HARDER-NARASIMHAN STRATA AND p-ADIC PERIOD DOMAINS

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Abstract. We revisit the Harder-Narasimhan stratification on a minuscule p-adic flag variety, by the theory of modifications of G-bundles on the Fargues-Fontaine curve. We compare the Harder-Narasimhan strata with the Newton strata introduced by Caraiani-Scholze. As a consequence, we get further equivalent conditions in terms of p-adic Hodge-Tate period domains for fully Hodge-Newton decomposable pairs. Moreover, we generalize these results to arbitrary cocharacters case by considering the associated B_{dR}^+-affine Schubert varieties. Applying Hodge-Tate period maps, our constructions give applications to p-adic geometry of Shimura varieties and their local analogues.

Contents

1. Introduction

This paper is a continuation and complement of our previous work [3]. We look at "p-adic period domains" from a different perspective (we refer to [3] and the references therein for more background on p-adic period domains). We also extend the main result of [3] to general (not necessarily minuscule) cocharacters.

More precisely, we revisit the Harder-Narasimhan stratifications on p-adic flag varieties, which were defined using the theory of filtered vector spaces with additional structures by Rapoport [34], and Dat-Orlik-Rapoport[6] Parts 1 and 2. In fact, in [34] only the maximal open strata were considered, while in [6] Parts 1 and 2 these Harder-Narasimhan stratifications were mainly investigated for reductive groups over finite fields. In this paper, we are interested in the p-adic setting, motivated by the work of Fargues [13] in the context of Harder-Narasimhan polygons for p-divisible groups. The pure linear algebra context here suggests that it should be easier to access than the usual context of filtered isocrystals with additional structures as [32, 37] and [6] Part 3. Under base change, filtered vector spaces can be viewed as filtered isocrystals with trivial underlying isocrystals. Thus we can study these p-adic Harder-Narasimhan strata by plugging them into the setting of Rapoport-Zink [37] chapter 1 and Dat-Orlik-Rapoport [6] Part 3, where the theory of filtered isocrystals with additional structures serves as the basic tool. In a different direction, the open Harder-Narasimhan strata were also

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defined and studied in certain cases by van der Put and Voskuil in [43].

Thanks to the recent developments in p-adic Hodge theory [14, 40], now we can apply the theory of modifications of G-bundles on the Fargues-Fontaine curve to study the Harder-Narasimhan strata. This new method has the advantage that it is easier and natural to compare the Harder-Narasimhan stratification with some other important stratifications on p-adic flag varieties, such as the Newton stratification[1] introduced by Caraiani-Scholze in [2] section 3, where the Fargues-Fontaine curve also plays the key role. From the point of view of period morphisms of local Shimura varieties, we consider these Harder-Narasimhan and Newton stratifications as constructions on the *Hodge-Tate side*. The purpose of this paper is to understand the relation between these two stratifications. In our previous work [3], we studied the Harder-Narasimhan strata and Newton strata on the *de Rham side* (although we mostly restricted to the open strata: the weakly admissible locus and the admissible locus). At the end, we will see the theories on both sides are very closely related, in the sense they are actually *dual* to each other.

To be more precise, let us fix some notations. Let G be a reductive group over[2] $\mathbb{Q}_p$ and $\{\mu\}$ be a conjugacy class of cocharacters $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$. Attached to $(G, \{\mu\})$, we have flag varieties $\mathscr{F}\ell(G,\mu)$ and $\mathscr{F}\ell(G,\mu^{-1})$, defined over a finite extension E of $\mathbb{Q}_p$. We view them as *adic spaces* over $\breve{E}$, the completion of maximal unramified extension of E. We assume that μ is *minuscule* at this moment for simplicity.

Consider the p-adic flag variety $\mathscr{F}\ell(G,\mu^{-1})$ first. Then by studying modifications of the trivial G-bundle over the Fargues-Fontaine curve, we can introduce two stratifications as follows. The first one is the Newton stratification introduced by Caraiani-Scholze in [2] section 3. Let $C|\breve{E}$ be any algebraically closed perfectoid field and $X = X_{C^\flat}$ be the Fargues-Fontaine curve over $\mathbb{Q}_p$ attached to the tilt $C^\flat$ equipped with a closed point ∞ with residue field C. To each point $x \in \mathscr{F}\ell(G,\mu^{-1})(C)$, we can attach a modified G-bundle at ∞

$$\mathcal{E}_{1,x}$$

of the trivial G-bundle $\mathcal{E}_1$ over X. The isomorphism class of $\mathcal{E}_{1,x}$ defines a point in $B(G)$ (the set of σ-conjugacy classes in $G(\breve{\mathbb{Q}}_p)$, cf. [23, 25]), which in fact lies in the Kottwitz set $B(G,\mu)$ ([25] section 6). This gives the Newton stratification

$$\mathscr{F}\ell(G,\mu^{-1}) = \coprod_{[b']\in B(G,\mu)} \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b']}.$$

Each stratum $\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b']}$ is a locally closed subspace of $\mathscr{F}\ell(G,\mu^{-1})$, therefore we can either view it as a pseudo-adic space in the sense of Huber ([21]) or a diamond in the sense of Scholze ([38]).

On the other hand, we can define the Harder-Narasimhan vector

$$\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f)$$

attached to the modification triple, which is an element in the set $\mathcal{N}(G)$ of [35] 1.7 attached to G. In the case of GL_n this has been studied by Cornut-Irissarry in [5]. It turns out that this vector $\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f)$ is identical to the Harder-Narasimhan vector $\nu(\mathcal{F}_x)$ defined in [6] chapter VI.3 for the "G-filtration" $\mathcal{F}_x$ attached to x. We can show that in fact $\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f)$ (in fact some normalization of it) lies in $\mathcal{N}(G,\mu)$, the image of

[1]Here to consider the Newton stratification on a p-adic flag variety, one has to assume that the corresponding cocharacter is minuscule. For general cocharacters, we need to work with the B_{dR}^+-affine Schubert cells, see the later discussions.

[2]Throughout this paper, the base field for our reductive groups is $\mathbb{Q}_p$. However one can replace it by any finite extension of $\mathbb{Q}_p$ and all the results are still true.

$B(G,\mu)$ under the Newton map $\nu : B(G) \to \mathcal{N}(G)$ (cf. [23] section 4). In this way we get the Harder-Narasimhan stratification

$$\mathscr{F}\ell(G,\mu^{-1}) = \coprod_{[b']\in B(G,\mu)} \mathscr{F}\ell(G,\mu^{-1})^{HN=[b']}.$$

Similarly as above, each Harder-Narasimhan stratum is a locally closed subspace of $\mathscr{F}\ell(G,\mu^{-1})$. For both stratifications, the maximal open strata are indexed by the *basic* element $[b] \in B(G,\mu)$ and we have an inclusion

$$\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]} \subset \mathscr{F}\ell(G,\mu^{-1})^{HN=[b]},$$

which comes from the inequality between the Harder-Narasimhan vector and the Newton vector in $\mathcal{N}(G)$

$$\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f) \leq \nu(\mathcal{E}_{1,x}),$$

cf. Proposition 3.4. See also [13] Proposition 14 and [5] Proposition 44 (both in the case of GL_n).

Now consider the side of $\mathscr{F}\ell(G,\mu)$. Let $b \in G(\breve{\mathbb{Q}}_p)$ be such that the associated class $[b] \in B(G,\mu)$ and it is the *basic* element. The triple $(G,\{\mu\},[b])$ then forms a basic local Shimura datum ([36]). Recall that by Fargues's main theorem in [11], we have a bijection $B(G) \xrightarrow{\sim} H^1_{\mathrm{et}}(X,G)$, $\quad [b'] \mapsto [\mathcal{E}_{b'}]$. Then we can also define the Newton stratification and Harder-Narasimhan stratification on $\mathscr{F}\ell(G,\mu)$ by considering the modifications

$$\mathcal{E}_{b,x}, \quad x \in \mathscr{F}\ell(G,\mu)(C)$$

of the G-bundle $\mathcal{E}_b$ on X similarly as above. The Newton stratification[3] in this setting was introduced in [3] 5.3, while the Harder-Narasimhan stratification was introduced in [6] chapter IX.6, where the more classical theory of filtered isocrystals with additional structures was used. The open Newton stratum is the admissible locus $\mathscr{F}\ell(G,\mu,b)^a$ ([39, 40, 34, 3]), while the open Harder-Narasimhan stratum is the weakly admissible locus $\mathscr{F}\ell(G,\mu,b)^{wa}$ ([37, 6]). By the theorem of Colmez-Fontaine (see [14] chapter 10), we have also the inclusion

$$\mathscr{F}\ell(G,\mu,b)^a \subset \mathscr{F}\ell(G,\mu,b)^{wa}.$$

The Newton and Harder-Narasimhan stratifications on the side of $\mathscr{F}\ell(G,\mu)$ also have the same index set, $B(G,0,\nu_b\mu^{-1})$, a generalized Kottwitz set which was introduced in [34] and [3] section 4.

