mathbb{Q}_p)$ denote a set which maps bijectively to a set of coset representatives for the image of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ in $\mathscr{A}(G)^\circ \backslash \mathscr{A}(G)/K_p^\circ$. Let $\mathsf{E}' = \mathsf{E}(G,X)\mathsf{E}(G_1,X_1)$. By [37, Corollary 4.6.15], the $O_{\mathsf{E}'^p,(v)} = O_{\mathsf{E}'^p} \otimes O_{(v)}$-scheme

$$\mathscr{S}_{K_p^\circ}(G,X) = \Big[[\mathscr{S}_{K_{1,p}^\circ}(G_1,X_1)^+ \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ \Big]^{|J|}$$

has a natural structure of a $O'_{(v)} = O_{\mathsf{E}'} \otimes O_{(v)}$-scheme with $G(\mathbb{A}_f^p)$-action, and is a model of $\mathrm{Sh}_{K_p^\circ}(G,X)$. Moreover, if we denote $E' = \mathsf{E}'_{v'}$ for any $v'|v|p$ place of E', by [37, Corollary 4.6.18], there is a diagram of $O_{E'}$-schemes with $G(\mathbb{A}_f^p)$-action

$$\begin{array}{ccc} & \widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p^\circ}(G,X) & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle q} \\ \mathscr{S}_{K_p^\circ}(G,X) & & \mathrm{M}^{\mathrm{loc}}_{G_1,X_1}, \end{array}$$

where $G(\mathbb{A}_f^p)$ acts trivially on $\mathrm{M}^{\mathrm{loc}}_{G_1,X_1}$, π is a $G^{\mathrm{ad}}_{1,\mathbb{Z}_p}$-torsor, and q is $G^{\mathrm{ad}}_{1,\mathbb{Z}_p}$-equivariant. Any sufficiently small open compact $K^p \subset G(\mathbb{A}_f^p)$ acts freely on $\widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p^\circ}(G,X)$, and the morphism $\widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p^\circ}(G,X)/K^p \to \mathrm{M}^{\mathrm{loc}}_{G_1,X_1}$, is smooth of relative dimension $\dim G_1^{\mathrm{ad}}$.

2.4.4. Suppose that G splits over a tamely ramified extension of $\mathbb{Q}_p$. Then by [37, Lemma 4.6.22], we can choose the Hodge type datum (G_1,X_1) as above such that it satisfies the following conditions:

(1) $\pi_1(G_1^{\mathrm{der}})$ is a 2-group, and is trivial if $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$.
(2) G_1 splits over a tamely ramified extension of $\mathbb{Q}_p$.
(3) Let $\mathsf{E} = E(G,X), \mathsf{E}_1 = E(G_1,X_1)$ and $\mathsf{E}' = \mathsf{E}\cdot\mathsf{E}_1$, then any primes $v|p$ of E splits completely in E'.
(4) Z_{G_1} is a torus.
(5) $X_*(G_1^{\mathrm{ab}})_{\Gamma_0}$ is torsion free, where $\Gamma_0 \subset \Gamma$ is the inertia subgroup.

2.4.5. Let (G,X) be as above such that G splits over a tamely ramified extension of $\mathbb{Q}_p$. Then we can choose a Hodge type datum (G_1,X_1) as in 2.4.4, such that we get a $G(\mathbb{A}_f^p)$-equivariant O_E-scheme $\mathscr{S}_{K_p^\circ}(G,X)$ as in 2.4.3. Moreover, we have

Theorem 2.4.6. ([37, Theorem 4.6.23])

(1) *$\mathscr{S}_{K_p^\circ}(G,X)$ is étale locally isomorphic to $\mathrm{M}^{\mathrm{loc}}_{G_1,X_1}$; if $p \nmid |\pi_1(G^{\mathrm{der}})|$, then $\mathscr{S}_{K_p^\circ}(G,X)$ is étale locally isomorphic to $\mathrm{M}^{\mathrm{loc}}_{G,X}$.*
(2) *For any discrete valuation ring R of mixed characteristic $(0,p)$, the map*

$$\mathscr{S}_{K_p^\circ}(G,X)(R) \to \mathscr{S}_{K_p^\circ}(G,X)(R[1/p])$$

is a bijection.
(3) *If $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$, then (G_1,X_1) can be chosen so that there exists a diagram of O_E-schemes with $G(\mathbb{A}_f^p)$-action*

$$\begin{array}{ccc} & \widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p^\circ}(G,X) & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle q} \\ \mathscr{S}_{K_p^\circ}(G,X) & & \mathrm{M}^{\mathrm{loc}}_{G_1,X_1}, \end{array}$$

where $G(\mathbb{A}_f^p)$ acts trivially on $\mathrm{M}^{\mathrm{loc}}_{G_1,X_1}$, π is a $G^{\mathrm{ad}\circ}_{\mathbb{Z}_p}$-torsor, q is $G^{\mathrm{ad}\circ}_{\mathbb{Z}_p}$-equivariant, and for any sufficiently small open compact $K^p \subset G(\mathbb{A}_f^p)$ acts freely on $\widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p^\circ}(G,X)$, and the morphism

$$\widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p^\circ}(G,X)/K^p \to \mathrm{M}^{\mathrm{loc}}_{G_1,X_1}$$

is smooth of relative dimension $\dim G^{\mathrm{ad}}$.

(4) *If* G *is unramified over* $\mathbb{Q}_p$*, and there exists* $x' \in \mathcal{B}(G, \mathbb{Q}_p)$ *with* $\mathcal{G}_{x'^{\mathrm{ad}}} = \mathcal{G}^\circ_{x^{\mathrm{ad}}}$*, then we can choose* (G_1, X_1) *such that the above construction applies with* x' *in place of* x *and gives rise to an* O_E*-scheme* $\mathscr{S}_{K_p^\circ}(G, X)$ *satisfying the conclusion of (3) above.*

We note that in the proof of (3) of the above theorem, (G_1, X_1) is choosen to satisfy all the conditions in 2.4.4 together with a compatible parahoric subgroup $K^\circ_{1,p} \subset G_1(\mathbb{Q}_p)$ such that $K^\circ_{1,p} = K_{1,p}$. In the diagram, the related group is

$$G^{\mathrm{ad}\circ}_{\mathbb{Z}_p} = G^{\mathrm{ad}\circ}_{1,\mathbb{Z}_p} = (G^{\mathrm{ad}}_{1,\mathbb{Z}_p})^\circ = G^{\mathrm{ad}}_{1,\mathbb{Z}_p},$$

as in the diagram in 2.4.3. Here the first two equalities follow from 2.4.2, and the last equality follows from [37, Proposition 1.1.4], see also 5.1.1. In particular, we can apply the above theorem to a Hodge type datum (G, X) with $\mathcal{G}^\circ \subsetneq \mathcal{G}$ not equal as in 2.2.5. In this case, we need to assume that $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$: we choose another Hodge type datum (G_1, X_1) as above with $\mathcal{G}_1^\circ = \mathcal{G}_1$ to get the $G^{\mathrm{ad}\circ}_{\mathbb{Z}_p}$-local model diagram for (G, X).

3. EKOR strata of Hodge type: local constructions

In this section, we will construct and study the EKOR stratification for Shimura varieties of Hodge type using the theory of G-zips. This will be a local construction, in the sense that we will construct EKOR strata by constructing an EO stratification on each KR stratum. In the next section we will give some global constructions for the whole special fiber.

We will assume $p > 2$ throughout the rest of the paper. For technical reasons, we will always assume condition (2.2.1). Let (G, X) be a Shimura datum of Hodge type, and $\mathscr{S}_{\mathsf{K}} = \mathscr{S}_{\mathsf{K}}(G, X)$ be the integral model introduced in 2.2 with $\mathsf{K} = K_pK^p, K_p = \mathcal{G}(\mathbb{Z}_p)$ and $\mathcal{G} = \mathcal{G}^\circ$. Let $\mathscr{S}_{K_p} = \varprojlim_{K^p} \mathscr{S}_{K_pK^p}$. We are interested in the special fibers $\mathscr{S}_{\mathsf{K},0}$ of $\mathscr{S}_{\mathsf{K}}$, and $\mathscr{S}_{K_p,0}$ of $\mathscr{S}_{K_p}$ respectively, over $k = \overline{\mathbb{F}}_p$. We sometimes denote them simply by $\mathscr{S}_0$ when the level is clear. We will simply write $K = K_p$. We keep the notations as in subsection 2.2. In particular, we have the $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$, the tensor $s \in V^{\otimes}_{\mathbb{Z}_{(p)}}$ defining $\mathcal{G}$ and the conjugacy class of cocharacter $\{\mu\}$.

3.1. **Definitions of some stratifications.** Notations as in 1.3, recall that we have the set $C(\mathcal{G}, \{\mu\})$ which is a subset of $C(\mathcal{G})$. As before, let $k = \overline{\mathbb{F}}_p$. Consider the set of k-valued points $\mathscr{S}_{\mathsf{K},0}(k) = \mathscr{S}_{\mathsf{K}}(k)$. We will construct a map

$$\Upsilon_{\mathsf{K}} : \mathscr{S}_{\mathsf{K}}(k) \to C(\mathcal{G}, \{\mu\}).$$

Let $x \in \mathscr{S}_{\mathsf{K}}(k)$ and $\widetilde{x} \in \mathscr{S}_{\mathsf{K}}(O_F)$ be a lifting of x, where F is a finite extension of $\breve{\mathbb{Q}}_p$. The starting point is the following Theorem 3.1.1, for which we need to fix some notations.

We set $\mathfrak{S} := \breve{\mathbb{Z}}_p[[u]]$, with a Frobenius φ which is the usual Frobenius on $\breve{\mathbb{Z}}_p$ and $u \mapsto u^p$ on indeterminate. Let $E(u)$ be the Eisenstein polynomial for a fixed uniformizer $\pi \in F$. Then $E(u) \in \mathfrak{S}$. Consider the Galois group $\Gamma_F = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$. Let $\mathrm{Rep}^{\mathrm{cris}\circ}_{\Gamma_F}$ be the category of Γ_F-stable $\mathbb{Z}_p$-lattices spanning a crystalline Γ_F-representation, and $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ be the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ with a Frobenius semi-linear isomorphism

$$1 \otimes \varphi : \varphi^*(\mathfrak{M})[1/E(u)] \to \mathfrak{M}[1/E(u)].$$

Note that the maximal ideal of $\mathfrak{S}$ is (p, u).

Theorem 3.1.1. ([37, Theorem 3.3.2]) *There is a fully faithful tensor functor*

$$\mathfrak{M} : \mathrm{Rep}^{\mathrm{cris}\circ}_{\Gamma_F} \to \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$$

which is compatible with formation of symmetric and exterior powers, and exact when restricting to $D^\times := \mathrm{Spec}\,\mathfrak{S}\backslash\{(p,u)\}$. *Moreover,*

(1) *for a p-divisible group* $\mathscr{G}$ *over* O_F, *set* $L = T_p\mathscr{G}^\vee := \mathrm{Hom}_{\mathbb{Z}_p}(T_p\mathscr{G}, \mathbb{Z}_p)$ *and* $\mathfrak{M} := \mathfrak{M}(L)$, *then there are canonical isomorphisms*

$$\mathbb{D}(\mathscr{G})(O_F) \simeq O_F \otimes_{\mathfrak{S}} \varphi^*(\mathfrak{M}), \quad D_{\mathrm{dR}}(L \otimes \mathbb{Q}_p) \simeq F \otimes_{\mathfrak{S}} \varphi^*(\mathfrak{M})$$

such that the induced injection

$$\mathbb{D}(\mathscr{G})(O_F) \to D_{\mathrm{dR}}(L \otimes \mathbb{Q}_p)$$

is compatible with filtrations;

(2) *for* $\mathscr{G}_0 := \mathscr{G} \otimes k$, $\mathbb{D}(\mathscr{G}_0)(\breve{\mathbb{Z}}_p)$ *is canonically identified with* $\varphi^*(\mathfrak{M}/u\mathfrak{M})$.

We consider the p-divisible group $\mathscr{G} = \mathcal{A}_{\widetilde{x}}[p^\infty]$ over O_F. Notations as in the above theorem, by [37, 4.1.7], the tensor $s \in V_{\mathbb{Z}_p}^{\otimes}$ induces a tensor $s_{\text{ét},\widetilde{x}} \in L^{\otimes}$ which is Γ_F-invariant. Applying the theorem, $s_{\text{ét},\widetilde{x}}$ induces a tensor $\widetilde{s}_{\widetilde{x}} \in \mathfrak{M}(L)^{\otimes}$, and thus a tensor $s_{\mathrm{cris},x} \in D_x^{\otimes}$, which is φ_x-invariant. Here $D_x := \mathbb{D}(\mathscr{G}_0)(\breve{\mathbb{Z}}_p)$, and φ_x is the Frobenius. We remark that, by [36, Proposition 1.3.9 (2)], $s_{\mathrm{cris},x} \in D_x^{\otimes}$ is independent of choices of $\widetilde{x}$. Now by [37, Corollary 3.3.6],

$$I_x := \mathbf{Isom}_{\breve{\mathbb{Z}}_p}((V_{\mathbb{Z}_{(p)}}^\vee \otimes \breve{\mathbb{Z}}_p, s), (D_x, s_{\mathrm{cris},x}))$$

is a trivial $\mathcal{G}$-torsor. Thus we can take a $t \in I_x(\breve{\mathbb{Z}}_p)$. Then the pull back of φ_x via

$$t : (V_{\mathbb{Z}_{(p)}}^\vee \otimes \breve{\mathbb{Z}}_p, s) \to (D_x, s_{\mathrm{cris},x})$$

is of the form

$$(\mathrm{id} \otimes \sigma) \circ g_{x,t},$$

for some $g_{x,t} \in G(\breve{\mathbb{Q}}_p)$. The image of $g_{x,t}$ in $C(\mathcal{G})$, denote by $g_x \in C(\mathcal{G})$, is independent of t. In particular,

$$x \mapsto g_x$$

gives a well defined map

$$\Upsilon_{\mathsf{K}} : \mathscr{S}_{\mathsf{K}}(k) \to C(\mathcal{G})$$

whose fibers are called *central leaves.*

Let us check that Υ_{K} factors through $C(\mathcal{G}, \{\mu\})$. It suffices to prove the following statement.

Lemma 3.1.2. *The element* $g_{x,t} \in G(\breve{\mathbb{Q}}_p)$ *as above is of the form* $\breve{K}\mathrm{Adm}(\{\mu\})\breve{K}$.

Proof. By the Bruhat-Tits decomposition, we know that $g_{x,t} \in \breve{K}w\breve{K}$, for some $w \in {}^K\widetilde{W}^K$. We view $g_{x,t}$ as a homomorphism $V_k^\vee \to V_k^\vee$. The Hodge filtration $t^{-1}(\mathrm{Ker}(\varphi_x)) = \mathrm{Ker}(g_{x,t})$ gives, by the local model diagram, an element in

$$\mathrm{Adm}(\{\mu\})_K = W_K\backslash \mathrm{Adm}(\{\mu\})^K/W_K.$$

In particular, $w \in \mathrm{Adm}(\{\mu\})^K$. □

We will also write Υ_{K} for the map $\mathscr{S}_{\mathsf{K}}(k) \to C(\mathcal{G}, \{\mu\})$. Composing Υ_{K} with the natural maps in 1.3.3, we get maps

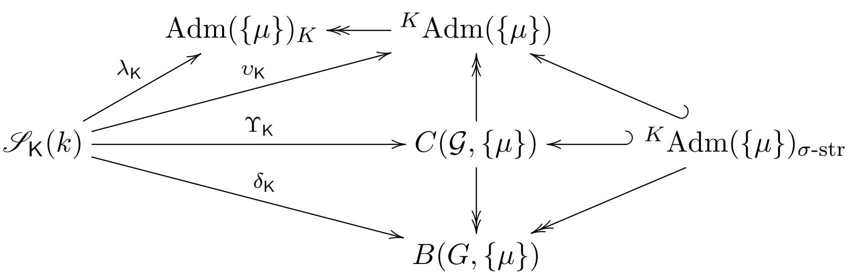
- $\delta_{\mathsf{K}} : \mathscr{S}_{\mathsf{K}}(k) \to B(G, \{\mu\})$ whose fibers are called *Newton strata*;
- $\lambda_{\mathsf{K}} : \mathscr{S}_{\mathsf{K}}(k) \to \mathrm{Adm}(\{\mu\})_K$ whose fibers are called *Kottwitz-Rapoport strata*, or KR strata for short;
- $\upsilon_{\mathsf{K}} : \mathscr{S}_{\mathsf{K}}(k) \to {}^K\mathrm{Adm}(\{\mu\})$ whose fibers are called *Ekedahl-Kottwitz-Oort-Rapoport strata*, or EKOR strata for short.

Recall that by the local diagram in Theorem 2.2.3 we can also define the KR strata (see [37, Corollary 4.2.12]). Now we explain that the two constructions for KR strata are compatible. Let the notations be as in Lemma 3.1.2. For $x \in \mathscr{S}_{\mathsf{K}}(k)$, a section $t \in I_x(\breve{\mathbb{Z}}_p)$ induces an element $g_{x,t} \in \breve{K}\mathrm{Adm}(\{\mu\})\breve{K}$. Let $M_0 = \mathrm{M}^{\mathrm{loc}}_{G,X} \otimes_{O_E} k$. Assigning to $g_{x,t}$ the point in $M_0(k)$ corresponding to the filtration $\mathrm{Ker}(g_{x,t}) \subset V_k^\vee$ induces a bijection

$$\breve{K}\backslash\breve{K}\mathrm{Adm}(\{\mu\})\breve{K}/\breve{K} \xrightarrow{\simeq} \mathcal{G}_0(k)\backslash M_0(k).$$

This identifies fibers of λ_K with the KR strata defined using the local model diagram as in Theorem 2.2.3.

Combined with the diagram for group theoretic data obtained in subsection 1.3, we have the following commutative diagram.



By construction, it is clear that when the prime to p level K^p varies, the maps $\Upsilon_{\mathsf{K}}, \delta_{\mathsf{K}}, \upsilon_{\mathsf{K}}$ and λ_{K} are compatible for the natural projections.

We will need geometric structures on the above sets:

- One can use the F-crystal attached to the universal abelian scheme $\mathcal{A} \to \mathscr{S}_0$ together with the G-additional structure to show that the fibers of Υ_{K} and δ_{K} are the sets of k-valued points of locally closed reduced subschemes of $\mathscr{S}_0$, see for example [20, Corollary 3.3.8]. We also call these locally closed reduced subschemes central leaves and Newton strata respectively.
- The local model diagram in Theorem 2.2.3 shows that the fibers of λ_{K} are the sets of k-valued points of locally closed reduced subschemes of $\mathscr{S}_0$, which we call equally the KR (Kottwitz-Rapoport) strata.
- Our main task in this section is to show that there exists also geometric structures on the fibers of υ_{K}, cf. Theorem 3.4.11.

As for the non-emptiness of the above various strata, we list the following results.

(1) In the hyperspecial case, Viehmann-Wedhorn proved the non-emptiness of Newton strata in the PEL type case ([81]), and Dong-Uk Lee proved the non-emptiness of Newton strata in the Hodge type case [49].
(2) Kisin-Madapusi Pera-Shin ([38]) and C.-F. Yu ([91]) proved in the Hodge type case for G quasi-split over $\mathbb{Q}_p$, that the Newton strata are non-empty.
(3) R. Zhou proved in [95, Proposition 8.2] that all KR strata are non-empty by applying the key input from [38] that the basic Newton strata are non-empty (see also [91]), and then all Newton strata are non-empty in this case.
(4) Based on these results (especially the non-emptiness of KR strata in [95]), we will show later all EKOR strata are non-empty, cf. Corollary 3.5.3.

3.2. **He-Rapoport axioms.** We turn to a different (abstract) setting. He and Rapoport introduced five axioms in [31] for integral models of Shimura varieties with parahoric levels to study stratifications in the special fibers. We will not recall all the axioms here, but just introduce some of them which will be used in this paper. Let (G, X) be a Shimura datum. Assume that the for each parahoric subgroup $K_p \subset G(\mathbb{Q}_p)$, we already have an integral model $\mathscr{S}_{\mathsf{K}} = \mathscr{S}_{\mathsf{K}}(G, X)$ for the associated Shimura variety $\mathrm{Sh}_{\mathsf{K}} = \mathrm{Sh}_{\mathsf{K}}(G, X)$ over O_E, the ring of integers of a local reflex field at p. Here $\mathsf{K} := K_p K^p$ with K^p small enough.

3.2.1. *Axiom 1.* Let $K_p \supset K'_p$ be another parahoric subgroup, and $\mathsf{K}' := K'_p K^p$. Then there is a natural morphism

$$\pi_{\mathsf{K}',\mathsf{K}} : \mathscr{S}_{\mathsf{K}'} \to \mathscr{S}_{\mathsf{K}}$$

which is proper surjective and extends the natural morphism on the generic fibers.

Let $\mathscr{S}_{\mathsf{K}',0}$ and $\mathscr{S}_{\mathsf{K},0}$ be their special fibers respectively, axiom 1 implies that the induced morphism $\pi_{\mathsf{K}',\mathsf{K},0} : \mathscr{S}_{\mathsf{K}',0} \to \mathscr{S}_{\mathsf{K},0}$ is proper surjective.

3.2.2. *Axiom 4 (c).* Let $\mathscr{S}^c_{\mathsf{K},0} \subset \mathscr{S}_{\mathsf{K},0}$ and $\mathscr{S}^{c'}_{\mathsf{K}',0} \subset \mathscr{S}_{\mathsf{K}',0}$ be central leaves with

$$\pi_{\mathsf{K},\mathsf{K}',0}(\mathscr{S}^c_{\mathsf{K},0}) \subset \mathscr{S}^{c'}_{\mathsf{K}',0}.$$

Then the induced morphism $\pi^c_{\mathsf{K},\mathsf{K}',0} : \mathscr{S}^c_{\mathsf{K},0} \to \mathscr{S}^{c'}_{\mathsf{K}',0}$ is surjective with finite fibers.

3.2.3. *Axiom 5.* Let $I \subset G(\mathbb{Q}_p)$ be an Iwahori subgroup, and τ be as in 1.2.5, then the intersection of the KR stratum $\mathscr{S}^{\tau}_{IK^p,0}$ with each geometrically connected component of $\mathscr{S}_{IK^p,0}$ is non empty.

3.2.4. *Known cases.*

- For Siegel modular varieties the axioms are verified by He-Rapoport in [31].
- For PEL-type Shimura varieties associated to unramified groups of type A and C and to odd ramified unitary groups, the axioms are verified by He-Zhou in [32].
- For Shimura varieties of Hodge type satisfying condition (2.2.1), it has been checked by R. Zhou, see [95], that all the five axioms except for the surjectivity in axiom 4 (c) hold for the Kisin-Pappas integral models as in 2.2. If $G_{\mathbb{Q}_p}$ is in addition residually split, then all the five axioms hold.

3.3. $\mathcal{G}_0^{\mathrm{rdt}}$**-zips attached to points.** We come back to the setting as that at the beginning of this section. We start with some constructions generalizing those in [92, 3.2.6]. For $w \in \widetilde{W}$, we simply write the same symbol for a representative in $G(\breve{\mathbb{Q}}_p)$. Let $\mathfrak{a}$ be the facet corresponding to $\mathcal{G}$, we set

$$\mathfrak{b} := \mathfrak{a} \cup w\mathfrak{a} \quad \text{and} \quad \mathfrak{c} := \mathfrak{a} \cup \sigma(w)^{-1}\mathfrak{a}.$$

Here $w\mathfrak{a}$ (resp. $\sigma(w)^{-1}\mathfrak{a}$) is the translation by w (resp. $\sigma(w)^{-1}$) of $\mathfrak{a}$. Let $\mathcal{G}_{\mathfrak{b}}$ and $\mathcal{G}_{\mathfrak{c}}$ be the corresponding Bruhat-Tits group schemes respectively, then working with reduced k-algebras, we see by Remark 2.1.2 that $\sigma' = \sigma \circ \mathrm{Ad}(w)$ induces a homomorphism

$$L^+\mathcal{G}_{\mathfrak{b}} \to L^+\mathcal{G}_{\mathfrak{c}}.$$

In particular, we have an isogeny of smooth group varieties over k

$$\mathcal{G}_{\mathfrak{b},0} \to \mathcal{G}_{\mathfrak{c},0},$$

which is again denoted by σ'. Recall that we have homomorphisms

$$\mathcal{G}_{\mathfrak{b},0} \to \mathcal{G}_{\mathfrak{a},0}, \quad \mathcal{G}_{\mathfrak{c},0} \to \mathcal{G}_{\mathfrak{a},0}.$$

We will also set $\mathcal{G}_{0,w}$ (resp. $\mathcal{G}_{0,\sigma(w)^{-1}}$) to be the image of $\mathcal{G}_{\mathfrak{b},0}$ (resp. $\mathcal{G}_{\mathfrak{c},0}$) in $\mathcal{G}_0 = \mathcal{G}_{\mathfrak{a},0}$. It is a quotient of $\mathcal{G}_{\mathfrak{b},0}$ (resp. $\mathcal{G}_{\mathfrak{c},0}$), and hence is a smooth group scheme over k.

Before moving into detailed constructions, we would like to say a few words about the main ideas. Our strategy here is the same as that in [92, §3.2], and we will also use F-zips introduced by Moonen and Wedhorn ([55]) freely. The definition of an F-zip will not be recalled, as it will be used in explicit forms here. Recall for each $w \in \mathrm{Adm}(\{\mu\})_K$, we have constructed an algebraic zip datum

$$\mathcal{Z}_w = (\mathcal{G}_0^{\mathrm{rdt}}, \overline{P}_{J_w}, \overline{P}_{\sigma'(J_w)}, \sigma' : \overline{L}_{J_w} \to \overline{L}_{\sigma'(J_w)})$$

in 1.3.6. We will construct first an F-zip from the element $w \in \mathrm{Adm}(\{\mu\})_K$, and use it as a standard F-zip attached to w. Properties of this standard F-zip will also be studied. There are also F-zips attached to points of $\mathscr{S}_0$. We will then show that one gets desired $\mathcal{G}_0^{\mathrm{rdt}}$-zips by "comparing" the F-zips at points with the standard ones.

3.3.1. *A standard F-zip.* For $w \in \mathrm{Adm}(\{\mu\})_K$, we will always fix a representative of it in $G(\breve{\mathbb{Q}}_p) \subset \mathrm{GL}(V)(\breve{\mathbb{Q}}_p)$ without changing notations. Then w has a left action on $V_{\breve{\mathbb{Q}}_p}$ which induces, by taking inverse dual definition, an endomorphism of $V_{\breve{\mathbb{Z}}_p}^\vee$ which is again denoted by w. Consider the reduction modulo p of $V_{\breve{\mathbb{Z}}_p}^\vee$, the k-vector space $V_k^\vee$. Let w_k be the induced map on $V_k^\vee$, we have an F-zip structure $(C_w^\bullet, D_\bullet^w, \varphi_\bullet^w)$ on $V_k^\vee$ given by

- $C_w^0 := V_k^\vee \supset C_w^1 := \mathrm{Ker}(w_k) \supset C_w^2 := 0$;
- $D_{-1}^w := 0 \subset D_0^w := \mathrm{Im}((\sigma(w))_k) \subset D_1^w := V_k^\vee$;
- $\varphi_0^w : C_w^0/C_w^1 \to D_0^w$ and $\varphi_1^w : C_w^1 \to D_1^w/D_0^w$, which are σ-linear isomorphisms whose linearizations are induced by $(\sigma(w))_k$ and the inverse of the isomorphism $(\frac{p}{\sigma(w)}) \otimes k : D_1^w/D_0^w \to C_w^{1,\sigma}$ respectively.

We remark that the filtration $C_w^\bullet$ (resp. $D_\bullet^w$) depends only on the image of w (resp. $\sigma(w)$) in $G(\breve{\mathbb{Q}}_p)/\breve{K}$ (resp. $\breve{K}\backslash G(\breve{\mathbb{Q}}_p)$).

Lemma 3.3.2. *Let $\mathrm{Aut}(C_w^\bullet, s)$ be the group scheme of automorphisms on $V_k^\vee$ respecting both the filtration $C_w^\bullet$ and the tensor $s \in V_k^\otimes$, similarly for $\mathrm{Aut}(D_\bullet^w, s)$. We have the following.*

(1) $\mathrm{Aut}(C_w^\bullet, s) = \mathcal{G}_{0,w}$ *and* $\mathrm{Aut}(D_\bullet^w, s) = \mathcal{G}_{0,\sigma(w)^{-1}}$.

(2) *Let $\mathcal{G}_{0,w}^L$ be the image of $\mathcal{G}_{0,w}$ in $\mathrm{Aut}(\oplus_i \mathrm{gr}_i(C_w^\bullet))$ and similarly for $\mathcal{G}_{0,\sigma(w)^{-1}}^L$, then σ' induces an isomorphism*
$$\mathcal{G}_{0,w}^{L,(p)} \to \mathcal{G}_{0,\sigma(w)^{-1}}^L$$
which is again denoted by σ'.

(3) *The isomorphism*
$$\oplus_i \mathrm{gr}_i(C_w^{\bullet,(p)}) \to \oplus_i \mathrm{gr}_i(D_\bullet^w)$$
induced by $\varphi_0^w \oplus \varphi_1^w$ is equivariant with respect to σ'.

Proof. For (1), it is more convenient to use equal-characteristic results. Let $G'/k((t))$ and $\mathcal{G}'/k[[t]]$ be as in subsection 2.1 (after Remark 2.1.10). In particular, w is identified with an element w' in the Iwahori Weyl group of G'. Let $\mathfrak{a}'$ be the facet corresponding to $\mathcal{G}'$, we set, as before
$$\mathfrak{b}' := \mathfrak{a}' \cup w'\mathfrak{a}' \quad \text{and} \quad \mathfrak{c}' := \mathfrak{a}' \cup \sigma(w')^{-1}\mathfrak{a},$$
and get Bruhat-Tits group schemes $\mathcal{G}'_{\mathfrak{b}'}$ and $\mathcal{G}'_{\mathfrak{c}'}$. The homomorphisms $\mathcal{G}_{\mathfrak{b},0} \to \mathcal{G}_0$ and $\mathcal{G}_{\mathfrak{c},0} \to \mathcal{G}_0$ are identified with $\mathcal{G}'_{\mathfrak{b}',0} \to \mathcal{G}'_0 = \mathcal{G}_0$ and $\mathcal{G}'_{\mathfrak{c}',0} \to \mathcal{G}'_0 = \mathcal{G}_0$, so $\mathcal{G}_{0,w}$ and $\mathcal{G}_{0,\sigma(w)^{-1}}$ are identified with $\mathcal{G}'_{0,w'}$ and $\mathcal{G}'_{0,\sigma(w')^{-1}}$ respectively. Let
$$\mathscr{C} := (\cdots \subset tV_{k[[t]]}^\vee \subset \frac{t}{w'}V_{k[[t]]}^\vee \subset V_{k[[t]]}^\vee \subset \cdots)$$
and
$$\mathscr{D} := (\cdots \subset tV_{k[[t]]}^\vee \subset \sigma(w')V_{k[[t]]}^\vee \subset V_{k[[t]]}^\vee \subset \cdots)$$
be the periodic lattice chain corresponding to $C_w^\bullet = C_{w'}^\bullet$ and $D_\bullet^w = D_\bullet^{w'}$ respectively. To simplify notations, we will write $\mathcal{H}$ for $\mathrm{GL}(V_{k[[t]]})$, $\breve{K}'^0$ for $\mathcal{H}(k[[t]])$ and P^0 (resp. Q^0) for the stabilizer of $C_w^\bullet$ (resp. $D_\bullet^w$) in $\mathrm{GL}(V_k)$. The Bruhat-Tits group scheme corresponding to $\breve{K}'^0 \cap w'\breve{K}'^0 w'^{-1}$ is then $\mathrm{GL}(\mathscr{C})$ which is parahoric. We have a commutative diagram
$$\begin{array}{ccccc}
L^+\mathcal{G}'w'L^+\mathcal{G}'/L^+\mathcal{G}' & \xrightarrow{\cong} & L^+\mathcal{G}'/L^+\mathcal{G}'_{\mathfrak{b}'} & \xrightarrow{\cong} & \mathcal{G}_0/\mathcal{G}_{0,w} \\
\downarrow & & \downarrow & & \downarrow \\
L^+\mathcal{H}w'L^+\mathcal{H}/L^+\mathcal{H} & \xrightarrow{\cong} & L^+H/L^+\,\mathrm{GL}(\mathscr{C}) & \xrightarrow{\cong} & \mathrm{GL}(V_k)/P^0.
\end{array}$$

By the arguments in the proof of [65, Proposition 8.1] (more precisely, in the middle of page 207), the induced morphism of affine Grassmannians $\mathrm{Gr}_{\mathcal{G}'} \to \mathrm{Gr}_{\mathrm{GL}(k[[t]])}$ is a closed immersion, and hence the left vertical arrow in the above diagram is an immersion. But then the right vertical arrow will also be an immersion, and hence the first identification in (1) follows.

The periodic lattice chain corresponding to $C_w^{\bullet,(p)}$ is

$$(\cdots \subset tV^\vee_{k[[t]]} \subset \frac{t}{\sigma(w')} V^\vee_{k[[t]]} \subset V^\vee_{k[[t]]} \subset \cdots).$$

Noting that the left action of $\sigma(w')$ maps the lattice chain of $C_w^{\bullet,(p)}$ to that of $D_\bullet^w$, we have, as in the previous case, a commutative diagram

$$\begin{array}{ccccc} L^+\mathcal{G}'\sigma(w')^{-1}L^+\mathcal{G}'/L^+\mathcal{G}' & \xrightarrow{\cong} & L^+\mathcal{G}'/L^+\mathcal{G}'_{\mathfrak{c}'} & \xrightarrow{\cong} & \mathcal{G}_0/\mathcal{G}_{0,\sigma(w)^{-1}} \\ \downarrow & & \downarrow & & \downarrow \\ L^+\mathcal{H}\sigma(w')^{-1}L^+\mathcal{H}/L^+\mathcal{H} & \xrightarrow{\cong} & L^+\mathcal{H}/L^+\,\mathrm{GL}(\mathscr{D}) & \xrightarrow{\cong} & \mathrm{GL}(V_k)/Q^0. \end{array}$$

By the same reason, the second identification in (1) follows.

For (2) and (3), both the mixed characteristic method and the equi characteristic one work. We will use the mixed one. Let $L^0 = \mathrm{Aut}(\oplus_i \mathrm{gr}_i(C_w^\bullet))$ (resp. $L'^0 = \mathrm{Aut}(\oplus_i \mathrm{gr}_i(D_\bullet^w))$) be the Levi quotient attached to P^0 (resp. Q^0). The action of $\sigma(w)$ induces an isomorphism from the parahoric subgroup (of $\mathrm{GL}(V)(\breve{\mathbb{Q}}_p)$) of $C_w^{\bullet,(p)}$ to that of $D_\bullet^w$, and hence induces an isomorphism $L^{0,(p)} \xrightarrow{\cong} L'^0$. Moreover, it induces an commutative diagram

$$\begin{array}{ccc} \mathcal{G}^{(p)}_{\mathfrak{b},0} & \longrightarrow & \mathcal{G}_{\mathfrak{c},0} \\ \downarrow & & \downarrow \\ L^{0,(p)} & \longrightarrow & L'^0, \end{array}$$

where the vertical arrows factor through $\mathcal{G}^{(p)}_{0,w}$ and $\mathcal{G}_{0,\sigma(w)^{-1}}$ respectively. In particular, (2) follows.

The isomorphism $\oplus_i \mathrm{gr}_i(C_w^{\bullet,(p)}) \to \oplus_i \mathrm{gr}_i(D_\bullet^w)$ induced by $\varphi_0^w \oplus \varphi_1^w$ is also induced by the action of $\sigma(w)$, and hence is equivariant with respect to the isomorphism $L^{0,(p)} \to L'^0$. In particular, (3) follows.

□

The natural homomorphisms $\mathcal{G}_{0,w} \to \overline{P}_{J_w}$ and $\mathcal{G}_{0,\sigma(w)^{-1}} \to \overline{P}_{\sigma'(J_w)}$ are compatible with the isomorphisms σ's. The tuple $\widetilde{\mathcal{Z}}_w := (\mathcal{G}_0, \mathcal{G}_{0,w}, \mathcal{G}_{0,\sigma(w)^{-1}}, \sigma')$ could be viewed as a generalization or "lifting" of the zip datum $\mathcal{Z}_w = (\mathcal{G}_0^{\mathrm{rdt}}, \overline{P}_{J_w}, \overline{P}_{\sigma'(J_w)}, \sigma')$. It is natural and necessary to work with structures related to $\widetilde{\mathcal{Z}}_w$.

3.3.3. *F-zips attached to points.* Fix a sufficiently small K^p and write $\mathscr{S}_0 = \mathscr{S}_{\mathsf{K},0}$ with $\mathsf{K} = KK^p$. Let $\mathcal{A} \to \mathscr{S}_0$ be the universal abelian scheme and consider the vector bundle $\mathcal{V}_0 = H^1_{\mathrm{dR}}(\mathcal{A}/\mathscr{S}_0)$ over $\mathscr{S}_0$. We have an F-zip structure $(C^\bullet, D_\bullet, \varphi_\bullet)$ on $\mathcal{V}_0$, given by

- $C^0 := \mathcal{V}_0 \supseteq C^1 := \mathcal{V}_0^1 \supseteq C^2 := 0$, which is the Hodge filtration;
- $D_{-1} := 0 \subset D_0 := \mathrm{Im}(F) \subset D_1 := \mathcal{V}_0$, which is the conjugate filtration;
- $\varphi_0 : C^0/C^1 \to D_0$ and $\varphi_1 : C^1 \to D_1/D_0$, which are induced by Frobenius and the inverse of Verschiebung respectively.

Now we will fix a $w \in \mathrm{Adm}(\{\mu\})_K$ and a k-point x in the KR-stratum $\mathscr{S}_0^w$, and state the main results of our pointwise constructions. We will denote by $(C_x^\bullet, D_{\bullet,x}, \varphi_{\bullet,x})$ the pull back to x of $(C^\bullet, D_\bullet, \varphi_\bullet)$. In the following we will talk about torsors over k, which are actually trivial since $k = \overline{\mathbb{F}}_p$.

Proposition 3.3.4. *Let $\widetilde{\mathbb{I}}_x$ be the $\mathcal{G}_0$-torsor $\mathbf{Isom}_k((V_k^\vee, s), (\mathcal{V}_{0,x}, s_{\mathrm{dR}}))$ over k.*

(1) *Let $\widetilde{\mathbb{I}}_{x,+} := \mathbf{Isom}_k((C_w^\bullet, s), (C_x^\bullet, s_{\mathrm{dR}})) \subset \widetilde{\mathbb{I}}_x$, then it is a $\mathcal{G}_{0,w}$-torsor over k.*
(2) *Let $\widetilde{\mathbb{I}}_{x,-} := \mathbf{Isom}_k((D_\bullet^w, s), (D_{\bullet,x}, s_{\mathrm{dR}})) \subset \widetilde{\mathbb{I}}_x$, then it is a $\mathcal{G}_{0,\sigma(w)^{-1}}$-torsor over k.*
(3) *Let $\mathcal{G}_{0,w}^U = \mathrm{Ker}(\mathcal{G}_{0,w} \twoheadrightarrow \mathcal{G}_{0,w}^L)$ and similarly for $\mathcal{G}_{0,\sigma(w)^{-1}}^U$. The Dieudonné module structure on $\mathcal{V}_x$ induces an isomorphism*

$$\widetilde{\iota} : \widetilde{\mathbb{I}}_{x,+}^{(p)} / \mathcal{G}_{0,w}^{U,(p)} \to \widetilde{\mathbb{I}}_{x,-} / \mathcal{G}_{0,\sigma(w)^{-1}}^U$$

which is equivariant with respect to the isomorphism $\sigma' : \mathcal{G}_{0,w}^{L,(p)} \to \mathcal{G}_{0,\sigma(w)^{-1}}^L$.

Proof. Let D_x be the Dieudonné module of $\mathcal{A}_x[p^\infty]$ and $\widetilde{\widetilde{\mathbb{I}}}_x$ be the (trivial) $\mathcal{G}$-torsor given by

$$\widetilde{\widetilde{\mathbb{I}}}_x := \mathbf{Isom}_{\breve{\mathbb{Z}}_p}((V_{\breve{\mathbb{Z}}_p}^\vee, s), (D_x, s_{\mathrm{cris}})).$$

We fix a section $t \in \widetilde{\widetilde{\mathbb{I}}}_x(\breve{\mathbb{Z}}_p)$ such that $t^*\varphi = \sigma \circ gw$ for some $g \in \mathcal{G}(\breve{\mathbb{Z}}_p)$. Let $\bar{t}$ be the image of t in $\widetilde{\mathbb{I}}_x$, one sees immediately that

$$\widetilde{\mathbb{I}}_{x,+} = \mathcal{G}_{0,w} \cdot \bar{t}, \quad \text{and} \quad \widetilde{\mathbb{I}}_{x,-} = \mathcal{G}_{0,\sigma(w)^{-1}} \cdot \overline{g \cdot t}.$$

In particular, (1) and (2) hold.

Let $\widetilde{\mathbb{I}}_{x,+}^L$ be the image of $\widetilde{\mathbb{I}}_{x,+}$ in $\mathbf{Isom}_k(\oplus_i \mathrm{gr}_i(C_w^\bullet), \oplus_i \mathrm{gr}_i(C_x^\bullet))$, then $\widetilde{\mathbb{I}}_{x,+}^L = \widetilde{\mathbb{I}}_{x,+}/\mathcal{G}_{0,w}^U$. Similarly, we have $\widetilde{\mathbb{I}}_{x,+}^{L,(p)}$ and $\widetilde{\mathbb{I}}_{x,-}^L$, as well as identifications $\widetilde{\mathbb{I}}_{x,+}^{L,(p)} = \widetilde{\mathbb{I}}_{x,+}^{(p)}/\mathcal{G}_{0,w}^{U,(p)}$ and $\widetilde{\mathbb{I}}_{x,-}^L = \widetilde{\mathbb{I}}_{x,-}/\mathcal{G}_{0,\sigma(w)^{-1}}^U$. Now we can define a morphism

$$\widetilde{\iota} : \widetilde{\mathbb{I}}_{x,+}^{L,(p)} \to \mathbf{Isom}_k(\oplus_i \mathrm{gr}_i(D_\bullet^w), \oplus_i \mathrm{gr}_i(D_{\bullet,x}))$$

as follows. It is induced by mapping $f : \oplus_i \mathrm{gr}_i(C_w^{\bullet,(p)}) \xrightarrow{\cong} \oplus_i \mathrm{gr}_i(C_x^{\bullet,(p)})$ to the composition

$$\widetilde{\iota}(f) : \oplus_i \mathrm{gr}_i(D_\bullet^w) \xrightarrow{\alpha} \oplus_i \mathrm{gr}_i(C_w^{\bullet,(p)}) \xrightarrow{f} \oplus_i \mathrm{gr}_i(C_x^{\bullet,(p)}) \xrightarrow{\oplus_i \varphi_\bullet^{\mathrm{lin}}} \oplus_i \mathrm{gr}_i(D_{\bullet,x}). \tag{3.3.5}$$

Here α is the inverse of the linearization of $\oplus_i \varphi_{w,\bullet}$, and $\varphi_\bullet^{\mathrm{lin}}$ is the linearization of $\varphi_\bullet$.

Noting that $\widetilde{\iota}$ is equivariant with respect to $\sigma' : \mathcal{G}_{0,w}^{L,(p)} \to \mathcal{G}_{0,\sigma(w)^{-1}}^L$ as α is so by Lemma 3.3.2 (3), so to prove statement (3) here, it suffices to check that $\widetilde{\iota}$ factors through $\widetilde{\mathbb{I}}_{x,-}^L$. To see this, one can simply take f to be the map induced by $\bar{t}^{(p)}$, then it is clear that $\widetilde{\iota}(f) \in \widetilde{\mathbb{I}}_{x,-}^L$. □

Recall that in 1.3.6 we have constructed an algebraic zip datum

$$\mathcal{Z}_w = (\mathcal{G}_0^{\mathrm{rdt}}, \overline{P}_{J_w}, \overline{P}_{\sigma'(J_w)}, \sigma' : \overline{L}_{J_w} \to \overline{L}_{\sigma'(J_w)}).$$

In this paper, we are mainly interested in structures related to $\mathcal{Z}_w$. One can pass easily from the tuple $(\widetilde{\mathbb{I}}_x, \widetilde{\mathbb{I}}_{x,+}, \widetilde{\mathbb{I}}_{x,-}, \widetilde{\iota})$ constructed in Proposition 3.3.4 to a certain structure related to $\mathcal{Z}_w$. More precisely, we take

$$\mathbb{I}_x := \widetilde{\mathbb{I}}_x \times^{\mathcal{G}_0} \mathcal{G}_0^{\mathrm{rdt}}, \quad \mathbb{I}_{x,+} := \widetilde{\mathbb{I}}_{x,+} \times^{\mathcal{G}_{0,w}} \overline{P}_{J_w}, \quad \mathbb{I}_{x,-} := \widetilde{\mathbb{I}}_{x,-} \times^{\mathcal{G}_{0,\sigma(w)^{-1}}} \overline{P}_{\sigma'(J_w)} \tag{3.3.6}$$

and

$$\iota : \mathbb{I}_{x,+}^{(p)} / U_{J_w}^{(p)} \to \mathbb{I}_{x,-} / U_{\sigma'(J_w)}$$

be the isomorphism induced by $\widetilde{\iota}$. The following statement is straightforward.

Corollary 3.3.7. *The tuple $(\mathbb{I}_x, \mathbb{I}_{x,+}, \mathbb{I}_{x,-}, \iota)$ is a $\mathcal{G}_0^{\mathrm{rdt}}$-zip of type J_w over k.*

Remark 3.3.8. Notations as in the previous proposition, let $\bar{t}^{\mathrm{rdt}}$ (resp. $\bar{g}^{\mathrm{rdt}}$) be the image of $\bar{t}$ (resp. $\bar{g}$) in $\mathbb{I}_x$ (resp. $\mathcal{G}_0^{\mathrm{rdt}}$). Let $E_{\mathcal{Z}_w}$ be the zip group attached to $\mathcal{Z}_w$, and $\mathbb{E}_x$ be the $E_{\mathcal{Z}_w}$-torsor attached to $(\mathbb{I}_x, \mathbb{I}_{x,+}, \mathbb{I}_{x,-}, \iota)$ (see e.g. our discussions after Definition 1.1.5). Then

$$(\bar{t}^{\mathrm{rdt}}, \overline{\sigma(g)}^{\mathrm{rdt}}\bar{t}^{\mathrm{rdt}}) \in \mathbb{E}_x(k),$$

and its image in $\mathcal{G}_0^{\mathrm{rdt}}$ is $\overline{\sigma(g)}^{\mathrm{rdt}}$.

3.4. **The EO stratification in a KR stratum.**

3.4.1. *The conjugate local model.* Let $G'/k((t))$ and $\mathcal{G}'/k[[t]]$ be as in subsection 2.1 (after Remark 2.1.10). In particular, w (resp. $\mathrm{Adm}(\{\mu\})$) is identified with an element (resp. a subset) in the Iwahori Weyl group of G', which will be denoted by the same symbol. Let

$$\mathrm{M}_0^\vee := \bigcup_{w\in \mathrm{Adm}(\{\mu\})} L^+\mathcal{G}'\backslash L^+\mathcal{G}' w L^+\mathcal{G}', \quad \text{and} \quad \mathrm{M}_0^{\mathrm{c}} := \bigcup_{w\in \mathrm{Adm}(\{\mu\})} L^+\mathcal{G}'\sigma(w)^{-1}L^+\mathcal{G}'/L^+\mathcal{G}'.$$

As in the right quotient case (cf. Corollary 2.1.11), $\mathrm{M}_0^\vee$ is a reduced closed subscheme of $L^+\mathcal{G}'\backslash LG'$ of dimension $\dim(\mathrm{M}_0)$. Noting that $\mathrm{M}_0^{\mathrm{c}}$, with the reduced scheme structure, is the image of $\mathrm{M}_0^\vee$ under the homeomorphism $L^+\mathcal{G}'\backslash LG' \to LG'/L^+\mathcal{G}', x \mapsto \sigma(x)^{-1}$, so it is also of dimension $\dim(\mathrm{M}_0)$. Moreover, as in the proof of Lemma 3.3.2, $\mathrm{M}_0^{\mathrm{c}}$ is a subscheme of $\mathrm{Gr}(V_k)$. The scheme $\mathrm{M}_0^{\mathrm{c}}$ will be called the *conjugate local model*.

We come back to notations introduced in 3.3.3. We start with the local model diagram

$$\begin{array}{ccc} & \widetilde{\mathscr{S}_0} & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle q} \\ \mathscr{S}_0 & & \mathrm{M}_0, \end{array}$$

which is obtained from the local model in Theorem 2.2.3 by taking the special fiber. We will look at a different map

$$q^c : \widetilde{\mathscr{S}_0} \to \mathrm{Gr}(V_k), \quad f \mapsto f^{-1}(D_0 \subset \mathcal{V}_0).$$

One sees easily that q^c factors through $\mathrm{M}_0^{\mathrm{c}}$ at the level of k-points, and hence factors through $\mathrm{M}_0^{\mathrm{c}}$, as $\widetilde{\mathscr{S}_0}$ is reduced. The induced morphism $\widetilde{\mathscr{S}_0} \to \mathrm{M}_0^{\mathrm{c}}$ will still be denoted by q^c, and the diagram

$$\begin{array}{ccc} & \widetilde{\mathscr{S}_0} & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle q^c} \\ \mathscr{S}_0 & & \mathrm{M}_0^{\mathrm{c}} \end{array}$$

will be called the *conjugate local model diagram*. One of our main tasks in this subsection is to study basic properties of the conjugate local model diagram.

Theorem 3.4.2. *The morphism $q^c : \widetilde{\mathscr{S}_0} \to \mathrm{M}_0^{\mathrm{c}}$ is smooth.*

Proof. We will show that, for each closed point $x \in \mathscr{S}_0(k)$, denoting by $O^\wedge_{\mathscr{S}_0,x}$ the attached complete local ring, there is a section $t : \operatorname{Spec} O^\wedge_{\mathscr{S}_0,x} \to \widetilde{\mathscr{S}_0}$ such that the composition $\operatorname{Spec} O^\wedge_{\mathscr{S}_0,x} \xrightarrow{t} \widetilde{\mathscr{S}_0} \xrightarrow{q^c} \mathrm{M}_0^{\mathrm{c}}$ induces an isomorphism of complete local rings.

Let R_G and R be as in the discussions after Lemma 2.3.7, and $R_{G,0}$ and R_0 be their reduction modulo p. Then there is an isomorphism $O^\wedge_{\mathscr{S}_0,x} \cong R_{G,0}$ such that $\mathcal{A}[p^\infty]|_{O^\wedge_{\mathscr{S}_0,x}} \cong \mathcal{B}|_{R_{G,0}}$, where $\mathcal{B}$ is the versal p-divisible group over R_E determined by the Dieudonné display $(M_{R_E}, M_{R_E,1}, \Psi)$ as in loc. cit. Let $\mathrm{C}^1 \subset M_{R_0}$ and $\mathrm{D}_0 \subset M_{R_0}$ be the Hodge filtration and conjugate filtration respectively. We fix an isomorphism

$t : V_k^\vee \to M_{R_0,x}$ respecting tensors, and view it as an isomorphism over R_0 by base-change. The filtration $t^{-1}(\mathrm{D}_0) \subset V_{R_0}^\vee$ induces a morphism $\operatorname{Spec} R_0 \to \mathrm{Gr}(V_k)$.

Let m_{R_0} be the maximal ideal of R_0, then by Remark 2.3.10 (2), we have

$$\big(t^{-1}(\mathrm{D}_0) \subset V_{R_0}^\vee\big) = \big((\mathfrak{u} \cdot t^{-1}(\mathrm{D}_{0,x})) \subset V_{R_0}^\vee\big)$$

over $R_0/m_{R_0}^2$. So $\operatorname{Spec} R_0 \to \mathrm{Gr}(V_k)$ is an isomorphism at the level of complete local rings, and hence the induced morphism $\operatorname{Spec} O^\wedge_{\mathscr{S}_0,x} \xrightarrow{t} \widetilde{\mathscr{S}_0} \xrightarrow{q^c} \mathrm{M}_0^c$ is an isomorphism onto its image after taking completion. Noting that $\mathscr{S}_0$ and M_0^c have the same dimension, the above morphism induces an isomorphism of complete local rings. □

Recall our notations in 1.3.6. Let $\mathcal{G}_0 = \mathcal{G} \otimes k$ and $\mathcal{G}_0^{\mathrm{rdt}}$ be the maximal reductive quotient of $\mathcal{G}_0 \otimes k$. Then it is a reductive group defined over $\mathbb{F}_p$. Let $\overline{B}$ be the image in $\mathcal{G}_0^{\mathrm{rdt}}$ of $\breve{I}$ and $\overline{T}$ be a maximal torus of $\overline{B}$. For $w \in {}^K\widetilde{W}^K$ which is a minimal length representative of a member in $\mathrm{Adm}(\{\mu\})_K$, we set

$$\sigma' = \sigma \circ \mathrm{Ad}(w), \quad \text{and} \quad J_w = J_K \cap \mathrm{Ad}(w^{-1})(J_K),$$

where $J_K \subset \widetilde{W}$ is the set of simple reflections in W_K. Let $\overline{L}_{J_w} \subset \mathcal{G}_0^{\mathrm{rdt}}$ (resp. $\overline{P}_{J_w} \subset \mathcal{G}_0^{\mathrm{rdt}}$) be the standard Levi subgroup (resp. parabolic subgroup) of type J_w, and $\overline{L}_{\sigma'(J_w)} \subset \mathcal{G}_0^{\mathrm{rdt}}$ (resp. $\overline{P}_{\sigma'(J_w)} \subset \mathcal{G}_0^{\mathrm{rdt}}$) be the standard Levi subgroup (resp. parabolic subgroup) of type $\sigma'(J_w)$. Then we have a natural isogeny $\overline{L}_{J_w} \to \overline{L}_{\sigma'(J_w)}$ which is again denoted by σ'. The tuple

$$\mathcal{Z}_w := (\mathcal{G}_0^{\mathrm{rdt}}, \overline{P}_{J_w}, \overline{P}_{\sigma'(J_w)}, \sigma' : \overline{L}_{J_w} \to \overline{L}_{\sigma'(J_w)})$$

is an algebraic zip datum. Set

$$E_{\mathcal{Z}_w} = (\overline{L}_{J_w})_\sigma (U_{J_w} \times U_{\sigma'(J_w)}),$$

where U_{J_w} (resp. $U_{\sigma'(J_w)}$) is the unipotent radical of $\overline{P}_{J_w}$ (resp. $\overline{P}_{\sigma'(J_w)}$). It has a left action on $\mathcal{G}_0^{\mathrm{rdt}}$, and hence induces a quotient stack $[E_{\mathcal{Z}_w} \backslash \mathcal{G}_0^{\mathrm{rdt}}]$.

Since we will work over k and with a fixed KR stratum $\mathscr{S}_0^w$, sometimes we will simply write L_{J_w} (resp. $L_{\sigma'(J_w)}$, P_{J_w}, $P_{\sigma'(J_w)}$) for $\overline{L}_{J_w}$ (resp. $\overline{L}_{\sigma'(J_w)}$, $\overline{P}_{J_w}$, $\overline{P}_{\sigma'(J_w)}$). We will construct a morphism

$$\zeta_w : \mathscr{S}_0^w \to [E_{\mathcal{Z}_w} \backslash \mathcal{G}_0^{\mathrm{rdt}}].$$

As explained in 1.1, it is equivalent to construct a $\mathcal{G}_0^{\mathrm{rdt}}$-zip of type J_w over $\mathscr{S}_0^w$. We will also assume, from now on, that $\mathscr{S}_0^w \neq \emptyset$, as otherwise, ζ exists automatically. In subsection 3.3 Corollary 3.3.7, for each $x \in \mathscr{S}_0^w(k)$, we have constructed a $\mathcal{G}_0^{\mathrm{rdt}}$-zip of type J_w. Here we will need a family version.

To construct $\mathcal{G}_0^{\mathrm{rdt}}$-zips, we will slightly change the notations and write $\widetilde{\mathbb{I}} = \widetilde{\mathscr{S}_0}$. We start with the local model diagram

$$\begin{array}{ccc} & \widetilde{\mathbb{I}} & \\ {}^{\pi}\swarrow & & \searrow^{q} \\ \mathscr{S}_0 & & \mathrm{M}_0, \end{array}$$

where π is a $\mathcal{G}_0$-torsor and q is $\mathcal{G}_0$-equivariant. The $\mathcal{G}_0$-orbit $\mathrm{M}_0^w \subset \mathrm{M}_0$ induces a $\mathcal{G}_0$-torsor

$$\widetilde{\mathbb{I}}^w := q^{-1}(\mathrm{M}_0^w) = \widetilde{\mathbb{I}}|_{\mathscr{S}_0^w}$$

over $\mathscr{S}_0^w$. We have the induced diagram

$$\begin{array}{ccc} & \widetilde{\mathbb{I}}^w & \\ {}^{\pi}\swarrow & & \searrow^{q} \\ \mathscr{S}_0^w & & \mathrm{M}_0^w. \end{array}$$

We get a $\mathcal{G}_0^{\mathrm{rdt}}$-torsor over $\mathscr{S}_0^w$ by simply taking

$$\mathbb{I}^w := \widetilde{\mathbb{I}}^w \times^{\mathcal{G}_0} \mathcal{G}_0^{\mathrm{rdt}}. \tag{3.4.3}$$

We remark that $\mathbb{I}^w$ is the pull back to $\mathscr{S}_0^w$ of the $\mathcal{G}_0^{\mathrm{rdt}}$-torsor $\mathbb{I} := \widetilde{\mathbb{I}} \times^{\mathcal{G}_0} \mathcal{G}_0^{\mathrm{rdt}} \to \mathscr{S}_0$.

Noting that w is an element in $\mathrm{M}_0^w(k)$, we consider $q^{-1}(w)$, the fiber of w in $\widetilde{\mathbb{I}}^w$. We also consider the variation of the structure defined in Proposition 3.3.4 (1), i.e. we set

$$\widetilde{\mathbb{I}}_+^w := \mathbf{Isom}_{\mathscr{S}_0^w}((C_w^\bullet, s), (C^\bullet, s_{\mathrm{dR}})) \subset \widetilde{\mathbb{I}}^w$$

Lemma 3.4.4. *We have $q^{-1}(w) = \widetilde{\mathbb{I}}_+^w$ which is a $\mathcal{G}_{0,w}$-torsor over $\mathscr{S}_0^w$. Here $\mathcal{G}_{0,w} \subset \mathcal{G}_0$ is the stabilizer of w as before.*

Proof. The natural morphism $\widetilde{\mathbb{I}}_+^w \to q^{-1}(w)$ is a closed immersion, as they are both closed subschemes of $\widetilde{\mathbb{I}}^w$. For $z \in \mathscr{S}_0^w(k)$, we set $\widetilde{\mathbb{I}}_{+,z}^w$ and $q^{-1}(w)_z$ to be its fibers in $\widetilde{\mathbb{I}}_+^w$ and $q^{-1}(w)$ respectively. It is easy to see that $\widetilde{\mathbb{I}}_{+,z}^w(k) = q^{-1}(w)_z(k)$, and hence $\widetilde{\mathbb{I}}_+^w = q^{-1}(w)$ as $q^{-1}(w)$ is a smooth variety.

The $\mathcal{G}_{0,w}$-action on $\widetilde{\mathbb{I}}_+^w$ is by definition faithful and free, so to see that it is a $\mathcal{G}_{0,w}$-torsor over $\mathscr{S}_0^w$, we only need to check the flatness of $\pi : \widetilde{\mathbb{I}}_+^w \to \mathscr{S}_0^w$. Noting that they are both smooth varieties, we only need to check that

$$\dim(\widetilde{\mathbb{I}}_+^w) = \dim(\mathscr{S}_0^w) + \dim(\widetilde{\mathbb{I}}_{+,z}^w), \qquad \text{for all } z \in \mathscr{S}_0^w(k).$$

We have

$$\dim(\mathscr{S}_0^w) = \dim(\mathrm{M}_0^w),\ \dim(\widetilde{\mathbb{I}}_{+,z}^w) = \dim(\mathcal{G}_{0,w}) \ \text{ and } \ \dim(\widetilde{\mathbb{I}}_+^w) = \dim(\mathrm{M}_0^w) + \dim(\mathcal{G}_{0,w}),$$

where the second equality follows from Proposition 3.3.4 (1), so the above equality is clear. □

The group $\mathcal{G}_{0,w}$ is smooth, and hence its image in $\mathcal{G}_0^{\mathrm{rdt}}$ lies in P_{J_w}, as it is so for k-points. We get as in (3.3.6) a P_{J_w}-torsor by taking

$$\mathbb{I}_+^w := q^{-1}(w) \times^{\mathcal{G}_{0,w}} P_{J_w} = \widetilde{\mathbb{I}}_+^w \times^{\mathcal{G}_{0,w}} P_{J_w}. \tag{3.4.5}$$

Remark 3.4.6. Let us consider the composition

$$\widetilde{\mathbb{I}}^w \to \mathrm{M}_0^w = \mathcal{G}_0/\mathcal{G}_{0,w} \twoheadrightarrow \mathcal{G}_0^{\mathrm{rdt}}/P_{J_w}.$$

It is equivariant with respect to the action of $\mathcal{G}_0$ on $\widetilde{\mathbb{I}}^w$ and that of $\mathcal{G}_0^{\mathrm{rdt}}$ on $\mathcal{G}_0^{\mathrm{rdt}}/P_{J_w}$, and hence induces a $\mathcal{G}_0^{\mathrm{rdt}}$-equivariant morphism

$$q_\# : \mathbb{I}^w = \widetilde{\mathbb{I}}^w \times^{\mathcal{G}_0} \mathcal{G}_0^{\mathrm{rdt}} \to \mathcal{G}_0^{\mathrm{rdt}}/P_{J_w}.$$

One checks immediately that $\mathbb{I}_+^w = q_\#^{-1}(\overline{w})$, where $\overline{w}$ is the image of w in $\mathcal{G}_0^{\mathrm{rdt}}/P_{J_w}$.

Similarly, we can consider the conjugate local model diagram

$$\begin{array}{ccc} & \widetilde{\mathbb{I}} & \\ {}^{\pi}\swarrow & & \searrow^{q^c} \\ \mathscr{S}_0 & & \mathrm{M}_0^{\mathrm{c}}. \end{array}$$

Let $\mathrm{M}_0^{c,w}$ be the $\mathcal{G}_0$-orbit of the point ϖ corresponding to the filtration $D_\bullet^w$. Then we have $\widetilde{\mathbb{I}}^w = q^{c,-1}(\mathrm{M}_0^{c,w})$. We also have $q^{c,-1}(\varpi)$, the fiber of ϖ in $\widetilde{\mathbb{I}}^w$ and

$$\widetilde{\mathbb{I}}_-^w := \mathbf{Isom}_{\mathscr{S}_0^w}((D_\bullet^w, s), (D_\bullet, s_{\mathrm{dR}})) \subset \widetilde{\mathbb{I}}^w.$$

Based on Theorem 3.4.2 and Proposition 3.3.4 (2), the same proof of Lemma 3.4.4 implies the following.

Lemma 3.4.7. *We have $q^{c,-1}(\varpi) = \widetilde{\mathbb{I}}_-^w$ which is a $\mathcal{G}_{0,\sigma(w)^{-1}}$-torsor over $\mathscr{S}_0^w$. Here $\mathcal{G}_{0,\sigma(w)^{-1}} \subset \mathcal{G}_0$ is as in the second paragraph of 3.3.*

We get, as above, a $P_{\sigma'(J_w)}$-torsor by taking

$$\mathbb{I}_-^w := q^{c,-1}(\varpi) \times^{\mathcal{G}_{0,\sigma(w)^{-1}}} P_{\sigma'(J_w)} = \widetilde{\mathbb{I}}_-^w \times^{\mathcal{G}_{0,\sigma(w)^{-1}}} P_{\sigma'(J_w)}. \tag{3.4.8}$$

Moreover, the composition

$$\widetilde{\mathbb{I}}^w \to \mathrm{M}_0^{c,w} = \mathcal{G}_0/\mathcal{G}_{0,\sigma(w)^{-1}} \twoheadrightarrow \mathcal{G}_0^{\mathrm{rdt}}/P_{\sigma'(J_w)}$$

induces a $\mathcal{G}_0^{\mathrm{rdt}}$-equivariant morphism

$$q_{\#}^c : \mathbb{I}^w = \widetilde{\mathbb{I}}^w \times^{\mathcal{G}_0} \mathcal{G}_0^{\mathrm{rdt}} \to \mathcal{G}_0^{\mathrm{rdt}}/P_{\sigma'(J_w)},$$

and we have $\mathbb{I}_-^w = q_{\#}^{c,-1}(\overline{\varpi})$, where $\overline{\varpi}$ is the image of ϖ in $\mathcal{G}_0^{\mathrm{rdt}}/P_{\sigma'(J_w)}$.

Notations as in Proposition 3.3.4, we consider the map

$$\widetilde{\iota} : \widetilde{\mathbb{I}}_+^{w,(p)}/\mathcal{G}_{0,w}^{U,(p)} \to \mathbf{Isom}_{\mathscr{S}_0^w}(\oplus_i \operatorname{gr}_i(D_\bullet^w), \oplus_i \operatorname{gr}_i(D_\bullet))$$

defined as in (3.3.5).

Lemma 3.4.9. *$\widetilde{\iota}$ induces an isomorphism $\widetilde{\mathbb{I}}_+^{w,(p)}/\mathcal{G}_{0,w}^{U,(p)} \to \widetilde{\mathbb{I}}_-^w/\mathcal{G}_{0,\sigma(w)^{-1}}^U$ which is equivariant with respect to the isomorphism $\sigma' : \mathcal{G}_{0,w}^{L,(p)} \to \mathcal{G}_{0,\sigma(w)^{-1}}^L$.*

Proof. By Proposition 3.3.4 (3), the induced map (by $\widetilde{\iota}$) on the sets of k-points factors through $\widetilde{\mathbb{I}}_-^w/\mathcal{G}_{0,\sigma(w)^{-1}}^U \subset \mathbf{Isom}_{\mathscr{S}_0^w}(\oplus_i \operatorname{gr}_i(D_\bullet^w), \oplus_i \operatorname{gr}_i(D_\bullet))$, so does $\widetilde{\iota}$. By the same reason as in the proof of [loc. cit], $\widetilde{\iota}$ is equivariant with respect to the isomorphism σ', and hence is an isomorphism. □

As in the pointwise case, the following statement is straightforward.

Corollary 3.4.10. *Let $\iota : \mathbb{I}_+^{w,(p)}/U_{J_w}^{(p)} \to \mathbb{I}_-^w/U_{\sigma'(J_w)}$ be the isomorphism induced by $\widetilde{\iota}$. Then the tuple $(\mathbb{I}^w, \mathbb{I}_+^w, \mathbb{I}_-^w, \iota)$ is a $\mathcal{G}_0^{\mathrm{rdt}}$-zip of type J_w.*

The tuple $(\mathbb{I}^w, \mathbb{I}_+^w, \mathbb{I}_-^w, \iota)$ then induces a morphism of stacks

$$\zeta_w : \mathscr{S}_0^w \to [E_{\mathcal{Z}_w} \backslash \mathcal{G}_0^{\mathrm{rdt}}],$$

whose fibers are precisely the EKOR strata in $\mathscr{S}_0^w$ by our previous discussions in 3.1 and 1.3.6.

Theorem 3.4.11. *The morphism ζ_w is smooth.*

Proof. Let $\mathbb{E}^w$ be formed by the following cartesian diagram:

$$\begin{array}{ccccccc}
\mathbb{E}^w & & \longrightarrow & & & & \mathbb{I}_-^w \\
\downarrow & & & & & & \downarrow \\
\mathbb{I}_+^w & \longrightarrow & \mathbb{I}_+^{w,(p)} & \longrightarrow & \mathbb{I}_+^{w,(p)}/U_{J_w}^{(p)} & \xrightarrow{\iota} & \mathbb{I}_-^w/U_{\sigma'(J_w)}
\end{array},$$

where the first map of the bottom line is the relative Frobenius map. Then $\mathbb{E}^w$ is an $E_{\mathcal{Z}_w}$-torsor over $\mathscr{S}_0^w$. In particular, $\mathbb{E}^w$ is smooth over k. The morphism ζ_w is equivalent to an $E_{\mathcal{Z}_w}$-equivariant morphism

$$\zeta_w^{\#} : \mathbb{E}^w \to \mathcal{G}_0^{\mathrm{rdt}},$$

and the smoothness of ζ_w is equivalent to that of $\zeta_w^{\#}$, which is, furthermore, equivalent to the surjectivity of the induced map on tangent spaces for all points of $\mathbb{E}^w(k)$. Noting that $\zeta_w^{\#}$ is $E_{\mathcal{Z}_w}$-equivariant, we only need to show that for any $y \in \mathscr{S}_0^w(k)$ with some lifting $z \in \mathbb{E}^w(k)$, the induced map of tangent spaces

$$T_z\mathbb{E}^w \to T_{\zeta_w^{\#}(z)}\mathcal{G}_0^{\mathrm{rdt}}$$

is surjective.

Let $(A, \mathfrak{m})$ be the complete local ring of $\mathscr{S}_0^w$ at y. The $E_{\mathcal{Z}_w}$-torsor $\mathbb{E}^w$ becomes trivial on $\operatorname{Spec} A$, and hence $\mathbb{E}_A^w \cong E_{\mathcal{Z}_w} \times_k \operatorname{Spec} A$ once we fix a section of it. Let D_x be the

Dieudonné module of the p-divisible group at x, and $t : V^\vee_{\breve{\mathbb{Z}}_p} \to D_x$ be an isomorphism respecting tensors such that the linearization of Frobenius is of the form $g \circ \sigma(w)$, where $g \in \mathcal{G}(\breve{\mathbb{Z}}_p)$ is certain element. We denote by $\bar{t}^{\mathrm{rdt}}$ (resp. $\overline{g}^{\mathrm{rdt}}$) the image of t (resp. g) in $\mathbb{I}$ (resp. $\mathcal{G}_0^{\mathrm{rdt}}$). By our choice, $z := (\bar{t}^{\mathrm{rdt}}, \overline{g}^{\mathrm{rdt}} \cdot \bar{t}^{\mathrm{rdt}})$ is in $\mathbb{E}^w(k)$, and its image in $\mathcal{G}_0^{\mathrm{rdt}}(k)$ is $\overline{g}^{\mathrm{rdt}}$.

By Remark 2.3.10 (2), viewing z as a section over A, the morphism

$$\zeta_w^\# : \mathbb{E}^w_{A/\mathfrak{m}^2} \cong E_{\mathcal{Z}_w} \times_k \operatorname{Spec} A/\mathfrak{m}^2 \to \mathcal{G}_0^{\mathrm{rdt}}$$

is given by $(h, \overline{\mathfrak{g}}) \mapsto h \cdot (\overline{\mathfrak{g}}^{\mathrm{rdt}} \overline{g}^{\mathrm{rdt}})$, here $\overline{\mathfrak{g}}$ is the image in $\mathcal{G}(A/\mathfrak{m}^2)$ of the element $\widetilde{\mathfrak{g}}^{-1}$ in [loc. cit], and $\overline{\mathfrak{g}}^{\mathrm{rdt}}$ is the image of $\overline{\mathfrak{g}}$ in $\mathcal{G}^{\mathrm{rdt}}(A/\mathfrak{m}^2)$. Let $U^-_{J_w} \subset \mathcal{G}_0^{\mathrm{rdt}}$ be the unipotent radical of the opposite parabolic subgroup of P_{J_w}. The natural morphism $\mathrm{M}_0^w \to \mathcal{G}_0^{\mathrm{rdt}}/P_{J_w}$ is a smooth covering, and hence the induced map on tangent spaces at identities is surjective. In particular, identifying $U^-_{J_w}$ as an open subscheme of $\mathcal{G}_0^{\mathrm{rdt}}/P_{J_w}$ containing the (image of) identity, $\overline{\mathfrak{g}}^{\mathrm{rdt}}$ induces a surjection $\mathfrak{m}/\mathfrak{m}^2 \to \mathrm{Lie}(U^-_{J_w})$. Noting that $\sigma' : L_{J_w} \to L_{\sigma'(J_w)}$ is trivial on elements of the form $\mathrm{id} + \mathrm{Lie}(L_{J_w})$, by the last paragraph of the proof of [92, Theorem 4.1.2], the induced map on tangent spaces $T_z\mathbb{E}^w \to T_{\overline{g}^{\mathrm{rdt}}}\mathcal{G}_0^{\mathrm{rdt}}$ is surjective. □

Now we can state many properties of EKOR strata. Recall that we have the surjection ${}^K\mathrm{Adm}(\{\mu\}) \twoheadrightarrow \mathrm{Adm}(\{\mu\})_K$.

Theorem 3.4.12. *For $x \in {}^K\mathrm{Adm}(\{\mu\})$, the corresponding EKOR stratum $\mathscr{S}_0^x$ is a locally closed subscheme of $\mathscr{S}_0$, which is smooth of dimension $\ell(x)$. Moreover, for $w \in \mathrm{Adm}(\{\mu\})_K$ (cf. 1.2.8)*

1. *there is a unique EKOR stratum, namely $\mathscr{S}_0^{{}^K w_K}$ with ${}^K w_K$ as in 1.2.6, in $\mathscr{S}_0^w$ which is open dense. This is called the w-ordinary locus,*
2. *there is a unique EKOR stratum, namely $\mathscr{S}_0^{x_w}$, in $\mathscr{S}_0^w$ which is of dimension $\ell(w)$. Here x_w is just w but viewed as an element of ${}^K\mathrm{Adm}(\{\mu\})$. This stratum is closed in $\mathscr{S}_0^w$, and is called the w-superspecial locus.*

Proof. For $x \in {}^K\mathrm{Adm}(\{\mu\})$, the non-emptiness of the EKOR stratum $\mathscr{S}_0^x$ follows from Corollary 3.5.3 (1), and the smoothness follows from Theorem 3.4.11. To compute the dimension, we have, by Theorem 3.4.11 again, $\dim(\mathscr{S}_0^x) = \dim(\mathscr{S}_0^w) - \mathrm{codim}(\mathcal{G}_0^{\mathrm{rdt},\overline{x}})$, where $\overline{x} \in {}^{J_w}W_K$ is the element corresponding to x. We also have $\dim(\mathscr{S}_0^w) = \ell({}^K w_K)$ by Corollary 2.1.11 (3), and $\mathrm{codim}(\mathcal{G}_0^{\mathrm{rdt},x}) = \dim(U_w) - \ell(\overline{x})$ by Theorem 1.1.2. Here U_w is the unipotent radical of the standard parabolic $P_w \subset \mathcal{G}_0^{\mathrm{rdt}}$ of type J_w. Noting that $\dim(U_w) = \ell(y_0)$ for y_0 the longest element in ${}^J W_K$, we have

$$\dim(\mathscr{S}_0^x) = \ell({}^K w_K) - \ell(y_0) + \ell(\overline{x}) = \ell(w) + \ell(\overline{x}) = \ell(x).$$

Statements (1) and (2) follow from the above results. □

Remark 3.4.13. (1) We can also define the *ordinary locus* $\mathscr{S}_0^{\mathrm{ord}}$ to be the union of EKOR strata $\mathscr{S}_0^x$s with $\ell(x) = \dim(\mathscr{S}_0)$. It is clear by definition that

$$\mathscr{S}_0^{\mathrm{ord}} = \coprod_{\substack{w \in \mathrm{Adm}(\{\mu\})_K, \\ \ell({}^K w_K) = \dim(\mathscr{S}_0)}} \mathscr{S}_0^{{}^K w_K},$$

and it is open in $\mathscr{S}_0$. Moreover it is dense in $\mathscr{S}_0$ by the smoothness of the morphisms $\zeta_w, w \in \mathrm{Adm}(\{\mu\})_K$ (Theorem 3.4.11). By Proposition 1.2.4, the elements in the index set of the above disjoint union are of the form $t^{\mu'}$, where μ' runs over elements in the W_0-orbit of $\underline{\mu}$ with $t^{\mu'} \in {}^K\widetilde{W}$.