To summarize, we have the open strata $\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]}$ and $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b]}$ inside $\mathscr{F}\ell(G,\mu^{-1})$ constructed starting from the triple $(G,\{\mu^{-1}\},[1])$ (the Hodge-Tate side), and the open strata $\mathscr{F}\ell(G,\mu,b)^a$ and $\mathscr{F}\ell(G,\mu,b)^{wa}$ inside $\mathscr{F}\ell(G,\mu)$ constructed from the local Shimura datum $(G,\{\mu\},[b])$ (the de Rham side). These open strata are related by the following diagram

$$\begin{array}{ccccc} & & \mathcal{M}(G,\mu,b)_\infty & & \\ & \swarrow^{\pi_{dR}} & & \searrow^{\pi_{HT}} & \\ \mathscr{F}\ell(G,\mu,b)^{a,\diamond} & & & & \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b],\diamond} \\ \big\downarrow & & & & \big\downarrow \\ \mathscr{F}\ell(G,\mu,b)^{wa,\diamond} & & & & \mathscr{F}\ell(G,\mu^{-1})^{HN=[b],\diamond}, \end{array}$$

where $\mathcal{M}(G,\mu,b)_\infty$ is the local Shimura variety with infinite level attached to the datum $(G,\{\mu\},[b])$ (cf. [3] Theorem 3.3 and [40] sections 23 and 24), π_{dR} and π_{HT} are the p-adic

[3] In [3] this was called the Harder-Narasimhan stratification, which should not be confused with the Harder-Narasimhan stratification here.

de Rham and Hodge-Tate period morphisms respectively. Thus it is more reasonable to call $\mathscr{F}\ell(G,\mu,b)^a$ and $\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]}$ *p-adic period domains*, although historically in [34, 6] it was the open Harder-Narasimhan strata $\mathscr{F}\ell(G,\mu,b)^{wa}$ and $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b]}$ that were called period domains. By construction, $\mathcal{M}(G,\mu,b)_\infty$ is a diamond over $\breve{E}$. This is why we pass to the diamonds associated to the above spaces.

Recall that Görtz-He-Nie have introduced the notion of *fully Hodge-Newton decomposability* for the Kottwitz set $B(G,\mu)$ (or the pair $(G,\{\mu\})$, cf. [17] Definition 2.1, where μ is a not necessarily minuscule cocharater). Roughly, this condition means that for any *non basic* element $[b'] \in B(G,\mu)$, the pair $([b'],\{\mu\})$ satisfies the *Hodge-Newton condition*. By [17] Theorem 2.5 we have a complete classification for fully Hodge-Newton decomposable pairs $(G,\{\mu\})$. Now we have the following theorem.

Theorem 1.1 (Theorem 5.1)**.** *Assume that μ is minuscule and $[b] \in B(G,\mu)$ is basic. The following statements are equivalent:*

(1) *$B(G,\mu)$ is fully Hodge-Newton decomposable,*
(2) *$\mathscr{F}\ell(G,\mu,b)^a = \mathscr{F}\ell(G,\mu,b)^{wa}$,*
(3) *$\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]} = \mathscr{F}\ell(G,\mu^{-1})^{HN=[b]}$.*

The equivalence (1) ⇔ (2) was proved in [3] Theorem 6.1. Here the novelty is to add the additional information (3). In fact, the equivalence (1) ⇔ (3) was conjectured by Fargues in [13] 9.7. In Theorem 5.1 we will give several further equivalent conditions.

The idea to prove the above theorem is to introduce the *dual* local Shimura datum $(J_b,\{\mu^{-1}\},[b^{-1}])$ (see subsection 4.1 or [36] Conjecture 5.8 and [40] Corollary 23.2.3) and consider the following similar statements:

(a) $B(J_b,\mu^{-1})$ is fully Hodge-Newton decomposable,
(b) $\mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^a = \mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^{wa}$,
(c) $\mathscr{F}\ell(J_b,\mu)^{Newt=[b^{-1}]} = \mathscr{F}\ell(J_b,\mu)^{HN=[b^{-1}]}$.

By [3] Corollary 4.15, we have shown (1) ⇔ (a) by purely group theoretical methods. Then by [3] Theorem 6.1, we get (a) ⇔ (b). The point here is to show (3) ⇔ (b) and (2) ⇔ (c), which can be viewed as certain dualities for the Newton and Harder-Narasimhan stratifications on the p-adic flag varieties $\mathscr{F}\ell(G,\mu)$ and $\mathscr{F}\ell(G,\mu^{-1})$. See Theorem 4.4 and Corollary 4.5. In fact, the duality for Newton stratifications already appeared implicitly in [3] 5.3, and the duality for Harder-Narasimhan stratifications appeared implicitly in [6] IX.6. The novelties here are:

- studying both dualities more systematically in the setting of twin towers principle (see [3] 5.1 and the following section 4),
- showing that how the combination of both dualities produces new information and sheds new lights on the other side of the whole story,
- extending both dualities to general not necessarily minuscule cocharacters μ by looking at the corresponding B_{dR}^+-affine Schubert cells, see below.

On the other hand, we can show *directly* the equivalence (1) ⇔ (3) by similar arguments as in the proof of [3] Theorem 6.1, see Remark 5.2. Then under the equivalences (1) ⇔ (a), (2) ⇔ (c) and (3) ⇔ (b), we get (a) ⇔ (c) ⇔ (b), and thus (1) ⇔ (2). In this way we give another proof for [3] Theorem 6.1, although essentially the two proofs are the same. As one has seen, the equivalence (1) ⇔ (a) is in fact the only key ingredient which we take from [3].

For a general *not necessarily minuscule* cocharacter μ, to have a similar picture as above, the correct objects to study are the B_{dR}^+-affine Schubert cells (cf. [40] sections 19 and 20)

$$\mathrm{Gr}_\mu \quad \text{and} \quad \mathrm{Gr}_{\mu^{-1}}$$

instead of the corresponding flag varieties. One of the main results of [40] says that Gr_μ and $\mathrm{Gr}_{\mu^{-1}}$ are *locally spatial diamonds* over E. They are related to flag varieties by the Bialynicki-Birula maps (cf. [2] Proposition 3.4.3 and [40] Proposition 19.4.2)

$$\pi_\mu : \mathrm{Gr}_\mu \to \mathscr{F}\ell(G,\mu)^\diamond, \quad \text{and} \quad \pi_{\mu^{-1}} : \mathrm{Gr}_{\mu^{-1}} \to \mathscr{F}\ell(G,\mu^{-1})^\diamond.$$