(2) We can define the *superspecial locus* to be the EKOR strata $\mathscr{S}_0^\tau$, where τ is as in 1.2.5. Then $\mathscr{S}_0^\tau$ is the unique closed EKOR stratum in $\mathscr{S}_0$, and it is of dimension 0.

Corollary 3.4.14. *Let ν be the Newton map as in 1.2.9, and $x \in {}^K\mathrm{Adm}(\{\mu\})$ be σ-straight, then*

(1) *the EKOR stratum $\mathscr{S}_0^x$ is a central leaf. In particular, it is contained in the Newton stratum $\mathscr{S}_0^{\nu(x)}$, and hence each Newton stratum contains an EKOR stratum;*
(2) *the dimension of $\mathscr{S}_0^x$ is $\langle \nu(x), \rho \rangle$, where ρ is the half sum of positive roots. In particular, central leaves given by σ-straight elements in a fixed Newton stratum are of the same dimension.*

Proof. Statement (1) follows directly from the commutative diagram before 3.2.

We have $\dim(\mathscr{S}_0^x) = \ell(x)$ by Theorem 3.4.12, and $\ell(x) = \langle \nu(x), \rho \rangle$ by 1.2.10. In particular, (2) follows. ☐

Remark 3.4.15. Statement (2) recovers W. Kim's formula for central leaves attached to σ-straight elements, cf. [34, Corollary 5.3.1].

3.5. **Change of parahorics and further properties.** It is sometimes helpful to look at the relations between EKOR strata for different parahoric subgroups. We will need sometimes the axioms (especially axiom 4 (c)) introduced by He and Rapoport (see subsection 3.2). We remind the readers that, as we have remarked in 3.2.4, in the current setting, by the work [95] of Zhou, all the axioms except for surjectivity in axiom 4 (c) hold under our assumption (2.2.1), and if in addition $G_{\mathbb{Q}_p}$ is residually split, axiom 4 (c) also holds.

We start with the following result of He and Rapoport.

Proposition 3.5.1. ([31, Proposition 6.6]) *For $w \in \widetilde{W}$, there is a subset*

$$\Sigma_K(w) \subset W_K w W_K \cap {}^K\widetilde{W}$$

such that

$$\breve{K}_\sigma(\breve{I} w \breve{I}) = \coprod_{x \in \Sigma_K(w)} \breve{K}_\sigma(\breve{I} x \breve{I}).$$

Moreover, if $w \in {}^K\widetilde{W}$, then $\Sigma_K(w) = \{w\}$.

As a consequence, one has the following. Fix a sufficiently small K^p and let $\mathsf{K} = KK^p$ $\mathsf{K}' = K'K^p$ for parahoric subgroups $K' \subset K$. Assume that $\mathscr{S}_{\mathsf{K}}$ and $\mathscr{S}_{\mathsf{K}'}$ are constructed by the same Siegel embedding, then by [95, Theorem 7.1], there exists a morphism $\pi_{\mathsf{K}',\mathsf{K}} : \mathscr{S}_{\mathsf{K}'} \to \mathscr{S}_{\mathsf{K}}$ satisfying He-Rapoport's axiom 1.

Proposition 3.5.2. ([31, Proposition 6.11], [95]) *Let $K' \subset K$ be standard parahoric subgroups with induced morphism $\pi_{\mathsf{K}',\mathsf{K}} : \mathscr{S}_{\mathsf{K}',0} \to \mathscr{S}_{\mathsf{K},0}$. Then*

(1) *for $x \in {}^{K'}\mathrm{Adm}(\{\mu\})$, we have*

$$\pi_{\mathsf{K}',\mathsf{K}}(\mathscr{S}_{\mathsf{K}',0}^x) \subset \coprod_{y \in \Sigma_K(x)} \mathscr{S}_{\mathsf{K},0}^y.$$

In particular, for $x \in {}^K\mathrm{Adm}(\{\mu\})$ viewed as an element in ${}^{K'}\mathrm{Adm}(\{\mu\})$, we have $\pi_{\mathsf{K}',\mathsf{K}}(\mathscr{S}_{\mathsf{K}',0}^x) \subset \mathscr{S}_{\mathsf{K},0}^x$.
(2) *if He-Rapoport's axiom 4 (c) holds, we have equalities in the above statement.*

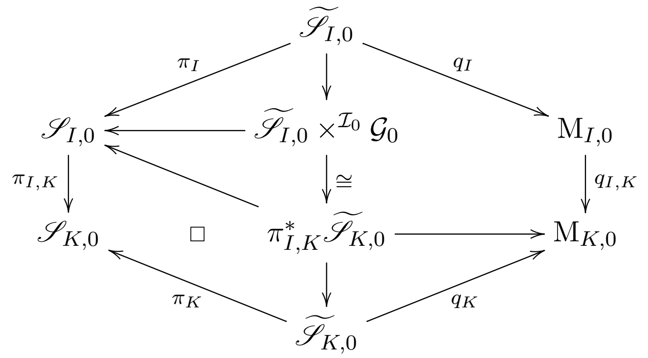
As applications, one deduces the following.

Corollary 3.5.3. *For $\mathscr{S}_{\mathsf{K},0}$ and $x \in {}^K\mathrm{Adm}(\{\mu\})$, we have*

(1) *the EKOR stratum $\mathscr{S}_{\mathsf{K},0}^x$ is non-empty;*
(2) ([31, Theorem 6.15]) *if He-Rapoport's axiom 4 (c) holds, the closure of $\mathscr{S}_{\mathsf{K},0}^x$ in $\mathscr{S}_{\mathsf{K},0}$ is $\coprod_{y \leq_{K,\sigma} x} \mathscr{S}_{\mathsf{K},0}^y$. Here $\leq_{K,\sigma}$ is as in 1.2.3.*

Proof. We only need to check (1). Since the prime to p level K^p is fixed, here and in the following, to simplify the notations we write $\mathscr{S}_{K,0}$ for $\mathscr{S}_{\mathsf{K},0}$, and similarly for the Iwahori level. Viewing x as an element of ${}^I\mathrm{Adm}(\{\mu\}) = \mathrm{Adm}(\{\mu\})$, we have, by Proposition 3.5.2 (1), that $\pi_{I,K}(\mathscr{S}^x_{I,0}) \subset \mathscr{S}^x_{K,0}$. As we have remarked in 3.2.4, non-emptiness of $\mathscr{S}^x_{I,0}$ has been proved by Zhou in [95], and hence $\mathscr{S}^x_{K,0}$ is non-empty. $\square$

To go further, we need compatibility of local model diagrams. More precisely, we will use the following commutative diagram. We use the above conventions. Let $\mathcal{I}$ be a parahoric model of G corresponding to I.



where the map $\widetilde{\mathscr{S}}_{I,0} \to \widetilde{\mathscr{S}}_{I,0} \times^{\mathcal{I}_0} \mathcal{G}_0$ is induced by the natural map $\mathcal{I}_0 \to \mathcal{G}_0$.

Lemma 3.5.4. *For $x \in {}^K\mathrm{Adm}(\{\mu\})$ viewed as an element of $\mathrm{Adm}(\{\mu\})$, $q_{I,K}$ induces a morphism $\mathrm{M}^x_{I,0} \to \mathrm{M}_{K,0}$ which is a finite étale covering onto its image.*

Proof. Let $\mathcal{G}'$ (resp. $\mathcal{I}'$) be the parahoric (resp. Iwahori) group scheme over $k[[t]]$ attached to $\breve{K}$ (resp. $\breve{I}$). We have

$$\mathrm{M}^x_{I,0} = L^+\mathcal{I}'xL^+\mathcal{I}'/L^+\mathcal{I}' \cong L^+\mathcal{I}'/(L^+\mathcal{I}' \cap xL^+\mathcal{I}'x^{-1}), \qquad \text{and}$$
$$\mathrm{M}^x_{K,0} := L^+\mathcal{I}'xL^+\mathcal{G}'/L^+\mathcal{G}' \cong L^+\mathcal{I}'/(L^+\mathcal{I}' \cap xL^+\mathcal{G}'x^{-1}) \subset L^+\mathcal{G}'xL^+\mathcal{G}'/L^+\mathcal{G}' = \mathrm{M}^{w_x}_{K,0}.$$

By the same proof as in Lemma 2.1.6, the quotient maps $\mathcal{I}'_0 \to \mathrm{M}^x_{I,0}$ and $\mathcal{I}'_0 \to \mathrm{M}^x_{K,0}$ are both smooth coverings. In particular, $q^x_{I,K} : \mathrm{M}^x_{I,0} \to \mathrm{M}^x_{K,0}$ is a smooth covering. We also have, by [74, Proposition 2.8 (ii)], that $\dim(\mathrm{M}^x_{I,0}) = \ell(x) = \dim(\mathrm{M}^x_{K,0})$, and hence $q^x_{I,K}$ is an étale covering.

To see that $q^x_{I,K}$ is finite, we only need to check that it is proper. Let $\mathrm{M}^{x,-}_{I,0}$ and $\mathrm{M}^{x,-}_{K,0}$ be the closures of $\mathrm{M}^x_{I,0}$ and $\mathrm{M}^x_{K,0}$ respectively, endowed with the reduced subscheme structures. Then $q_{I,K}$ induces a proper morphism

$$q^{x,-}_{I,K} : \mathrm{M}^{x,-}_{I,0} \longrightarrow q_{I,K}(\mathrm{M}^{x,-}_{I,0}) = \mathrm{M}^{x,-}_{K,0}.$$

Noting that $\mathrm{M}^{x,-}_{I,0} = \coprod_{x' \leq x} \mathrm{M}^{x'}_{I,0}$ and $q_{I,K}(\mathrm{M}^{x'}_{I,0})$ is either $\mathrm{M}^x_{K,0}$ or disjoint with it, we have

$$q_{I,K}(\mathrm{M}^{x'}_{I,0}) \cap \mathrm{M}^x_{K,0} = \emptyset, \qquad \text{for any } x' \leq x \text{ with } x' \neq x$$

since $\dim q_{I,K}(\mathrm{M}^{x'}_{I,0}) = \ell(x') < \dim \mathrm{M}^x_{K,0} = \ell(x)$. In particular, $q^x_{I,K}$ is the base-change to $\mathrm{M}^x_{K,0}$ of $q^{x,-}_{I,K}$, and hence is proper. $\square$

Theorem 3.5.5. *For $x \in {}^K\mathrm{Adm}(\{\mu\})$ viewed as an element of $\mathrm{Adm}(\{\mu\})$, the morphism $\pi^x_{I,K} : \mathscr{S}^x_{I,0} \to \mathscr{S}^x_{K,0}$ induced by $\pi_{I,K}$ is finite étale. If in addition He-Rapoport's axiom 4 (c) is satisfied, $\pi^x_{I,K}$ is a finite étale covering.*

Proof. For a point $z \in \mathscr{S}^x_{I,0}(k)$, we fix an identification $V^\vee_{\breve{\mathbb{Z}}_p} \cong \mathbb{D}(\mathcal{A}_z[p^\infty])$ respecting tensors and consider its image $t \in \widetilde{\mathscr{S}}_{I,0}(k)$. Let $O_{I,z}$ (resp. $O^x_{I,z}$) be the complete local ring at z of $\mathscr{S}_{I,0}$ (resp. $\mathscr{S}^x_{I,0}$), and R_0 (resp. $R_{I,0}$) be the special fiber of the ring R_E

(resp. R_G) in the discussions after Lemma 2.3.7, then $O_{I,z} \cong R_{I,0}$, and it induces an isomorphism

$$\mathcal{A}[p^\infty]|_{O_{I,z}} \cong \mathcal{B}_I|_{R_{I,0}},$$

where $\mathcal{B}_I$ is the (special fiber of the) versal p-divisible group as in loc. cit. Let $\mathcal{V}^1 \subset \mathcal{V}$ be the Hodge filtration of $\mathcal{A}/\mathscr{S}_{I,0}$, and $\mathfrak{u}$ be restriction to $\mathrm{M}^x_{I,0}$ of the universal element of $\mathrm{Gr}(V_k)(R_0)$ at the point corresponding to $t^{-1}(\mathcal{V}^1_x) \subset V_k^\vee$, then $\mathfrak{u} \cdot t \in \widetilde{\mathscr{S}}_{I,0}(O^x_{I,z})$, and it induces, by pulling back the Hodge filtration, a morphism $\operatorname{Spec} O^x_{I,z} \to \mathrm{M}^x_{I,0}$. By our choices as in loc. cit, $\operatorname{Spec} O^x_{I,z} \to \mathrm{M}^x_{I,0}$ induces a bijection of tangent spaces, and hence is an isomorphism at the level of complete local rings.

For $z' := \pi_{I,K}(z) \in \mathscr{S}^x_{K,0}$, let w_x the KR type of x (equivalently, z') and $t' \in \widetilde{\mathscr{S}}_{K,0}(k)$ be the image of t. With $\mathfrak{u}'$ defined similarly, we have $\mathfrak{u}' \cdot t' \in \widetilde{\mathscr{S}}_{K,0}(O^{w_x}_{K,z'})$, and it induces a morphism $\operatorname{Spec} O^{w_x}_{K,z'} \to \mathrm{M}^{w_x}_{K,0}$ which is an isomorphism at the level of complete local rings.

Let $O^x_{K,z'}$ be the complete local ring at z' of the EKOR stratum, then we have a commutative diagram

$$\begin{array}{ccccc} & & \operatorname{Spec} O^x_{I,z} & \xrightarrow{\mathfrak{u}\cdot t} & \mathrm{M}^x_{I,0} \\ & & \downarrow{\scriptstyle \pi_{I,K}} & & \downarrow{\scriptstyle q_{I,K}} \\ \operatorname{Spec} O^x_{K,z'} & \hookrightarrow & \operatorname{Spec} O^{w_x}_{K,z'} & \xrightarrow{\mathfrak{u}'\cdot t'} & \mathrm{M}^{w_x}_{K,0}. \end{array}$$

The morphism $\operatorname{Spec} O^x_{I,z} \to \operatorname{Spec} O^{w_x}_{K,z'}$ is a closed immersion by Lemma 3.5.4, and it factors through $\operatorname{Spec} O^x_{K,z'}$. Noting that $\operatorname{Spec} O^x_{I,z}$ and $\operatorname{Spec} O^x_{K,z'}$ are of the same dimension, the morphism $\pi_{I,K} : \operatorname{Spec} O^x_{I,z} \to \operatorname{Spec} O^x_{K,z'}$ is an isomorphism. □

Remark 3.5.6. Notations as above, and let $\mathrm{M}^x_{K,0} \subset \mathrm{M}^{w_x}_{K,0}$ be as in Lemma 3.5.4. One can prove, without using smoothness or dimension formula, that $(\mathfrak{u}'\cdot t')^{-1}(\mathrm{M}^x_{K,0}) = \operatorname{Spec} O^x_{K,z'}$.

3.5.7. *Quasi-affineness of EKOR strata.* Let us recall some facts about quasi-affineness and ampleness. The following lemma can be found easily in text books and hence and we omit the proofs and references.

Lemma 3.5.8. (1) *Let X be a scheme, then X is quasi-affine if and only if O_X is ample.*

(2) *Let X and Y be schemes that are separated and quasi-compact, $\mathcal{L}$ be a line bundle on Y.*
   (a) *Let $f : X \to Y$ be a composition of immersions and finite morphisms. If $\mathcal{L}$ is ample, then $f^*\mathcal{L}$ is ample.*
   (b) *Let $g : X \to Y$ be faithfully flat, then $\mathcal{L}$ is ample if $g^*\mathcal{L}$ is ample.*

Theorem 3.5.9. *Every KR stratum in $\mathscr{S}_{I,0}$ is quasi-affine. If in addition axiom 4 (c) is satisfied, every EKOR stratum in $\mathscr{S}_{K,0}$ is quasi-affine.*

Proof. Notations as before, for $x \in {}^K\mathrm{Adm}(\{\mu\})$ viewed as an element in $\mathrm{Adm}(\{\mu\})$, the morphism $\mathscr{S}^x_{I,0} \to \mathscr{S}^x_{K,0}$ is a finite étale covering if axiom 4 (c) is satisfied, and hence by Lemma 3.5.8, it suffices to show that $\mathscr{S}^w_{I,0}$ is quasi-affine for all $w \in \mathrm{Adm}(\{\mu\})$.

We work with a symplectic embedding $\mathcal{I} \to \mathrm{GSp}(V_{\mathbb{Z}_p}, \psi)$ as in Remark 2.2.4, the induced morphism

$$f : \mathscr{S}_{I,0} \to \mathscr{S}_{H,0}(\mathrm{GSp}, S^\pm)$$

is finite. Let $\mathcal{V}^1 \subset \mathcal{V}$ be the Hodge filtration attached to the universal abelian scheme on $\mathscr{S}_{H,0}(\mathrm{GSp}, S^\pm)$, then $\omega := \det(\mathcal{V}^1)$ is ample, and hence $f^*\omega \cong \det(\mathcal{V}^1_{\mathscr{S}_{I,0}})$ is ample. Let P^0 be the stabilizer in $\mathrm{GL}(V_k)$ attached to the filtration $C^\bullet_w$, it is well known that ω comes from a character of P^0, denoted by η. The induced character of $\mathcal{I}_{0,w}$ factors through the quotient $\mathcal{I}_{0,w}^{\mathrm{rdt}} = \mathcal{I}_0^{\mathrm{rdt}}$ which is a torus (over k and defined over $\mathbb{F}_p$) denoted by T.

The $\mathcal{I}_0$-zip on $\mathscr{S}_{I,0}^w$ induces a morphism

$$\zeta_I : \mathscr{S}_{I,0}^w \to [T_{\sigma'}\backslash T],$$

and $f^*\omega|_{\mathscr{S}_{I,0}^w}$ is the pullback via ζ_I of the line bundle $\mathcal{L}_\eta$ on $[T_{\sigma'}\backslash T]$ induced by the character $\eta : T \to \mathbb{G}_m$. Noting that $f^*\omega|_{\mathscr{S}_{I,0}^w}$ is ample, to prove that $\mathscr{S}_{I,0}^w$ is quasi-affine, it suffices to show that $\mathcal{L}_\eta \in \mathrm{Pic}([T_{\sigma'}\backslash T])$ is of finite order.

The action of T on itself is induced by the homomorphism $h : T \to T$, $x \mapsto \sigma'(x)/x$. Let $\mathrm{H} := \mathrm{Ker}(h)$, we have an isomorphism of stacks $[T_{\sigma'}\backslash T] \cong [\mathrm{H}\backslash \mathrm{Spec}\ k]$. So it suffices to show that $\mathrm{Pic}([\mathrm{H}\backslash \mathrm{Spec}\ k])$ is finite.

Recall that $\sigma' = \sigma \circ \mathrm{Ad}(w)$, under the invertible substitution $x = w^{-1}yw$, $h : T_{\sigma'} \to T$ becomes $y \mapsto \sigma(y)/(w^{-1}yw)$. We compute its tangent map at the identity. Let $m : T \times T \to T$ be the multiplication map, and $i_w : T \to T$ be the isomorphism induced by $y \mapsto (w^{-1}yw)^{-1}$. Then

$$dh|_e = d\big(m \circ (\sigma, i_w)\big) = dm|_{(e,e)} \circ d\big(\sigma, i_w\big)|_e = d\sigma|_e + di_w|_e.$$

Noting that i_w is an isomorphism, so $di_w|_e$ is invertible, while $d\sigma|_e = 0$, so $dh|_e$ is invertible. In particular, $\mathrm{H} = \mathrm{Ker}(h)$ is finite étale.

The (trivial) action of H on Spec k induces the trivial action of $\mathrm{H}(k)$ on $k^\times = O(\mathrm{Spec}\ k)^\times$. By the second exact sequence of [40, Theorem 2.2.1], $\mathrm{Pic}([\mathrm{H}\backslash \mathrm{Spec}\ k]) \cong H^1_{\mathrm{alg}}(\mathrm{H}, k^\times)$, where $H^1_{\mathrm{alg}}(\mathrm{H}, k^\times) \subset H^1(\mathrm{H}(k), k^\times)$ is the subgroup of classes of cocycles induced by a morphism $\mathrm{H} \to \mathbb{G}_{m,k}$. Noting that $H^1(\mathrm{H}(k), k^\times) = \mathrm{Hom}(\mathrm{H}(k), k^\times)$ as we are using the trivial action, we have $H^1_{\mathrm{alg}}(\mathrm{H}, k^\times) \cong X^*(\mathrm{H})$. So $\mathrm{Pic}([\mathrm{H}\backslash \mathrm{Spec}\ k]) \cong X^*(\mathrm{H})$, and hence is finite.

□

Remark 3.5.10. The element $\mathcal{L}_\eta \in \mathrm{Pic}([E_{\mathcal{Z}_w}\backslash \mathcal{G}_0^{\mathrm{rdt}}])$ is not of finite order in general, since otherwise the whole KR stratum (which is the whole special fiber if K is hyperspecial) would be quasi-affine, which is absurd.

3.5.11. We can also talk about the relations between KR strata for different parahoric subgroups. Consider the projection morphism $\pi_{I,K} : \mathscr{S}_{I,0} \to \mathscr{S}_{K,0}$. Let $w \in \mathrm{Adm}(\{\mu\})_K$ which we view as an element of ${}^K\widetilde{W}^K$. Then

$$\pi_{I,K}^{-1}(\mathscr{S}_{K,0}^w) = \coprod_{x \in W_K w W_K \cap \mathrm{Adm}(\{\mu\})} \mathscr{S}_{I,0}^x.$$

Thus we get an induced morphism

$$\coprod_{x \in W_K w W_K \cap \mathrm{Adm}(\{\mu\})} \mathscr{S}_{I,0}^x \longrightarrow \mathscr{S}_{K,0}^w = \coprod_{y \in W_K w W_K \cap {}^K\widetilde{W}} \mathscr{S}_{K,0}^y.$$

Note that by Theorem 1.2.2, $W_K w W_K \cap {}^K\widetilde{W} \subset {}^K\mathrm{Adm}(\{\mu\}) = \mathrm{Adm}(\{\mu\}) \cap {}^K\widetilde{W}$, thus

$$W_K w W_K \cap {}^K\widetilde{W} \subset W_K w W_K \cap \mathrm{Adm}(\{\mu\}).$$

By Proposition 3.5.2, we have

- for $x \in W_K w W_K \cap {}^K\widetilde{W}$, $\pi_{I,K}(\mathscr{S}_{I,0}^x) \subset \mathscr{S}_{K,0}^x$;
- for $x \in W_K w W_K \cap \mathrm{Adm}(\{\mu\}) \smallsetminus W_K w W_K \cap {}^K\widetilde{W}$, $\pi_{I,K}(\mathscr{S}_{I,0}^x) \subset \coprod_{y \in \Sigma_K(w)} \mathscr{S}_{K,0}^y$.

If axiom 4 (c) holds, then both inclusions above are in fact equalities.

3.6. **Affine Deligne-Lusztig varieties.** In this subsection, we discuss the local counterparts of our previous constructions and some local-global compatibilities of our constructions.

3.6.1. We turn to a general local setting to discuss affine Deligne-Lusztig varieties. Let $(G, [b], \{\mu\})$ be a triple where G is a connected reductive group over $\mathbb{Q}_p$, $\{\mu\}$ is a conjugacy class of cocharacters $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$, $[b]$ is a σ-conjugacy class such that $[b] \in B(G, \{\mu\})$. Fix a representative $b \in G(\breve{\mathbb{Q}}_p)$ of $[b]$ and a parahoric subgroup $K \subset G(\mathbb{Q}_p)$. Let $\breve{K} \subset G(\breve{\mathbb{Q}}_p)$ be the associated parahoric subgroup. We get the associated affine Deligne-Lusztig set

$$X(\mu, b)_K = \{g \in G(\breve{\mathbb{Q}}_p) |\, g^{-1}b\sigma(g) \in \bigcup_{w \in \mathrm{Adm}(\{\mu\})_K} \breve{K}w\breve{K}\}/\breve{K},$$

which admits a geometric structure as a perfect scheme over $k = \overline{\mathbb{F}}_p$ by [93] and [1]. See also the following 4.3. On the other hand, for any $w \in \mathrm{Adm}(\{\mu\})_K$, consider

$$X_w(b)_K = \{g \in G(\breve{\mathbb{Q}}_p) |\, g^{-1}b\sigma(g) \in \breve{K}w\breve{K}\}/\breve{K}.$$

Then we have the KR decomposition

$$X(\mu, b)_K = \coprod_{w \in \mathrm{Adm}(\{\mu\})_K} X_w(b)_K.$$

We remark that in the above decomposition it can happen that some $X_w(b)_K$ is empty. We view $X(\mu, b)_K$ as a perfect scheme over k. Let $M^{\mathrm{loc}} = \bigcup_{w \in \mathrm{Adm}^K(\{\mu\})} \breve{K}w\breve{K}/\breve{K}$ which we view as a perfect scheme over k, cf. the following 4.1. By construction, we have a tautological ("local model") diagram (cf. [69] p. 299)

$$\begin{array}{ccc} & \widetilde{X}(\mu, b)_K & \\ {}^{\pi}\swarrow & & \searrow^{q} \\ X(\mu, b)_K & & M^{\mathrm{loc}}. \end{array}$$

Therefore, for any $X_w(b)_K \neq \emptyset$, we can view it as a locally closed perfect subscheme of $X(\mu, b)_K$.

Following [11] 3.4 and [12] 1.4, for $x \in {}^K\widetilde{W}$ we set

$$X_{K,x}(b) = \{g \in G(\breve{\mathbb{Q}}_p) |\, g^{-1}b\sigma(g) \in \breve{K} \cdot_\sigma \breve{I}x\breve{I}\}/\breve{K}.$$

Then we have the EKOR decomposition

$$X(\mu, b)_K = \coprod_{x \in {}^K\mathrm{Adm}(\{\mu\})} X_{K,x}(b),$$

which is finer than the above KR decomposition.

3.6.2. Now we come back to the setting as in the beginning of this section. Let (G, X) be a Shimura datum of Hodge type. After fixing an element $[b] \in B(G, \{\mu\})$ and a representative $b \in G(\breve{\mathbb{Q}}_p)$ of $[b]$, we have the affine Deligne-Lusztig variety $X(\mu, b)_K$ as above. Our construction in subsection 3.4 has a local analogue for $X(\mu, b)_K$, so that for each non empty KR stratum $X_w(b)_K$, we have a $\mathcal{G}_0^{\mathrm{rdt}}$-zip on it and thus a morphism of algebraic stacks

$$\zeta_w : X_w(b)_K \to [E_{\mathcal{Z}_w} \backslash \mathcal{G}_0^{\mathrm{rdt}}]^{pf}.$$

Here $[E_{\mathcal{Z}_w} \backslash \mathcal{G}_0^{\mathrm{rdt}}]^{pf}$ is the perfection of the quotient stack $[E_{\mathcal{Z}_w} \backslash \mathcal{G}_0^{\mathrm{rdt}}]$, see [84] Appendix (and also the following 4.4). This gives us a geometric meaning for the above EKOR decomposition. In the case that K is hyperspecial and $X(\mu, b)_K$ is isomorphic to the perfection of the special fiber of a formal Rapoport-Zink space, there is a related construction in [77, Theorem 7.1].

Fix a sufficiently small K^p and write $\mathscr{S}_0 = \mathscr{S}_{\mathsf{K},0}$ with $\mathsf{K} = KK^p$. Let $\mathscr{S}_0^b$ be the Newton stratum attached to $[b]$. Now the links between the local and global KR and EKOR strata are as follows.

Proposition 3.6.3. *We have:*

(1) $X_w(b)_K \neq \emptyset \Leftrightarrow \mathscr{S}_0^w(k) \cap \mathscr{S}_0^b(k) \neq \emptyset$.
(2) $X_{K,x}(b) \neq \emptyset \Leftrightarrow \mathscr{S}_0^x(k) \cap \mathscr{S}_0^b(k) \neq \emptyset$.

Proof. For (1), we have

$$X_w(b)_K \neq \emptyset \Leftrightarrow \breve{K}w\breve{K} \cap [b] \neq \emptyset$$

by definition. By [95, Theorem 8.1], the map $\Upsilon_K : \mathscr{S}_0(k) \to C(\mathcal{G}, \{\mu\})$ is surjective. Let C^w and C^b be the inverse images of w and $[b]$ under the natural surjections $C(\mathcal{G}, \{\mu\}) \to \mathrm{Adm}(\{\mu\})_K$ and $C(\mathcal{G}, \{\mu\}) \to B(G, \{\mu\})$ respectively. Then the surjectivity of Υ_K implies that

$$\mathscr{S}_0^w(k) \cap \mathscr{S}_0^b(k) \neq \emptyset \Leftrightarrow C^w \cap C^b \neq \emptyset.$$

Finally, one sees easily that

$$C^w \cap C^b \neq \emptyset \Leftrightarrow \breve{K}w\breve{K} \cap [b] \neq \emptyset.$$

The proof for (2) is similar. □

Fix a point $x_0 \in \mathscr{S}_0^b(k)$ and assume further *either G is residually split or $[b]$ is basic.* Then by [95, Proposition 6.5], there exists a unique uniformization map

$$\iota_{x_0} : X(\mu, b)_K \to \mathscr{S}_0^b(k).$$

One can check directly the following local-global compatibilities under the uniformization map:

Corollary 3.6.4. *The above morphism ι_{x_0} induces*

(1) *for any $w \in \mathrm{Adm}(\{\mu\})_K$ a map*

$$X_w(b)_K \to \mathscr{S}_0^w(k) \cap \mathscr{S}_0^b(k),$$

(2) *for any $x \in {}^K\mathrm{Adm}(\{\mu\})$ a map*

$$X_{K,x}(b) \to \mathscr{S}_0^x(k) \cap \mathscr{S}_0^b(k).$$

4. EKOR strata of Hodge type: global constructions

In this section, we will give some global constructions of the EKOR strata by adapting and generalizing some ideas of Xiao-Zhu in [84]. More precisely, we will introduce the notions of local $(\mathcal{G}, \mu)$-Shtukas and their truncations in level 1: the so called $(m, 1)$-restricted local $(\mathcal{G}, \mu)$-Shtukas, generalizing those in [84] in the case of good reductions. We will construct a smooth morphism from the perfection $\mathscr{S}_{\mathsf{K},0}^{pf}$ of $\mathscr{S}_{\mathsf{K},0}$ to the moduli stack of $(m, 1)$-restricted local $(\mathcal{G}, \mu)$-Shtukas, such that its fibers give the EKOR strata on $\mathscr{S}_{\mathsf{K},0}^{pf}$. This global construction will enable us to prove the closure relation for the EKOR strata, which is independent of the He-Rapoport axioms in [31]. In the hyperspecial case, see [85] for a different construction of the EO stratification using classical affine Grassmannians.

4.1. **Local $(\mathcal{G}, \mu)$-Shtukas and their moduli.** We return to the setting of subsection 1.2. Let G be a reductive group over $\mathbb{Q}_p$, and $\mathcal{G}$ be a parahoric group scheme over $\mathbb{Z}_p$ with generic fiber G. Let $K = \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ be the associated parahoric subgroup.

4.1.1. *Witt vector affine flag varieties.* Let $k = \overline{\mathbb{F}}_p$ and Aff_k^{pf} be the category of perfect k-algebras. We will work with the fpqc topology on Aff_k^{pf}, and we refer the reader to the appendices of [93] and [84] to some generalities on perfect algebraic geometry. Recall that we have the Witt vector loop groups $L^+\mathcal{G}$, LG, and the Witt vector affine flag variety $\mathrm{Gr}_\mathcal{G}$, such that for any $R \in \mathrm{Aff}_k^{pf}$, $L^+\mathcal{G}(R) = \mathcal{G}(W(R))$, $LG(R) = \mathcal{G}(W(R)[1/p]) = G(W(R)[1/p])$, and

$$\mathrm{Gr}_\mathcal{G}(R) = \{(\mathcal{E}, \beta) |\, \mathcal{E} : \mathcal{G}\text{-torsor on } W(R), \beta : \mathcal{E}[1/p] \xrightarrow{\sim} \mathcal{E}_0[1/p]\}$$

where $\mathcal{E}_0$ is the trivial $\mathcal{G}$-torsor.

Let $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ be a cocharacter and $\{\mu\}$ be the associated conjugacy class. After choosing a Borel subgroup B of $G_{\overline{\mathbb{Q}}_p}$, we may assume that μ is dominant with respective to B. Recall that we have the finite subset $\mathrm{Adm}(\{\mu\})_K \subset W_K \backslash \widetilde{W} / W_K$. For any $w \in W_K \backslash \widetilde{W} / W_K$, we have the associated affine Schubert cell $\mathrm{Gr}_w \subset \mathrm{Gr}_\mathcal{G}$. Let

$$M^{\mathrm{loc}} = \bigcup_{w \in \mathrm{Adm}(\{\mu\})_K} \mathrm{Gr}_w \subset \mathrm{Gr}_\mathcal{G}$$

be the closed subscheme associated to $\mathrm{Adm}(\{\mu\})_K$. In subsection 4.4, it will be the (perfection of the) special fiber of the Pappas-Zhu local model.

4.1.2. *Local $\mathcal{G}$-Shtukas in mixed characteristic.* Motivated by [76] Definitions 11.4.1 and 23.1.2 and [59], we have the following generalization of local Shtukas in mixed characteristic of [84, Definition 5.2.1]. Recall that we write $\sigma : W(R) \to W(R)$ for the Frobenius map, as in subsection 2.3.

Definition 4.1.3. Let $R \in \mathrm{Aff}_k^{pf}$. A local $(\mathcal{G}, \mu)$-Shtuka (or, a local $\mathcal{G}$-Shtuka of type μ) over R is a pair $(\mathcal{E}, \beta)$, where

- $\mathcal{E}$ is a $\mathcal{G}$-torsor over $\mathrm{Spec}\, W(R)$,
- β is a modification of the $\mathcal{G}$-torsors $\sigma^*\mathcal{E}$ and $\mathcal{E}$ over $\mathrm{Spec}\, W(R)$, i.e. an isomorphism
$$\beta : \sigma^*\mathcal{E}|_{\mathrm{Spec}\, W(R)[1/p]} \xrightarrow{\sim} \mathcal{E}|_{\mathrm{Spec}\, W(R)[1/p]},$$

such that for any geometric point x of $\mathrm{Spec}\, R$, we have $Inv_x(\beta) \in \mathrm{Adm}(\{\mu\})_K$. Sometimes we will also write the modification β as $\beta : \sigma^*\mathcal{E} \dashrightarrow \mathcal{E}$.

The last condition may need a few explanations: take any trivializations $\alpha : \mathcal{E}_{0,x} \xrightarrow{\sim} \sigma^*\mathcal{E}_x$ and $\gamma : \mathcal{E}_x \xrightarrow{\sim} \mathcal{E}_{0,x}$, then the composition $\gamma \circ \beta \circ \alpha$ defines an element $g \in G(W(k(x))_\mathbb{Q})$. For different choices of α' and γ', we get $g' = h_1 g h_2$ for some $h_1, h_2 \in \mathcal{G}(W(k(x)))$. Thus the modification β defines a well defined element

$$Inv_x(\beta) := [g] \in \mathcal{G}(W(k(x)))\backslash G(W(k(x))_\mathbb{Q})/\mathcal{G}(W(k(x))) \simeq \breve{K}\backslash G(\breve{\mathbb{Q}}_p)/\breve{K} \simeq W_K\backslash\widetilde{W}/W_K.$$

The above condition is that for all geometric points x of $\mathrm{Spec}\, R$, we require the elements $Inv_x(\beta)$ to be in the finite subset $\mathrm{Adm}(\{\mu\})_K$.

Let $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}}$ be the prestack of $\mathcal{G}$-Shtukas of type μ. To describe it, first recall that by construction we have $\mathrm{Gr}_\mathcal{G} = LG/L^+\mathcal{G}$. In the following we will need to consider the σ-conjugation action of $L^+\mathcal{G}$ on LG: for any $R \in \mathrm{Aff}_k^{pf}, g \in LG(R) = G(W(R)_\mathbb{Q})$, and $h \in L^+\mathcal{G}(R) = \mathcal{G}(W(R))$,

$$h \cdot g := hg\sigma(h^{-1}) \in LG(R).$$

Similar to [84] (5.3.7) and [94] (4.1.1), we can describe $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}}$ as follows.