For any algebraically closed field $C|\breve{E}$ and any point $x \in \mathrm{Gr}_{\mu^{-1}}(C,\mathcal{O}_C)$, we have a modification $\mathcal{E}_{1,x}$ of the trivial G-bundle $\mathcal{E}_1$ on the Fargues-Fontaine curve $X = X_{C^\flat}$. By considering the Newton vector (resp. Harder-Narasimhan vector) attached to $\mathcal{E}_{1,x}$ (resp. the triple $(\mathcal{E}_1,\mathcal{E}_{1,x},f)$), we can construct the Newton (resp. Harder-Narasimhan) stratification on $\mathrm{Gr}_{\mu^{-1}}$ similarly as before. It turns out the Harder-Narasimhan stratification is the pullback of the corresponding stratification on the flag variety under $\pi_{\mu^{-1}} : \mathrm{Gr}_{\mu^{-1}} \to \mathscr{F}\ell(G,\mu^{-1})^\diamond$. Let $b \in G(\breve{\mathbb{Q}}_p)$ be such that $[b] \in B(G,\mu)$. For any $C|\breve{E}$ as above and any point $x \in \mathrm{Gr}_\mu(C,\mathcal{O}_C)$, we have a modification $\mathcal{E}_{b,x}$ of the G-bundle $\mathcal{E}_b$ over $X = X_{C^\flat}$ attached to $[b]$. By considering the Newton vector of $\mathcal{E}_{b,x}$, We can construct the Newton stratification on Gr_μ. We define the Harder-Narasimhan stratification on Gr_μ as the pullback of the corresponding stratification on $\mathscr{F}\ell(G,\mu)$ (defined in [6] Part 3) under the map $\pi_\mu : \mathrm{Gr}_\mu \to \mathscr{F}\ell(G,\mu)^\diamond$. Moreover, the dualities for Newton and Harder-Narasimhan stratifications on $\mathrm{Gr}_{\mu^{-1}}$ and Gr_μ also hold in this general setting (see Theorem 6.10). When μ is minuscule, the Bialynicki-Birula maps $\pi_{\mu^{-1}} : \mathrm{Gr}_{\mu^{-1}} \to \mathscr{F}\ell(G,\mu^{-1})^\diamond$ and $\pi_\mu : \mathrm{Gr}_\mu \to \mathscr{F}\ell(G,\mu)^\diamond$ are isomorphisms (cf. [2] Theorem 3.4.5 and [40] Proposition 19.4.2), and we recover the Newton and Harder-Narasimhan stratifications on the flag varieties $\mathscr{F}\ell(G,\mu^{-1})$ and $\mathscr{F}\ell(G,\mu)$.

Let $b \in G(\breve{\mathbb{Q}}_p)$ be such that $[b] \in B(G,\mu)$. Starting from the datum $(G,\{\mu\},[b])$, we get the admissible locus Gr_μ^a (the open Newton stratum) and the weakly admissible locus Gr_μ^{wa} (the open Harder-Narasimhan stratum) inside Gr_μ. Both of them are *open* sub diamonds of Gr_μ. The theorem of Colmez-Fontaine ([14] chapter 10) implies that we have the inclusion of locally spatial diamonds over $\breve{E}$:

$$\mathrm{Gr}_\mu^a \subset \mathrm{Gr}_\mu^{wa}.$$

Now assume that $[b] \in B(G,\mu)$ is *basic*. On the Hodge-Tate side $\mathrm{Gr}_{\mu^{-1}}$, by the inequality $\nu(\mathcal{E}_1,\mathcal{E}_{1,x},f) \leq \nu(\mathcal{E}_{1,x})$ as above, we have the inclusion for *open* Newton and Harder-Narasimhan strata:

$$\mathrm{Gr}_{\mu^{-1}}^{Newt=[b]} \subset \mathrm{Gr}_{\mu^{-1}}^{HN=[b]}.$$

Here is the generalization of Theorem 1.1, where we remove the minuscule condition (see [13] 9.7, Conjecture 1 (1)):

Theorem 1.2 (Theorem 6.16, Corollary 6.17)**.** *Let $[b] \in B(G,\mu)$ be basic. The following statements are equivalent:*

1. *$B(G,\mu)$ is fully Hodge-Newton decomposable,*
2. *$\mathrm{Gr}_\mu^a = \mathrm{Gr}_\mu^{wa}$,*
3. *$\mathrm{Gr}_{\mu^{-1}}^{Newt=[b]} = \mathrm{Gr}_{\mu^{-1}}^{HN=[b]}$.*

As before, once we prove the equivalence (1) ⇔ (2), which is the generalized version of [3] Theorem 6.1, the remaining equivalence (1) ⇔ (3) follows by the dualities for Newton and Harder-Narasimhan stratifications. The key new idea is to study the geometry of Gr_μ in terms of B_{dR}^+-affine Grassmannians of parabolic and Levi subgroups of G, which is in some sense a theory of (generalized) semi-infinite orbits in the current setting. More precisely, we have the following new information:

- We prove the dimension formula and closure relation for the B_{dR}^+-affine Schubert cells (same as the classical setting, see Proposition 6.2).
- Let M be a Levi subgroup inside a parabolic P of G over $\mathbb{Q}_p$. Then we have a stratification $\mathrm{Gr}_\mu = \coprod_{\lambda \in S_M(\mu)} \mathrm{Gr}_{G,\lambda}$, induced on Gr_μ by the natural diagram of

the B_{dR}^+-affine Grassmannians of M, P and G respectively (the strata $\mathrm{Gr}_{G,\lambda}$ are intersections of the generalized semi-infinite orbits S_λ with Gr_μ, see subsection 6.6 for more details). Moreover, we know the closure relation for this stratification and we give some description for the index set $S_M(\mu)$ (which is in fact related to the geometric Satake equivalence for B_{dR}^+-affine Grassmannians, cf. [15]).

- The above stratification naturally arises when we study reductions of modifications of G-bundles to P-bundles (resp. M-bundles) on the Fargues-Fontaine curve, cf. Lemma 6.14. Using this, we give an interpretation of the weakly admissible locus Gr_μ^{wa} in terms of the Fargues-Fontaine curve, cf. Proposition 6.15, which is a generalization of [3] Proposition 2.7.

With these ingredients at hand, the arguments in the proof of [3] Theorem 6.1 apply here to establish the above equivalence (1) $\Leftrightarrow$ (2), see Theorem 6.16 for more details.

The pullbacks under the Hodge-Tate period morphisms define Harder-Narasimhan stratifications on moduli of local G-Shtukas (cf. [40] section 23) and on Shimura varieties, see sections 7 and 8. We hope these constructions will be found useful for further arithmetic applications.

We briefly describe the structure of this article. In section 2, we review some basics about modifications of G-bundles on the Fargues-Fontaine curve, which will be our tool in the following. In section 3, we define and study the Newton and Harder-Narasimhan strata on the flag variety $\mathscr{F}\ell(G,\mu^{-1})$. In section 4, we explain how to transfer the point of view by using modifications of J_b-bundles. More precisely, we explain how to identify the Newton and Harder-Narasimhan strata on the Hodge-Tate (resp. de Rham) side for G by the corresponding strata on the de Rham (resp. Hodge-Tate) side for J_b. These are the dualities of the Newton and Harder-Narasimhan strata established in Theorem 4.4 and Corollary 4.5. In section 5, we summarize several various equivalent conditions for a fully Hodge-Newton decomposable pair, using the results in sections 3 and 4. In section 6, we explain how to generalize the previous constructions and results to not necessarily minuscule cocharacters μ by studying the B_{dR}^+-affine Schubert cells $\mathrm{Gr}_{\mu^{-1}}$ and Gr_μ. In particular, we choose to work on the de Rham side Gr_μ and prove the generalized Fargues-Rapoport conjecture (Theorem 6.16). Then we transfer back to the Hodge-Tate side $\mathrm{Gr}_{\mu^{-1}}$ (Corollary 6.17) by dualities (Theorem 6.10). In sections 7 and 8, we give applications to moduli of local G-Shtukas and Shimura varieties respectively.

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2. Modifications of G-bundles on the Fargues-Fontaine curve

Let $C|\mathbb{Q}_p$ be a fixed algebraically closed perfectoid field, with $C^\flat$ its tilt. We have the Fargues-Fontaine curve $X = X_{C^\flat}$ over $\mathbb{Q}_p$, together with the canonical point $\infty \in X$ with completed local ring $\widehat{\mathcal{O}}_{X,\infty} = B_{dR}^+(C)$. We refer the reader to [14] for a detailed study of this curve, and to [3] section 1 for a brief summary. We will simply write $B_{dR} = B_{dR}(C)$ and $B_{dR}^+ = B_{dR}^+(C)$ in the following. Let $\xi \in B_{dR}^+$ be a fixed uniformizer.

Let $\varphi\mathrm{-Mod}_{\breve{\mathbb{Q}}_p}$ be the category of F-isocrystals over $\overline{\mathbb{F}}_p$, and Bun_X be the category of vector bundles on X. A basic result of [14] says that we have a natural functor

$$\mathcal{E}(-) : \varphi\mathrm{-Mod}_{\breve{\mathbb{Q}}_p} \longrightarrow \mathrm{Bun}_X$$

which is essentially surjective. For $\mathcal{E} \in \mathrm{Bun}_X$, we have the Harder-Narasimhan filtration of $\mathcal{E}$ with the associated Harder-Narasimhan vector $\nu(\mathcal{E}) \in \mathbb{Q}^n_+$ where $n = \operatorname{rank} \mathcal{E}$. To avoid confusion, we will call it the Newton filtration, since later we will introduce several further Harder-Narasimhan filtrations. We refer to [1, 4] for some generalities on the Harder-Narasimhan formalism.