Lemma 4.1.4. *We have the following isomorphism of prestacks*

$$\mathrm{Sht}_{\mu,K}^{\mathrm{loc}} \simeq \Big[\frac{M^{\mathrm{loc},\infty}}{Ad_\sigma L^+\mathcal{G}}\Big],$$

where $M^{\mathrm{loc},\infty} \subset LG$ is the pre-image of $M^{\mathrm{loc}} \subset \mathrm{Gr}_{\mathcal{G}} = LG/L^+\mathcal{G}$, which is stable by the σ-conjugation action of $L^+\mathcal{G}$ on LG, and the quotient means that we take the σ-conjugation action of $L^+\mathcal{G}$ on $M^{\mathrm{loc},\infty}$.

Proof. This is similar to [94] 4.1. We briefly sketch the arguments. First we note $M^{\mathrm{loc},\infty}$ is stable under the σ-conjugation action of $L^+\mathcal{G}$. Let $\mathrm{Sht}_{\mu,K}^{\mathrm{loc},\square}$ denote the $L^+\mathcal{G}$-torsor over $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}}$ classifying local $(\mathcal{G},\mu)$-Shtukas together with a trivialization $\epsilon : \mathcal{E} \simeq \mathcal{E}_0$. Then the composition $\epsilon \circ \beta \circ \sigma(\epsilon^{-1})$ defines an element $g \in M^{\mathrm{loc},\infty}$ and induces an isomorphism $\mathrm{Sht}_{\mu,K}^{\mathrm{loc},\square} \simeq M^{\mathrm{loc},\infty}$, under which the $L^+\mathcal{G}$-action on $\mathrm{Sht}_{\mu,K}^{\mathrm{loc},\square}$ is identified with the σ-conjugation action of $L^+\mathcal{G}$ on $M^{\mathrm{loc},\infty}$. This gives the desired isomorphism of prestacks. □

Recall that in 1.3.2 we have introduced the set $C(\mathcal{G},\{\mu\}) = \breve{K}\mathrm{Adm}(\{\mu\})\breve{K}/\breve{K}_\sigma$. Since $k = \overline{\mathbb{F}}_p$, we get $L^+\mathcal{G}(k) = \mathcal{G}(W(k)) = \breve{K}$, $M^{\mathrm{loc},\infty}(k) = \breve{K}\mathrm{Adm}(\{\mu\})\breve{K}$. In particular, by Lemma 4.1.4 we have

$$\mathrm{Sht}_{\mu,K}^{\mathrm{loc}}(k) = C(\mathcal{G},\{\mu\}).$$

4.1.5. *An alternative description.* Let $(\mathcal{E},\beta)$ be a local $(\mathcal{G},\mu)$-Shtuka. Set $\overleftarrow{\mathcal{E}} := \sigma^*\mathcal{E}$, $\overrightarrow{\mathcal{E}} := \mathcal{E}$, then $(\mathcal{E},\beta)$ is equivalent to the following data $(\gamma : \overleftarrow{\mathcal{E}} \dashrightarrow \overrightarrow{\mathcal{E}}, \psi)$, where

- $\gamma : \overleftarrow{\mathcal{E}} \dashrightarrow \overrightarrow{\mathcal{E}}$ is a modification of $\mathcal{G}$-torsors over $\mathrm{Spec}\, W(R)$ of type μ,
- $\psi : \sigma^*\overrightarrow{\mathcal{E}} \simeq \overleftarrow{\mathcal{E}}$ is an isomorphism of $\mathcal{G}$-torsors over $\mathrm{Spec}\, W(R)$.

Consider the associated local Hecke stack $Hk_{\mu,K}^{\mathrm{loc}}$: for any $R \in \mathrm{Aff}_k^{pf}$, $Hk_{\mu,K}^{\mathrm{loc}}(R)$ classifies the modifications $\gamma : \overleftarrow{\mathcal{E}} \dashrightarrow \overrightarrow{\mathcal{E}}$ of $\mathcal{G}$-torsors over $\mathrm{Spec}\, W(R)$ of type μ. We have the isomorphism of stacks

$$Hk_{\mu,K}^{\mathrm{loc}} \simeq [L^+\mathcal{G}\backslash M^{\mathrm{loc}}].$$

Let $BL^+\mathcal{G}$ be the stack of $L^+\mathcal{G}$-torsors, and

$$\overleftarrow{t} : Hk_{\mu,K}^{\mathrm{loc}} \to BL^+\mathcal{G} \quad (\text{resp. } \overrightarrow{t} : Hk_{\mu,K}^{\mathrm{loc}} \to BL^+\mathcal{G})$$

be the functor which sends γ to $\overleftarrow{\mathcal{E}}$ (resp. $\overrightarrow{\mathcal{E}}$). Then we have the following cartesian digram

$$\begin{array}{ccc} \mathrm{Sht}_{\mu,K}^{\mathrm{loc}} & \longrightarrow & Hk_{\mu,K}^{\mathrm{loc}} \\ \downarrow & & \downarrow{\scriptstyle \overleftarrow{t} \times \overrightarrow{t}} \\ BL^+\mathcal{G} & \xrightarrow{\sigma \times 1} & BL^+\mathcal{G} \times BL^+\mathcal{G}. \end{array}$$

For later use, we will also view the functors $\overleftarrow{t}$ and $\overrightarrow{t}$ in the following way. Recall that $M^{\mathrm{loc},\infty} \to M^{\mathrm{loc}}$ is a $L^+\mathcal{G}$-torsor. Then $\overrightarrow{t}$ is given by the $L^+\mathcal{G}$-torsor

$$M^{\mathrm{loc}} \to [L^+\mathcal{G}\backslash M^{\mathrm{loc}}] = Hk_{\mu,K}^{\mathrm{loc}},$$

and $\overleftarrow{t}$ is given by the $L^+\mathcal{G}$-torsor

$$[L^+\mathcal{G}\backslash M^{\mathrm{loc},\infty}] \to [L^+\mathcal{G}\backslash M^{\mathrm{loc},\infty}/L^+\mathcal{G}] = [L^+\mathcal{G}\backslash M^{\mathrm{loc}}] = Hk_{\mu,K}^{\mathrm{loc}}.$$

4.2. **Moduli of $(m,1)$-restricted local $(\mathcal{G},\mu)$-Shtukas.** To construct EKOR strata, we need "truncation in level 1" of local $(\mathcal{G},\mu)$-Shtukas.

4.2.1. $(\infty,1)$-*restricted local Shtukas.* Consider the reductive 1-truncation group $L^{1-\mathrm{rdt}}\mathcal{G}$, i.e. for any $R \in \mathrm{Aff}_k^{pf}$,

$$L^{1-\mathrm{rdt}}\mathcal{G}(R) = \mathcal{G}_0^{\mathrm{rdt}}(R).$$

Let

$$L^+\mathcal{G}^{(1)-\mathrm{rdt}} := \ker(L^+\mathcal{G} \to L^{1-\mathrm{rdt}}\mathcal{G})$$

and

$$M^{\mathrm{loc},(1)-\mathrm{rdt}} \subset LG/L^+\mathcal{G}^{(1)-\mathrm{rdt}}$$

be the image of $M^{\mathrm{loc},\infty} \subset LG$ under the projection $LG \to LG/L^+\mathcal{G}^{(1)-\mathrm{rdt}}$. Therefore, the natural morphism $LG/L^+\mathcal{G}^{(1)-\mathrm{rdt}} \to LG/L^+\mathcal{G}$ induces a morphism

$$M^{\mathrm{loc},(1)-\mathrm{rdt}} \to M^{\mathrm{loc}},$$

which is a $\mathcal{G}_0^{\mathrm{rdt}}$-torsor. Motivated by Lemma 4.1.4, we consider the prestack

$$\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)} = \Big[\frac{M^{\mathrm{loc},(1)-\mathrm{rdt}}}{Ad_\sigma L^+\mathcal{G}}\Big].$$

Let $B\mathcal{G}_0^{\mathrm{rdt}}$ be the stack of $\mathcal{G}_0^{\mathrm{rdt}}$-torsors. To describe $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)}$, let us first note that there are morphisms

$$(\overleftarrow{t}^{1-\mathrm{rdt}}, \overrightarrow{t}) : Hk_{\mu,K}^{\mathrm{loc}} \to B\mathcal{G}_0^{\mathrm{rdt}} \times BL^+\mathcal{G},$$

where $\overrightarrow{t}$ is given by the $L^+\mathcal{G}$-torsor

$$M^{\mathrm{loc}} \to [L^+\mathcal{G}\backslash M^{\mathrm{loc}}] = Hk_{\mu,K}^{\mathrm{loc}},$$

and $\overleftarrow{t}^{1-\mathrm{rdt}}$ is given by the $\mathcal{G}_0^{\mathrm{rdt}}$-torsor

$$[L^+\mathcal{G}\backslash M^{\mathrm{loc},(1)-\mathrm{rdt}}] \to [L^+\mathcal{G}\backslash M^{\mathrm{loc},(1)-\mathrm{rdt}}/\mathcal{G}_0^{\mathrm{rdt}}] = [L^+\mathcal{G}\backslash M^{\mathrm{loc}}] = Hk_{\mu,K}^{\mathrm{loc}}.$$

Then we have a similar cartesian digram

$$\begin{array}{ccc}
\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)} & \longrightarrow & Hk_{\mu,K}^{\mathrm{loc}} \\
\big\downarrow & & \big\downarrow {\scriptstyle \overleftarrow{t}^{1-\mathrm{rdt}} \times res_1^\infty \circ \overrightarrow{t}} \\
B\mathcal{G}_0^{\mathrm{rdt}} & \xrightarrow{\sigma\times 1} & B\mathcal{G}_0^{\mathrm{rdt}} \times B\mathcal{G}_0^{\mathrm{rdt}},
\end{array}$$

where

$$res_1^\infty : BL^+\mathcal{G} \to B\mathcal{G}_0^{\mathrm{rdt}}$$

is the map induced by the composition of projections $L^+\mathcal{G} \to L^1G = \mathcal{G}_0 \to \mathcal{G}_0^{\mathrm{rdt}}$. In particular, $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)}$ has the following moduli interpretation. For any $R \in \mathrm{Aff}_k^{pf}$, $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)}(R)$ classifies

- a modification $\beta : \overleftarrow{\mathcal{E}} \dashrightarrow \overrightarrow{\mathcal{E}}$ of $\mathcal{G}$-torsors over $\mathrm{Spec}\, W(R)$ of type μ, i.e. an element $\beta \in Hk_{\mu,K}^{\mathrm{loc}}(R)$,
- an isomorphism $\psi : \sigma^*(\overrightarrow{\mathcal{E}})|_R^{\mathrm{rdt}} \simeq \overleftarrow{\mathcal{E}}|_R^{\mathrm{rdt}}$ of $\mathcal{G}_0^{\mathrm{rdt}}$-torsors over $\mathrm{Spec}\, R$.

Unfortunately, $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)}$ is not algebraic, since $L^+\mathcal{G}$ is infinite dimensional.

4.2.2. *$(m,1)$-restricted local Shtukas.* We need some variants of the above construction. For any integer $m \geq 1$, consider the m-truncation group $L^m\mathcal{G}$, i.e. for any $R \in \mathrm{Aff}_k^{pf}$, $L^m\mathcal{G}(R) = \mathcal{G}(W_m(R))$. We have the natural projection morphism

$$\pi_{m,1-\mathrm{rdt}} : L^m\mathcal{G} \to L^{1-\mathrm{rdt}}\mathcal{G}.$$

Before proceeding further, let us note that the action of $L^+\mathcal{G}$ on M^{loc} factors through $L^m\mathcal{G}$ for some sufficiently large integer m. Indeed, recall $M^{\mathrm{loc}} = \bigcup_{w\in \mathrm{Adm}(\{\mu\})_K} \mathrm{Gr}_w$ and for each $w \in \mathrm{Adm}(\{\mu\})_K$, we have

$$\mathrm{Gr}_w \simeq L^+\mathcal{G}/(L^+\mathcal{G} \cap n_w L^+\mathcal{G} n_w^{-1}),$$

where $n_w \in LG(k)$ are some representatives for all the $w \in \mathrm{Adm}(\{\mu\})_K$. Consider the intersection $\bigcap_{w\in \mathrm{Adm}(\{\mu\})_K} L^+\mathcal{G} \cap n_w L^+\mathcal{G} n_w^{-1}$. Let

$$L^+\mathcal{G}^{(m)} = \ker(L^+\mathcal{G} \to L^m\mathcal{G}).$$

Since the descending chain of subgroups $L^+\mathcal{G}^{(m)}$ forms a topological basis of neighborhoods of the identity, we find that for some sufficiently large integer m,

$$L^+\mathcal{G}^{(m)} \subset \bigcap_{w\in \mathrm{Adm}(\{\mu\})_K} L^+\mathcal{G} \cap n_w L^+\mathcal{G} n_w^{-1},$$

i.e. the action of $L^+\mathcal{G}$ on M^{loc} factors through $L^m\mathcal{G}$.

Let m_0 be the minimal integer satisfying the above. Take any integer $m \geq m_0 + 1$. Now we consider the Ad_σ-action of $L^m\mathcal{G}$ on $M^{\mathrm{loc},(1)-\mathrm{rdt}}$: for any $R \in \mathrm{Aff}^{pf}$, $g \in L^m\mathcal{G}(R)$ and $x \in M^{\mathrm{loc},(1)-\mathrm{rdt}}(R)$,

$$g \cdot x = gx\sigma(\pi_{m,1-\mathrm{rdt}}(g))^{-1}.$$

Definition 4.2.3. For any sufficiently large integer m, set

$$\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)} = \Big[\frac{M^{\mathrm{loc},(1)-\mathrm{rdt}}}{Ad_\sigma L^m\mathcal{G}}\Big].$$

This is an algebraic stack over k.

Let m be as above and $Hk_{\mu,K}^{\mathrm{loc}(m)} = [L^m\mathcal{G}\backslash M^{\mathrm{loc}}]$, which is called the m-restricted local Hecke stack. Similarly as before, we have morphisms

$$(\overleftarrow{t}^{1-\mathrm{rdt}}, \overrightarrow{t}) : Hk_{\mu,K}^{\mathrm{loc}(m)} \to B\mathcal{G}_0^{\mathrm{rdt}} \times BL^m\mathcal{G},$$

where $\overrightarrow{t}$ is given by the $L^m\mathcal{G}$-torsor

$$M^{\mathrm{loc}} \to [L^m\mathcal{G}\backslash M^{\mathrm{loc}}] = Hk_{\mu,K}^{\mathrm{loc}},$$

which we denote by $\overrightarrow{\mathcal{E}}|_{D_m}$, and $\overleftarrow{t}^{1-\mathrm{rdt}}$ is given by the $\mathcal{G}_0^{\mathrm{rdt}}$-torsor

$$[L^m\mathcal{G}\backslash M^{\mathrm{loc},(1)-\mathrm{rdt}}] \to [L^m\mathcal{G}\backslash M^{\mathrm{loc},(1)-\mathrm{rdt}}/\mathcal{G}_0^{\mathrm{rdt}}] = [L^m\mathcal{G}\backslash M^{\mathrm{loc}}] = Hk_{\mu,K}^{\mathrm{loc}(m)},$$

which we denote by $\overleftarrow{\mathcal{E}}|_{D_1}^{\mathrm{rdt}}$. Then we have a similar cartesian digram

$$\begin{array}{ccc}
\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)} & \longrightarrow & Hk_{\mu,K}^{\mathrm{loc}(m)} \\
\downarrow & & \downarrow {\scriptstyle \overleftarrow{t}^{1-\mathrm{rdt}} \times res_1^m \circ \overrightarrow{t}} \\
B\mathcal{G}_0^{\mathrm{rdt}} & \xrightarrow{\sigma\times 1} & B\mathcal{G}_0^{\mathrm{rdt}} \times B\mathcal{G}_0^{\mathrm{rdt}},
\end{array}$$

where $res_1^m : BL^m\mathcal{G} \to B\mathcal{G}_0^{\mathrm{rdt}}$ is the map induced by the projection $L^m\mathcal{G} \to \mathcal{G}_0^{\mathrm{rdt}}$. $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)}$ has the following moduli interpretation. For any $R \in \mathrm{Aff}_k^{pf}$, $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)}(R)$ classifies

- a point of $Hk_{\mu,K}^{\mathrm{loc}(m)}(R)$,
- an isomorphism $\psi : \sigma^*(\overrightarrow{\mathcal{E}}|_{D_m})|_R^{\mathrm{rdt}} \simeq \overleftarrow{\mathcal{E}}|_R^{\mathrm{rdt}}$ of $\mathcal{G}_0^{\mathrm{rdt}}$-torsors over $\mathrm{Spec}\, R$.

Lemma 4.2.4. *For any m sufficiently large as above, we have homeomorphisms of underlying topological spaces*

$$|\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)}| \simeq |\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)}| \simeq {}^K\mathrm{Adm}(\{\mu\}).$$

Proof. As always, set $k = \overline{\mathbb{F}}_p$. Let $\breve{K}_1$ be the pro-unipotent radical of $\breve{K}$, then by definition $L^+\mathcal{G}^{(1)-\mathrm{rdt}}(k) = \breve{K}_1$. Since $M^{\mathrm{loc},\infty}(k) = \breve{K}\mathrm{Adm}(\{\mu\})\breve{K}$, we have

$$M^{\mathrm{loc},(1)-\mathrm{rdt}}(k) = \breve{K}\mathrm{Adm}(\{\mu\})\breve{K}/\breve{K}_1.$$

Then

$$\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)}(k) = \frac{\breve{K}\mathrm{Adm}(\{\mu\})\breve{K}/\breve{K}_1}{\breve{K}_\sigma} = \frac{\breve{K}\mathrm{Adm}(\{\mu\})\breve{K}}{\breve{K}_\sigma(\breve{K}_1\times\breve{K}_1)}.$$

By Theorem 1.3.4 (works of Lusztig and He), we have

$$\breve{K}\mathrm{Adm}(\{\mu\})\breve{K}/\breve{K}_\sigma(\breve{K}_1\times\breve{K}_1) \simeq \mathrm{Adm}(\{\mu\})^K \cap {}^K\widetilde{W} = {}^K\mathrm{Adm}(\{\mu\}).$$

This gives a bijiection

$$|\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)}| \simeq {}^K\mathrm{Adm}(\{\mu\}).$$

To show it is a homeomorphism of topological spaces, we apply [24, Proposition 3.5] (see also loc. cit. page 3248, paragraph 3.5), which implies that the closure relation on $|\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)}|$ is exactly given by the partial order $\leq_{K,\sigma}$. Since for any m sufficiently large, the Ad_σ-action of $L^+\mathcal{G}$ on $M^{\mathrm{loc},(1)-\mathrm{rdt}}$ factors through $L^m\mathcal{G}$, we have the homeomorphism $|\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)}| \simeq |\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)}|$. □

Consider the 1-restricted local Hecke stack $Hk_{\mu,K}^{\mathrm{loc}(1)} = [\mathcal{G}_0\backslash M^{\mathrm{loc}}]$. Sometimes we also write it as $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(1,0)}$. The underlying topological space is

$$|[\mathcal{G}_0\backslash M^{\mathrm{loc}}]| = \mathrm{Adm}(\{\mu\})_K.$$

We have natural maps

$$\mathrm{Sht}_{\mu,K}^{\mathrm{loc}} \to \mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)} \to \mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)} \xrightarrow{\pi_K} [\mathcal{G}_0\backslash M^{\mathrm{loc}}].$$

Proposition 4.2.5. *All the arrows above are perfectly smooth.*

Proof. The first arrow is perfectly smooth, since $M^{\mathrm{loc},\infty} \to M^{\mathrm{loc},1-\mathrm{rdt}}$ is a $L^{(1)-\mathrm{rdt}}\mathcal{G}$-torsor; the morphism $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(\infty,1)} \to \mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)}$ is a $L^{(m)}\mathcal{G}$-gerbe, thus it is perfectly smooth; the morphism $\mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)} \xrightarrow{\pi_K} [\mathcal{G}_0\backslash M^{\mathrm{loc}}]$ is a composition of a $\mathcal{G}_0^{\mathrm{rdt}}$-torsor and a $\ker(L^m\mathcal{G} \to L^1\mathcal{G} = \mathcal{G}_0)$-gerbe, thus it is also perfectly smooth. □

When $K = I$ is an Iwahori subgroup, then by [31, Corollary 6.2], the morphism

$$\pi_I : \mathrm{Sht}_{\mu,I}^{\mathrm{loc}(m,1)} \to [\mathcal{G}_0\backslash M^{\mathrm{loc}}]$$

induces a homeomorphism of underlying topological spaces

$$|\mathrm{Sht}_{\mu,I}^{\mathrm{loc}(m,1)}| \simeq |[\mathcal{G}_0\backslash M^{\mathrm{loc}}]| \simeq \mathrm{Adm}(\{\mu\}).$$

The following proposition gives the links between moduli stacks of $(m,1)$-restricted local $(\mathcal{G},\mu)$-Shtukas and $\mathcal{G}_0^{\mathrm{rdt}}$-zips. For any $w \in \mathrm{Adm}(\{\mu\})_K$, recall that in subsection 3.4 we have the algebraic stack $\mathcal{G}_0^{rdt}\text{-}\mathrm{Zip}_{J_w}$ of $\mathcal{G}_0^{rdt}$-Zips of type J_w. Let $\mathcal{G}_0^{rdt}\text{-}\mathrm{Zip}_{J_w}^{pf}$ be its perfection.

Proposition 4.2.6. *For any $w \in [\mathcal{G}_0\backslash M^{\mathrm{loc}}]$, there exists a natural perfectly smooth map of algebraic stacks*

$$\pi_K^{-1}(w) \longrightarrow \mathcal{G}_0^{rdt}\text{-}Zip_{J_w}^{pf}$$

inducing a homeomorphism of the underlying topological spaces $|\pi_K^{-1}(w)| \simeq |\mathcal{G}_0^{rdt}\text{-}Zip_{J_w}^{pf}|$.

Proof. For any $w \in |[\mathcal{G}_0\backslash M^{\rm loc}]| = \mathrm{Adm}(\{\mu\})_K$, let $\widetilde{\mathrm{Gr}_w} \to \mathrm{Gr}_w$ be the $\mathcal{G}_0^{\rm rdt}$-torsor induced by $M^{\mathrm{loc},(1)-\mathrm{rdt}} \to M^{\rm loc}$ over $\mathrm{Gr}_w \subset M^{\rm loc}$. Then we have

$$\pi_K^{-1}(w) \simeq \Big[\frac{\widetilde{\mathrm{Gr}_w}}{Ad_\sigma L^m\mathcal{G}}\Big].$$

Let J_w be as above. We get parabolic subgroups P_{J_w} and $P_{\sigma'(J_w)}$ of $\mathcal{G}_0^{\rm rdt}$ with corresponding unipotent radicals U_{J_w} and $U_{\sigma'(J_w)}$ as in 1.3.6. Let L_{J_w} be the common Levi subgroup of P_{J_w} and $P_{\sigma'(J_w)}$. Then similar to [84, Lemma 5.3.6], we have a perfectly smooth morphism

$$\widetilde{\mathrm{Gr}_w} \to \Big[(\mathcal{G}_0^{\rm rdt}/U_{J_w} \times \mathcal{G}_0^{\rm rdt}/U_{\sigma'(J_w)})/L_{J_w}\Big]^{pf},$$

which intertwines the $L^m\mathcal{G}$-action on the left hand side to the $\mathcal{G}_0^{\rm rdt}$-action on the right hand side. Hence we get a perfectly smooth morphism

$$\Big[\frac{\widetilde{\mathrm{Gr}_w}}{Ad_\sigma L^m\mathcal{G}}\Big] \to \mathcal{G}_0^{\rm rdt}\backslash\Big[(\mathcal{G}_0^{\rm rdt}/U_{J_w} \times \mathcal{G}_0^{\rm rdt}/U_{\sigma'(J_w)})/L_{J_w}\Big]^{pf} = \mathcal{G}_0^{rdt}\text{-}\mathrm{Zip}_{J_w}^{pf},$$

which induces a homeomorphism between the underlying topological spaces by 1.3.6. Here the last equality comes from [66, Theorem 12.7]. □

When K is a hyperspecial subgroup, then $|[\mathcal{G}_0\backslash M^{\rm loc}]|$ consists of a single point, thus we have $\pi_K^{-1}(w) = \mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)}$, and the above proposition recovers [84, Lemma 5.3.6].

4.3. **Application to affine Deligne-Lusztig varieties.** Let $(G,[b],\{\mu\})$ be a triple as in subsection 3.6. Fix a representative $b \in G(\breve{\mathbb{Q}}_p)$ and a parahoric subgroup $K \subset G(\mathbb{Q}_p)$. Let $\mathcal{G}$ be the parahoric model over $\mathbb{Z}_p$ of G corresponding to K. We get the associated affine Deligne-Lusztig variety $X(\mu,b)_K$ as in subsection 3.6. This is a closed subscheme of $\mathrm{Gr}_\mathcal{G}$ which can be described as follows. For any $R \in \mathrm{Aff}_k^{pf}$, we have

$$\begin{aligned} X(\mu,b)_K(R) = \{(\mathcal{E},\beta) \in \mathrm{Gr}_\mathcal{G}(R) \,|\, \forall\, \text{geometric point}\, x \,\text{of}\, \mathrm{Spec}\, R,\\ Inv_x(\beta^{-1}b\sigma(\beta)) \in \mathrm{Adm}(\{\mu\})_K\}.\end{aligned}$$

Fix an element $w \in \mathrm{Adm}(\{\mu\})_K$, then the locally closed subscheme $X_w(b)_K$ introduced in 3.6 can be described similarly: for any $R \in \mathrm{Aff}_k^{pf}$, we have

$$\begin{aligned} X_w(b)_K(R) = \{(\mathcal{E},\beta) \in \mathrm{Gr}_\mathcal{G}(R) \,|\, \forall\, \text{geometric point}\, x \,\text{of}\, \mathrm{Spec}\, R,\\ Inv_x(\beta^{-1}b\sigma(\beta)) = w\}.\end{aligned}$$

The KR stratification in this setting is the following

$$X(\mu,b)_K = \coprod_{w\in \mathrm{Adm}(\{\mu\})_K} X_w(b)_K.$$

The element b defines a local $(\mathcal{G},\mu)$-Shtuka by the modification

$$b : \sigma^*\mathcal{E}_0 = \mathcal{E}_0 \dashrightarrow \mathcal{E}_0.$$

For any point $(\mathcal{E},\beta) \in X(\mu,b)_K(R)$, we get another local $(\mathcal{G},\mu)$-Shtuka by the modification

$$\beta^{-1}b\sigma(\beta) : \sigma^*\mathcal{E} \dashrightarrow \mathcal{E}.$$

In this way we get a natural morphism of prestacks

$$X(\mu,b)_K \to \mathrm{Sht}_{\mu,K}^{\rm loc}.$$

Fix an integer m which is sufficiently large as in the last subsection. Composing the above morphism with the restriction map $\mathrm{Sht}_{\mu,K}^{\rm loc} \to \mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)}$, we get a morphism of algebraic stacks

$$\upsilon_K : X(\mu,b)_K \to \mathrm{Sht}_{\mu,K}^{\mathrm{loc}(m,1)},$$

which interpolates the morphisms ζ_w in 3.6.2 when w varies. The fibers of υ_K are then the EKOR strata of $X(\mu, b)_K$. This gives a geometric meaning of the EKOR decomposition in [11] 3.4 and [12] 1.4.

4.4. **Application to Shimura varieties.** Now we come back to Shimura varieties and the setting at the beginning of section 3. Let (G, X) be a Shimura datum of Hodge type, and $\mathscr{S}_{\mathsf{K}} = \mathscr{S}_{\mathsf{K}}(G, X)$ be the integral model introduced in 2.2 with $\mathsf{K} = K_pK^p, K_p = \mathcal{G}(\mathbb{Z}_p)$ and $\mathcal{G} = \mathcal{G}^\circ$. We are interested in the special fibers $\mathscr{S}_{\mathsf{K},0}$ of $\mathscr{S}_{\mathsf{K}}$, and we consider its perfection

$$\mathrm{Sh}_{\mathsf{K}} := (\mathscr{S}_{\mathsf{K},0})^{pf} = \varprojlim_{\sigma} \mathscr{S}_{\mathsf{K},0}.$$

In the following, since K^p is fixed, we will simply write Sh_{K} as Sh_K, and the subscript of related morphisms with source Sh_K will also be written simply as K. By [76, Corollary 21.6.10] and [30, Theorem 2.15], the perfection of the special fiber of the Pappas-Zhu local model can be identified with the closed subscheme M^{loc} in subsection 4.1. As usual, we get the conjugacy class of minuscule characters $\{\mu\}$.

Proposition 4.4.1. *There exists a $\mathcal{G}$-Shtuka of type μ over Sh_K. In particular, we get a morphism of prestacks*

$$\mathrm{Sh}_K \to \mathrm{Sht}^{\mathrm{loc}}_{\mu,K}.$$

Proof. Recall that in 2.3.11 we have the display $(\underline{M}, \underline{M}_1, \underline{\Psi})$ attached to the p-divisible group $\mathcal{A}[p^\infty] \mid_{\mathscr{S}_{\mathsf{K}}^\wedge}$, and by Proposition 2.3.12 there is a tensor $\underline{s}_{\mathrm{cris}} \in \underline{M}^\otimes$. For any $R \in \mathrm{Aff}_k^{pf}$, by Corollary 2.3.14

$$\mathcal{E} := \sigma^{-1,*}\mathbf{Isom}_{W(R)}\big((L^\vee, s) \otimes W(R), (\underline{M}, \underline{s}_{\mathrm{cris}})\big)$$

is a $\mathcal{G}$-torsor over $W(R)$. Then $\sigma^*\mathcal{E}$ is also a $\mathcal{G}$-torsor over $W(R)$, and the linearization of the Frobenius on $\underline{M}$ induces a modification $\beta : \sigma^*\mathcal{E} \dashrightarrow \mathcal{E}$. The local model diagram over k implies that for any geometric point $x \in \operatorname{Spec} R$, we have $Inv_x(\beta) \in \mathrm{Adm}(\{\mu\})_K$. Thus we get a $\mathcal{G}$-Shtuka of type μ over Sh_K.

□

Now consider the moduli stack $\mathrm{Sht}^{\mathrm{loc}(m,1)}_{\mu,K}$ of $(m, 1)$-restricted local $(\mathcal{G}, \mu)$-Shtukas as in Definition 4.2.3 for the current setting.

Lemma 4.4.2. *The minimal m for the definition of the above $\mathrm{Sht}^{\mathrm{loc}(m,1)}_{\mu,K}$ is $m = 2$.*

Proof. We will prove that the left action of $L^+\mathcal{G}$ on $M^{\mathrm{loc},(1)-\mathrm{rdt}}$ factors through $L^2\mathcal{G}$, and the Ad_σ-action is similar.

For any $m \geq 2$, as before, let $L^+\mathcal{G}^{(m)} = \ker(L^+\mathcal{G} \to L^m\mathcal{G})$. We need to check that $L^+\mathcal{G}^{(2)}$ acts trivially on $M^{\mathrm{loc},(1)-\mathrm{rdt}}$. This comes from the fact the $L^+\mathcal{G}^{(1)}$ acts trivially on M^{loc}, since by construction of the Pappas-Zhu local model, the action of $L^+\mathcal{G}$ on M^{loc} factors through $\mathcal{G}_0$.

More precisely, let $k = \overline{\mathbb{F}}_p$ and $\mathfrak{g}$ be the Lie algebra of $\mathcal{G}_{W(k)}$, then $L^+\mathcal{G}^{(n)}(k) = 1+p^n\cdot\mathfrak{g}$ for all positive integer n. The action of $L^+\mathcal{G}^{(1)}$ on M^{loc} is trivial, so for any $w \in \mathrm{Adm}(\{\mu\})_K$,

$$L^+\mathcal{G}^{(1)}(k) \subset w \cdot L^+\mathcal{G}(k) \cdot w^{-1},$$

and hence $p \cdot \mathfrak{g} \subset w\mathfrak{g}w^{-1}$. In particular, we have

$$L^+\mathcal{G}^{(n+1)}(k) \subset w \cdot L^+\mathcal{G}^{(n)}(k) \cdot w^{-1}, \text{ for all positive integer } n \text{ and all } w \in \mathrm{Adm}(\{\mu\})_K.$$

We use the notation in the proof of Proposition 4.2.6. For any $w \in \mathrm{Adm}(\{\mu\})_K$,

$$\widetilde{\mathrm{Gr}_w}(k) = \breve{K}w\breve{K}/\breve{K}_1 = L^+\mathcal{G}(k)wL^+\mathcal{G}(k)/L^+\mathcal{G}^{(1)-\mathrm{rdt}}(k).$$

For any $g \in L^+\mathcal{G}^{(2)}(k), w \in \mathrm{Adm}(\{\mu\})_K$ and $g_1, g_2 \in L^+\mathcal{G}(k)$, we have

$$g \cdot g_1wg_2 = g_1wg_2 \cdot (g_1wg_2)^{-1} \cdot g \cdot g_1wg_2.$$

Noting that $L^+\mathcal{G}^{(2)} \subset L^+\mathcal{G}$ is normal, we see by the above inclusion that $g \cdot g_1 w g_2$ is of the form $g_1 w g_2 \cdot g_3$ for some $g_3 \in L^+\mathcal{G}^{(1)}(k) \subset L^+\mathcal{G}^{(1)-\mathrm{rdt}}(k)$. This proves that the usual action of $L^+\mathcal{G}^{(2)}$ on $M^{\mathrm{loc},(1)-\mathrm{rdt}}$ is trivial. □

Fix an integer $m \geq 2$. Composing the morphism $\mathrm{Sh}_K \to \mathrm{Sht}^{\mathrm{loc}}_{\mu,K}$ in Proposition 4.4.1 with $\mathrm{Sht}^{\mathrm{loc}}_{\mu,K} \to \mathrm{Sht}^{\mathrm{loc}(m,1)}_{\mu,K}$, we get a morphism of stacks

$$\upsilon_K : \mathrm{Sh}_K \to \mathrm{Sht}^{\mathrm{loc}(m,1)}_{\mu,K}.$$

Recall that by the local model diagram, we have the morphism of stacks

$$\lambda_K : \mathrm{Sh}_K \to [\mathcal{G}_0 \backslash M^{\mathrm{loc}}],$$

which is perfectly smooth.

Theorem 4.4.3. *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Sh}_K & \xrightarrow{\upsilon_K} & \mathrm{Sht}^{\mathrm{loc}(m,1)}_{\mu,K} \\ & \searrow^{\lambda_K} & \downarrow{\scriptstyle \pi_K} \\ & & [\mathcal{G}_0 \backslash M^{\mathrm{loc}}]. \end{array}$$

Moreover, υ_K is perfectly smooth.

Proof. The above commutative diagram comes from the fact that the perfection of the special fiber of the Pappas-Zhu local model can be identified with the closed subscheme M^{loc} in subsection 4.1, by [76, Corollary 21.6.10] and [30, Theorem 2.15].

Now we prove that υ_K is perfectly smooth. The arguments in the proof of [84, Proposition 7.2.4] apply here. For the reader's convenience, we recall how to adapt the arguments of loc. cit. to our situation as follows. First, similarly as in the proof of Lemma 4.1.4, let $\mathrm{Sht}^{\mathrm{loc}(m,1),\square}_{\mu,K}$ be the $L^m\mathcal{G}$-torsor over $\mathrm{Sht}^{\mathrm{loc}(m,1)}_{\mu,K}$ such that for any $R \in \mathrm{Aff}^{pf}_k$, $\mathrm{Sht}^{\mathrm{loc}(m,1),\square}_{\mu,K}(R)$ classifies the trivialization $\epsilon : \overrightarrow{\mathcal{E}}|_{D_m(R)} \simeq \mathcal{E}_{0,D_m(R)}$, where $D_m(R) = \mathrm{Spec}\, W_m(R)$. Then standard arguments show that

$$\mathrm{Sht}^{\mathrm{loc}(m,1),\square}_{\mu,K} \simeq M^{\mathrm{loc},(1)-\mathrm{rdt}}$$

such that the $L^m\mathcal{G}$-action on the left hand side corresponds to the $Ad_\sigma L^m\mathcal{G}$-action on the right hand side. On the other hand, we can consider the $L^m\mathcal{G}$-torsor $\mathrm{Sh}^{(m,1)\square}_K$ over Sh_K such that for any $R \in \mathrm{Aff}^{pf}_k$, $\mathrm{Sh}^{(m,1)\square}_K(R)$ classifies the trivialization $\epsilon : \overrightarrow{\mathcal{E}}|_{D_m(R)} \simeq \mathcal{E}_{0,D_m(R)}$. Clearly we have

$$\mathrm{Sh}^{(m,1)\square}_K = \mathrm{Sh}_K \times_{\mathrm{Sht}^{\mathrm{loc}(m,1)}_{\mu,K}} \mathrm{Sht}^{\mathrm{loc}(m,1),\square}_{\mu,K}.$$

Let $\pi(m,1) : \mathrm{Sh}^{(m,1)\square}_K \to \mathrm{Sh}_K$ be the projection, which is a $L^m\mathcal{G}$-torsor. Consider the composition map

$$q(m,1) : \mathrm{Sh}^{(m,1)\square}_K \to \mathrm{Sht}^{\mathrm{loc}(m,1),\square}_{\mu,K} \simeq M^{\mathrm{loc},(1)-\mathrm{rdt}}.$$

Then the map $\upsilon_K : \mathrm{Sh}_K \to \mathrm{Sht}^{\mathrm{loc}(m,1)}_{\mu,K}$ is equivalent to the following diagram

$$\begin{array}{ccccc} & & \mathrm{Sh}^{(m,1)\square}_K & & \\ & \swarrow^{\pi(m,1)} & & \searrow^{q(m,1)} & \\ \mathrm{Sh}_K & & & & M^{\mathrm{loc},(1)-\mathrm{rdt}}, \end{array}$$

which is $L^m\mathcal{G}$-equivariant for the actions of $L^m\mathcal{G}$ on $\mathrm{Sh}^{(m,1)\square}_K$ and $M^{\mathrm{loc},(1)-\mathrm{rdt}}$ as above. To prove υ_K is perfectly smooth, we need only to show $q(m,1)$ is perfectly smooth.

Let $x^\square$ be a k-point of $\mathrm{Sh}_K^{(m,1)\square}$ with image $x \in \mathrm{Sh}_K$. Then we can find an étale neighborhood $a : U \to \mathrm{Sh}_K$ of x such that the pullback of the $L^m\mathcal{G}$-torsor $\overrightarrow{\mathcal{E}}|_{D_m}$ to U is trivial. Fix such a trivialization $\epsilon : \overrightarrow{\mathcal{E}}|_{D_m} \simeq \mathcal{E}_{0,D_m}$, which is equivalent to a lifting $a^{(m)} : U \to \mathrm{Sh}_K^{(m,1)\square}$ of a. Then the map

$$a^\square : U \times L^m\mathcal{G} \to \mathrm{Sh}_K^{(m,1)\square}, \quad (u,g) \mapsto (a(u), g\epsilon)$$

is étale and gives an étale neighborhood of $x^\square$. It suffices to show the composition

$$a^\square(m,1) = q(m,1) \circ a^\square : U \times L^m\mathcal{G} \to \mathrm{Sh}_K^{(m,1)\square} \to M^{\mathrm{loc},(1)-\mathrm{rdt}}$$

is perfectly smooth. We can make the above map more explicit. Consider the composition

$$a(m,1) = q(m,1) \circ a^{(m)} : U \to \mathrm{Sh}_K^{(m,1)\square} \to M^{\mathrm{loc},(1)-\mathrm{rdt}}.$$

Then for any $(u,g) \in U \times L^m\mathcal{G}$, we have

$$a^\square(m,1)(u,g) = ga(m,1)(u)\pi_{m,1-\mathrm{rdt}}(\sigma(g)^{-1}).$$

The perfection of the local model diagram over k gives us

$$\begin{array}{ccc} & \mathrm{Sh}_K^{(1,0)\square} & \\ {}^{\pi(1,0)}\swarrow & & \searrow^{q(1,0)} \\ \mathrm{Sh}_K & & M^{\mathrm{loc}}, \end{array}$$

with $\mathrm{Sh}_K^{(1,0)\square} = \widetilde{\mathrm{Sh}_K} := (\widetilde{\mathscr{S}_{K,0}})^{pf}, \pi(1,0) = \pi^{pf}$, and $q(1,0) = q^{pf}$. The morphism $a^{(m)} : U \to \mathrm{Sh}_K^{(m,1)\square}$ naturally induces a morphism $a^{(1)} : U \to \mathrm{Sh}_K^{(1,0)\square}$. The local model diagram implies that the composition

$$U \to \mathrm{Sh}_K^{(1,0)\square} \to M^{\mathrm{loc}}$$

is étale. Now we have the following commutative diagram

$$\begin{array}{ccc} U \times L^m\mathcal{G} & \xrightarrow{a^\square(m,1)} & M^{\mathrm{loc},(1)-\mathrm{rdt}} \\ \downarrow & & \downarrow \\ U & \longrightarrow & M^{\mathrm{loc}}, \end{array}$$

where $U \times L^m\mathcal{G} \to U$ is the natural projection. Since the bottom line is étale, to show $a^\square(m,1)$ is perfectly smooth, it suffices to show the induced map

$$\widetilde{a^\square(m,1)} : U \times L^m\mathcal{G} \to U \times_{M^{\mathrm{loc}}} M^{\mathrm{loc},(1)-\mathrm{rdt}}$$

is perfectly smooth. Note that the left hand side is a trivial $L^m\mathcal{G}$-torsor over U, the right hand side is a $\mathcal{G}_0^{\mathrm{rdt}}$-torsor over U, and the above map is morphism over U.

Let $L_p^m\mathcal{G}$ be the usual Greenberg transform of $\mathcal{G} \otimes \mathbb{Z}/p^m\mathbb{Z}$, then $L^m\mathcal{G}$ is the perfection of $L_p^m\mathcal{G}$. The morphism $a : U \to M^{\mathrm{loc}}$ descends to an étale morphism $a' : U' \to M^{\mathrm{loc}}$. Let $U'^{(1)}$ be the trivial $\mathcal{G}_0^{\mathrm{rdt}}$-torsor over U'. Then $(U'^{(1)})^{pf} \simeq U \times_{M^{\mathrm{loc}}} M^{\mathrm{loc},(1)-\mathrm{rdt}}$, and the above morphism $\widetilde{a^\square(m,1)}$ is the perfection of

$$f : U' \times L_p^m\mathcal{G} \to U'^{(1)}, \quad (u,g) \mapsto ga(m,1)(u)\pi_{m,1-\mathrm{rdt}}(\sigma(g)^{-1}),$$

which is a morphism over U'. It suffices to show this morphism is smooth over U'. Noting that $L_p^m\mathcal{G}$ is smooth and that $U'^{(1)}$ is smooth over U', we only need to show that for each $x \in U'(k)$, the induced map on fibers $f_x : L_p^m\mathcal{G} \to U_x'^{(1)}$ is smooth. On the level of tangent spaces, we can ignore the action of $\sigma(g)$ and the infinitesimal action induced by f_x is identified with the projection $L_p^m\mathcal{G} \to \mathcal{G}_0^{\mathrm{rdt}}$. So f_x and hence f is smooth.

□

Consider the morphism of stacks

$$\upsilon_K : \mathrm{Sh}_K \to \mathrm{Sht}^{\mathrm{loc}(m,1)}_{\mu,K}.$$

By Lemma 4.2.4 $|\mathrm{Sht}^{\mathrm{loc}(m,1)}_{\mu,K}| \simeq {}^K\mathrm{Adm}(\{\mu\})$ and by Proposition 4.2.6, the fibers of υ_K are then the EKOR strata of Sh_K. Note that we have the identification of underling topological spaces

$$|\mathrm{Sh}_K| = |\mathscr{S}_{\mathsf{K},0}|.$$

Since by Theorem 4.4.3 υ_K is perfectly smooth, we get the closure relation for EKOR strata on $\mathscr{S}_{\mathsf{K},0}$, which is independent of the axioms of [31] (compare Corollary 3.5.3 (2)):

Corollary 4.4.4. *For any $x \in {}^K\mathrm{Adm}(\{\mu\})$, the Zariski closure of the EKOR stratum $\mathscr{S}^x_{\mathsf{K},0}$ is given by*

$$\overline{\mathscr{S}^x_{\mathsf{K},0}} = \coprod_{x' \leq_{K,\sigma} x} \mathscr{S}^{x'}_{\mathsf{K},0}.$$

5. EKOR strata of abelian type

In this section, we will extend the construction of the EKOR stratification to Shimura varieties of abelian type.

5.1. Some group theory.

5.1.1. Let $\alpha : G_1 \to G_2$ be a central extension between connected reductive groups over $\mathbb{Q}_p$ with kernel Z. Then α induces a canonical $G_1(\mathbb{Q}_p)$-equivariant map

$$\alpha_* : \mathcal{B}(G_1, \mathbb{Q}_p) \to \mathcal{B}(G_2, \mathbb{Q}_p).$$

Let $x \in \mathcal{B}(G_1, \mathbb{Q}_p)$ with associated group schemes $\mathcal{G}_{1,x}$ and $\mathcal{G}^\circ_{1,x}$. Set $y = \alpha_*(x)$. Then α extends to group scheme homomorphisms

$$\alpha : \mathcal{G}_{1,x} \to \mathcal{G}_{2,y}, \quad \alpha : \mathcal{G}^\circ_{1,x} \to \mathcal{G}^\circ_{2,y}.$$

Let $\mathcal{Z}$ be the schematic closure of Z in $\mathcal{G}^\circ_{1,x}$.

Proposition 5.1.2. ([37, Proposition 1.1.4]) *Suppose that G_1 splits over a tamely ramified extension of $\mathbb{Q}_p$ and that Z is either a torus or is finite of rank prime to p. Then $\mathcal{Z}$ is smooth over $\mathbb{Z}_p$ and it fits in an exact sequence*

$$1 \to \mathcal{Z} \to \mathcal{G}^\circ_{1,x} \to \mathcal{G}^\circ_{2,y} \to 1$$

of group schemes over $\mathbb{Z}_p$. If Z is a torus which is a direct summand of an induced torus, then $\mathcal{Z} = \mathcal{Z}^\circ$ is the connected Neron model of Z.

In particular, if G splits over a tamely ramified extension of $\mathbb{Q}_p$, $Z = Z_G$ is either a torus or $Z_{G^{\mathrm{der}}}$ has rank prime to p, for $x \in \mathcal{B}(G, \mathbb{Q}_p)$ with associated integral model $\mathcal{G} = \mathcal{G}^\circ$, $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}/\mathcal{Z}$ is connected, and it can be identified with the parahoric model of $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ attached to x^{ad}, i.e. $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}^{\mathrm{ad}\circ}$.

5.1.3. Let G be a connected reductive group over $\mathbb{Q}_p$ with a parahoric model $\mathcal{G}$ over $\mathbb{Z}_p$. Let $\{\mu\}$ be the conjugacy class of a cocharacter $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$. As in 1.2, we get the associated $\{\mu\}$-admissible set $\mathrm{Adm}(\{\mu\}) \subset \widetilde{W}$. Let $K = \mathcal{G}(\mathbb{Z}_p)$. We have also the sets ${}^K\mathrm{Adm}(\{\mu\}), \mathrm{Adm}(\{\mu\})_K$ together with the surjection ${}^K\mathrm{Adm}(\{\mu\}) \twoheadrightarrow \mathrm{Adm}(\{\mu\})_K$.

Let $(G^{\mathrm{ad}}, \{\mu^{\mathrm{ad}}\})$ be the associated adjoint group with the induced conjugacy class of cocharacter. Let $K^{\mathrm{ad}} = \mathcal{G}^{\mathrm{ad}\circ}(\mathbb{Z}_p)$, then we have the associated sets $\mathrm{Adm}(\{\mu^{\mathrm{ad}}\})$, $\mathrm{Adm}(\{\mu^{\mathrm{ad}}\})_{K^{\mathrm{ad}}}$ and ${}^{K^{\mathrm{ad}}}\mathrm{Adm}(\{\mu^{\mathrm{ad}}\})$.

Lemma 5.1.4. *The natural map $(G, \{\mu\}, K) \to (G^{\mathrm{ad}}, \{\mu^{\mathrm{ad}}\}, K^{\mathrm{ad}})$ induces bijections*

(1) $\mathrm{Adm}(\{\mu\}) \xrightarrow{\sim} \mathrm{Adm}(\{\mu^{\mathrm{ad}}\})$,
(2) $\mathrm{Adm}(\{\mu\})_K \xrightarrow{\sim} \mathrm{Adm}(\{\mu^{\mathrm{ad}}\})_{K^{\mathrm{ad}}}$,

(3) ${}^K\mathrm{Adm}(\{\mu\}) \xrightarrow{\sim} {}^{K^{\mathrm{ad}}}\mathrm{Adm}(\{\mu^{\mathrm{ad}}\})$.

Proof. Let $\widetilde{W}$ and $\widetilde{W}^{\mathrm{ad}}$ be the Iwahori Weyl group of G and G^{ad} respectively. Then the natural map $G \to G^{\mathrm{ad}}$ induces a map $\widetilde{W} \to \widetilde{W}^{\mathrm{ad}}$, which restricts to a bijection on the affine Weyl groups $W_a(G) \xrightarrow{\sim} W_a(G^{\mathrm{ad}})$. By definition, $\mathrm{Adm}(\{\mu\}) \subset W_a(G)\tau$, $\mathrm{Adm}(\{\mu^{\mathrm{ad}}\}) \subset W_a(G^{\mathrm{ad}})\tau^{\mathrm{ad}}$, with the elements τ and τ^{ad} attached to $\{\mu\}$ and $\{\mu^{\mathrm{ad}}\}$ in 1.2.5 respectively. Since $W_0(G_{\breve{\mathbb{Q}}_p}) \simeq W_0(G^{\mathrm{ad}}_{\breve{\mathbb{Q}}_p})$, one sees that $W_a(G)\tau \xrightarrow{\sim} W_a(G^{\mathrm{ad}})\tau^{\mathrm{ad}}$ restricts to a bijection

$$\mathrm{Adm}(\{\mu\}) \xrightarrow{\sim} \mathrm{Adm}(\{\mu^{\mathrm{ad}}\}).$$

On the other hand, the natural map $K \to K^{\mathrm{ad}}$ induces a bijection on finite Weyl groups $W_K \xrightarrow{\sim} W_{K^{\mathrm{ad}}}$. Thus we get a bijection

$$W_K\backslash W_K\mathrm{Adm}(\{\mu\})W_K/W_K \xrightarrow{\sim} W_{K^{\mathrm{ad}}}\backslash W_{K^{\mathrm{ad}}}\mathrm{Adm}(\{\mu^{\mathrm{ad}}\})W_{K^{\mathrm{ad}}}/W_{K^{\mathrm{ad}}}.$$

The maps $\widetilde{W} \to \widetilde{W}^{\mathrm{ad}}$ and $W_K \xrightarrow{\sim} W_{K^{\mathrm{ad}}}$ induce a map ${}^K\widetilde{W} \to {}^{K^{\mathrm{ad}}}\widetilde{W}$. Thus we get a map

$$\mathrm{Adm}(\{\mu\}) \cap {}^K\widetilde{W} \to \mathrm{Adm}(\{\mu^{\mathrm{ad}}\}) \cap {}^{K^{\mathrm{ad}}}\widetilde{W},$$

which is a bijection since the two sides admit surjections to $\mathrm{Adm}(\{\mu\})_K$ and $\mathrm{Adm}(\{\mu\})_{K^{\mathrm{ad}}}$ respectively, but we have just shown $\mathrm{Adm}(\{\mu\})_K \simeq \mathrm{Adm}(\{\mu\})_{K^{\mathrm{ad}}}$, and the above map induces bijections between the fibers on both hand sides. We can conclude since ${}^K\mathrm{Adm}(\{\mu\}) = \mathrm{Adm}(\{\mu\}) \cap {}^K\widetilde{W}$ and similarly ${}^{K^{\mathrm{ad}}}\mathrm{Adm}(\{\mu^{\mathrm{ad}}\}) = \mathrm{Adm}(\{\mu^{\mathrm{ad}}\}) \cap {}^{K^{\mathrm{ad}}}\widetilde{W}$ by Theorem 1.2.2. □

5.2. **The adjoint group action on KR strata.** In this and the next subsection, let (G, X) be a Shimura datum of Hodge type such that G splits over a tamely ramified extension of $\mathbb{Q}_p$ and the center $Z = Z_G$ is a torus.

5.2.1. Fix a Siegel embedding

$$i : (G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$$

as in subsection 2.2. Fix a point $x \in \mathcal{B}(G, \mathbb{Q}_p)$. We write $\mathcal{G} = \mathcal{G}_x$ for the Bruhat-Tits group scheme attached to x. We assume $\mathcal{G} = \mathcal{G}^\circ$. Then $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}^{\mathrm{ad}\circ}$ and $G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}} = G^{\mathrm{ad}\circ}_{\mathbb{Z}_{(p)}}$ (cf. 5.1.1). Let $K_p := \mathcal{G}(\mathbb{Z}_p)$, for $K^p \subset G(\mathbb{A}_f^p)$ small enough, we set $\mathsf{K} := K_pK^p$. Let $K'_p \subset \mathrm{GSp}(\mathbb{Q}_p)$ be the stabilizer of the lattice $V_{\mathbb{Z}_p}$ as in 2.2. For K^p above, we can find an open compact subgroup $K'^p \subset \mathrm{GSp}(\mathbb{A}_f^p)$ and set $\mathsf{K}' = K'_pK'^p$, such that we have a morphism of schemes over O_E

$$\mathscr{S}_{\mathsf{K}}(G, X) \to \mathscr{S}_{\mathsf{K}'}(\mathrm{GSp}, S^{\pm})_{O_E},$$

which induces a closed embedding $\mathrm{Sh}_{\mathsf{K}}(G, X) \hookrightarrow \mathrm{Sh}_{\mathsf{K}'}(\mathrm{GSp}, S^{\pm})_E$ on generic fibers. Taking the limit over K^p, we get a morphism $\mathscr{S}_{K_p}(G, X) \to \mathscr{S}_{K'_p}(\mathrm{GSp}, S^{\pm})_{O_E}$.

By [37, Lemma 4.5.9], we have the following diagram

$$\begin{array}{ccc} & \widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p}(G, X) & \\ \swarrow^{\pi} & & \searrow^{q} \\ \mathscr{S}_{K_p}(G, X) & & \mathrm{M}^{\mathrm{loc}}_{G,X}, \end{array}$$

where π is a $\mathcal{G}^{\mathrm{ad}}$-torsor, and q is $\mathcal{G}^{\mathrm{ad}}$-equivariant. The $\mathcal{G}^{\mathrm{ad}}$-torsor $\widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p}(G, X)$ is induced from the $\mathcal{G}$-torsor $\widetilde{\mathscr{S}}_{K_p}(G, X)$ in Theorem 2.2.3 by the natural map $\mathcal{G} \to \mathcal{G}^{\mathrm{ad}}$. For any sufficiently small K^p, the induced map $\widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p}(G, X)/K^p \to \mathrm{M}^{\mathrm{loc}}_{G,X}$ is smooth of relative dimension $\dim G^{\mathrm{ad}}$.

5.2.2. By [37] subsections 4.4 and 4.5, we can describe the action of $G^{\mathrm{ad}\circ}(\mathbb{Z}_{(p)})^+ = G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ on $\mathscr{S}_{K_p}(G,X)$ as follows. Let $\gamma \in G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ and $\mathcal{P}$ the fibre of $G_{\mathbb{Z}_{(p)}} \to G^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$ over γ. Then $\mathcal{P}$ is a $Z_{G_{\mathbb{Z}_{(p)}}}$-torsor. The element γ induces a morphism on generic fibers

$$\gamma : \mathrm{Sh}_{K_p}(G,X) \to \mathrm{Sh}_{K_p}(G,X).$$

Let T be a O_E-scheme and $x \in \mathscr{S}_{K_p}(G,X)(T)$. As in [37] 4.5.1, we get a triple

$$(\mathcal{A}_x, \lambda_x, \varepsilon^p_x)$$

by the morphism $\mathscr{S}_{K_p}(G,X) \to \mathscr{S}_{K'_p}(\mathrm{GSp}, S^\pm)_{O_E}$, where $\mathcal{A}_x$ is an abelian scheme over T up to $\mathbb{Z}_{(p)}$-isogeny, equipped with a weak $\mathbb{Z}_{(p)}$-polarization λ_x, and

$$\varepsilon^p_x \in \varprojlim_{K^p} \Gamma(T, \mathrm{Isom}(V_{\mathbb{A}^p_f}, \widehat{V}^p(\mathcal{A}_x)_{\mathbb{Q}})).$$

Then by [37] Lemmas 4.4.6, 4.4.8 and 4.5.4, we get another triple $(\mathcal{A}^{\mathcal{P}}_x, \lambda^{\mathcal{P}}_x, \varepsilon^{p,\mathcal{P}}_x)$, and by loc. cit. Lemma 4.5.7, the assignment

$$(\mathcal{A}_x, \lambda_x, \varepsilon^p_x) \mapsto (\mathcal{A}^{\mathcal{P}}_x, \lambda^{\mathcal{P}}_x, \varepsilon^{p,\mathcal{P}}_x)$$

induces a map

$$\gamma : \mathscr{S}_{K_p}(G,X) \to \mathscr{S}_{K_p}(G,X)$$

whose generic fibre agrees with the map induced by conjugation by γ.

Combining the $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action with the natural action of $G(\mathbb{A}^p_f)$ on $\mathscr{S}_{K_p}(G,X)$ induces an action of $\mathscr{A}(G_{\mathbb{Z}_{(p)}})$ (cf. 2.4.2) on $\mathscr{S}_{K_p}(G,X)$.

Now, following [37, Lemma 4.5.9], we explain how to lift the $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\mathscr{S}_{K_p}(G,X)$ to an action on $\widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p}(G,X)$. Fix a Galois extension $F|\mathbb{Q}$ such that $\mathcal{P}$ admits an $O_{F,(p)} = O_F \otimes \mathbb{Z}_{(p)}$-point $\tilde{\gamma}$. We have an isomorphism of abelian schemes

$$\alpha_{\tilde{\gamma}} : \mathcal{A}^{\mathcal{P}}_x \otimes O_F \to \mathcal{A}_x \otimes O_F,$$

which is O_F-linear for the natural O_F-actions on both side. Passing to the de Rham cohomology, we get an O_F-linear isomorphism

$$\alpha^{-1}_{\tilde{\gamma}} : H^1_{\mathrm{dR}}(\mathcal{A}_x/T) \otimes O_F \xrightarrow{\sim} H^1_{\mathrm{dR}}(\mathcal{A}^{\mathcal{P}}_x/T) \otimes O_F.$$

Let $(x,f) \in \widetilde{\mathscr{S}}_{K_p}(G,X)(T)$ be a point which lifts $x \in \mathscr{S}_{K_p}(G,X)(T)$. Then

$$f : V^\vee_{\mathbb{Z}_p} \otimes O_T \to H^1_{\mathrm{dR}}(\mathcal{A}_x/T)$$

is an isomorphism with $f^\otimes(s_\alpha) = s_{\alpha,\mathrm{dR}}$. The composition

$$\alpha^{-1}_{\tilde{\gamma}} \circ (f \otimes 1) : V^\vee_{\mathbb{Z}_p} \otimes O_T \otimes O_F \xrightarrow{f\otimes 1} H^1_{\mathrm{dR}}(\mathcal{A}_x/T) \otimes O_F \xrightarrow{\alpha^{-1}_{\tilde{\gamma}}} H^1_{\mathrm{dR}}(\mathcal{A}^{\mathcal{P}}_x/T) \otimes O_F$$

induces a well defined element in $\widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p}(G,X)(T)$ (cf. the proof of [37, Lemma 4.5.9]), which depends only on the image of (x,f) in $\widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p}(G,X)(T)$ and on γ.

Proposition 5.2.3. *Let $\mathscr{S}_0$ be the special fiber of $\mathscr{S}_{K_p}(G,X)$ over $k = \overline{\mathbb{F}}_p$ and $K = K_p$. Consider the KR stratification $\mathscr{S}_0 = \coprod_{w \in \mathrm{Adm}(\{\mu\})_K} \mathscr{S}^w_0$. Then each KR stratum $\mathscr{S}^w_0$ is stable under the action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ on $\mathscr{S}_0$.*

Proof. Let $\widetilde{\mathscr{S}}^{\mathrm{ad}}_0$ and M_0 be the special fibers of $\widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p}(G,X)$ and $\mathrm{M}^{\mathrm{loc}}_{G,X}$ respectively. Consider the following diagram

$$\mathscr{S}_0 \xleftarrow{\ \pi\ } \widetilde{\mathscr{S}}^{\mathrm{ad}}_0 \xrightarrow{\ q\ } \mathrm{M}_0$$

induced from that in 5.2.1 on the special fibers. By the above, the $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\mathscr{S}_0$ can be lifted to an action on $\widetilde{\mathscr{S}}_0^{\mathrm{ad}}$. Let $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ act trivially on M_0. Then π and q in the above diagram are $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ -equivariant. Therefore each KR stratum $\mathscr{S}_0^w$ is stable under the action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ on $\mathscr{S}_0$. □

5.3. **The adjoint group action on Zips.** As before, let $\mathscr{S}_0$ be the special fiber of $\mathscr{S}_{K_p}(G,X)$ over k. We change our notations slightly and write $\tilde{\mathbb{I}} = \widetilde{\mathscr{S}_0}$ and $\tilde{\mathbb{I}}^{\mathrm{ad}} = \widetilde{\mathscr{S}}_0^{\mathrm{ad}}$. Consider the following diagram

$$\begin{array}{ccc} & \widetilde{\mathbb{I}}^{\mathrm{ad}} & \\ \swarrow & & \searrow \\ \mathscr{S}_0 & & \mathrm{M}_0 \end{array}$$

introduced in 3.4.1 and in the proof of Proposition 5.2.3. As $\mathcal{G}_0^{\mathrm{ad}}$-torsors over $\mathscr{S}_0$, we have $\tilde{\mathbb{I}}^{\mathrm{ad}} = \widetilde{\mathbb{I}} \times^{\mathcal{G}_0} \mathcal{G}_0/Z_k$.

We write $K = K_p$. Fix an element

$$w \in \mathrm{Adm}(\{\mu\})_K = W_K\backslash W_K\mathrm{Adm}(\{\mu\})W_K/W_K = [\mathcal{G}_0\backslash M_0](k),$$

and consider the associated KR stratum $\mathscr{S}_0^w \subset \mathscr{S}_0$. Let

$$\tilde{\mathbb{I}}^w \to \mathscr{S}_0^w, \quad \text{and} \quad \tilde{\mathbb{I}}^{\mathrm{ad},w} \to \mathscr{S}_0^w$$

be the pullbacks of $\tilde{\mathbb{I}} \to \mathscr{S}_0$ and $\tilde{\mathbb{I}}^{\mathrm{ad}} \to \mathscr{S}_0$ respectively under the inclusion of the KR stratum $\mathscr{S}_0^w \subset \mathscr{S}_0$. As in 3.4, we get the map

$$\tilde{\mathbb{I}}_+^w \to \mathscr{S}_0^w,$$

which is a $\mathcal{G}_{0,w}$-torsor. Set $\mathcal{G}_{0,w}^{\mathrm{ad}} = \mathcal{G}_{0,w}/Z_k$ and

$$\tilde{\mathbb{I}}_+^{\mathrm{ad},w} = \tilde{\mathbb{I}}_+^w \times^{\mathcal{G}_{0,w}} \mathcal{G}_{0,w}^{\mathrm{ad}}.$$

Then $\tilde{\mathbb{I}}_+^{\mathrm{ad},w} \to \mathscr{S}_0^w$ is a $\mathcal{G}_{0,w}^{\mathrm{ad}}$-torsor .

By Proposition 5.2.3 we have a $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\mathscr{S}_0^w$. We get an induced $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\tilde{\mathbb{I}}^{\mathrm{ad},w}$, which is by definition $f \mapsto \alpha_{\tilde{\gamma}}^{-1} \circ (f \otimes 1)$, for $f \in \tilde{\mathbb{I}}^{\mathrm{ad},w}$.

Similarly, we can consider the conjugate local model diagram on special fibers (cf. 3.4.1)

$$\begin{array}{ccc} & \widetilde{\mathbb{I}}^{\mathrm{ad}} & \\ \swarrow & & \searrow \\ \mathscr{S}_0 & & \mathrm{M}_0^c, \end{array}$$

which is $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-equivariant (where $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ acts trivially on M_0^c). Similarly as above, we get

$$\tilde{\mathbb{I}}_-^w \to \mathscr{S}_0^w, \quad \text{and} \quad \tilde{\mathbb{I}}_-^{\mathrm{ad},w} \to \mathscr{S}_0^w.$$

Consider the universal abelian scheme $\mathcal{A} \to \mathscr{S}_0^w$ and its de Rham cohomology $H^1_{\mathrm{dR}}(\mathcal{A}/\mathscr{S}_0^w)$. The F-V-module structure on $H^1_{\mathrm{dR}}(\mathcal{A}/\mathscr{S}_0^w)$ induces an isomorphism

$$\widetilde{\iota} : \widetilde{\mathbb{I}}_+^{w,(p)}/\mathcal{G}_{0,w}^{U,(p)} \to \widetilde{\mathbb{I}}_-^w/\mathcal{G}_{0,\sigma(w)^{-1}}^U,$$

which is equivariant with respective to the isomorphism

$$\mathcal{G}_{0,w}/\mathcal{G}_{0,w}^{U,(p)} \xrightarrow{\sim} \mathcal{G}_{0,\sigma(w)^{-1}}/\mathcal{G}_{0,\sigma(w)^{-1}}^U,$$

where $\mathcal{G}_{0,w}^U$ and $\mathcal{G}_{0,\sigma(w)^{-1}}^U$ are the groups defined in Proposition 3.3.4 (3). Passing to $\mathcal{G}_0^{\mathrm{rdt}}$ and $\mathcal{G}_0^{\mathrm{rdt,ad}}$ we get a $\mathcal{G}_0^{\mathrm{rdt}}$-zip of type J_w over $\mathscr{S}_0^w$ (cf. Corollary 3.4.10):

$$(\mathbb{I}^w, \mathbb{I}_+^w, \mathbb{I}_-^w, \iota),$$

and a $\mathcal{G}_0^{\mathrm{rdt,ad}}$-zip of type J_w over $\mathscr{S}_0^w$:

$$(\mathbb{I}^{\mathrm{ad},w}, \mathbb{I}_+^{\mathrm{ad},w}, \mathbb{I}_-^{\mathrm{ad},w}, \iota).$$

In particular, as in the proof of Theorem 3.4.11, we get a diagram

$$\begin{array}{ccc} \mathbb{E}^{\mathrm{ad},w} & \longrightarrow & \mathcal{G}_0^{\mathrm{rdt,ad}} \\ \downarrow & & \downarrow \\ \mathscr{S}_0^w & \longrightarrow & [E_{\mathcal{Z}_w^{\mathrm{ad}}} \backslash \mathcal{G}_0^{\mathrm{rdt,ad}}], \end{array}$$

with $\mathbb{E}^{\mathrm{ad},w} \to \mathscr{S}_0^w$ an $E_{\mathcal{Z}_w^{\mathrm{ad}}}$-torsor, where $\mathcal{Z}_w^{\mathrm{ad}}$ is the algebraic zip datum induced from $\mathcal{Z}_w$ by $\mathcal{G}_0^{\mathrm{rdt}} \to \mathcal{G}_0^{\mathrm{rdt,ad}}$. By Proposition 5.2.3, $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ acts on $\mathscr{S}_0^w$.

Proposition 5.3.1. *The $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\mathscr{S}_0^w$ lifts to an action on $\mathbb{E}^{\mathrm{ad},w}$. Moreover, the above diagram is $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-equivariant, where $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ acts trivially on $\mathcal{G}_0^{\mathrm{rdt,ad}}$.*

Proof. Notations as in Lemma 3.4.9, we work with the tuple $(\widetilde{\mathbb{I}}^w, \widetilde{\mathbb{I}}_+^w, \widetilde{\mathbb{I}}_-^w, \widetilde{\iota})$. Let $\widetilde{\mathbb{E}}^w$ be formed by the following cartesian diagram.

$$\begin{array}{ccccccc} \widetilde{\mathbb{E}}^w & \longrightarrow & & & & & \widetilde{\mathbb{I}}_-^w \\ \downarrow & & & & & & \downarrow \\ \widetilde{\mathbb{I}}_+^w & \longrightarrow & \widetilde{\mathbb{I}}_+^{w,(p)} & \longrightarrow & \widetilde{\mathbb{I}}_+^{w,(p)}/\mathcal{G}_{0,w}^{U,(p)} & \xrightarrow{\widetilde{\iota}} & \widetilde{\mathbb{I}}_-^w/\mathcal{G}_{0,\sigma(w)^{-1}}^U \end{array}$$

There is a morphism $\widetilde{\mathbb{E}}^w \to \mathcal{G}_0$ induced by $(f_+, f_-) \mapsto g$ where g is the unique element in $\mathcal{G}_0$ mapping f_+ to f_-. Passing to quotients by Z_k, we have $(\widetilde{\mathbb{I}}^{\mathrm{ad},w}, \widetilde{\mathbb{I}}_+^{\mathrm{ad},w}, \widetilde{\mathbb{I}}_-^{\mathrm{ad},w}, \widetilde{\iota}^{\mathrm{ad}})$, and hence $\widetilde{\mathbb{E}}^{\mathrm{ad},w}$ with a morphism to $\mathcal{G}_0^{\mathrm{ad}} := \mathcal{G}_0/Z_k$.

It suffices to show that the $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\mathscr{S}_0^w$ lifts to an action on $\widetilde{\mathbb{E}}^{\mathrm{ad},w}$, and the above morphism is $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-equivariant, where $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ acts trivially on $\mathcal{G}_0^{\mathrm{ad}}$.

For $\gamma \in G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$, T a scheme over $\mathscr{S}_0^w$ and $(x, f) \in \widetilde{\mathbb{I}}^w(T)$ with $x \in \mathscr{S}_0^w(T)$, notations as in 5.2.2, we have an isomorphism of abelian schemes

$$\alpha_{\tilde{\gamma}} : \mathcal{A}_x^{\mathcal{P}} \otimes O_F \to \mathcal{A}_x \otimes O_F,$$

which is O_F-linear for the natural O_F-actions on both sides. Passing to the de Rham cohomology, we get an O_F-linear isomorphism respecting Hodge-Tate tensors

$$\alpha_{\tilde{\gamma}}^{-1} : H^1_{\mathrm{dR}}(\mathcal{A}_x/T) \otimes O_F \xrightarrow{\sim} H^1_{\mathrm{dR}}(\mathcal{A}_x^{\mathcal{P}}/T) \otimes O_F.$$

Moreover, $\big(\gamma(x), \alpha_{\tilde{\gamma}}^{-1} \circ (f \otimes 1)\big) \in \widetilde{\mathbb{I}}^w(T \otimes O_F)$ descends to an element in $\widetilde{\mathbb{I}}^{\mathrm{ad},w}(T)$ which depends only on the image of (x, f) in $\widetilde{\mathbb{I}}^{\mathrm{ad},w}(T)$ and on γ.

The isomorphism $\alpha_{\tilde{\gamma}}^{-1}$ is induced by an isomorphism of abelian schemes, and hence commutes with Frobenius and Verschiebung. So, if $(x, f) \in \widetilde{\mathbb{I}}_+^w(T)$, then

$$\big(\gamma(x), \alpha_{\tilde{\gamma}}^{-1} \circ (f \otimes 1)\big) \in \widetilde{\mathbb{I}}^w(T \otimes O_F)$$

descends to an element in $\widetilde{\mathbb{I}}_+^{\mathrm{ad},w}(T)$. The same holds if we change $+$ to $-$. Moreover, as $\widetilde{\iota}$ is induced by Frobenius and Verschiebung, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{I}}_+^{w,\mathrm{ad},(p)}/\mathcal{G}_{0,w}^{U,(p)} & \xrightarrow{\widetilde{\iota}^{\mathrm{ad}}} & \widetilde{\mathbb{I}}_-^{\mathrm{ad},w}/\mathcal{G}_{0,\sigma(w)^{-1}}^U \\ \downarrow{\scriptstyle \alpha_{\tilde{\gamma}}^{-1}} & & \downarrow{\scriptstyle \alpha_{\tilde{\gamma}}^{-1}} \\ \widetilde{\mathbb{I}}_+^{w,\mathrm{ad},(p)}/\mathcal{G}_{0,w}^{U,(p)} & \xrightarrow{\widetilde{\iota}^{\mathrm{ad}}} & \widetilde{\mathbb{I}}_-^{\mathrm{ad},w}/\mathcal{G}_{0,\sigma(w)^{-1}}^U, \end{array}$$

and hence a $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\widetilde{\mathbb{E}}^{\mathrm{ad},w}$. Noting that

$$\widetilde{\mathbb{E}}^{w}(T) \ni (x, f_+, f_-) \mapsto (\gamma(x),\ \alpha_{\tilde{\gamma}}^{-1} \circ (f_+ \otimes 1),\ \alpha_{\tilde{\gamma}}^{-1} \circ (f_- \otimes 1))$$

via γ, they have the same image in $\mathcal{G}_0^{\mathrm{ad}}$, as it is the unique $g \in \mathcal{G}_0^{\mathrm{ad}}(T)$ such that $f_+ \circ g = f_- \circ g$ in $\widetilde{\mathbb{I}}^{\mathrm{ad}}$.

□

5.4. **EKOR strata of abelian type.** We return to the setting of subsection 2.4. Let (G, X) be a Shimura datum of abelian type such that

- either $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$,
- or G is unramified over $\mathbb{Q}_p$ and K_p is contained in some hyperspecial subgroup of $G(\mathbb{Q}_p)$.

We take an associated Hodge type datum (G_1, X_1) as in Theorem 2.4.6 (3) or (4) according to the above cases. We will apply the constructions in the last two subsections to (G_1, X_1).

5.4.1. Let $x \in \mathcal{B}(G, \mathbb{Q}_p)$ and $x_1 \in \mathcal{B}(G_1, \mathbb{Q}_p)$ such that $x^{\mathrm{ad}} = x_1^{\mathrm{ad}} \in \mathcal{B}(G^{\mathrm{ad}}, \mathbb{Q}_p)$. We denote the model of G (resp. G_1) defined as the stabilizer of x (resp. x_1) by $\mathcal{G}$ (resp. $\mathcal{G}_1$), with connected model $\mathcal{G}^\circ$ (resp. $\mathcal{G}_1^\circ$). As in the proof of [37, Theorem 4.6.23], we can and we do choose (G_1, X_1) and x_1 such that Z_{G_1} is a torus and $\mathcal{G}_1 = \mathcal{G}_1^\circ$. We have group schemes $G_{\mathbb{Z}_{(p)}}$, $G^\circ_{\mathbb{Z}_{(p)}}$ and $G_{1,\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ corresponding to $\mathcal{G}$, $\mathcal{G}^\circ$ and $\mathcal{G}_1$ respectively. Write $K_p = \mathcal{G}(\mathbb{Z}_p), K_p^\circ = \mathcal{G}^\circ(\mathbb{Z}_p)$ and $K_{1,p} = \mathcal{G}_1(\mathbb{Z}_p)$. By the discussion in 5.1.1, we have

$$G^{\mathrm{ad}\circ}_{\mathbb{Z}_{(p)}} = G^{\mathrm{ad}}_{1,\mathbb{Z}_{(p)}}$$

as group schemes over $\mathbb{Z}_{(p)}$. In particular, we have $G^{\mathrm{ad}\circ}(\mathbb{Z}_{(p)})^+ = G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$, and

$$\mathcal{G}_0^{\circ,\mathrm{rdt},\mathrm{ad}} = \mathcal{G}_0^{\mathrm{ad}\circ,\mathrm{rdt}} = \mathcal{G}_{1,0}^{\mathrm{ad},\mathrm{rdt}} = \mathcal{G}_{1,0}^{\mathrm{rdt},\mathrm{ad}}$$

as reductive adjoint groups over k.