2.1. **Modifications of vector bundles.** We are interested in the category of modifications[4] of vector bundles on X, which we denote by

$$\mathrm{Modif}_X.$$

Recall that a modification of vector bundles is a triple $\underline{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, f)$, where

- $\mathcal{E}_1, \mathcal{E}_2$ are vector bundles on X,
- $f : \mathcal{E}_1|_{X\setminus\{\infty\}} \xrightarrow{\sim} \mathcal{E}_2|_{X\setminus\{\infty\}}$ is an isomorphism.

A morphism $F : \underline{\mathcal{E}} \to \underline{\mathcal{E}}'$ is a pair of morphisms $F_i : \mathcal{E}_i \to \mathcal{E}'_i$ with $F_2 \circ f = f' \circ F_1$. This category Modif_X is a quasi-abelian $\mathbb{Q}_p$-linear rigid $\otimes$-category with a Tate twist, cf. [5] 3.1.4. For a modification $(\mathcal{E}_1, \mathcal{E}_2, f)$, let

$$\mathcal{E}^+_{i,dR} = \mathcal{E}^\wedge_{i,\infty}$$

be the completed local stalk at ∞ of $\mathcal{E}_i$, and

$$f_{dR} : \mathcal{E}^+_{1,dR}[\xi^{-1}] \to \mathcal{E}^+_{2,dR}[\xi^{-1}]$$

be the induced isomorphism of B_{dR}-vector spaces. We have the Newton filtrations $\mathcal{F}_{N,i}(\underline{\mathcal{E}}) := \mathcal{F}(\mathcal{E}_i)$ for $i = 1, 2$. Moreover, we have the Hodge filtrations $\mathcal{F}_{H,i}(\underline{\mathcal{E}})$, which are the $\mathbb{Z}$-filtrations on the residues $\mathcal{E}_i(\infty) := \mathcal{E}^+_{i,dR}/\xi\mathcal{E}^+_{i,dR}$ of $\mathcal{E}_i$ induced by $\mathcal{E}^+_{3-i,dR}$: for any $j \in \mathbb{Z}$,

$$\mathcal{F}^j_{H,1} = \frac{f^{-1}_{dR}(\xi^j\mathcal{E}^+_{2,dR}) \cap \mathcal{E}^+_{1,dR} + \xi\mathcal{E}^+_{1,dR}}{\xi\mathcal{E}^+_{1,dR}}, \quad \mathcal{F}^j_{H,2} = \frac{f_{dR}(\xi^j\mathcal{E}^+_{1,dR}) \cap \mathcal{E}^+_{2,dR} + \xi\mathcal{E}^+_{2,dR}}{\xi\mathcal{E}^+_{2,dR}}.$$

Let $n = \operatorname{rank}(\mathcal{E}_1) = \operatorname{rank}(\mathcal{E}_2)$. Then $\mathcal{F}_{H,1}$ and $\mathcal{F}_{H,2}$ define opposed types $\nu_{H,i}(\underline{\mathcal{E}}) \in \mathbb{Z}^n_+$.

We have the following natural functors

$$\begin{array}{ccc} & \mathrm{Modif}_X & \\ \overleftarrow{h} \swarrow & & \searrow \overrightarrow{h} \\ \mathrm{Bun}_X & & \mathrm{Bun}_X, \end{array}$$

with

$$\overleftarrow{h}(\mathcal{E}_1, \mathcal{E}_2, f) = \mathcal{E}_2, \quad \overrightarrow{h}(\mathcal{E}_1, \mathcal{E}_2, f) = \mathcal{E}_1.$$

These functors $\overleftarrow{h}$ and $\overrightarrow{h}$ will be related to the de Rham periods and the Hodge-Tate periods respectively.

2.2. **Filtered F-isocrystals.** For any extension $K|\breve{\mathbb{Q}}_p$ (not necessary finite), let

$$\varphi\text{−}\mathrm{FilMod}_{K/\breve{\mathbb{Q}}_p}$$

be the category of filtered F-isocrystals with respect to $K|\breve{\mathbb{Q}}_p$. A filtered F-isocrystal $\mathcal{D} = (D, \varphi, \mathcal{F}) \in \varphi\text{−}\mathrm{FilMod}_{K/\breve{\mathbb{Q}}_p}$ consists of a underlying F-isocrystal $(D, \varphi) \in \varphi\text{−}\mathrm{Mod}_{\breve{\mathbb{Q}}_p}$ together with a $\mathbb{Q}$-filtration $\mathcal{F}$ on $D \otimes_{\breve{\mathbb{Q}}_p} K$. We have the rank and degree functions

$$\operatorname{rank} : \varphi\text{−}\mathrm{FilMod}_{K/\breve{\mathbb{Q}}_p} \to \mathbb{N}, \quad \deg : \varphi\text{−}\mathrm{FilMod}_{K/\breve{\mathbb{Q}}_p} \to \mathbb{Z}$$

defined by

$$\operatorname{rank} \mathcal{D} = \dim D, \quad \deg \mathcal{D} = t_H(\mathcal{D}) - t_N(\mathcal{D}),$$

where $t_H(\mathcal{D}) = \sum_i i \dim gr^i_{\mathcal{F}} D_K$ and $t_N(\mathcal{D}) = v_p(\det \varphi)$. These functions induce a Harder-Narasimhan filtration on $\varphi\text{−}\mathrm{FilMod}_{K/\breve{\mathbb{Q}}_p}$.

[4]In this paper we only consider modifications at the canonical point $\infty \in X$.

Consider the case $K = C$. By Fargues's de Rham classification for modifications of vector bundles in [10] 4.2.2, we can construct a functor

$$\pi : \mathrm{Modif}_X \longrightarrow \varphi\text{−FilMod}_{C/\breve{\mathbb{Q}}_p}$$

as follows.

Recall that by [10] 4.2.2, given a modification $(\mathcal{E}_1, \mathcal{E}_2, f)$ is equivalent to given the pair $(\mathcal{E}_2, \Lambda)$, where $\Lambda \subset \widehat{\mathcal{E}_{2,\infty}}$ is the B_{dR}^+-lattice defined by $\mathcal{E}_1$. Let $B^+ = \bigcap_{n\geq 0} \varphi^n(B_{cris}^+)$, and $\varphi\text{−Mod}_{B^+}$ is the category of φ-modules over B^+. By [14] Théorème 11.1.9, we have an equivalence of categories

$$\mathrm{Bun}_X \simeq \varphi\text{−Mod}_{B^+},$$

which then induces an equivalence of categories

$$\mathrm{Modif}_X \simeq \varphi\text{−Mod}Ja_{B^+},$$

where $\varphi\text{−Mod}Ja_{B^+}$ is the category of gauged φ-modules over B^+. A gauged φ-module (D, φ, Λ) over B^+ consists of (cf. [10] Définition 4.15)

- a φ-module (D, φ) over B^+,
- and a φ-stable B_{dR}^+-lattice $\Lambda \subset D \otimes B_{dR}$.