Let $\mathscr{S}_{K_p^\circ,0}$ (resp. $\mathscr{S}_{K_{1,p},0}$) be the special fiber of $\mathscr{S}_{K_p^\circ}(G, X)$ (resp. $\mathscr{S}_{K_{1,p}}(G_1, X_1)$) over $k = \overline{\mathbb{F}}_p$. Recall that we have (cf. 2.4.3)

$$\mathscr{S}_{K_p^\circ,0} = \left[[\mathscr{S}^+_{K_{1,p},0} \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ\right]^{|J|},$$

and the following diagram (cf. Theorem 2.4.6 (3))

$$\begin{array}{ccc} & \widetilde{\mathscr{S}}^{\mathrm{ad}}_{K_p^\circ,0} & \\ {}^{\pi}\swarrow & & \searrow^{q} \\ \mathscr{S}_{K_p^\circ,0} & & \mathrm{M}^{\mathrm{loc}}_{G_1,X_1,0}, \end{array}$$

where π is a $\mathcal{G}_0^{\mathrm{ad}\circ} = \mathcal{G}_{1,0}^{\mathrm{ad}}$-torsor and q is $\mathcal{G}_0^{\mathrm{ad}\circ}$-equivariant.

Corollary 5.4.2. *We have the KR stratification* $\mathscr{S}_{K_p^\circ,0} = \coprod_{w \in \mathrm{Adm}(\{\mu\})_K} \mathscr{S}^w_{K_p^\circ,0}$, *such that for each w, the stratum $\mathscr{S}^w_{K_p^\circ,0}$ is non empty, smooth, equi-dimensional of dimension* $\dim \mathscr{S}^w_{K_p^\circ,0} = \ell({}^K w_K)$.

Proof. Since $\mathrm{M}^{\mathrm{loc}}_{G_1,X_1,0}$ only depends on the associated adjoint datum, and the $\mathcal{G}_{1,0}^{\mathrm{ad}}$-orbits on $\mathrm{M}^{\mathrm{loc}}_{G_1,X_1,0}$ are parametrized by $\mathrm{Adm}(\{\mu_1^{\mathrm{ad}}\})_{K_{1,p}^{\mathrm{ad}}} = \mathrm{Adm}(\{\mu\})_K$, the above diagram gives a KR stratification $\mathscr{S}_{K_p^\circ,0} = \coprod_{w \in \mathrm{Adm}(\{\mu\})_K} \mathscr{S}^w_{K_p^\circ,0}$, with each stratum $\mathscr{S}^w_{K_p^\circ,0}$ smooth, of equi-dimension with $\dim \mathscr{S}^w_{K_p^\circ,0} = \ell({}^K w_K)$.

Moreover, for $w \in \mathrm{Adm}(\{\mu\})_K = \mathrm{Adm}(\{\mu_1^{\mathrm{ad}}\})_{K_{1,p}^{\mathrm{ad}}}$, we can make the link between the KR strata $\mathscr{S}^w_{K_p^\circ,0}$ and $\mathscr{S}^w_{K_{1,p},0}$. Let $\mathscr{S}^{w,+}_{K_{1,p},0} \subset \mathscr{S}^+_{K_{1,p},0}$ be the pullback of $\mathscr{S}^w_{K_{1,p},0}$

to the connected component $\mathscr{S}^{+}_{K_{1,p},0} \subset \mathscr{S}_{K_{1,p},0}$. By Proposition 5.2.3, the $G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\mathscr{S}^{+}_{K_{1,p},0}$ stabilizes $\mathscr{S}^{w,+}_{K_{1,p},0}$, we get an extended action of $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$ on it. The construction in 2.4.3 gives us

$$\mathscr{S}^{w}_{K_p^\circ,0} = \Big[[\mathscr{S}^{w,+}_{K_{1,p},0} \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ\Big]^{|J|}.$$

We get the non emptiness since each $\mathscr{S}^{w,+}_{K_{1,p},0}$ is non empty. ☐

5.4.3. Consider the diagram

$$\begin{array}{ccc} \mathbb{E}^{\mathrm{ad},w}_{K_{1,p}} & \longrightarrow & \mathcal{G}^{\mathrm{rdt,ad}}_{1,0} \\ \big\downarrow & & \\ \mathscr{S}^{w}_{K_{1,p},0} & & \end{array}$$

for the KR stratum $\mathscr{S}^{w}_{K_{1,p},0}$ as in the paragraph above Proposition 5.3.1 (see also the proof of Theorem 3.4.11). Here $\mathbb{E}^{\mathrm{ad},w}_{K_{1,p}} \to \mathscr{S}^{w}_{K_{1,p},0}$ is an $E_{\mathcal{Z}^{\mathrm{ad}}_{1,w}}$-torsor, with $\mathcal{Z}^{\mathrm{ad}}_{1,w}$ the algebraic zip datum induced from $\mathcal{Z}_{1,w}$ by $\mathcal{G}^{\mathrm{rdt}}_{1,0} \to \mathcal{G}^{\mathrm{rdt,ad}}_{1,0}$. We get

$$\mathbb{E}^{\mathrm{ad},w,+}_{K_{1,p}} \to \mathscr{S}^{w,+}_{K_{1,p},0}$$

by pulling back $\mathbb{E}^{\mathrm{ad},w}_{K_{1,p}} \to \mathscr{S}^{w}_{K_{1,p},0}$ along the inclusion $\mathscr{S}^{w,+}_{K_{1,p},0} \subset \mathscr{S}^{w}_{K_{1,p},0}$. Since the $G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action stabilizes $\mathscr{S}^{w,+}_{K_{1,p},0}$, by Proposition 5.3.1, we have a lift of this action to $\mathbb{E}^{\mathrm{ad},w,+}_{K_{1,p}}$. Then we get an induced action of $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$ on $\mathbb{E}^{\mathrm{ad},w,+}_{K_{1,p}}$. Set

$$\mathbb{E}^{\mathrm{ad},w}_{K_p^\circ} = \Big[[\mathbb{E}^{\mathrm{ad},w,+}_{K_{1,p}} \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ\Big]^{|J|}.$$

By (the proof of) the above Corollary 5.4.2, we get a diagram

$$\begin{array}{ccc} \mathbb{E}^{\mathrm{ad},w}_{K_p^\circ} & \longrightarrow & \mathcal{G}^{\mathrm{rdt,ad}}_{1,0} \\ \big\downarrow & & \\ \mathscr{S}^{w}_{K_p^\circ,0}. & & \end{array}$$

Here $\mathbb{E}^{\mathrm{ad},w}_{K_p^\circ} \to \mathscr{S}^{w}_{K_p^\circ,0}$ is an $E_{\mathcal{Z}^{\mathrm{ad}}_{w}}$-torsor, with $\mathcal{Z}^{\mathrm{ad}}_{w}$ the algebraic zip datum induced from $\mathcal{Z}_w$ by $G_1 \to G_1^{\mathrm{ad}}$. We have $\mathcal{G}^{\circ,\mathrm{rdt,ad}}_{0} = \mathcal{G}^{\mathrm{rdt,ad}}_{1,0}$, $\mathcal{Z}^{\mathrm{ad}}_{w} = \mathcal{Z}^{\mathrm{ad}}_{1,w}$ and thus $E_{\mathcal{Z}^{\mathrm{ad}}_{w}} = E_{\mathcal{Z}^{\mathrm{ad}}_{1,w}}$. In particular, as in 3.4 we have a morphism of stacks

$$\zeta_w : \mathscr{S}^{w}_{K_p^\circ,0} \to [E_{\mathcal{Z}^{\mathrm{ad}}_{w}} \backslash \mathcal{G}^{\mathrm{rdt,ad}}_{1,0}],$$

from which we can define an EO stratification to get the EKOR strata in $\mathscr{S}^{w}_{K_p^\circ,0}$. Letting $w \in \mathrm{Adm}(\{\mu\})_K$ vary, we get the EKOR stratification on $\mathscr{S}_{K_p^\circ,0}$.

5.4.4. By Lemma 5.1.4 we can identify the sets $\mathrm{Adm}(\{\mu\})_K = \mathrm{Adm}(\{\mu_1^{\mathrm{ad}}\})_{K^{\mathrm{ad}}_{1,p}}$ and ${}^K\mathrm{Adm}(\{\mu\}) = {}^{K^\circ_{1,p}}\mathrm{Adm}(\{\mu_1\})$. For $w \in \mathrm{Adm}(\{\mu\})_K$, we have the following diagram

$$\begin{array}{ccccc} \mathscr{S}^{w,+}_{K_{1,p},0} & \longrightarrow & \mathscr{S}^{w}_{K_{1,p},0} & \longrightarrow & [E_{\mathcal{Z}_{1,w}} \backslash \mathcal{G}^{\mathrm{rdt}}_{1,0}] \\ \big\downarrow & & & & \big\downarrow \\ \mathscr{S}^{w,+}_{K_p^\circ,0} & \longrightarrow & \mathscr{S}^{w}_{K_p^\circ,0} & \longrightarrow & [E_{\mathcal{Z}^{\mathrm{ad}}_{w}} \backslash \mathcal{G}^{\mathrm{rdt,ad}}_{1,0}], \end{array}$$

where $\mathscr{S}^{w,+}_{K_{1,p},0} \to \mathscr{S}^{w,+}_{K_p^\circ,0}$ is a pro-étale cover and

$$\mathscr{S}^{w,+}_{K_p^\circ,0} = \mathscr{S}^{w,+}_{K_{1,p},0}/\Delta,$$

with

$$\Delta = \ker(\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ \to \mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ).$$

The map $[E_{\mathcal{Z}_{1,w}}\backslash \mathcal{G}^{\mathrm{rdt}}_{1,0}] \to [E_{\mathcal{Z}^{\mathrm{ad}}_w}\backslash \mathcal{G}^{\mathrm{rdt,ad}}_{1,0}]$ is a homeomorphism. For $x \in {}^K\mathrm{Adm}(\{\mu\})$, we have also a pro-étale cover

$$\mathscr{S}^{x,+}_{K_{1,p},0} \to \mathscr{S}^{x,+}_{K_p^\circ,0}, \quad \mathscr{S}^{x,+}_{K_p^\circ,0} = \mathscr{S}^{x,+}_{K_{1,p},0}/\Delta,$$

and

$$\mathscr{S}^{x}_{K_p^\circ,0} = \Big[[\mathscr{S}^{x,+}_{K_{1,p},0} \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ\Big]^{|J|}.$$

Therefore, we can deduce results for EKOR strata of abelian type from those in the Hodge type case. We summarize the results as follows.

Theorem 5.4.5. *Let (G, X) be a Shimura datum of abelian type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$. Let $\mathsf{K} = K_p^\circ K^p$ and $\mathscr{S}_0 = \mathscr{S}_{\mathsf{K},0}$.*

(1) *We have the EKOR stratification*

$$\mathscr{S}_0 = \coprod_{x\in {}^K\mathrm{Adm}(\{\mu\})} \mathscr{S}_0^x,$$

where for each $x \in {}^K\mathrm{Adm}(\{\mu\})$, the stratum $\mathscr{S}_0^x$ is a non-empty, locally closed smooth subscheme of $\mathscr{S}_0$, which is equi-dimensional of dimension $\ell(x)$. Moreover, we have the closure relation

$$\overline{\mathscr{S}_0^x} = \coprod_{x'\leq_{K,\sigma} x} \mathscr{S}_0^{x'}.$$

(2) *Every KR stratum in $\mathscr{S}_{I,0}$ is quasi-affine. If in addition axiom 4 (c) is satisfied, every EKOR in $\mathscr{S}_{K,0}$ is quasi-affine.*

(3) *For $x \in {}^K\mathrm{Adm}(\{\mu\})$ viewed as an element of* $\mathrm{Adm}(\{\mu\})$, *the morphism*

$$\pi^x_{I,K} : \mathscr{S}^x_{I,0} \to \mathscr{S}^x_{K,0}$$

induced by $\pi_{I,K}$ is finite étale. If in addition axiom 4 (c) is satisfied, $\pi^x_{I,K}$ is a finite étale covering.

We believe that our global constructions in section 4 can be generalized to the abelian type case, except we should work with the associated $\mathcal{G}^{\mathrm{ad}\circ}$-torsors instead, as [37] subsection 4.6. In case of good reductions, see [50] and [78] for some related constructions. We leave the general case to the interested readers.

5.5. **Relations with central leaves and Newton strata.** We continue the assumptions and notations of the last subsection.

5.5.1. Recall that in subsection 3.1 we have defined

$$\Upsilon_{K_{1,p}} : \mathscr{S}_{K_{1,p}}(k) \to C(\mathcal{G}_1, \{\mu_1\})$$

and

$$\delta_{K_{1,p}} : \mathscr{S}_{K_{1,p}}(k) \to B(G_1, \{\mu_1\}).$$

The fibers of $\Upsilon_{K_{1,p}}$ and $\delta_{K_{1,p}}$ are called central leaves and Newton strata respectively, both of which are the sets of k-valued points of some locally closed reduced subschemes of $\mathscr{S}_{K_{1,p},0}$.

By the method of [78] sections 3 and 4, we can define central leaves and Newton strata for the abelian type case $\mathscr{S}_{K_p^\circ,0}$. More precisely, consider the composition of $\Upsilon_{K_{1,p}}$ with the natural map

$$C(\mathcal{G}_1, \{\mu_1\}) \to C(\mathcal{G}_1^{\mathrm{ad}}, \{\mu_1^{\mathrm{ad}}\}).$$

We call the fibers of

$$\mathscr{S}_{K_{1,p}}(k) \to C(\mathcal{G}_1^{\mathrm{ad}}, \{\mu_1^{\mathrm{ad}}\})$$

adjoint central leaves, which are finite (set theoretically) disjoint unions of the central leaves defined above, by the following lemma. Therefore, we may consider them as locally closed reduced subschemes of $\mathscr{S}_{K_{1,p},0}$, which we will also call adjoint central leaves.

Lemma 5.5.2. *We write simply $G = G_1$, $\mathcal{G} = \mathcal{G}_1$ and $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}_1^{\mathrm{ad}}$. The natural map $C(\mathcal{G}) \to C(\mathcal{G}^{\mathrm{ad}})$ is finite to one. If $Z_{\mathbb{Z}_p}$, the closure of Z_G in $\mathcal{G}$, has connected fibers, then it is bijective.*

Proof. Recall that we always assume that $Z_{\mathbb{Z}_{(p)}}$ is smooth. Let $h \in G(\breve{\mathbb{Q}}_p)$ with image $h^{\mathrm{ad}} \in G^{\mathrm{ad}}(\breve{\mathbb{Q}}_p)$. Denote by $[h]$ and $[h^{\mathrm{ad}}]$ the associated classes in $C(\mathcal{G})$ and $C(\mathcal{G}^{\mathrm{ad}})$. Up to σ-$\breve{K}$-conjugacy, we can assume that $h = gw$ with $g \in \breve{K}$ and $w \in \widetilde{W}$. Then $h^{\mathrm{ad}} = g^{\mathrm{ad}} w^{\mathrm{ad}}$. The preimage of $[h^{\mathrm{ad}}]$ under the map $C(\mathcal{G}) \to C(\mathcal{G}^{\mathrm{ad}})$ is the set

$$\{[zh] | \, z \in Z_{\mathbb{Z}_p}(\breve{\mathbb{Z}}_p)\}.$$

If $Z_{\mathbb{Z}_p}$ has connected special fiber, for any $m \geq 1$, consider the level-m Greenberg transformation Z_m of $Z_{\mathbb{Z}_p}$, which is a connected smooth scheme over $\mathbb{F}_p$. Noting that Z_m is connected, the morphism

$$\phi_m : Z_m \to Z_m, \quad x \mapsto x^{-1}\sigma(x)$$

is a finite étale cover. Let $\phi'_m : Z'_m \to Z_{m+1}$ be the pullback of $\phi_m : Z_m \to Z_m$ under the natural projection $Z_{m+1} \to Z_m$. Since the following diagram

$$\begin{array}{ccc} Z_{m+1} & \longrightarrow & Z_m \\ {\scriptstyle \phi_{m+1}}\downarrow & & \downarrow{\scriptstyle \phi_m} \\ Z_{m+1} & \longrightarrow & Z_m \end{array}$$

is commutative, we get a morphism $\phi_{m+1,m} : Z_{m+1} \to Z'_m$ such that $\phi_{m+1} = \phi'_m \circ \phi_{m+1,m}$. The morphisms ϕ_{m+1} and ϕ'_m are both finite étale covers, thus so is $\phi_{m+1,m}$. We deduce that for any $z \in Z_{\mathbb{Z}_p}(\breve{\mathbb{Z}}_p)$, there exists $x \in Z_{\mathbb{Z}_p}(\breve{\mathbb{Z}}_p)$ such that $z = x^{-1}\sigma(x)$. Therefore the set $\{[zh] | \, z \in Z_{\mathbb{Z}_p}(\breve{\mathbb{Z}}_p)\}$ consists of one class. On the other hand, by Steinberg's theorem $G(\breve{\mathbb{Q}}_p) \to G^{\mathrm{ad}}(\breve{\mathbb{Q}}_p)$ is surjective if $Z_{\mathbb{Z}_p}$ has connected generic fiber, and $\breve{K} \to \breve{K}^{\mathrm{ad}}$ is surjective since $Z_{\mathbb{Z}_p}$ is smooth. We conclude that if $Z_{\mathbb{Z}_p}$ has connected fibers, $C(\mathcal{G}) \to C(\mathcal{G}^{\mathrm{ad}})$ is bijective.

In general, let $Z^{\circ}_{\mathbb{Z}_p} \subset Z_{\mathbb{Z}_p}$ be the neutral connected component. Then $Z^{\circ}_{\mathbb{Z}_p}(\breve{\mathbb{Z}}_p)$ is of finite index in $Z_{\mathbb{Z}_p}(\breve{\mathbb{Z}}_p)$. From the above paragraph we deduce that $C(\mathcal{G}) \to C(\mathcal{G}^{\mathrm{ad}})$ is finite to one. □

5.5.3. *Adjoint central leaves and Newton strata of abelian type.* Consider the map

$$\mathscr{S}_{K_{1,p}}(k) \to C(\mathcal{G}_1^{\mathrm{ad}}, \{\mu_1^{\mathrm{ad}}\})$$

as above. For each $c \in C(\mathcal{G}_1^{\mathrm{ad}}, \{\mu_1^{\mathrm{ad}}\})$, let $\mathscr{S}^c_{K_{1,p},0}$ be the fiber of the above map at c, considered as a subscheme of $\mathscr{S}_{K_{1,p},0}$. Consider the connected component $\mathscr{S}^+_{K_{1,p},0} \subset \mathscr{S}_{K_{1,p},0}$. Let

$$\mathscr{S}^{c,+}_{K_{1,p},0} \subset \mathscr{S}^c_{K_{1,p},0}$$

be the pullback of the adjoint leaf $\mathscr{S}^c_{K_{1,p},0}$ under the inclusion $\mathscr{S}^+_{K_{1,p},0} \subset \mathscr{S}_{K_{1,p},0}$.

Proposition 5.5.4. *The $G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\mathscr{S}^+_{K_{1,p},0}$ stabilizes $\mathscr{S}^{c,+}_{K_{1,p},0}$.*

Proof. For $x \in \mathscr{S}_{K_{1,p},0}(k)$, we denote by D_x the Dieudonné module of the p-divisible group at x, and I_x the *set* of trivializations respecting the crystalline tensors. In particular, I_x is a $\breve{K}_1$-torsor. Let $\xi : V^\vee_{\breve{\mathbb{Z}}_p} \to V^{\vee,\sigma}_{\breve{\mathbb{Z}}_p}$ be the the isomorphism given by $v \otimes k \mapsto v \otimes 1 \otimes k$, then the assignment

$$t \mapsto t^*(\varphi_x^{\mathrm{lin}}) := (V^\vee_{\breve{\mathbb{Z}}_p} \xrightarrow{\xi} V^{\vee,(\sigma)}_{\breve{\mathbb{Z}}_p} \xrightarrow{t^{(\sigma)}} D_x^{(\sigma)} \xrightarrow{\varphi_x^{\mathrm{lin}}} D_x \xrightarrow{t^{-1}} V^\vee_{\breve{\mathbb{Z}}_p})$$

induces a $\breve{K}_1$-equivariant map $I_x \to \breve{G}_1$. Here φ_x^{lin} is the linearization of φ_x, and the $\breve{K}_1$-action on $\breve{G}_1$ is via σ-conjugacy.

When x varies, we have a diagram

$$\begin{array}{ccc} \mathcal{S} := \coprod_{x \in \mathscr{S}_{K_{1,p},0}(k)} I_x & \xrightarrow{\widetilde{\Upsilon}} & \breve{G}_1 \\ \downarrow & & \\ \mathscr{S}_{K_{1,p},0}(k) & & \end{array}$$

which, after passing to adjoint, induces

$$\begin{array}{ccc} \mathcal{S}^{\mathrm{ad}} := \coprod_{x \in \mathscr{S}_{K_{1,p},0}(k)} I_x^{\mathrm{ad}} & \xrightarrow{\widetilde{\Upsilon}^{\mathrm{ad}}} & \breve{G}_1^{\mathrm{ad}} \\ \downarrow & & \\ \mathscr{S}_{K_{1,p},0}(k). & & \end{array}$$

We claim that the $G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\mathscr{S}_{K_{1,p},0}(k)$ lifts to $\mathcal{S}^{\mathrm{ad}}$, and $\widetilde{\Upsilon}^{\mathrm{ad}}$ is $G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-equivariant. Here $\breve{G}_1^{\mathrm{ad}}$ is equipped with the trivial $G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action.

Notations as in 5.2.2, we have an isomorphism of abelian schemes

$$\alpha_{\tilde{\gamma}} : \mathcal{A}_x^{\mathcal{P}} \otimes O_F \to \mathcal{A}_x \otimes O_F,$$

which is O_F-linear for the natural O_F-actions on both sides. Passing to Dieudonné modules, we get an O_F-linear isomorphism respecting Hodge-Tate tensors

$$\alpha_{\tilde{\gamma}}^{-1} : D_x \otimes O_F \xrightarrow{\sim} D_{\gamma(x)} \otimes O_F.$$

Noting that $\alpha_{\tilde{\gamma}}^{-1}$ commutes with Frobenius, we have a commutative diagram

$$\begin{array}{ccccccccc} V^\vee_{\breve{\mathbb{Z}}_p} \otimes O_F & \xrightarrow{\xi} & V^{\vee,(\sigma)}_{\breve{\mathbb{Z}}_p} \otimes O_F & \xrightarrow{t^{(\sigma)}} & D_x^{(\sigma)} \otimes O_F & \xrightarrow{\varphi_x^{\mathrm{lin}}} & D_x \otimes O_F & \xrightarrow{t^{-1}} & V^\vee_{\breve{\mathbb{Z}}_p} \otimes O_F \\ & & & & \downarrow{\scriptstyle \alpha_{\tilde{\gamma}}^{-1}} & & \downarrow{\scriptstyle \alpha_{\tilde{\gamma}}^{-1}} & & \\ & & & & D_x^{(\sigma)} \otimes O_F & \xrightarrow{\varphi_{\gamma(x)}^{\mathrm{lin}}} & D_x \otimes O_F. & & \end{array}$$

The images in $\breve{G}_1^{\mathrm{ad}}$ of t and $\gamma(t)$ are precisely the images of the two compositions of isomorphisms from left to right, and hence coincide. This proves the claim, and the proposition follows formally as before. $\square$

Therefore, we can extend the $G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action to an action of $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$ on $\mathscr{S}^{c,+}_{K_{1,p},0}$. Recall that $\mathcal{G}^{\mathrm{ad}\circ} = \mathcal{G}_1^{\mathrm{ad}}$, $\{\mu^{\mathrm{ad}}\} = \{\mu_1^{\mathrm{ad}}\}$ and thus $C(\mathcal{G}^{\mathrm{ad}\circ}, \{\mu^{\mathrm{ad}}\}) = C(\mathcal{G}_1^{\mathrm{ad}}, \{\mu_1^{\mathrm{ad}}\})$. For any $c \in C(\mathcal{G}^{\mathrm{ad}\circ}, \{\mu^{\mathrm{ad}}\})$, we define the associated adjoint central leaf of abelian type

$$\mathscr{S}^c_{K_p^\circ,0} = \Big[[\mathscr{S}^{c,+}_{K_{1,p},0} \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ\Big]^{|J|},$$

with

$$\mathscr{S}^{c,+}_{K_p^\circ,0} = \mathscr{S}^{c,+}_{K_{1,p},0}/\Delta,$$

where as before $\Delta = \ker(\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ \to \mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ)$. In particular, we have a map

$$\mathscr{S}_{K_p^\circ}(k) \to C(\mathcal{G}^{\mathrm{ad}\circ}, \{\mu^{\mathrm{ad}}\}),$$

which is surjective.

Since $B(G, \{\mu\}) \simeq B(G^{\mathrm{ad}}, \{\mu^{\mathrm{ad}}\}) \simeq B(G_1, \{\mu_1\})$ (cf. [43] 6.5.1), for each $b \in B(G, \{\mu\})$ (which we identify with an element of $B(G_1, \{\mu_1\})$), we consider the associated Newton strata of Hodge type on the connected component $\mathscr{S}^{b,+}_{K_{1,p}^\circ,0} \subset \mathscr{S}^{+}_{K_{1,p}^\circ,0}$. By the above Proposition 5.5.4, the $G_1^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\mathscr{S}^{+}_{K_{1,p}^\circ,0}$ stabilizes $\mathscr{S}^{b,+}_{K_{1,p}^\circ,0}$. We can then extend it to an action of $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$ on $\mathscr{S}^{b,+}_{K_{1,p}^\circ,0}$, and define similarly Newton strata of abelian type

$$\mathscr{S}^{b}_{K_p^\circ,0} = \left[[\mathscr{S}^{b,+}_{K_{1,p},0} \times \mathscr{A}(G_{\mathbb{Z}_{(p)}})]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ\right]^{|J|},$$

with

$$\mathscr{S}^{b,+}_{K_p^\circ,0} = \mathscr{S}^{b,+}_{K_{1,p},0}/\Delta,$$

for the above Δ. We have then a map

$$\delta_K : \mathscr{S}_{K_p^\circ}(k) \to B(G, \{\mu\}),$$

which factors through the above $\mathscr{S}_{K_p^\circ}(k) \twoheadrightarrow C(\mathcal{G}^{\mathrm{ad}\circ}, \{\mu^{\mathrm{ad}}\})$ under the projection

$$C(\mathcal{G}^{\mathrm{ad}\circ}, \{\mu^{\mathrm{ad}}\}) \twoheadrightarrow B(G^{\mathrm{ad}}, \{\mu^{\mathrm{ad}}\}) \simeq B(G, \{\mu\}).$$

We get the following decomposition

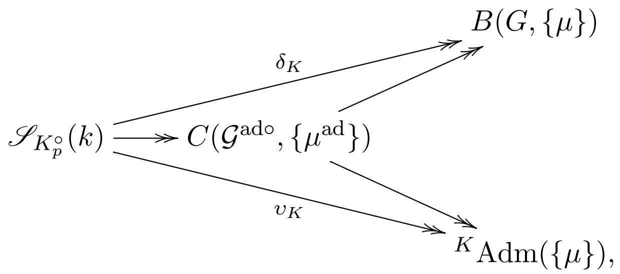
$$\mathscr{S}_{K_p^\circ,0} = \coprod_{b \in B(G,\{\mu\})} \mathscr{S}^{b}_{K_p^\circ,0},$$

which we call the Newton stratification of $\mathscr{S}_{K_p^\circ,0}$.

5.5.5. Let $K = K_p^\circ$. By Lemma 5.1.4 we have ${}^K\mathrm{Adm}(\{\mu\}) \xrightarrow{\sim} {}^{K^{\mathrm{ad}}}\mathrm{Adm}(\{\mu^{\mathrm{ad}}\})$. The map

$$\upsilon_K : \mathscr{S}_{K_p^\circ}(k) \to {}^K\mathrm{Adm}(\{\mu\})$$

factors through $C(\mathcal{G}^{\mathrm{ad}\circ}, \{\mu^{\mathrm{ad}}\})$. We get the following commutative diagram



and the composition

$$\lambda_K : \mathscr{S}_{K_p^\circ}(k) \to {}^K\mathrm{Adm}(\{\mu\}) \twoheadrightarrow \mathrm{Adm}(\{\mu\})_K$$

gives the KR stratification. Recall the subset of σ-straight elements

$${}^K\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}} = \mathrm{Adm}(\{\mu\})_{\sigma\text{-str}} \cap {}^K\widetilde{W} \subset {}^K\mathrm{Adm}(\{\mu\})$$

was introduced in 1.2.10. By Theorem 1.3.5, we have the following analogue of Corollary 3.4.14.

Corollary 5.5.6. (1) *For $x \in {}^K\mathrm{Adm}(\{\mu\})_{\sigma\text{-str}}$, $\mathscr{S}^{x}_{K_p^\circ,0}$ is an adjoint central leaf.*

(2) *For any $b \in B(G, \{\mu\})$, the Newton stratum $\mathscr{S}^{b}_{K_p^\circ,0}$ contains an EKOR stratum $\mathscr{S}^{x}_{K_p^\circ,0}$ such that x is σ-straight.*

5.5.7. *Fully Hodge-Newton decomposable Shimura varieties.* We will change our setting to include also our results in section 3. Let (G, X) be a Shimura datum of abelian type, $\mathsf{K} = K_pK^p \subset G(\mathbb{A}_f)$ an open compact subgroup with $K^p \subset G(\mathbb{A}_f^p)$ sufficently small and $K_p \subset G(\mathbb{Q}_p)$ a parahoric subgroup. Let x be a point of the Bruhat-Tits building $\mathcal{B}(G, \mathbb{Q}_p)$, with the attached Bruhat-Tits stabilizer group scheme $\mathcal{G} = \mathcal{G}_x$ and its neutral connected component $\mathcal{G}^\circ = \mathcal{G}_x^\circ$, such that $K_p = \mathcal{G}^\circ(\mathbb{Z}_p)$. We will consider the following cases:

- (G, X) is of Hodge type and $\mathcal{G} = \mathcal{G}^\circ$.
- (G, X) is of abelian type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$.
- (G, X) is of abelian type such that G is unramified over $\mathbb{Q}_p$ and K_p is contained in some hyperspecial subgroup of $G(\mathbb{Q}_p)$.

We write $\mathscr{S}_0 = \mathscr{S}_{\mathsf{K},0}$ for the special fiber of the associated Kisin-Pappas integral model.

Recall that the notion of fully Hodge-Newton decomposable pairs $(G, \{\mu\})$ is introduced in [12]. Roughly speaking, it says that for any non basic element $[b'] \in B(G, \{\mu\})$, the pair $([b'], \{\mu\})$ satisfies the Hodge-Newton condition. We refer the readers to [12] Definition 2.1 for the precise definition of a fully Hodge-Newton decomposable pair $(G, \{\mu\})$ and loc. cit. Theorem 2.5 for a complete classification of all such pairs. Under the assumption that the pair attached to the Shimura datum is fully Hodge-Newton decomposable, with our geometric constructions at hand, we have the following results (see also [12] section 6, where the results are conditional on the He-Rapoport axioms), which generalizes the corresponding results of [78] in the good reduction case.

Theorem 5.5.8. *Let the notations be as above. Assume that the attached pair $(G, \{\mu\})$ is fully Hodge-Newton decomposable. Then*

(1) *each Newton stratum of $\mathscr{S}_0$ is a union of EKOR strata;*
(2) *each EKOR stratum in a non-basic Newton stratum is an adjoint central leaf, and it is open and closed in the Newton stratum, in particular, non-basic Newton strata are smooth;*
(3) *the basic Newton stratum is a union of certain Deligne-Lusztig varieties.*

Proof. With our geometric constructions of EKOR strata at hand, the above statements (1) and (2) follow from [12, Theorem 2.3], see [78] section 6 for example (which also works in the parahoric level case).

The assertion (3) also follows from [12, Theorem 2.3], but we use the informal version (5) as in loc. cit. Theorem B and the paragraph above 4.11 there. Indeed, in the Hodge type case, we can use the uniformization morphism constructed in [95, Proposition 6.4] (see also our Proposition 3.6.4) to prove that all irreducible components of the basic Newton stratum are certain Deligne-Lusztig varieties. For the general abelian type case, one can use arguments as in [77] section 6 and [37] subsection 4.6 to deduce it from the Hodge type case. □

In fact, by [12] Theorems 2.3 and 6.4, the above assertions (1)-(3) are equivalent to each other, and any of them characterizes the condition that $(G, \{\mu\})$ is fully Hodge-Newton decomposable.

6. EKOR strata for Siegel modular varieties

In this section, we shall discuss the case of Siegel modular varieties in more details. Namely, we consider the Shimura variety attached to $(\mathrm{GSp}_{2g}, S^{\pm}, \mathsf{K})$ with $\mathsf{K} = KK^p$ and $K \subset \mathrm{GSp}_{2g}(\mathbb{Q}_p)$ a parahoric subgroup. In the case $g = 2$, we describe explicitly the geometry.

6.1. **Moduli spaces of polarized abelian varieties with parahoric level structure.** Let $g \geq 1$ be a positive integer, p a rational prime, and $N \geq 3$ an integer with $(p, N) = 1$. For a primitive Nth root of unity $\zeta_N \in \overline{\mathbb{Q}}$, let $\mathcal{A}_{g,1,\zeta_N}$ be the moduli space of g-dimensional principally polarized abelian varieties with a symplectic level-N structure over $\overline{\mathbb{F}}_p$. This is a smooth connected scheme over $\overline{\mathbb{F}}_p$. We have the following decomposition

$$\mathscr{S}_{K(N),0}(\mathrm{GSp}_{2g}, S^{\pm}) = \coprod_{\zeta_N} \mathcal{A}_{g,1,\zeta_N},$$

where $\mathscr{S}_{K(N),0}(\mathrm{GSp}_{2g}, S^{\pm})$ is the special fiber (over $\overline{\mathbb{F}}_p$) of the canonical integral model of the Shimura varieties associated to $(\mathrm{GSp}_{2g}, S^{\pm}, K(N))$ with $K(N)$ the principal level-N congruence subgroup, and ζ_N runs through the set of primitive Nth root of unity. From now on, fix a choice of primitive Nth root of unity $\zeta_N \in \overline{\mathbb{Q}} \subset \mathbb{C}$ and fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We simply write $\mathcal{A}_{g,1,N} = \mathcal{A}_{g,1,\zeta_N}$. More generally, for any integer $d \geq 0$, we can consider $\mathcal{A}_{g,p^d,N}$, the moduli space of g-dimensional abelian varieties with a degree p^d polarization and a symplectic level-N structure over $\overline{\mathbb{F}}_p$.

Let $I := \{0, 1, \dots, g\}$. Let $\mathcal{A}_I$ denote the Siegel moduli space with Iwahori level structure over $\overline{\mathbb{F}}_p$ with respect to ζ_N. It parametrizes the equivalence classes of objects

$$(A_0 \overset{\alpha}{\to} A_1 \overset{\alpha}{\to} \cdots \overset{\alpha}{\to} A_g, \lambda_0, \lambda_g, \eta),$$

where

- each A_i is a g-dimensional abelian variety,
- α is an isogeny of degree p,
- λ_0 and λ_g are principal polarizations on A_0 and A_g, respectively, such that $(\alpha^g)^*\lambda_g = p\lambda_0$.
- η is a symplectic level-N structure on A_0 with respect to ζ_N.

Put $\eta_0 := \eta$, $\eta_i := \alpha_*\eta_{i-1}$ for $i = 1, \dots, g$, and $\lambda_{i-1} := \alpha^*\lambda_i$ for $i = g, \dots, 2$. Let $\underline{A}_i := (A_i, \lambda_i, \eta_i)$. Then $\mathcal{A}_I$ parametrizes equivalence classes of objects

$$(\underline{A}_0 \overset{\alpha}{\to} \underline{A}_1 \overset{\alpha}{\to} \cdots \overset{\alpha}{\to} \underline{A}_g),$$

where $\underline{A}_0 \in \mathcal{A}_{g,1,N}$, and for $i \neq 0$,

$$\underline{A}_i \in \mathcal{A}'_{g,p^{g-i},N} := \{\underline{A} \in \mathcal{A}_{g,p^{g-i},N} \mid \ker \lambda \subset A[p]\}.$$

For any non-empty subset $J = \{i_0, \dots, i_r\} \subset I$ with $i_0 < \cdots < i_r$, let $\mathcal{A}_J$ be the moduli space over $\overline{\mathbb{F}}_p$ parameterizing equivalence classes of objects

$$(\underline{A}_{i_0} \overset{\alpha}{\to} \underline{A}_{i_1} \overset{\alpha}{\to} \cdots \overset{\alpha}{\to} \underline{A}_{i_r}),$$

where $\underline{A}_{i_0} \in \mathcal{A}_{g,1,N}$ if $i_0 = 0$, and $\underline{A}_{i_j} \in \mathcal{A}'_{g,p^{g-i_j},N}$ for others which satisfies the natural compatibility condition. The moduli space $\mathcal{A}_J$ is the Siegel moduli space (over $\overline{\mathbb{F}}_p$) with parahoric level structure of type J. For $J_2 \subset J_1$, let

$$\pi_{J_1,J_2} : \mathcal{A}_{J_1} \to \mathcal{A}_{J_2}$$

be the natural projection, which is proper and surjective.

We also write $\mathcal{A}_J$ as $\mathcal{A}_{J,\zeta_N}$ if we want to emphasize that it is relative to the choice of ζ_N. When ζ_N varies, we get a similar decomposition as above

$$\mathscr{S}_{K_J K^p(N),0}(\mathrm{GSp}_{2g}, S^{\pm}) = \coprod_{\zeta_N} \mathcal{A}_{J,\zeta_N},$$

where $K_J \subset \mathrm{GSp}_{2g}(\mathbb{Q}_p)$ is the parahoric subgroup corresponding to the lattice chains of type J, $K^p(N) \subset \mathrm{GSp}_{2g}(\mathbb{A}_f^p)$ is the principal level-N congruence subgroup outside p, and $\mathscr{S}_{K_J K^p(N),0}(\mathrm{GSp}_{2g}, S^{\pm})$ is the special fiber (over $\overline{\mathbb{F}}_p$) of the integral model of the Shimura varieties associated to $(\mathrm{GSp}_{2g}, S^{\pm}, K_J K^p(N))$ defined by the moduli problem as [71, chapter 6] (see also the following appendix).

In the following we fix our choice of ζ_N as before. We will study the geometry of $\mathcal{A}_J = \mathcal{A}_{J,\zeta_N}$.

Theorem 6.1.1.

(1) *The ordinary locus* $\mathcal{A}_J^{\mathrm{ord}} \subset \mathcal{A}_J$ *is dense.*
(2) $\mathcal{A}_J$ *is equi-dimensional of dimension* $g(g+1)/2$.
(3) $\mathcal{A}_J$ *is irreducible if* $|J| = 1$, *and for* $|J| \geq 2$, $\mathcal{A}_J$ *has* $(k_1+1)\dots(k_r+1)$ *irreducible components, where* $k_j := i_j - i_{j-1}$.
(4) $\mathcal{A}_J$ *is connected.*

Proof. For statements (1)-(3), see [9, 57, 87] and also see [90, Theorem 2.1]. (4) It suffices to show that $\mathcal{A}_I$ is connected because the map $\pi_{I,J}$ is surjective. The moduli space $\mathcal{A}_I$ is a union of 2^g irreducible components and each irreducible component is the closure of a maximal KR stratum, because every maximal KR stratum is irreducible [87]. Since the closure of every maximal KR stratum contains the the minimal KR stratum, every two irreducible components intersect. It follows that $\mathcal{A}_I$ is connected. □

6.2. **The KR and EKOR stratifications.** Let $(V = \mathbb{Q}_p^{2g}, \psi)$ a symplectic space of dimension $2g$, where the alternating pairing $\psi : V \times V \to \mathbb{Q}_p$ is represented by $\begin{bmatrix} 0 & \widetilde{I}_g \\ -\widetilde{I}_g & 0 \end{bmatrix}$ and $\widetilde{I}_g = \text{anti-diag}(1, \dots, 1)$. Let $G := \mathrm{GSp}_{2g} \subset \mathrm{GL}_{2g}$ and $T \subset G$ be the diagonal maximal torus. Let $\widetilde{W} := N_G(T)(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ be the Iwahori-Weyl group of GSp_{2g} with respect to T. The torus T is contained in the diagonal maximal torus T_0 of GL_{2g} and the cocharacter group $X_*(T)$ is contained in $X_*(T_0)$, which is equal to $\mathbb{Z}^{2g}$. Then $X_*(T)_{\mathbb{R}} = \{(u_i) \in X_*(T_0)_{\mathbb{R}} = \mathbb{R}^{2g} \mid u_1 + u_{2g} = \dots = u_g + u_{g+1}\}$. We fix a base point in the apartment corresponding to the maximal torus T and identify it with $X_*(T)_{\mathbb{R}}$. Then $\widetilde{W} = X_*(T) \rtimes W$, where W is the Weyl group of G. We fix the base alcove

$$\mathfrak{a} := \{(u_i) \in X_*(T)_{\mathbb{R}} \mid 1 + u_1 > u_{2g} > \dots > u_{g+1} > u_g\},$$

and let $s_0, \dots, s_g$ be the simple reflections with respect to the facets of $\mathfrak{a}$. The affine Weyl group W_a then is the Coxeter group generated by $s_0, \dots, s_g$ and we have $\widetilde{W} = W_a \rtimes \Omega$, where $\Omega \subset \widetilde{W}$ is the stabilizer of $\mathfrak{a}$. The length function and Bruhat order on W_a are naturally extended to those on $\widetilde{W}$.

Let $\mu = (1^{(g)}, 0^{(g)}) \in X_*(T)$ be the standard minuscule coweight. We denote by $\mathrm{Adm}(\mu) := \mathrm{Adm}(\{\mu\}) \subset \widetilde{W}$ the set of μ-admissible elements. Let τ be the unique minimal element in $\mathrm{Adm}(\mu)$. We have

$$s_i = (i, i+1)(2g+1-i, 2g-i), \quad i = 1, \dots, g-1,$$

$$s_g = (g, g+1), \quad s_0 = \big((-1, 0, \dots, 0, 1), (1, 2g)\big),$$

$$\tau = \big((0, \dots, 0, 1, \dots, 1), (1, g+1)(2, g+2)\dots(g, 2g)\big).$$

For any non-empty subset $J \subset I$, let

$$\mathcal{A}_J = \coprod_{y \in \mathrm{Adm}(\mu)_J} \mathcal{A}_{J,y}$$

be the KR stratification, where $\mathrm{Adm}(\mu)_J := W_{J^c}\backslash W_{J^c}\,\mathrm{Adm}(\mu) W_{J^c}/W_{J^c}$ is the image of $\mathrm{Adm}(\mu)$ in $W_{J^c}\backslash\widetilde{W}/W_{J^c}$ under the map $\widetilde{W} \to W_{J^c}\backslash\widetilde{W}/W_{J^c}$ and $W_{J^c} \subset \widetilde{W}$ is the subgroup generated by s_i for $i \in J^c := I - J$. Put ${}^J\mathrm{Adm}(\mu) := \mathrm{Adm}(\mu) \cap {}^{J^c}\widetilde{W}$, where ${}^{J^c}\widetilde{W} \subset \widetilde{W}$ is the set of minimal length coset representatives for $W_{J^c}\backslash\widetilde{W}$. Let

$$\mathcal{A}_J = \coprod_{x \in {}^J\mathrm{Adm}(\mu)} \mathcal{A}_J^x \tag{6.2.1}$$

be the EKOR stratification. If $J = I$, then $\mathcal{A}_I^x = \mathcal{A}_{I,x}$ and (6.2.1) is the KR stratification.

Theorem 6.2.1. *For all $x \in {}^J\mathrm{Adm}(\mu)$, one has the following statements.*

(1) *The corresponding EKOR stratum $\mathcal{A}_J^x$ is quasi-affine, smooth and equi-dimensional of dimension $\ell(x)$. Every point in $\mathcal{A}_J^x$ has the same p-rank.*

(2) *If the EKOR stratum $\mathcal{A}_J^x$ is not supersingular (not contained in the supersingular locus of $\mathcal{A}_J$), then it is irreducible.*

(3) *Every non-supersingular KR stratum $\mathcal{A}_{J,y}$, where $y \in \mathrm{Adm}(\mu)_J$, is irreducible.*

Proof. (1) The case where $J = I$ is proved in [57] and [16, Theorem 1.5]. Since $\pi_{I,J} : \mathcal{A}_{I,x} \to \mathcal{A}_J^x$ is finite, étale and surjective, the stratum $\mathcal{A}_J^x$ also share the same properties.

(2) For the case where $J = I$ the irreducibility of $\mathcal{A}_{I,x}$ is proved in [16, Theorem 1.4 and Theorem 1.5]. For arbitrary J, since $\mathcal{A}_{I,x}$ is irreducible and $\pi_{I,J} : \mathcal{A}_{I,x} \to \mathcal{A}_J^x$ is surjective, $\mathcal{A}_J^x$ is irreducible.

(3) This is [14, Proposition 4.4]. We give a different proof using EKOR strata. The KR stratum $\mathcal{A}_{J,y}$ is a union of EKOR strata, and by Theorem 3.4.12 (1), it contains a unique maximal EKOR stratum, called the y-ordinary locus, which is open and dense in the KR stratum $\mathcal{A}_{J,y}$. Since this EKOR stratum is non-supersingular, it is irreducible by (2). Therefore, the KR stratum $\mathcal{A}_{J,y}$ is irreducible. □

6.3. **Geometry of Siegel threefolds.** We restrict ourselves to the case $g = 2$. Let $K_J \subset G(\mathbb{Q}_p)$ denote the parahoric subgroup corresponding to the lattice chains of type J. The group K_J is conjugate to $K_{J^\vee}$ in $G(\mathbb{Q}_p)$, where $J^\vee := \{g - j | j \in J\}$. Thus, one can only consider the cases $J = \{0\}, \{1\}, \{0, 1\}, \{0, 2\}, \{0, 1, 2\}$, which correspond to hyperspecial, paramodular, Klingen parahoric, Siegel parahoric and Iwahori open compact subgroups, respectively.

Recall that the p-rank function is constant on each KR stratum of $\mathcal{A}_I$, so it induces a map $p\text{-rank} : \mathrm{Adm}(\mu) \to \mathbb{Z}_{\geq 0}$. For each integer $0 \leq f \leq g = 2$, set

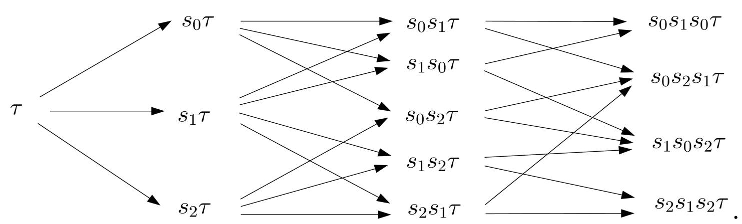
$$\mathrm{Adm}^f(\mu) := \{x \in \mathrm{Adm}(\mu) \mid p\text{-rank}(x) = f\}.$$

We denote by $\bar{x} = (\bar{x})_J$ the image of $x \in \mathrm{Adm}(\mu)$ in $\mathrm{Adm}(\mu)_J$, and

$$[x] = [x]_J := \{z \in \mathrm{Adm}(\mu) \mid \bar{z} = \bar{x}\} = W_{J^c} x W_{J^c} \cap \mathrm{Adm}(\mu).$$

By abuse of notation, we also write $\mathcal{A}_{[x]_J}$ for the KR stratum $\mathcal{A}_{J,\bar{x}}$. For each $0 \leq f \leq g = 2$, let $\mathcal{A}_J^f \subset \mathcal{A}_J$ (resp. $\mathcal{A}_J^{\leq f} \subset \mathcal{A}_J$) be the subvariety consisting of objects with p-rank f (resp. p-rank less than or equal to f). For any locally closed subvariety $X \subset \mathcal{A}_J$, denote by $\overline{X}$ the Zariski closure of X in $\mathcal{A}_J$. Let $\mathcal{S}_2 = \mathcal{S}_{\{0\}} \subset \mathcal{A}_2 = \mathcal{A}_{\{0\}}$ be the supersingular locus of $\mathcal{A}_2$. Note that for $g = 2$, the Newton strata coincide with the p-rank strata. The relationship between the Newton strata and EKOR strata can be described by that of the p-rank strata and EKOR strata.

(1) Case $J = I = \{0, 1, 2\}$ (Iwahori level). In this case the EKOR strata coincide with the KR strata. The following are the elements in the set $\mathrm{Adm}(\mu)$ together with the Bruhat order (cf. [89, Section 6] and [90, Section 4.1]):



Here we write $x \to y$ for two elements $x, y \in W_a\tau$ if $x \le y$ in the Bruhat order. Using the p-rank formula [57] we obtain

$$\begin{aligned} \mathrm{Adm}^2(\mu) &= \{s_0s_1s_0\tau,\ s_1s_0s_2\tau,\ s_2s_1s_2\tau,\ s_0s_2s_1\tau\}, \\ \mathrm{Adm}^1(\mu) &= \{s_0s_1\tau,\ s_1s_2\tau,\ s_2s_1\tau,\ s_1s_0\tau\}, \\ \mathrm{Adm}^0(\mu) &= \{\tau,\ s_1\tau,\ s_0\tau,\ s_2\tau,\ s_0s_2\tau\}. \end{aligned} \tag{6.3.1}$$

In the following we shall write $s_{j_1j_2\cdots j_r}$ for the element $s_{j_1}s_{j_2}\cdots s_{j_r}$ in the affine Weyl group W_a.

Proposition 6.3.1.

(1) *The p-rank two stratum $\mathcal{A}_I^2$ is smooth of pure dimension 3. It is a disjoint union of 4 irreducible components indexed by $s_{010}\tau$, $s_{102}\tau$, $s_{212}\tau$ and $s_{021}\tau$. It is the smooth locus of $\mathcal{A}_I$. The closure $\overline{\mathcal{A}_I^2}$ is equal to $\mathcal{A}_I$ and is connected.*

(2) *The p-rank one stratum $\mathcal{A}_I^1$ is smooth of pure dimension 2. It is a disjoint union of 4 irreducible components indexed by $s_{01}\tau$, $s_{12}\tau$, $s_{21}\tau$ and $s_{10}\tau$. The closure $\overline{\mathcal{A}_I^1}$ of $\mathcal{A}_I^1$ is connected. One has the decomposition $\mathcal{A}_I^{\le 1} = \overline{\mathcal{A}_I^1} \coprod \mathcal{A}_{I,s_{02}\tau}$. In particular, the p-rank one stratum $\mathcal{A}_I^1 \subset \mathcal{A}_I^{\le 1}$ is not dense.*

(3) *The supersingular locus $\mathcal{S}_I = \mathcal{A}_I^0 \subset \mathcal{A}_I$ consists of one-dimensional components (the closure $\overline{\mathcal{A}_{s_1\tau}}$) and two-dimensional components (the closure $\overline{\mathcal{A}_{s_{02}\tau}}$). Each connected component of $\overline{\mathcal{A}_{s_1\tau}}$ is isomorphic to $\mathbf{P}^1$, and each connected component of $\overline{\mathcal{A}_{s_{02}\tau}}$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. The supersingular locus $\mathcal{S}_I$ is connected.*

(4) *The projection morphism $\mathcal{S}_I \to \mathcal{S}_2 \subset \mathcal{A}_2$ is not finite. Therefore, the morphism $\mathcal{A}_I \to \mathcal{A}_2$ is proper but not finite.*

Proof. Statements (1), (2), (4) and the first part of (3) follow directly from Theorems 6.1.1 and 6.2.1, (6.3.1) and the Bruhat order. (3) The second part follows from the description of $\mathcal{S}_I$; see [89, Theorem 8.1]. We show the connectedness of $\mathcal{S}_I$. By [16, Theorem 7.3], the union $\mathcal{A}_{I,\le 1} := \cup_{\ell(x)\le 1}\mathcal{A}_{I,x}$ of KR strata of dimension ≤ 1 is connected. The supersingular locus $\mathcal{S}_I$ is the union of $\mathcal{A}_{I,\le 1}$ and $\mathcal{S}_{I,s_{20}\tau}$. Since each connected component of $\mathcal{A}_{I,s_{20}\tau}$ is quasi-affine, its closure in $\mathcal{A}_I$ is proper and intersects with $\mathcal{A}_{I,\le 1}$. Thus, $\mathcal{S}_I$ is connected. □

Remark 6.3.2. Proposition 6.3.1 (3) rules out the possibility of equi-dimensionality of p-rank strata and of Newton strata. Proposition 6.3.1 (2) shows that the p-rank strata (and the Newton strata) do not form a stratification on $\mathcal{A}_I$, that is, the closure of each stratum is a union of strata.

(2) Case $J = \{0\}$ (hyperspecial level) and $W_{J^c} = \langle s_1, s_2\rangle$. In this case the EKOR strata coincide with the EO strata and the whole moduli space is a KR stratum. Using $\tau s_2 = s_0\tau$ and $\tau s_1 = s_1\tau$, one easily computes the set ${}^J\mathrm{Adm}(\mu)$ with closure relation

$${}^J\mathrm{Adm}(\mu) = \{\tau, s_0\tau, s_{01}\tau, s_{010}\tau\}, \quad \tau \to s_0\tau \to s_{01}\tau \to s_{010}\tau.$$

Proposition 6.3.3. *Let $J = \{0\}$.*

(1) *There are 4 EKOR strata. The p-prank two stratum $\mathcal{A}_2^2$ is smooth and irreducible. The p-prank one stratum $\mathcal{A}_2^1$ is smooth and irreducible. The supersingular locus $\mathcal{S}_J$ is connected and each irreducible component is isomorphic to $\mathbf{P}^1$.*

(2) *We have*

$$\pi_{I,J}(\mathcal{A}_{I,x}) = \begin{cases} \mathcal{A}_J^{s_{010}\tau} & \text{for } x \in \mathrm{Adm}^2(\mu); \\ \mathcal{A}_J^{s_{01}\tau} & \text{for } x \in \mathrm{Adm}^1(\mu), \end{cases} \tag{6.3.2}$$

and

$$\pi_{I,J}(\mathcal{A}_{I,s_0\tau} \coprod \mathcal{A}_{I,s_2\tau}) = \mathcal{A}_J^{s_0\tau},$$
$$\pi_{I,J}(\mathcal{A}_{I,\tau} \coprod \mathcal{A}_{I,s_1\tau}) = \mathcal{A}_J^{\tau}, \quad (6.3.3)$$
$$\pi_{I,J}(\mathcal{A}_{I,s_{02}\tau}) = \mathcal{A}_J^{\tau} \coprod \mathcal{A}_J^{s_0\tau}.$$

Proof. (1) The connectedness of S_J follows from Proposition 6.3.1 (3). The other results are well-known. See Katsura-Oort [33] for a detailed description of $\mathcal{S}_J$.

(2) Since the map $\pi_{I,J}$ preserves p-ranks, the first relation follows. The second relation follows from the explicit description of the supersingular locus S_I; see [89, Theorem 8.1]. □

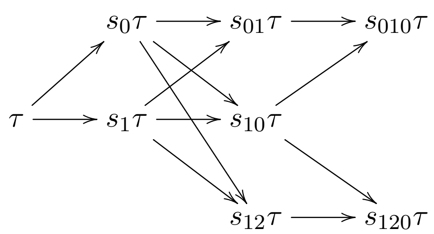
Remark 6.3.4. By Proposition 6.3.3, the intersection $\pi_{I,J}^{-1}(\mathcal{A}_J^{\tau}) \cap \mathcal{A}_{I,s_{02}\tau}$ is nonempty and $\pi_{I,J}^{-1}(\mathcal{A}_J^{\tau})$ does not contain the KR stratum $\mathcal{A}_{I,s_{02}\tau}$. Therefore, the preimage $\pi_{I,J}^{-1}(\mathcal{A}_J^{\tau})$ of $\mathcal{A}_J^{\tau}$ is not a union of KR strata.

(3) Case $J = \{0,1\}$ (Klingen parahoric level) and $W_{J^c} = \langle s_2 \rangle$. Using $\tau s_2 = s_0\tau$, we get

KR types	EKOR	dim	p-rank
$[\tau]_J = [s_0\tau]_J = \{\tau, s_2\tau, s_0\tau, s_{02}\tau\}$,	τ, $s_0\tau$	0, 1	0, 0
$[s_1\tau]_J = [s_{10}\tau]_J = \{s_1\tau, s_{10}\tau, s_{21}\tau\}$,	$s_1\tau$, $s_{10}\tau$	1, 2	0, 1
$[s_{12}\tau]_J = [s_{120}\tau]_J = \{s_{12}\tau, s_{120}\tau, s_{212}\tau\}$,	$s_{12}\tau$, $s_{120}\tau$	2, 3	1, 2
$[s_{01}\tau]_J = [s_{010}\tau]_J = \{s_{01}\tau, s_{010}\tau, s_{201}\tau\}$,	$s_{01}\tau$, $s_{010}\tau$	2, 3	1, 2.

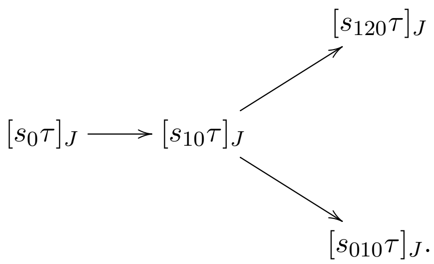
By [31, Theorem 6.15], we obtain the closure relation for EKOR strata:

(6.3.4)



Notice the new EKOR order $s_0\tau \to s_{12}\tau$ because of $s_2(s_0\tau)s_2^{-1} = s_2\tau \leq s_{12}\tau$ in the Bruhat order. The KR closure relation is as follows:

(6.3.5)



There are 8 EKOR strata and 4 KR strata.

Proposition 6.3.5.

1. *The p-rank two stratum $\mathcal{A}_J^2$ is smooth and has two irreducible components which are the EKOR strata $\mathcal{A}_J^{s_{010}\tau}$ and $\mathcal{A}_J^{s_{120}\tau}$; they are properly contained in the KR strata $\mathcal{A}_{[s_{010}\tau]_J}$ and $\mathcal{A}_{[s_{120}\tau]_J}$, respectively.*
2. *The p-rank one stratum $\mathcal{A}_J^1$ is smooth and has three irreducible components which are the EKOR strata $\mathcal{A}_J^{s_{01}\tau}, \mathcal{A}_J^{s_{10}\tau}$ and $\mathcal{A}_J^{s_{12}\tau}$; they are properly contained in the KR strata $\mathcal{A}_{[s_{01}\tau]_J}$, $\mathcal{A}_{[s_{10}\tau]_J}$ and $\mathcal{A}_{[s_{12}\tau]_J}$, respectively.*

(3) *The supersingular stratum $\mathcal{S}_J$ is connected and it consists of two types of irreducible components: those in $\overline{\mathcal{A}_J^{s_0\tau}} = \mathcal{A}_{[\tau]_J}$ ("horizontal" ones) and those in $\overline{\mathcal{A}_J^{s_1\tau}}$ ("vertical" ones)*[3]. *The intersection $\mathcal{S}_J \cap \mathcal{A}_{[s_{10}\tau]_J}$ is equal to the EKOR stratum $\mathcal{A}_J^{s_1\tau}$, which consists of open "vertical" components of $\mathcal{S}_J$.*

(4) *The union $\mathcal{A}_{[s_{010}\tau]_J} \cup \mathcal{A}_{[s_{120}\tau]_J}$ is the smooth locus of $\mathcal{A}_J$.*

(5) *Two irreducible components $\overline{\mathcal{A}_{[s_{010}\tau]_J}}$ and $\overline{\mathcal{A}_{[s_{120}\tau]_J}}$ are smooth and they intersect transversally at an irreducible smooth surface, which is equal to the closure $\overline{\mathcal{A}_{[s_{10}\tau]_J}}$ of the KR stratum $\mathcal{A}_{[s_{10}\tau]_J}$.*

(6) *(The transition relation) We have $\pi_{I,J}(\mathcal{A}_I^x) = \mathcal{A}_J^x$ for all $x \in {}^J\operatorname{Adm}(\mu)$,*

$$\pi_{I,J}(\mathcal{A}_I^{s_{212}\tau}) = \mathcal{A}_J^{s_{120}\tau}, \quad \pi_{I,J}(\mathcal{A}_I^{s_{201}\tau}) = \mathcal{A}_J^{s_{010}\tau}, \quad \pi_{I,J}(\mathcal{A}_I^{s_{21}\tau}) = \mathcal{A}_J^{s_{10}\tau}, \tag{6.3.6}$$

and

$$\begin{aligned} &\pi_{I,J}(\mathcal{A}_I^{s_{20}\tau}) = \mathcal{A}_J^{\tau} \coprod \mathcal{A}_J^{s_0\tau}, \quad \pi_{I,J}(\mathcal{A}_I^{s_2\tau}) = \mathcal{A}_J^{s_0\tau}, \\ &\pi_{J,\{0\}}(\mathcal{A}_J^{\tau} \coprod \mathcal{A}_J^{s_1\tau}) = \mathcal{A}_{\{0\}}^{\tau}, \quad \pi_{J,\{0\}}(\mathcal{A}_J^{s_0\tau}) = \mathcal{A}_{\{0\}}^{s_0\tau}. \end{aligned} \tag{6.3.7}$$

Proof. Statements (1), (2) and (3) follow directly from the EKOR stratification and their relation with p-rank strata; see also [90, p. 2346] and [88, Proposition 4.5] and for the description of $\mathcal{S}_J$. (4) From the closure relation of KR strata (6.3.5), the complement $\overline{\mathcal{A}_{[s_{10}\tau]_J}}$ of $\mathcal{A}_{[s_{010}\tau]_J} \cup \mathcal{A}_{[s_{120}\tau]_J}$ is contained in both irreducible components, and hence it is the singular locus of the moduli space $\mathcal{A}_J$. (5) This follows from Theorem 3 of [79]. (6) The transition relation (6.3.6) follows from $\pi_{I,J}(\mathcal{A}_I^{s_{212}\tau}) = \mathcal{A}_{[s_{120}\tau]_J}$, $\pi_{I,J}(\mathcal{A}_I^{s_{201}\tau}) = \mathcal{A}_{[s_{010}\tau]_J}$ and $\pi_{I,J}(\mathcal{A}_I^{s_{21}\tau}) = \mathcal{A}_{[s_{10}\tau]_J}$. The relation (6.3.7) follows from the description of the supersingular locus $\mathcal{S}_I$ and $\mathcal{S}_J$; see [89, Theorem 8.1] and [88, Proposition 4.5].

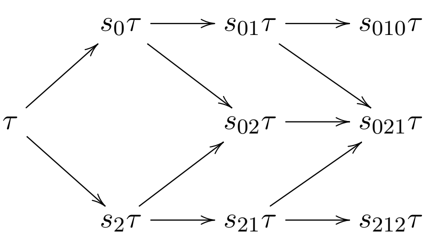
□

(4) Case $J = \{0, 2\}$ (Siegel parahoric level) and $W_{J^c} = \langle s_1 \rangle$. Using $\tau s_1 = s_1\tau$, we get

KR types	EKOR	dim	p-rank
$[\tau]_J = \{\tau, s_1\tau, s_0\tau, s_{02}\tau\}$,	τ, $s_0\tau$	0	0
$[s_{21}\tau]_J = [s_2\tau]_J = \{s_2\tau, s_{21}\tau, s_{12}\tau\}$,	$s_2\tau$, $s_{21}\tau$	1, 2	0, 1
$[s_{01}\tau]_J = [s_0\tau]_J = \{s_0\tau, s_{10}\tau, s_{01}\tau\}$,	$s_0\tau$, $s_{01}\tau$	1, 2	0, 1
$[s_{021}\tau]_J = [s_{02}\tau]_J = \{s_{02}\tau, s_{021}\tau, s_{102}\tau\}$,	$s_{02}\tau$, $s_{021}\tau$	2, 3	1, 2
$[s_{212}\tau]_J = \{s_{212}\tau\}$,	$s_{212}\tau$	3	2
$[s_{010}\tau]_J = \{s_{010}\tau\}$,	$s_{010}\tau$	3	2.

By [31, Theorem 6.15], we obtain the closure relation for EKOR strata:

(6.3.8)



[3] We refer to [88, Remark 1.3] for a detailed description of the role of horizontal and vertical components of $\mathcal{S}_J$.

and the KR closure relation:

(6.3.9)
$$\begin{array}{ccccc} & & [s_{01}\tau]_J & \longrightarrow & [s_{010}\tau]_J \\ & \nearrow & & \searrow & \\ [\tau]_J & & & & [s_{021}\tau]_J \\ & \searrow & & \nearrow & \\ & & [s_{21}\tau]_J & \longrightarrow & [s_{212}\tau]_J. \end{array}$$

There are 9 EKOR strata and 6 KR strata.

Proposition 6.3.6. *Let $J = \{0, 2\}$.*

1. *The p-rank two stratum $\mathcal{A}_J^2$ is smooth and has three irreducible components. Two of them are $\mathcal{A}_J^{s_{212}\tau} = \mathcal{A}_{[s_{212}\tau]_J}$ and $\mathcal{A}_J^{s_{010}\tau} = \mathcal{A}_{[s_{010}\tau]_J}$, and the other is $\mathcal{A}_J^{s_{021}\tau}$, which is properly contained in the KR stratum $\mathcal{A}_{[s_{021}\tau]_J}$.*
2. *The p-rank one stratum $\mathcal{A}_J^1$ is smooth and has two irreducible components. They are EKOR strata $\mathcal{A}_J^{s_{01}\tau}$ and $\mathcal{A}_J^{s_{21}\tau}$, which are properly contained in $\mathcal{A}_{[s_{01}\tau]_J}$ and $\mathcal{A}_{[s_{21}\tau]_J}$, respectively.*
3. *The supersingular locus $\mathcal{S}_J$ has pure dimension 2. It is contained in the 3-dimensional closed KR stratum $\overline{\mathcal{A}_{[s_{021}\tau]_J}}$. Each irreducible component of $\mathcal{S}_J$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.*
4. *The zero dimensional stratum $\mathcal{A}_{[\tau]_J}$ consists of points $(\underline{A}_0 \xrightarrow{F} \underline{A}_0^{(p)})$, where A_0 is superspecial.*
5. *The union $\mathcal{A}_{[s_{212}\tau]_J} \cup \mathcal{A}_{[s_{010}\tau]_J} \cup \mathcal{A}_{[s_{02}\tau]_J}$ is the smooth locus of $\mathcal{A}_J$.*
6. *We have $\pi_{I,J}(\mathcal{A}_I^x) = \mathcal{A}_J^x$ for all $x \in {}^J\mathrm{Adm}(\mu)$ and*

(6.3.10)
$$\begin{aligned} \pi_{I,J}(\mathcal{A}_I^{s_{102}\tau}) = \mathcal{A}_J^{s_{021}\tau}, \quad \pi_{I,J}(\mathcal{A}_I^{s_{12}\tau}) = \mathcal{A}_J^{s_{21}\tau}, \\ \pi_{I,J}(\mathcal{A}_I^{s_{10}\tau}) = \mathcal{A}_J^{s_{01}\tau}, \quad \pi_{I,J}(\mathcal{A}_I^{s_1\tau}) = \mathcal{A}_J^{\tau}. \end{aligned}$$

Proof. Statements (1), (2) and (5) follow directly from the closure relations (6.3.8) and (6.3.9). The supersingular locus $\mathcal{S}_I$ is the union of $\overline{\mathcal{A}_I^{s_1\tau}}$ and $\overline{\mathcal{A}_I^{s_{20}\tau}}$. The transition map $\tau_{I,J}$ gives an isomorphism $\overline{\mathcal{A}_I^{s_{20}\tau}} \xrightarrow{\sim} \mathcal{S}_J$. Thus, we can describe $\mathcal{S}_J$ by $\overline{\mathcal{A}_I^{s_{20}\tau}}$, whose description is included in [89, Theorem 8.1]. From this, statements (3) and (4) follow. (6) The first three relations follow from the relations

$$\pi_{I,J}(\mathcal{A}_I^{s_{102}\tau}) \subset \mathcal{A}_{[s_{021}\tau]_J}, \quad \pi_{I,J}(\mathcal{A}_I^{s_{12}\tau}) = \mathcal{A}_{[s_{21}\tau]_J}, \quad \pi_{I,J}(\mathcal{A}_I^{s_{10}\tau}) = \mathcal{A}_{[s_{01}\tau]_J}.$$

The last one follows from the description of $\mathcal{S}_J$ via (3) and (4). $\square$

In fact, in the moduli space $\mathcal{A}_I$ with Iwahori level structure, we have

$$\mathcal{S}_I = \overline{\mathcal{A}_{s_{021}\tau}} \cap \overline{\mathcal{A}_{s_{102}\tau}}.$$

These two components are mapped, through the transition map $\pi_{I,J}$, onto the component $\overline{\mathcal{A}_{[s_{102}\tau]_J}}$.

(5) Case $J = \{1\}$ (paramodular level) and $W_{J^c} = \langle s_0, s_2 \rangle$. Using $\tau s_2 = s_0\tau$ and $\tau s_0 = s_2\tau$, we get

KR types	EKOR	dim	p-rank
$[\tau]_J = \{\tau, s_2\tau, s_0\tau, s_{02}\tau\} = W_{\{0,2\}}\tau$	τ	0	0
$[s_{102}\tau]_J = W_{\{0,2\}}s_1\tau \cup W_{\{0\}}s_{10}\tau$	$s_1\tau$, $s_{10}\tau$	1, 2	0, 1
$\cup W_{\{2\}}s_{12}\tau \cup s_{102}\tau$	$s_{12}\tau$, $s_{120}\tau$	2, 3	1, 2

and the closure relation for EKOR strata and KR strata:

$$\tau \to s_1\tau \to s_{10}\tau, s_{12}\tau \to s_{120}\tau, \quad [\tau]_J \to [s_{120}\tau]_J. \tag{6.3.11}$$

There are 5 EKOR strata and 2 KR strata. We have the follow result (cf. [90, Theorem 4.4]).

Proposition 6.3.7. *Let $J = \{1\}$.*

(1) *The p-rank two stratum $\mathcal{A}_J^2$ is smooth and has one irreducible component. This component is the EKOR stratum $\mathcal{A}_J^{s_{102}\tau}$ and it is properly contained in the maximal KR stratum $\mathcal{A}_{[s_{102}\tau]_J}$.*
(2) *The p-rank one stratum $\mathcal{A}_J^1$ is smooth and has two irreducible components. They are the EKOR strata $\mathcal{A}_J^{s_{10}\tau}$ and $\mathcal{A}_J^{s_{12}\tau}$ properly contained in the maximal KR stratum $\mathcal{A}_{[s_{102}\tau]_J}$.*
(3) *The supersingular locus has pure dimension 1. Each irreducible component is isomorphic to $\mathbf{P}^1$. The intersection $S_J \cap \mathcal{A}_{[s_{102}\tau]_J}$ is the smooth locus of S_J.*
(4) *The zero dimensional stratum $\mathcal{A}_{[\tau]_J}$ is the singular locus of $\mathcal{A}_J$, and it is also the singular locus of $\mathcal{S}_J$.*
(5) *The stratum $\mathcal{A}_{[s_{102}\tau]_J}$ is the smooth locus.*
(6) *We have*

$$\pi_{I,J}(\mathcal{A}_I^x) = \begin{cases} \mathcal{A}_J^{s_{120}\tau} & \text{for } x \in \mathrm{Adm}^2(\mu); \\ \mathcal{A}_J^{s_{10}\tau} & \text{for } x \in \{s_{10}\tau, s_{21}\tau\}; \\ \mathcal{A}_J^{s_{12}\tau} & \text{for } x \in \{s_{12}\tau, s_{01}\tau\}; \\ \mathcal{A}_J^{s_{1}\tau} & \text{for } x = s_1\tau; \\ \mathcal{A}_J^{\tau} & \text{for } x \in W_{\{0,2\}}\tau. \end{cases} \tag{6.3.12}$$

Proof. Statements (1)-(5) follow from the closure relation and the description of the supersingular locus $\mathcal{S}_J$ [88]. (6) We only need to show the case of p-rank one strata. Using the geometric characterization of KR types [90, Section 4.2], for a p-rank one point (A, λ) in $\mathcal{A}_J^1$, the kernel $\ker \lambda$ is isomorphic to $\mu_p \times \mathbb{Z}/p\mathbb{Z}$ if $(A, \lambda) \in \mathcal{A}_J^{s_{12}\tau}$, or is local-local if $(A, \lambda) \in \mathcal{A}_J^{s_{10}\tau}$. From this, we obtain $\pi_{I,J}(\mathcal{A}_I^{s_{21}\tau}) = \mathcal{A}_J^{s_{10}\tau}$ and $\pi_{I,J}(\mathcal{A}_I^{s_{01}\tau}) = \mathcal{A}_J^{s_{12}\tau}$. □

Appendix A. He-Rapoport axiom 4 (c) for Shimura varieties of PEL-type

In this appendix we verify He-Rapoport's axiom 4 (c) ([31] 3.3) for Shimura varieties of PEL-type in the case where the parahoric level subgroup K at p is the stabilizer group, i.e. $K = K^\circ$. This extends earlier results of Zhou [95] and He-Zhou [32] mainly in the ramified cases and the case which contains a simple factor of type D. This also improves our main results (Theorem C (2) and Theorem D) for the PEL-type case.

We follow Rapoport-Zink [71] for the construction of the "naive" integral model with "parahoric" level structure.

A.1. Moduli spaces of PEL-type.

Definition A.1.1. *A* PEL-datum *is a tuple $(B, *, V, \psi, h)$, where*

- *$(B, *)$ is a finite dimensional semi-simple $\mathbb{Q}$-algebra with a positive involution,*
- *(V, ψ) is a finite faithful non-degenerate $\mathbb{Q}$-valued skew-Hermitian B-module, and*
- *$h : \mathbb{C} \to \mathrm{End}_{B_\mathbb{R}}(V_\mathbb{R})$ is an $\mathbb{R}$-algebra homomorphism such that $h(z)' = h(\bar{z})$ and $(x, y) := \psi(h(i)x, y)$ is a positive definite symmetric form, where $B_\mathbb{R} := B \otimes_\mathbb{Q} \mathbb{R}$, $V_\mathbb{R} := V \otimes_\mathbb{Q} \mathbb{R}$ and $'$ is the adjoint with respect to the pairing ψ.*

For each PEL-datum, we associate an algebraic $\mathbb{Q}$-group $G = GU_B(V, \psi)$ of unitary similitudes on (V, ψ). Let X be the $G(\mathbb{R})$-conjugacy class of $h : \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{\mathrm{m}\mathbb{C}} \to G_\mathbb{R}$.