Let $\overline{B} = (B^+/[\varpi])_{red}$ with $\varpi \in C^\flat$ a pseudo-uniformizer, which is a local ring with residue field $W(k_C)_{\mathbb{Q}}$ (where k_C is the residue field of C), see [11] 5.2.1. Let $\varphi\text{−Mod}_{B^+}$ and $\varphi\text{−Mod}_{\overline{B}}$ be the categories of φ-modules over B^+ and $\overline{B}$ respectively. Then the reduction of scalar induces an equivalence of categories

$$\varphi\text{−Mod}_{B^+} \simeq \varphi\text{−Mod}_{\overline{B}}.$$

Reduction to the residue field of $\overline{B}$ together with the equivalence $\varphi\text{−Mod}_{W(k_C)_{\mathbb{Q}}} \simeq \varphi\text{−Mod}_{\breve{\mathbb{Q}}_p}$ induce a functor

$$\varphi\text{−Mod}_{\overline{B}} \longrightarrow \varphi\text{−Mod}_{\breve{\mathbb{Q}}_p}.$$

Now for $(V, \varphi) \in \varphi\text{−Mod}_{\breve{\mathbb{Q}}_p}$ and $\Lambda \subset V \otimes B_{dR}$, we can define a filtration on V_C by

$$\mathcal{F}^i V_C = \frac{\xi^i \Lambda \cap V_{dR}^+ + \xi V_{dR}^+}{\xi V_{dR}^+},$$

where $V_{dR}^+ = V \otimes B_{dR}^+$. Putting all the things together, we get a functor $\pi : \mathrm{Modif}_X \to \varphi\text{−FilMod}_{C/\breve{\mathbb{Q}}_p}$ by the composition of

$$\mathrm{Modif}_X \to \varphi\text{−Mod}Ja_{B^+} \to \varphi\text{−FilMod}_{C/\breve{\mathbb{Q}}_p}.$$

If $K|\breve{\mathbb{Q}}_p$ is a finite totally ramified extension and $C = \widehat{\overline{K}}$, there is also a functor $\varphi\text{−FilMod}_{K/\breve{\mathbb{Q}}_p} \to \mathrm{Modif}_X$ for which we refer the interested reader to [10] Exemple 4.19.

2.3. **Admissible modifications.** Consider the full subcategory of admissible modifications

$$\mathrm{Modif}_X^{ad}$$

inside Modif_X. Recall that a modification $\underline{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, f)$ is called admissible if $\mathcal{E}_1$ is a semi-stable vector bundle of slope 0 (i.e. $\mathcal{E}_1$ is the trivial vector bundle). This is again a quasi-abelian $\mathbb{Q}_p$-linear rigid $\otimes$-category with a Tate twist. For an admissible modification $\underline{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, f)$, we set

$$\mathcal{F}_N(\underline{\mathcal{E}}) := \mathcal{F}(\mathcal{E}_2), \quad \mathcal{F}_H(\underline{\mathcal{E}}) := \mathcal{F}_{H,1}(\underline{\mathcal{E}}), \quad \nu_N(\underline{\mathcal{E}}) := \nu(\mathcal{E}_2), \quad \text{and} \quad \nu_H(\underline{\mathcal{E}}) := \nu_{H,1}(\underline{\mathcal{E}}).$$

We have an exact $\mathbb{Q}_p$-linear faithful $\otimes$-functor

$$\omega : \mathrm{Modif}_X^{ad} \to \mathrm{Vect}_{\mathbb{Q}_p}, \quad \omega(\underline{\mathcal{E}}) = \Gamma(X, \mathcal{E}_1),$$

which induces a bijection between the poset of strict subobjects of $\underline{\mathcal{E}}$ in Modif_X^{ad} and the poset of $\mathbb{Q}_p$-subspaces of $\omega(\underline{\mathcal{E}})$. We have the rank and degree functions

$$\mathrm{rank}: \mathrm{Modif}_X^{ad} \to \mathbb{N}, \quad \deg: \mathrm{Modif}_X^{ad} \to \mathbb{Z}$$

defined by

$$\mathrm{rank}(\underline{\mathcal{E}}) = \mathrm{rank}(\mathcal{E}_1) = \mathrm{rank}(\mathcal{E}_2) = \dim \omega(\underline{\mathcal{E}})$$

and

$$\deg(\underline{\mathcal{E}}) = \deg \mathcal{E}_2.$$

They induce a Harder-Narasimhan[5] filtration on Modif_X^{ad} with slopes $\mu = \deg/\mathrm{rank}$ in $\mathbb{Q}$. We denote it by $\mathcal{F}(\underline{\mathcal{E}})$ with the associated Harder-Narasimhan vector $\nu(\underline{\mathcal{E}})$.

Later we will need the following variant. Let $\mathrm{Modif}_X^{ad'}$ be the subcategory of modifications $\underline{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, f)$ with E_2 trivial. On this category we have the rank and degree functions defined by $rank(\underline{\mathcal{E}}) = \mathrm{rank}(\mathcal{E}_1) = \mathrm{rank}(\mathcal{E}_2)$ and $\deg(\underline{\mathcal{E}}) = \deg \mathcal{E}_1$. One checks similarly as above that they induce a Harder-Narasimhan filtration on $\mathrm{Modif}_X^{ad'}$. Moreover, we have the equivalence

$$\mathrm{Modif}_X^{ad'} \xrightarrow{\sim} \mathrm{Modif}_X^{ad}, \quad \underline{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, f) \mapsto \underline{\mathcal{E}'} = (\mathcal{E}_2, \mathcal{E}_1, f^{-1})$$

and $\nu(\underline{\mathcal{E}}) = \nu(\underline{\mathcal{E}'})$.

Let $\mathrm{HT}^{B_{dR}}$ be the category of pairs (V, Ξ), where

- V is a finite dimensional $\mathbb{Q}_p$-vector space,
- Ξ is a B_{dR}^+-lattice in $V_{dR} = V \otimes B_{dR}$.

A morphism $F: (V, \Xi) \to (V', \Xi')$ is a $\mathbb{Q}_p$-linear morphism $f: V \to V'$ such that the induced morphism $f_{dR}: V_{dR} \to V'_{dR}$ satisfies $f(\Xi) \subset \Xi'$. This defines a quasi-abelian rigid $\mathbb{Q}_p$-linear $\otimes$-category. For $(V, \Xi) \in \mathrm{HT}^{B_{dR}}$, as above Ξ induces a Hodge filtration $\mathcal{F}_H(V, \Xi)$, which is the $\mathbb{Z}$-filtration on the residue $V_C = V \otimes C$ of the lattice $V_{dR}^+ = V \otimes B_{dR}^+ \subset V_{dR}$. Moreover, we have the rank and degree functions

$$\mathrm{rank}: \mathrm{HT}^{B_{dR}} \to \mathbb{N}, \quad \deg: \mathrm{HT}^{B_{dR}} \to \mathbb{Z}$$

defined by

$$\mathrm{rank}(V, \Xi) = \dim V = \mathrm{rank}_{B_{dR}^+}(\Xi),$$

and

$$\deg(V, \Xi) = \sum_i i \dim gr^i_{\mathcal{F}_H} V_C.$$

They induce a Harder-Narasimhan filtration on $\mathrm{HT}^{B_{dR}}$. We denote it by $\mathcal{F}(V, \Xi)$ with Harder-Narasimhan vector $\nu(V, \Xi)$.

By Fargues's Hodge-Tate classification in [10] 4.2.3, we have an exact $\otimes$-equivalence of $\otimes$-categories

$$\mathrm{HT}: \mathrm{Modif}_X^{ad} \to \mathrm{HT}^{B_{dR}}, \quad \underline{\mathcal{E}} \mapsto (\Gamma(X, \mathcal{E}_1), f_{dR}^{-1}(\mathcal{E}_{2,dR}^+)).$$

The inverse functor is given by $(V, \Xi) \mapsto (\mathcal{E}_1, \mathcal{E}_2, f)$, where

- $\mathcal{E}_1 = V \otimes \mathcal{O}_X$
- $\mathcal{E}_2$ and f are given by the modification of $\mathcal{E}_1$ at ∞ corresponding to the B_{dR}^+-lattice Ξ of $\mathcal{E}_{1,\infty}^\wedge[\xi^{-1}] = V \otimes B_{dR}$ under the Beauville-Laszlo correspondence (cf. [14] 5.3.1).

The functor HT preserves the rank and deg functions on the two categories, and it induces a bijection between the posets of strict subobjects of $\underline{\mathcal{E}}$ and $\mathrm{HT}(\underline{\mathcal{E}})$. Thus it preserves the Harder-Narasimhan filtrations and types on both sides.

[5]In [5] this filtration is called the Fargues filtration.

2.4. **Filtered vector spaces.** Consider the category $\mathrm{Fil}^C_{\mathbb{Q}_p}$ of pairs $(V,\mathcal{F})$, where

- V is a finite $\mathbb{Q}_p$-vector space.
- $\mathcal{F}$ is a descending $\mathbb{Q}$-filtration on $V_C=V\otimes C$.