For each open compact subgroup $\mathsf{K} \subset G(\mathbb{A}_f)$, denote by $\mathrm{Sh}_{\mathsf{K}}(G, X)$ the Shimura variety[4] defined over the reflex field E, whose $\mathbb{C}$-points are given by

$$\mathrm{Sh}_{\mathsf{K}}(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/\mathsf{K},$$

even G may not be connected, i.e. if its adjoint group contains a $\mathbb{Q}$-simple factor of type D. When there is no confusion, we simply write Sh_{K} for $\mathrm{Sh}_{\mathsf{K}}(G, X)$.

Recall that the canonical model $\mathrm{Sh}_{\mathsf{K}}(G, X)$ is constructed using a moduli interpretation. Indeed, one first constructs an E-scheme M_{K} of finite type using a moduli interpretation (see below) and obtains the inclusion $\mathrm{Sh}_{\mathsf{K}}(G, X)(\mathbb{C}) \subset M_{\mathsf{K}}(\mathbb{C})$ (the inclusion may be strict due to the failure of the Hasse principle, see [42] section 8, which holds in the current setting; see also the proof of Proposition A.3.5). Then one defines $\mathrm{Sh}_{\mathsf{K}}(G, X)$ to be the corresponding E-subscheme of M_{K}. Using the main theorem of complex multiplication, one shows that $\mathrm{Sh}_{\mathsf{K}}(G, X)$ is the canonical model of the complex Shimura variety $\mathrm{Sh}_{\mathsf{K}}(G, X)(\mathbb{C})$; see [3] for more details.

Let $p > 2$ be a prime. Assume that there exists an order $O_B \subset B$ which is stable under $*$ and maximal at p, i.e. $O_{B_p} := O_B \otimes \mathbb{Z}_p$ is a maximal order in $B_p := B \otimes \mathbb{Q}_p$, and we fix such an order.

Suppose that there exists a self-dual O_{B_p}-lattice in $V_p := V \otimes \mathbb{Q}_p$. Choose an O_B-lattice Λ in V such that $\Lambda_p = \Lambda \otimes \mathbb{Z}_p$ is self-dual. Let

$$K = \mathrm{Stab}_{G(\mathbb{Q}_p)}(\Lambda_p), \quad \text{and} \quad K^p \subset \mathrm{Stab}_{G(\mathbb{A}_f^p)}(\Lambda \otimes \hat{\mathbb{Z}}^{(p)})$$

be a sufficiently small open compact subgroup, and put $\mathsf{K} = KK^p$.

Let $v|p$ be a place of E over p, and O_E the ring of integers of the v-adic completion $E = \mathsf{E}_v$ of E. Using the modular interpretation, we construct the naive integral model over O_E denoted by $\mathbf{M}_{\mathsf{K}}^{\mathrm{naiv}}$. It classifies the isomorphism classes of tuples $(A, \lambda, \iota, \bar{\eta})$, where

- (A, λ, ι) is a prime-to-p degree polarized O_B-abelian variety of dimension $\frac{1}{2}\dim_{\mathbb{Q}} V$,
- $\bar{\eta}$ is a K^p-orbit of $O_B \otimes \hat{\mathbb{Z}}^{(p)}$-linear isomorphisms

$$\eta : \Lambda \otimes \hat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} T^{(p)}(A) = \prod_{\ell \neq p} T_\ell(A)$$

  which preserve the pairings for a suitable isomorphism $\mathbb{Z}^{(p)} \simeq \mathbb{Z}^{(p)}(1)$.

We also require that objects $(A, \lambda, \iota, \bar{\eta})$ in $\mathbf{M}_{\mathsf{K}}^{\mathrm{naiv}}$ satisfy the Kottwitz determinant condition, cf. [71] chapter 6 or [42].

We have defined an O_E-scheme $\mathbf{M}_{\mathsf{K}}^{\mathrm{naiv}}$. By construction, it is an integral model of M_{K} over O_E. In particular, this gives an inclusion

$$\mathrm{Sh}_{\mathsf{K}}(G, X)_E := \mathrm{Sh}_{\mathsf{K}}(G, X) \otimes_{\mathsf{E}} E \subset \mathbf{M}_{\mathsf{K}}^{\mathrm{naiv}} \otimes_{O_E} E.$$

Define $\mathscr{S}_{\mathsf{K}}$ as the flat closure of $\mathrm{Sh}_{\mathsf{K}}(G, X)_E$ in $\mathbf{M}_{\mathsf{K}}^{\mathrm{naiv}}$.

A.2. **Parahoric data at p.** We define the notion of self-dual multichains of lattices $\mathcal{L}$ in V_p following [71]. This allows us to extend the above level subgroup K at p to more general forms.

To simplify the notation, we write $(B, *, V, \psi, O_B)$ for $(B_p, *, V_p, \psi_p, O_{B_p})$ in this subsection.

Definition A.2.1. *Suppose that B is a simple $\mathbb{Q}_p$-algebra.* A chain of O_B-lattices *in V is a set of totally ordered O_B-lattices $\mathcal{L}$ such that for every element $x \in B^\times$ which normalizes O_B, one has*

$$\Lambda \in \mathcal{L} \implies x\Lambda \in \mathcal{L}.$$

[4] Since the group G may be non connected, we have to modify a little the formalism of Shimura varieties in [4] to include the current setting.

Explicitly, suppose we write $B = M_n(D)$ and let $O_B = M_n(O_D)$, where D is a division $\mathbb{Q}_p$-algebra and O_D is the maximal order of D. The group $N_{B^\times}(O_B)$ of normalizers of O_B in $B^\times$ is equal to $\Pi^{\mathbb{Z}}\,\mathrm{GL}_n(O_D)$, where Π is a prime element of O_D. Then $\mathcal{L}$ is a chain of O_B-lattices in V if and only if $\mathcal{L}$ is totally ordered and for any member $\Lambda \in \mathcal{L}$, one has $\Pi^{\pm 1}\Lambda \in \mathcal{L}$. If we fix a member $\Lambda_0 \in \mathcal{L}$, then there is an unique integer $r \geq 1$, called the *period*, and a sequence of lattices $\{\Lambda_i\}_{0\leq i\leq r}$ such that

$$\Lambda_0 \subsetneq \Lambda_1 \subsetneq \dots \Lambda_r = \Pi^{-1}\Lambda_0,$$

and every member $\Lambda \in \mathcal{L}$ is $\Pi^n\Lambda_i$ for some $n \in \mathbb{Z}$ and some i with $0 \leq i \leq r$. We extend the set of lattices by putting $\Lambda_{i+r} = \Pi^{-1}\Lambda_i$ for $i \in \mathbb{Z}$ and we have $\mathcal{L} = \{\Lambda_i\}_{i\in\mathbb{Z}}$. So $\mathcal{L}$ is determined by the period r and the sequence $\{\Lambda_i\}_{0\leq i\leq r}$ of $r+1$ O_B-lattices.

In general, let $B = B_1 \times \cdots \times B_m$ and $O_B = O_{B_1} \times \cdots \times O_{B_m}$, where $B_i = M_{n_i}(D_i)$ and $O_{B_i} = M_{n_i}(O_{D_i})$ for some division $\mathbb{Q}_p$-algebra D_i whose maximal order is denoted by O_{D_i}. With respect to this decomposition, one has a decomposition $V = V_1 \oplus \cdots \oplus V_m$ and for each O_B-lattice Λ in V a decomposition $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_m$. Let $e_1, \dots, e_m$ be the central primitive idempotents of B. Write $\mathrm{pr}_i\Lambda = e_i\Lambda = \Lambda_i$.

Definition A.2.2. (1) *A set $\mathcal{L}$ of O_B-lattices in V is said to be* a multichain of O_B-lattices *if there exist a chain of O_{B_i}-lattices $\mathcal{L}_i$ in V_i for each $i = 1, \dots, m$ such that for any member $\Lambda \in \mathcal{L}$ one has $\mathrm{pr}_i\Lambda \in \mathcal{L}_i$ for all $i = 1, \dots, m$.*

(2) *A multichain of O_B-lattices $\mathcal{L}$ is called* self-dual *if for every member $\Lambda \in \mathcal{L}$, its dual lattice $\Lambda^\vee$ also belongs to $\mathcal{L}$, where*

$$\Lambda^\vee := \{x \in V \mid \psi(x, \Lambda) \subset \mathbb{Z}_p\}.$$

For a multichain of O_B-lattices $\mathcal{L}$, let $\mathcal{G} = \mathcal{G}_{\mathcal{L}} \subset \prod_{\Lambda\in\mathcal{L}} \mathcal{G}_\Lambda$ be the group scheme over $\mathbb{Z}_p$ whose S-valued points, for each $\mathbb{Z}_p$-scheme S, are the S-points $(g_\Lambda) \in \mathcal{G}_\Lambda(S)$ which are compatible with all transition maps, where $\mathcal{G}_\Lambda$ is the integral model of $G_{\mathbb{Q}_p}$ defined by the lattice Λ. Put $K = K_{\mathcal{L}} := \mathcal{G}(\mathbb{Z}_p) = \cap_{\Lambda\in\mathcal{L}} K_\Lambda$, where $K_\Lambda = \mathcal{G}_\Lambda(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$. Denote by $\mathcal{G}^\circ$ the connected component of $\mathcal{G}$, which is the maximal open subscheme whose fibers are all connected. Put $K^\circ = K^\circ_{\mathcal{L}} := \mathcal{G}^\circ(\mathbb{Z}_p)$.

Note that $\mathcal{L}$ is not determined by $\mathcal{L}_1, \dots, \mathcal{L}_m$. Therefore, the input datum $\mathcal{L}$ is not determined by $K_{\mathcal{L}}$ in general.

A.3. **Moduli spaces with parahoric type $\mathcal{L}$.** We retain the notation as in subsection A.1. Let AV denote the category of O_B-abelian varieties up to prime-to-p isogeny. The objects of AV consist of pairs (A, ι), where A is an abelian variety and

$$\iota : O_B \otimes \mathbb{Z}_{(p)} \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$$

is a ring monomorphism. For any two objects $\underline{A}_1 = (A_1, \iota_1)$ and $\underline{A}_2 = (A_2, \iota_2)$, the set of morphisms from $\underline{A}_1$ to $\underline{A}_2$ is $\mathrm{Hom}_{AV}(\underline{A}_1, \underline{A}_2) := \mathrm{Hom}_{O_B}(A_1, A_2) \otimes \mathbb{Z}_{(p)}$. If there is no risk of confusion, we write A for $\underline{A}$.

Suppose $\rho : A_1 \to A_2$ is an isogeny in AV. Then $\rho_p : A_1[p^\infty] \to A_2[p^\infty]$ is an O_{B_p}-linear isogeny. With respect to the decomposition $O_{B_p} = O_{B_1} \times \cdots \times O_{B_m}$, one has the decomposition

$$\ker \rho_p = H_1 \times \cdots \times H_m \tag{A.3.1}$$

as the product of finite flat group schemes H_i with O_{B_i}-action. The height of ρ is the (height) function $h = h_H : \{1, \dots, m\} \to \mathbb{N}$ given by

$$h(i) = h_H(i) := \log_p(\mathrm{ord}(H_i)),$$

where $\mathrm{ord}(H_i)$ denotes the order of the finite group scheme H_i.

Let $A = (A, \iota)$ be an object in AV. The dual abelian variety is defined to be $A^t = (A, \iota)^t := (A^t, \iota^t)$, where $\iota^t(b) := \iota(b^*)^t$ for $b \in O_B \otimes \mathbb{Z}_{(p)}$. For any element $a \in N_{B^\times}(O_B \otimes \mathbb{Z}_{(p)})$, the normalizer of $O_B \otimes \mathbb{Z}_{(p)}$ in $B^\times$, define $A^a = (A, \iota^a)$, where $\iota^a(b) = \iota(a^{-1}ba)$ for all $b \in O_B \otimes \mathbb{Z}_{(p)}$. The multiplication by a gives a quasi-isogeny

$a : A^a \to A$ in AV. A polarization on A in AV is a quasi-isogeny $\lambda : A \to A^t$ in AV such that the morphism $n\lambda$ comes from an ample line bundle on A for some $n \in \mathbb{N}$.

Definition A.3.1. *Let $\mathcal{L}$ be a multichain of O_{B_p}-lattices in V_p. A $\mathcal{L}$-set of abelian varieties over a $\mathbb{Z}_{(p)}$-scheme S is a functor*

$$\mathcal{L} \to AV,$$
$$\Lambda \mapsto A_\Lambda,$$
$$\Lambda \subset \Lambda' \mapsto \rho_{\Lambda,\Lambda'} : A_\Lambda \to A_{\Lambda'}$$

satisfying

(a) *For any $\Lambda \subset \Lambda'$ in $\mathcal{L}$, write $\rho_{\Lambda,\Lambda'} : H_1^{DR}(A_\Lambda/S) \to H_1^{DR}(A_{\Lambda'}/S)$ for the induced map on de Rham homologies, then locally on S for the Zariski topology, the $\mathcal{O}_S$-module $H_1^{DR}(A_{\Lambda'}/S)/\rho_{\Lambda,\Lambda'}H_1^{DR}(A_\Lambda/S)$ is isomorphic to $(\Lambda'/\Lambda)\otimes\mathcal{O}_S$ as $O_B\otimes\mathcal{O}_S$-modules.*
In particular, the morphism $\rho_{\Lambda,\Lambda'} : A_\Lambda \to A_{\Lambda'}$ is an isogeny of height
$$h(i) = \log_p |\mathrm{pr}_i\Lambda'/\mathrm{pr}_i\Lambda|.$$

(b) *For any element $a \in B^\times \cap (O_B \otimes \mathbb{Z}_{(p)})$ which normalizes $O_B \otimes \mathbb{Z}_{(p)}$ and any member $\Lambda \in \mathcal{L}$ (so $a\Lambda \subset \Lambda$), there exists an isomorphism $\theta_{a,\Lambda} : A_\Lambda^a \xrightarrow{\sim} A_{a\Lambda}$ such that the following diagram*

$$\begin{array}{ccc} A_\Lambda^a & \xrightarrow[\sim]{\theta_{a,\Lambda}} & A_{a\Lambda} \\ & \searrow^{a} & \downarrow{\scriptstyle \rho_{a\Lambda,\Lambda}} \\ & & A_\Lambda \end{array}$$

commutes.

The map $\theta_{a,\Lambda}$ is unique if it exists, and it is functorial in Λ.

Definition A.3.2. *Let $\mathcal{L}$ be a self-dual multichain of O_{B_p}-lattices in V_p, and*

$$A_{\mathcal{L}} = ((A_\Lambda)_{\Lambda\in\mathcal{L}}, \rho_{\Lambda,\Lambda'})$$

be an $\mathcal{L}$-set of abelian varieties over S in AV. Define an $\mathcal{L}$-set of abelian varieties $\widetilde{A}_{\mathcal{L}} = ((\widetilde{A}_\Lambda), \widetilde{\rho}_{\Lambda,\Lambda'})$ over S in AV as follows: for each Λ and $\Lambda \subset \Lambda'$ in $\mathcal{L}$,

- $\widetilde{A}_\Lambda := (A_{\Lambda^\vee})^t$;
- $\widetilde{\rho}_{\Lambda,\Lambda'} := (\rho_{\Lambda'^\vee,\Lambda^\vee})^t : \widetilde{A}_\Lambda = A^t_{\Lambda^\vee} \to \widetilde{A}_{\Lambda'} = A^t_{\Lambda'^\vee}$;
- $\widetilde{\theta}_{a,\Lambda} := [\theta^t_{(a^*)^{-1},\Lambda^\vee}]^{-1} : (\widetilde{A}_\Lambda)^a \xrightarrow{\sim} \widetilde{A}_{a\Lambda}$.

We explain the last item. One easily computes $(a\Lambda)^\vee = (a^*)^{-1}\Lambda^\vee$. Also,

$$(\iota^t)^a(b) = \iota^t(a^{-1}ba) = \iota(a^*b^*(a^*)^{-1})^t = [\iota^{(a^*)^{-1}}(b^*)]^t = [\iota^{(a^*)^{-1}}]^t(b).$$

This shows that $(A_{\Lambda^\vee}^{(a^*)^{-1}})^t = (A^t_{\Lambda^\vee})^a = (\widetilde{A}_\Lambda)^a$ and $\widetilde{A}_{a\Lambda} = (A_{a^{*-1}\Lambda^\vee})^t$. By definition, $\widetilde{\theta}_{a,\Lambda}$ is the inverse of the isomorphism $\theta^t_{(a^*)^{-1},\Lambda^\vee} : (A_{(a^*)^{-1}\Lambda^\vee})^t \to (A_{\Lambda^\vee}^{(a^*)^{-1}})^t$.

Definition A.3.3. *Let $\mathcal{L}$, $A = A_{\mathcal{L}}$ and $\widetilde{A} = \widetilde{A}_{\mathcal{L}}$ be as in Definition A.3.2.*

(1) *A* polarization *on $A = (A_\Lambda)$ is a quasi-isogeny $\lambda : A \to \widetilde{A}$, (i.e. there exists $n \in \mathbb{N}_{>0}$ such that $n\lambda_\Lambda : A_\Lambda \to \widetilde{A}_\Lambda$ is an isogeny in AV for all $\Lambda \in \mathcal{L}$) such that for all $\Lambda \in \mathcal{L}$, the composition*
$$\lambda'_\Lambda : A_\Lambda \xrightarrow{\lambda_\Lambda} \widetilde{A}_\Lambda = A^t_{\Lambda^\vee} \xrightarrow{(\rho_{\Lambda,\Lambda^\vee})^t} A^t_\Lambda$$
is a polarization on A_Λ.

(2) *A polarization λ on A in AV is called* principal *if λ_Λ is an isomorphism in AV for all $\Lambda \in \mathcal{L}$.*

Suppose $\mathcal{L}$ contains a self-dual member Λ_0, then $\lambda_{\Lambda_0} : A_{\Lambda_0} \to A^t_{\Lambda_0}$ is an isomorphism in AV, i.e. there exists $n \in \mathbb{N}_{>0}$, $(n,p) = 1$ such that $n\lambda_{\Lambda_0}$ is a polarization on A_Λ in the usual sense. For $\Lambda, \Lambda' \in \mathcal{L}$, we have the following commutative diagram

$$\begin{array}{ccc} A_\Lambda & \xrightarrow{\rho_{\Lambda,\Lambda'}} & A_{\Lambda'} \\ \downarrow{\scriptstyle \lambda_\Lambda} & & \downarrow{\scriptstyle \lambda_{\Lambda'}} \\ A^t_{\Lambda^\vee} & \xrightarrow{(\rho_{\Lambda'^\vee,\Lambda^\vee})^t} & A^t_{\Lambda'^\vee}. \end{array}$$

Therefore, a polarization λ is uniquely determined by λ_Λ for one member $\Lambda \in \mathcal{L}$.

Let $K^p \subset G(\mathbb{A}^p_f)$ be a sufficiently small open compact subgroup. Let $\mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}}$ denote the moduli scheme over O_E classifying the objects $(A_{\mathcal{L}}, \bar{\lambda}, \bar{\eta})_S$, where

- $A_{\mathcal{L}} = (A_\Lambda)_{\Lambda \in \mathcal{L}}$ is an $\mathcal{L}$-set of abelian varieties over S in AV;
- $\bar{\lambda} = \mathbb{Q}^\times \cdot \lambda$ is a $\mathbb{Q}$-homogeneous principal polarization on $A_{\mathcal{L}}$;
- $\bar{\eta}$ is a $\pi_1(S, \bar{s})$-invariant K^p-orbit of isomorphisms
$$\eta : V \otimes \mathbb{A}^p_f \xrightarrow{\sim} V^{(p)}(A_{\bar{s}})$$
which preserve the pairings to up a scalar in $(\mathbb{A}^p_f)^\times$. Here for simplicity we assume that S is connected.

Put $K = K_{\mathcal{L}}, \mathsf{K} = K \cdot K^p$ and $\mathsf{K}^\circ = K^\circ K^p$. Similarly as before, $\mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}}$ is an integral model over O_E of the rational moduli scheme M_{K} (which is defined over E), and one has $\mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}} \otimes_{O_E} E \supset \mathrm{Sh}_{\mathsf{K},E}$. Let $\mathscr{S}_{\mathcal{L}} = \mathscr{S}^{\mathrm{can}}_{\mathcal{L}}$ be the flat closure of $\mathrm{Sh}_{\mathsf{K},E}$ in $\mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}}$. As $\mathsf{K}^\circ \subset \mathsf{K}$, we have the natural cover $\mathrm{Sh}_{\mathsf{K}^\circ} \to \mathrm{Sh}_{\mathsf{K}}$ over E. We define $\mathscr{S}_{\mathsf{K}^\circ}$ as the normalization of $\mathscr{S}_{\mathcal{L}}$ in $\mathrm{Sh}_{\mathsf{K}^\circ}$. If G splits over a tamely ramified extension of $\mathbb{Q}_p$, then the main results of Pappas-Zhu [65] subsection 8.2 imply that $\mathscr{S}_{\mathsf{K}^\circ}$ fits into a local model diagram.

Let $\mathcal{L}'$ be another set of self-dual multichain of O_{B_p}-lattices in V_p containing $\mathcal{L}$. Put $K' := K_{\mathcal{L}'}$ and $K'^\circ := K^\circ_{\mathcal{L}'}$. Then $K' \subset K$, $\mathsf{K}' \subset \mathsf{K}$, $K'^\circ \subset K^\circ$ and $\mathsf{K}'^\circ \subset \mathsf{K}^\circ$. Since $\mathrm{Sh}_{\mathsf{K}'}$ and Sh_{K} are dense in $\mathscr{S}_{\mathcal{L}'}$ and $\mathscr{S}_{\mathcal{L}}$, respectively, the morphism $\pi_{\mathsf{K}',\mathsf{K}} : \mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}'} \to \mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}}$ maps $\mathscr{S}_{\mathcal{L}'}$ into $\mathscr{S}_{\mathcal{L}}$, and we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Sh}_{\mathsf{K}'} & \longrightarrow & \mathscr{S}_{\mathcal{L}'} & \longrightarrow & \mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}'} \\ \downarrow{\scriptstyle \pi_{\mathsf{K}',\mathsf{K}}} & & \downarrow{\scriptstyle \pi_{\mathsf{K}',\mathsf{K}}} & & \downarrow{\scriptstyle \pi_{\mathsf{K}',\mathsf{K}}} \\ \mathrm{Sh}_{\mathsf{K}} & \longrightarrow & \mathscr{S}_{\mathcal{L}} & \longrightarrow & \mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}}. \end{array}$$

Definition A.3.4. *Let k be an algebraically closed field containing the residue field $k(v)$ of E. Let $x \in \mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}}(k)$ be a k-point. Denote by $\mathcal{C}^{\mathrm{naiv}}_{\mathcal{L}}(x) \subset \mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}}(k)$ the subset consisting of the points $y \in \mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}}(k)$ such that the associated $\mathcal{L}$-set of p-divisible groups $A_{y,\mathcal{L}}[p^\infty]$ is isomorphic to $A_{x,\mathcal{L}}[p^\infty]$, where $A_{x,\mathcal{L}}$ (resp. $A_{y,\mathcal{L}}$) is the $\mathcal{L}$-set of abelian varieties in AV corresponding to the point x (resp. y) in $\mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}}(k)$. The set $\mathcal{C}^{\mathrm{naiv}}_{\mathcal{L}}(x)$ is locally closed in $\mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}} \otimes_{O_E} k$ and we regard it as the locally closed subscheme with the induced reduced scheme structure, and call it the* central leaf passing through *x in $\mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}} \otimes_{O_E} k$.*

If $x \in \mathscr{S}_{\mathcal{L}}(k)$, then write $\mathcal{C}_{\mathcal{L}}(x)$ for the intersection $\mathcal{C}^{\mathrm{naiv}}_{\mathcal{L}}(x) \cap (\mathscr{S}_{\mathcal{L}} \otimes_{O_E} k)$ and call it the central leaf passing through *x in $\mathscr{S}_{\mathcal{L}} \otimes_{O_E} k$.*

It is known that $\mathcal{C}^{\mathrm{naiv}}_{\mathcal{L}}(x)$ is smooth and quasi-affine. The smoothness follows from the properties that $\mathcal{C}^{\mathrm{naiv}}_{\mathcal{L}}(x)$ is reduced and the completions $\mathcal{C}^{\mathrm{naiv}}_{\mathcal{L}}(x)^\wedge_y$ at all $y \in \mathcal{C}^{\mathrm{naiv}}_{\mathcal{L}}(x)(k)$ are isomorphic. The quasi-affineness follows from that of central leaves in the Siegel modular varieties. It follows that if x lies in $\mathscr{S}_{\mathcal{L}}(k)$, then the subscheme $\mathcal{C}_{\mathcal{L}}(x) \subset \mathscr{S}_{\mathcal{L}} \otimes_{O_E} k$ is smooth and quasi-affine.

Proposition A.3.5. *Let k be as in Definition A.3.4, $x' \in \mathscr{S}_{\mathcal{L}'}(k)$ and x be its image in $\mathscr{S}_{\mathcal{L}}(k)$. Then the morphism $\pi_{\mathsf{K}',\mathsf{K}} : \mathcal{C}_{\mathcal{L}'}(x') \to \mathcal{C}_{\mathcal{L}}(x)$ is surjective with finite fibers.*

Proof. We first show this statement for the morphism $\pi_{\mathsf{K}',\mathsf{K}}: \mathcal{C}^{\rm naiv}_{\mathcal{L}'}(x') \to \mathcal{C}^{\rm naiv}_{\mathcal{L}}(x)$. Let $x' = [((A_{\Lambda'})_{\Lambda' \in \mathcal{L}'}, \bar{\lambda}, \bar{\eta})]$, where λ is a principal representative. Then

$$x = [((A_\Lambda)_{\Lambda \in \mathcal{L}}, \bar{\lambda}, \bar{\eta})].$$

Let $y = [((B_\Lambda)_\Lambda, \bar{\lambda}_B, \bar{\eta}_B)] \in \mathcal{C}^{\rm naiv}_{\mathcal{L}}(x)$, where λ_B is a principal representative. We fix an isomorphism

$$((B_\Lambda)_\Lambda, \lambda_B)[p^\infty] \simeq ((A_\Lambda)_\Lambda, \lambda)[p^\infty]$$

of $\mathcal{L}$-sets of p-divisible groups. Then there is an extension $((B_{\Lambda'})_{\Lambda'}, \lambda_B)$ such that

$$((B_{\Lambda'})_{\Lambda'}, \lambda_B)[p^\infty] \simeq ((A_{\Lambda'})_{\Lambda'}, \lambda)[p^\infty].$$

It is enough to extend $(B_{\Lambda'})_{\Lambda' \in \mathcal{L}'}$ for $\Lambda' \in \mathcal{L}'^{\Box}$, where

$$\mathcal{L}'^{\Box} := \{\Lambda \in \mathcal{L}' \mid \Lambda \subset \Lambda^\vee \subset p^{-1}\Lambda\},$$

and use the periodic property to extend $(B_{\Lambda'})_{\Lambda'}$ for all $\Lambda' \in \mathcal{L}'$. Choose $\Lambda \in \mathcal{L}$ such $\Lambda \subset \Lambda'$ for all $\Lambda' \in \mathcal{L}'^{\Box}$. Suppose $\Lambda' \notin \mathcal{L}$. We define $B_{\Lambda'}$ as the quotient $B_\Lambda / H_{\Lambda'}$, where $H_{\Lambda'} \subset B_\Lambda[p^\infty]$ is the finite flat subgroup scheme which corresponds to the finite subgroup scheme $\ker\big(\rho_{\Lambda,\Lambda'}[p^\infty] : A_\Lambda[p^\infty] \to A_{\Lambda'}[p^\infty]\big)$ via the isomorphism $B_\Lambda[p^\infty] \simeq A_\Lambda[p^\infty]$. As explained, the polarization λ_B is uniquely determined by $\lambda_{B,\Lambda}$ for one member $\Lambda \in \mathcal{L}'$. Therefore, there is a unique extension of the polarization λ_B on $(B_{\Lambda'})_{\Lambda' \in \mathcal{L}'}$. So we construct a member $((B_{\Lambda'})_{\Lambda' \in \mathcal{L}'}, \bar{\lambda}_B, \bar{\eta}_B)$ such that there is an isomorphism $((B_{\Lambda'}), \lambda_B)[p^\infty] \simeq ((A_{\Lambda'}), \lambda)[p^\infty]$. This shows the surjectivity of $\pi_{\mathsf{K}',\mathsf{K}} : \mathcal{C}^{\rm naiv}_{\mathcal{L}'}(x') \to \mathcal{C}^{\rm naiv}_{\mathcal{L}}(x)$.

Let $\mathcal{L}_{\rm GSp}$ denote the set of $\mathbb{Z}_p$-lattices in $\mathcal{L}$ by ignoring the O_{B_p}-module structure, and let $\mathbf{A}_{\mathcal{L}_{\rm GSp}}$ denote the Siegel moduli space over O_E with parahoric level $\mathcal{L}_{\rm GSp}$ at p. For any $z \in \mathbf{A}_{\mathcal{L}_{\rm GSp}}(k)$, we denote by $\mathcal{C}_{\mathcal{L}_{\rm GSp}}(z)$ the central leaves passing through z in $\mathbf{A}_{\mathcal{L}_{\rm GSp}} \otimes k$. Forgetting the endomorphism structure gives a finite morphism

$$f_{\mathcal{L}} : \mathbf{M}^{\rm naiv}_{\mathcal{L}} \to \mathbf{A}_{\mathcal{L}_{\rm GSp}}$$

[86, Proposition 1.1]. Similarly, we have a finite morphism $f_{\mathcal{L}'} : \mathbf{M}^{\rm naiv}_{\mathcal{L}'} \to \mathbf{A}_{\mathcal{L}'_{\rm GSp}}$ and a commutative diagram

$$\begin{array}{ccc} \mathbf{M}^{\rm naiv}_{\mathcal{L}'} & \xrightarrow{f_{\mathcal{L}'}} & \mathbf{A}_{\mathcal{L}'_{\rm GSp}} \\ \downarrow{\scriptstyle \pi_{\mathsf{K}',\mathsf{K}}} & & \downarrow{\scriptstyle \pi^{\rm GSp}_{\mathcal{L}',\mathcal{L}}} \\ \mathbf{M}^{\rm naiv}_{\mathcal{L}} & \xrightarrow{f_{\mathcal{L}}} & \mathbf{A}_{\mathcal{L}_{\rm GSp}}. \end{array}$$

Restricting the forgetting morphisms $f_{\mathcal{L}'}$ and $f_{\mathcal{L}}$ to the central leaves we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{C}^{\rm naiv}_{\mathcal{L}'}(x') & \xrightarrow{f_{\mathcal{L}'}} & \mathcal{C}_{\mathcal{L}'_{\rm GSp}}(f_{\mathcal{L}'}(x')) \\ \downarrow{\scriptstyle \pi_{\mathsf{K}',\mathsf{K}}} & & \downarrow{\scriptstyle \pi^{\rm GSp}_{\mathcal{L}',\mathcal{L}}} \\ \mathcal{C}^{\rm naiv}_{\mathcal{L}}(x) & \xrightarrow{f_{\mathcal{L}}} & \mathcal{C}_{\mathcal{L}_{\rm GSp}}(f_{\mathcal{L}}(x)). \end{array}$$

Note that the morphism $\pi^{\rm GSp}_{\mathcal{L}',\mathcal{L}}$ is finite [31, Section 7] and so is the composition $\pi^{\rm GSp}_{\mathcal{L}',\mathcal{L}} \circ f_{\mathcal{L}'}$. From the above commutative diagram, any fiber of $\pi_{\mathsf{K}',\mathsf{K}}$ is contained in a fiber of the finite morphism $\pi^{\rm GSp}_{\mathcal{L}',\mathcal{L}} \circ f_{\mathcal{L}'}$, which has only finitely many elements. Thus, the morphism $\pi_{\mathsf{K}',\mathsf{K}} : \mathcal{C}^{\rm naiv}_{\mathcal{L}'}(x') \to \mathcal{C}^{\rm naiv}_{\mathcal{L}}(x)$ has finite fibers. It follows that the morphism $\mathcal{C}_{\mathcal{L}'}(x') \to \mathcal{C}_{\mathcal{L}}(x)$ has finite fibers. It remains to show the surjectivity.

Note that the generic fiber of $\mathbf{M}^{\rm naiv}_{\mathcal{L}}$ is the disjoint union of $\mathrm{Sh}^{(1)}_{\mathsf{K},E} = \mathrm{Sh}_{\mathsf{K},E}, \dots, \mathrm{Sh}^{(m)}_{\mathsf{K},E}$, where m is the cardinality of the finite set $\ker^1(\mathbb{Q}, G) := \ker\big(H^1(\mathbb{Q}, G) \to \prod_v H^1(\mathbb{Q}_v, G)\big)$, for each i, $\mathrm{Sh}^{(i)}_{\mathsf{K},E} = \mathrm{Sh}^{(i)}_{\mathsf{K}} \otimes_{\mathsf{E}} E$, and $\mathrm{Sh}^{(i)}_{\mathsf{K}}$ is the corresponding model of $\mathrm{Sh}^{(i)}_{\mathsf{K}}(\mathbb{C})$ in the decomposition $\mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}}(\mathbb{C}) = \coprod_{i=1}^{m} \mathrm{Sh}^{(i)}_{\mathsf{K}}(\mathbb{C})$. This is described in [42, Section 8] under the unramified setting but because this is a statement in characteristic zero and we have assumed $p \neq 2$, it also holds for the present setting. Similarly, $\mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}'} \otimes E = \coprod_{i=1}^{m} \mathrm{Sh}^{(i)}_{\mathsf{K}',E}$ and the transition map sends $\mathrm{Sh}^{(i)}_{\mathsf{K}',E} \twoheadrightarrow \mathrm{Sh}^{(i)}_{\mathsf{K},E}$.

Let $y = [((B_\Lambda)_\Lambda, \bar{\lambda}_B, \bar{\eta}_B)] \in \mathcal{C}_{\mathcal{L}}(x)$, and $y' = [((B_{\Lambda'})_{\Lambda'}, \bar{\lambda}_B, \bar{\eta}_B)] \in \mathcal{C}^{\mathrm{naiv}}_{\mathcal{L}'}(x')$ be a point we constructed over y as before. As $x' \in \mathscr{S}_{\mathcal{L}'}(k)$, one fixes a lifting $\widetilde{x}' \in \mathscr{S}_{\mathcal{L}'}(R)$ of x' over a complete DVR R with residue field k; the base change $\widetilde{x}'_{\mathrm{Frac}(R)}$ lands in $\mathrm{Sh}_{\mathsf{K}'}(\mathrm{Frac}(R))$. By the Serre-Tate theorem, deforming abelian varieties is the same as deforming the associated p-divisible groups. Thus, the way of lifting x' to $\widetilde{x}'$ produces a lifting $\widetilde{y}' \in \mathbf{M}^{\mathrm{naiv}}_{\mathcal{L}'}(R)$ of y' over R. Since its image $\pi_{\mathsf{K}',\mathsf{K}}(\widetilde{y}') =: \widetilde{y}$ is a lifting of y over R, the point $\widetilde{y}_{\mathrm{Frac}(R)}$ lands in Sh_{K}. Therefore, the point $\widetilde{y}'_{\mathrm{Frac}(R)}$ lands in $\mathrm{Sh}_{\mathsf{K}'}$ and $y' \in \mathcal{C}_{\mathcal{L}'}(x')$. □

Suppose that $K' = K'^{\circ}$ and $K = K^{\circ}$, which are parahoric subgroups by definition. Put $\mathscr{S}_{\mathsf{K}} := \mathscr{S}_{\mathcal{L}}$ and $\mathscr{S}_{\mathsf{K}'} := \mathscr{S}_{\mathcal{L}'}$. Let $k = \overline{\mathbb{F}}_p$. For each point $x \in \mathscr{S}_{\mathsf{K}}(k)$, there exists an $O_{B_p} \otimes_{\mathbb{Z}_p} \breve{\mathbb{Q}}_p$-linear isomorphism

$$M(A_{x,\Lambda}) \otimes \breve{\mathbb{Q}}_p \simeq V \otimes \breve{\mathbb{Q}}_p$$

which preserves the pairings up to a scalar in $\breve{\mathbb{Q}}_p^{\times}$, where $M(A_{x,\Lambda})$ is the covariant Dieudonné module of $A_{x,\Lambda}$, for one $\Lambda \in \mathcal{L}$. Transporting the Frobenius map on $M(A_{x,\Lambda})$ to $V \otimes \breve{\mathbb{Q}}_p$, we obtain an element $G(\breve{\mathbb{Q}}_p)$, which is well-defined up to σ-$\breve{K}$-conjugate. Thus, we have defined a map

$$\Upsilon_{\mathsf{K}} : \mathscr{S}_{\mathsf{K}}(k) \to G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma,$$

whose fibers are the central leaves introduced in Definition A.3.4. Similarly, we also have a map $\Upsilon_{\mathsf{K}'} : \mathscr{S}_{\mathsf{K}'}(k) \to G(\breve{\mathbb{Q}}_p)/\breve{K}'_\sigma$ and have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{S}_{\mathsf{K}'}(k) & \xrightarrow{\Upsilon_{\mathsf{K}'}} & G(\breve{\mathbb{Q}}_p)/\breve{K}'_\sigma \\ \downarrow{\scriptstyle \pi_{K',K}} & & \downarrow{\scriptstyle \pi^G} \\ \mathscr{S}_{\mathsf{K}}(k) & \xrightarrow{\Upsilon_{\mathsf{K}}} & G(\breve{\mathbb{Q}}_p)/\breve{K}_\sigma. \end{array}$$

Proposition A.3.5 confirms axiom 4 (c) of [31] for the canonical Rapoport-Zink integral models $\mathscr{S}_{\mathcal{L}}$ under the assumption $p > 2$, $K' = K'^{\circ}$ and $K = K^{\circ}$.

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