For $(V,\mathcal{F})\in \mathrm{Fil}^C_{\mathbb{Q}_p}$, the rank and deg functions are defined by

$$\mathrm{rank}(V,\mathcal{F})=\dim V,\quad \deg(V,\mathcal{F})=\sum_i i\dim gr^i_{\mathcal{F}}V_C$$

induce a Harder-Narasimhan filtration on $\mathrm{Fil}^C_{\mathbb{Q}_p}$. We have a natural functor

$$\pi:\mathrm{HT}^{B_{dR}}\longrightarrow \mathrm{Fil}^C_{\mathbb{Q}_p},\quad (V,\Xi)\mapsto (V,\mathcal{F}_H(V,\Xi)).$$

Composed with the equivalence functor $\mathrm{HT}:\mathrm{Modif}^{ad}_X\to \mathrm{HT}^{B_{dR}}$ we get

$$\pi:\mathrm{Modif}^{ad}_X\longrightarrow \mathrm{Fil}^C_{\mathbb{Q}_p}.$$

We remark that we can also construct the functor $\pi:\mathrm{Modif}^{ad}_X\longrightarrow \mathrm{Fil}^C_{\mathbb{Q}_p}$ by using the functor in 2.2 and [10] Proposition 4.17. Now one checks directly the following fact (compare [13] Proposition 10):

Proposition 2.1. *The functor π preserves the rank and deg functions on the two categories, and it induces a bijection between the posets of strict subobjects of $\underline{\mathcal{E}}$ and $\pi(\underline{\mathcal{E}})$. Thus it preserves the Harder-Narasimhan filtrations and vectors on both sides.*

2.5. G**-structures.** Let G be a connected reductive group over $\mathbb{Q}_p$. We would like to add "G-structures" to our previous discussions.

Let us first fix some notations. We fix a minimal parabolic subgroup P_0 of G defined over $\mathbb{Q}_p$ and a Levi subgroup M_0 of P_0. Then a standard parabolic subgroup is a parabolic P with $P\supset P_0$. There is a unique Levi subgroup M of P containing M_0, which we call a standard Levi subgroup. We write U_P for the unipotent radical of P. Let $A\subset M_0$ be the maximal split torus over $\mathbb{Q}_p$, and $T\subset M_0$ be a maximal torus of M_0 defined over $\mathbb{Q}_p$ which contains A. Then $T=M_0$ if and only if G is quasi-split over $\mathbb{Q}_p$.

For a parabolic subgroup $P\subset G$ with Levi subgroup $M\subset P$ over $\mathbb{Q}_p$, let $W_P:=W_M$ be the absolute Weyl group of M. Assume that $P\supset P_0$ is standard with associated standard Levi M. Let A_M be the maximal split torus contained in the center Z_M of M, and A'_M be the maximal split quotient torus of M. Then we have a natural isogeny $A_M\to A'_M$. We also write $A_P=A_M$ and $A'_P=A'_M$. In particular $A=A_{P_0}=A_{M_0}$. If $Q\supset P$, then we have an inclusion $A_Q\subset A_P$ and a quotient $A'_P\to A'_Q$. Let $B\subset G_{\overline{\mathbb{Q}}_p}$ be a Borel subgroup such that $B\subset P_{0,\overline{\mathbb{Q}}_p}$. Let $T\subset B$ be a maximal torus such that $A\subset T\subset M_0$. Then we get the absolute based root datum

$$(X^*(T),\Phi,X_*(T),\Phi^\vee,\Delta)$$

and the relative based root datum

$$(X^*(A),\Phi_0,X_*(A),\Phi_0^\vee,\Delta_0).$$

Let Δ_P (resp. $\Delta_{0,P}$) be the set of non-trivial restrictions of elements of Δ (resp. Δ_0) to Z_M (resp. A_P) (recall $Z_M\subset T$ resp. $A_P\subset A$). If we replace G by M and let[6] Δ_M (resp. $\Delta_{0,M}$) be the set of simple roots (resp. relative roots) of M, then Δ_P (resp. $\Delta_{0,P}$) is in bijection with $\Delta\setminus\Delta_M$ (resp. $\Delta_0\setminus\Delta_{0,M}$). Let $\Delta^\vee$ be the set of simple coroots of G, then we have $\Delta_P^\vee$ corresponding to P. Similarly we have the relative version $\Delta_0^\vee$ and $\Delta_{0,P}^\vee$.

[6]Note that the notation here is compatible with the notation of [3].

Let W and W_0 be the absolute and relative Weyl groups of G respectively. We can identify
$$X_*(A)_{\mathbb{Q}}/W_0 = X_*(A)_{\mathbb{Q}}^+ := \{x \in X_*(A)_{\mathbb{Q}} |\, \langle x, \alpha\rangle \geq 0,\ \forall\, \alpha \in \Delta_0\}.$$
On the other hand, consider
$$\mathcal{N}(G) = \left[\mathrm{Hom}(\mathbb{D}_{\overline{\mathbb{Q}}_p}, G_{\overline{\mathbb{Q}}_p}) \,/\, G(\overline{\mathbb{Q}}_p)\text{-conjugacy}\right]^{\Gamma},$$
with $\mathbb{D}$ the pro-torus over $\mathbb{Q}_p$ whose character group is $\mathbb{Q}$ and $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. As in [6], let $X_*(G)$ denote the set of cocharacters of G defined over $\overline{\mathbb{Q}}_p$. Then
$$X_*(G)_{\mathbb{Q}} = \mathrm{Hom}(\mathbb{D}_{\overline{\mathbb{Q}}_p}, G_{\overline{\mathbb{Q}}_p}),$$
on which $G(\overline{\mathbb{Q}}_p)$ acts by conjugation and we will write $\mathcal{N}(G) = (X_*(G)_{\mathbb{Q}}/G)^{\Gamma}$, which we identify with $(X_*(T)_{\mathbb{Q}}/W)^{\Gamma}$. We have
$$X_*(A)_{\mathbb{Q}}^+ = X_*(A)_{\mathbb{Q}}/W_0 = X_*(G)_{\mathbb{Q}}^{\Gamma}/G(\mathbb{Q}_p) \subset (X_*(G)_{\mathbb{Q}}/G)^{\Gamma} = \mathcal{N}(G).$$
Then
$$G \text{ is quasi-split over } \mathbb{Q}_p \quad \Longleftrightarrow \quad X_*(A)_{\mathbb{Q}}^+ = \mathcal{N}(G).$$
We identify
$$X_*(T)_{\mathbb{Q}}/W = X_*(T)_{\mathbb{Q}}^+ = \{x \in X_*(T)_{\mathbb{Q}} |\, \langle x, \alpha\rangle \geq 0,\ \forall\, \alpha \in \Delta\}.$$
Moreover, the choice of B defines a partial order $\leq$ on $X_*(T)$ by $\mu_1 \leq \mu_2$ if $\mu_2 - \mu_1$ is a sum of positive coroots with non negative integral coefficients. We get an induced partial order $\leq$ on $X_*(T)_{\mathbb{Q}}$ and thus on $\mathcal{N}(G) \subset X_*(T)_{\mathbb{Q}}^+ \subset X_*(T)_{\mathbb{Q}}$. By [42] 15.5.8, we get an involution $x \mapsto w_0(-x)$ on $\mathcal{N}(G)$, where w_0 is the element of longest length in W acting on $X_*(T)_{\mathbb{Q}}$.

Let $\omega^G : \mathrm{Rep}\, G \to \mathrm{Vect}_{\mathbb{Q}_p}$ be the standard fiber functor for the category $\mathrm{Rep}\, G$ of algebraic representations of G. For any field extension $K|\mathbb{Q}_p$, let $\mathrm{Fil}_K(\omega^G)$ be the set of filtrations of ω^G over K. An element $\mathcal{F} \in \mathrm{Fil}_K(\omega^G)$ is given by a tensor functor
$$F : \mathrm{Rep}\, G \longrightarrow \mathrm{Fil}_{\mathbb{Q}_p}^K$$
such that $\omega^G = \omega_0 \circ F$ and the induced tensor functor
$$\mathrm{gr} \circ F : \mathrm{Rep}\, G \longrightarrow \mathrm{Grad}_K$$
is exact. Here $\omega_0 : \mathrm{Fil}_{\mathbb{Q}_p}^K \to \mathrm{Vect}_{\mathbb{Q}_p}$ is the natural functor and $\mathrm{gr} : \mathrm{Fil}_{\mathbb{Q}_p}^K \to \mathrm{Grad}_K$ is the graded functor from $\mathrm{Fil}_{\mathbb{Q}_p}^K$ to the category of graded K-vector spaces. We refer the reader to [6] chapter IV.2 for more discussions on these objects. We have natural maps
$$\mathrm{Fil}_{\mathbb{Q}_p}(\omega^G) \to X_*(G)_{\mathbb{Q}}^{\Gamma}/G(\mathbb{Q}_p) = X_*(A)_{\mathbb{Q}}^+ \hookrightarrow (X_*(G)_{\mathbb{Q}}/G)^{\Gamma} = \mathcal{N}(G).$$

Recall that an F-isocrystal with G-structure is an exact tensor functor
$$N : \mathrm{Rep}\, G \longrightarrow \varphi\text{−}\mathrm{Mod}_{\breve{\mathbb{Q}}_p}.$$
An element $b \in G(\breve{\mathbb{Q}}_p)$ defines an isocrystal with G-structure
$$\begin{aligned} N_b : \mathrm{Rep}\, G &\longrightarrow \varphi\text{−}\mathrm{Mod}_{\breve{\mathbb{Q}}_p} \\ V &\longmapsto (V_{\breve{\mathbb{Q}}_p}, b\sigma). \end{aligned}$$
Its isomorphism class only depends on the σ-conjugacy class $[b] \in B(G)$ of b, where $B(G)$ is the set of σ-conjugacy classes in $G(\breve{\mathbb{Q}}_p)$, cf. [23, 25, 35]. By Steinberg's theorem, any isocrystal with G-structure arises in this way. Thus $B(G)$ is the set of isomorphism classes of isocrystals with G-structure, cf. [35] Remarks 3.4 (i). We have the Newton map ([23] section 4) and Kottwitz map ([24] section 6 and [25] 4.9, 7.5)
$$\nu : B(G) \to \mathcal{N}(G), \quad \kappa : B(G) \to \pi_1(G)_{\Gamma},$$

where

$$\pi_1(G) = X_*(T)/\langle \Phi^\vee \rangle$$

(by our previous group theoretic notations, and it does not depend on the choice of T) and $\pi_1(G)_\Gamma$ is its Galois coinvariant. In fact, ν is induced by a map $\nu : G(\breve{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathbb{D}_{\breve{\mathbb{Q}}_p}, G_{\breve{\mathbb{Q}}_p})$, while κ is induced by a map $\kappa : G(\breve{\mathbb{Q}}_p) \to \pi_1(G)_\Gamma$. For this reason we also write $\nu([b]) = [\nu_b]$ and $\kappa([b]) = \kappa(b)$ for $b \in G(\breve{\mathbb{Q}}_p)$ with the induced class $[b] \in B(G)$. The partial order on $\mathcal{N}(G)$ induces a partial order $\leq$ on $B(G)$ (cf. [35] section 2).

We explain $B(G)$ in terms of G-bundles on the Fargues-Fontaine curve X as follows. Recall that we have the following two equivalent definitions of a G-bundle on X:

- an exact tensor functor $\mathrm{Rep}\, G \to \mathrm{Bun}_X$, where $\mathrm{Rep}\, G$ is the category of rational algebraic representations of G,
- a G-torsor on X locally trivial for the étale topology.

Attached to a G-bundle $\mathcal{E}$ on X, we have the Newton vector $\nu(\mathcal{E}) \in \mathcal{N}(G)$ and the G-equivariant first Chern class $c_1^G(\mathcal{E}) \in \pi_1(G)_\Gamma$. For $b \in G(\breve{\mathbb{Q}}_p)$, let $\mathcal{E}_b$ be the composition of the above functor N_b and

$$\mathcal{E}(-) : \varphi\text{−}\mathrm{Mod}_{\breve{\mathbb{Q}}_p} \longrightarrow \mathrm{Bun}_X.$$

In this way, the set $B(G)$ also classifies G-bundles on X. In fact, we have

Theorem 2.2 ([11])**.** *There is a bijection of pointed sets*

$$B(G) \xrightarrow{\sim} H^1_{\acute{e}t}(X, G)$$
$$[b] \longmapsto [\mathcal{E}_b].$$

Under this bijection, we have

(1) $\nu(\mathcal{E}_b) = w_0(-\nu([b]))$,
(2) $c_1^G(\mathcal{E}_b) = -\kappa([b])$.

A modification of G-bundles is given by

- either an exact tensor functor $\mathrm{Rep}\, G \to \mathrm{Modif}_X$,
- or a triple $(\mathcal{E}_1, \mathcal{E}_2, f)$, where $\mathcal{E}_1, \mathcal{E}_2$ are G-bundles on X and $f : \mathcal{E}_1|_{X \setminus \{\infty\}} \xrightarrow{\sim} \mathcal{E}_2|_{X \setminus \{\infty\}}$ is an isomorphism.

Applying the functor $\pi : \mathrm{Modif}_X \to \varphi\text{−}\mathrm{FilMod}_{C/\breve{\mathbb{Q}}_p}$ in subsection 2.2, a modification of G-bundles $\underline{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, f)$ gives rise to a filtered F-isocrystal with G-structure

$$\pi(\underline{\mathcal{E}}) : \mathrm{Rep}\, G \longrightarrow \varphi\text{−}\mathrm{FilMod}_{C/\breve{\mathbb{Q}}_p},$$

which is in turn equivalent to a pair $(N, \mathcal{F})$, where (cf. [6] p. 239)

- $N : \mathrm{Rep}\, G \to \varphi\text{−}\mathrm{Mod}_{\breve{\mathbb{Q}}_p}$ is the underlying F-isocrystal with G-structure induced by the natural functor (forgetting filtrations) $\varphi\text{−}\mathrm{FilMod}_{C/\breve{\mathbb{Q}}_p} \to \varphi\text{−}\mathrm{Mod}_{\breve{\mathbb{Q}}_p}$,
- $\mathcal{F} \in \mathrm{Fil}_C(\omega^G)$.

Theorem 2.3 ([6] Theorem 9.2.18)**.** *For the pair $(N, \mathcal{F})$, there exists a unique $\mathbb{Q}$-filtration ${}^\bullet N_{\mathcal{F}}$ of N, such that for any $(V, \rho) \in \mathrm{Rep}\, G$, the induced filtration ${}^\bullet N_{\mathcal{F}}(V)$ on $N(V)$ is the Harder-Narasimhan filtration of the filtered isocrystal $(N(V), \mathcal{F}^\bullet N(V))$.*

In particular, the $\mathbb{Q}$-filtration ${}^\bullet N_{\mathcal{F}} \in \mathrm{Fil}_{\breve{\mathbb{Q}}_p}(\omega^G)$ defines a Harder-Narasimhan vector $\nu(N, \mathcal{F}) \in (X_*(G)_{\mathbb{Q}}/G)^{\Gamma_0}$ where $\Gamma_0 = \mathrm{Gal}(\overline{\mathbb{Q}}/\breve{\mathbb{Q}}_p)$. By [6] IX.4, we have in fact

$$\nu(N, \mathcal{F}) \in \mathcal{N}(G) = (X_*(G)_{\mathbb{Q}}/G)^\Gamma.$$

In the following we will write $\nu(\mathcal{E}_1, \mathcal{E}_2, f) := \nu(N, \mathcal{F})$ for a modification of G-bundles $(\mathcal{E}_1, \mathcal{E}_2, f)$ with the associated $(N, \mathcal{F})$.

An admissible modification of G-bundles is given by

- either an exact tensor functor $\mathrm{Rep}\, G \to \mathrm{Modif}_X^{ad}$,
- or a triple $(\mathcal{E}_1, \mathcal{E}_2, f)$, where $\mathcal{E}_1, \mathcal{E}_2$ are G-bundles on X such that $\mathcal{E}_1$ is the trivial G-bundle and $f : \mathcal{E}_1|_{X\setminus\{\infty\}} \xrightarrow{\sim} \mathcal{E}_2|_{X\setminus\{\infty\}}$ is an isomorphism.

By the equivalence of categories $\mathrm{HT} : \mathrm{Modif}_X^{ad} \xrightarrow{\sim} \mathrm{HT}^{B_{dR}}$, given an admissible modification $\underline{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, f)$ is equivalent to given an exact functor

$$\mathrm{HT}(\underline{\mathcal{E}}) : \mathrm{Rep}\, G \longrightarrow \mathrm{HT}^{B_{dR}}.$$

Composing with the functor $\pi : \mathrm{HT}^{B_{dR}} \to \mathrm{Fil}_{\mathbb{Q}_p}^C$ in 2.4, we then get a functor

$$\pi(\underline{\mathcal{E}}) : \mathrm{Rep}\, G \to \mathrm{Fil}_{\mathbb{Q}_p}^C,$$

which defines an element $\mathcal{F} \in \mathrm{Fil}_C(\omega^G)$. Recall on all the categories $\mathrm{Modif}_X^{ad}, \mathrm{HT}^{B_{dR}}$ and $\mathrm{Fil}_{\mathbb{Q}_p}^C$, there exist Harder-Narasimhan filtrations, which are compatible under the functors

$$\mathrm{Modif}_X^{ad} \to \mathrm{HT}^{B_{dR}} \to \mathrm{Fil}_{\mathbb{Q}_p}^C.$$

Theorem 2.4. *Let $\mathcal{C}$ be one of the categories $\mathrm{Modif}_X^{ad}, \mathrm{HT}^{B_{dR}}, \mathrm{Fil}_{\mathbb{Q}_p}^C$, and $N : \mathrm{Rep}\, G \to \mathcal{C}$ be an exact functor. There exists a $\mathbb{Q}$-filtration ${}^\bullet N$ of N, such that for any $(V, \rho) \in \mathrm{Rep}\, G$, the induced filtration ${}^\bullet N(V)$ on $N(V)$ is the Harder-Narasimhan filtration of $N(V)$.*

Proof. For $\mathcal{C} = \mathrm{Fil}_{\mathbb{Q}_p}^C$, this follows from [6] Theorem 5.3.1. For $\mathcal{C} = \mathrm{Modif}_X^{ad}$ or $\mathcal{C} = \mathrm{HT}^{B_{dR}}$, by [5] Proposition 47, since the Harder-Narasimhan filtrations are compatible with tensor products, duals, symmetric and exterior powers, one sees that the arguments in the proof of [6] Theorem 5.3.1 work here. See also [4] Theorem 5.8, Proposition 5.9 and Proposition 4.2. □

Let $\underline{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, f)$ be an admissible modification of G-bundles on X. Then the associated Harder-Narasimhan filtration defines an element in $\mathrm{Fil}_{\mathbb{Q}_p}(\omega^G)$, thus we get a vector

$$\nu(\underline{\mathcal{E}}) \in X_*(G)_{\mathbb{Q}}^{\Gamma}/G(\mathbb{Q}_p) = X_*(A)_{\mathbb{Q}}/W_0 = X_*(A)_{\mathbb{Q}}^+ \subset \mathcal{N}(G).$$

We get a standard parabolic P of G such that the associated standard Levi M is the centralizer of $\nu(\underline{\mathcal{E}})$. By Proposition 2.1, we have

$$\nu(\underline{\mathcal{E}}) = \nu(\mathrm{HT}(\underline{\mathcal{E}})) = \nu(\pi(\underline{\mathcal{E}})) = \nu(\mathcal{F}),$$

where $\mathcal{F} \in \mathrm{Fil}_C(\omega^G)$ is the element corresponding to $\pi(\underline{\mathcal{E}})$.

2.6. **Moduli of local G-Shtukas.** As before, G is a connected reductive group over $\mathbb{Q}_p$. Let $\{\mu\}$ be the conjugacy class of cocharacters $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$. Fixing a Borel subgroup $B \subset G_{\overline{\mathbb{Q}}_p}$ containing a maximal torus T. The class $\{\mu\}$ defines an element $\mu \in X_*(T)^+$ for the choice of B. We view it as an element in $X_*(G)_{\mathbb{Q}}/G$. Then we have the associated flag variety $\mathscr{F}\ell(G, \mu)$ over a finite extension $E = E(G, \{\mu\})$ of $\mathbb{Q}_p$. Recall that we have a natural map $\mathrm{Fil}_{\overline{\mathbb{Q}}_p}(\omega^G) \to X_*(G)_{\mathbb{Q}}/G$, sending a filtration to its type. By construction,

$$\mathscr{F}\ell(G, \mu)(\overline{\mathbb{Q}}_p) = G(\overline{\mathbb{Q}}_p)/P_\mu(\overline{\mathbb{Q}}_p) = \{\mathcal{F} \in \mathrm{Fil}_{\overline{\mathbb{Q}}_p}(\omega^G) \text{ of type } \mu\},$$

where P_μ is the parabolic subgroup of $G_{\overline{\mathbb{Q}}_p}$ associated to μ by the formula

$$P_\mu = \{g \in G_{\overline{\mathbb{Q}}_p} \,|\, \lim_{t \to 0} \mu(t) g \mu(t)^{-1} \text{ exists}\}.$$

In particular $P_\mu \supset B$.

In the following sections 3-5, we will assume that μ is minuscule and work with the associated p-adic flag varieties $\mathscr{F}\ell(G, \mu)$ and $\mathscr{F}\ell(G, \mu^{-1})$. For an arbitrary μ, we will need the B_{dR}^+-affine Schubert cell Gr_μ, which is a *diamond* over E, see [40] 19.2, 20.2 and the following subsection 6.1. There is a morphism of diamonds

$$\pi_\mu : \mathrm{Gr}_\mu \to \mathscr{F}\ell(G,\mu)^\diamond,$$

which is an isomorphism if μ is minuscule, see [2] Proposition 3.4.3, Theorem 3.4.5, [40] Proposition 19.4.2 and the following 6.1. We have also the B_{dR}^+-affine Schubert variety $\mathrm{Gr}_{\leq\mu} = \coprod_{\mu'\leq\mu} \mathrm{Gr}_{\mu'}$, which is a proper diamond over E.

A local Shtuka datum[7] (cf. [40] 23.1) is a triple $(G, \{\mu\}, [b])$, where

- G is a connected reductive group over $\mathbb{Q}_p$,
- $\{\mu\}$ is a conjugacy class of cocharacter $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$,
- $[b] \in B(G)$ is a σ-conjugacy class of $b \in G(\breve{\mathbb{Q}}_p)$ such that $[b] \in B(G,\mu)$.

If moreover μ is minuscule, then $(G, \{\mu\}, [b])$ is called a local Shimura datum (cf. [36] Definition 5.1).

Let $(G, \{\mu\}, [b])$ be a local Shtuka datum and fix a representative $b \in G(\breve{\mathbb{Q}}_p)$. Attached to the triple $(G, \{\mu\}, b)$, we have the moduli space of local G-Shtukas with one leg (cf. [40] sections 12-14 and the appendix to section 19) with infinite level (cf. [40] section 23)

$$\mathrm{Sht}(G,\mu,b)_\infty,$$

which is a diamond over $\breve{E}$, and up to isomorphism, all of which depend only on $(G, \{\mu\}, [b])$. By construction, there exist two natural morphisms of diamonds

$$\pi_{dR} : \mathrm{Sht}(G,\mu,b)_\infty \to \mathrm{Gr}_\mu, \quad \text{and} \quad \pi_{HT} : \mathrm{Sht}(G,\mu,b)_\infty \to \mathrm{Gr}_{\mu^{-1}},$$

which factor through certain subspaces $\mathrm{Gr}_\mu^a \subset \mathrm{Gr}_\mu$ (see subsection 6.4) and $\mathrm{Gr}_{\mu^{-1}}^{Newt=[b]} \subset \mathrm{Gr}_{\mu^{-1}}$ (see subsection 6.3) respectively. We call π_{dR} (resp. π_{HT}) the de Rham (resp. Hodge-Tate) period morphism. By [40] subsection 23.3, we can view that $\mathrm{Sht}(G,\mu,b)_\infty$ classifies

- either modifications of G-bundles of type μ between $\mathcal{E}_b$ and $\mathcal{E}_1$ over Gr_μ^a,
- or modifications of$G$-bundles of type μ^{-1} between $\mathcal{E}_1$ and $\mathcal{E}_b$ over $\mathrm{Gr}_{\mu^{-1}}^{Newt=[b]}$.

We get the following diagram of de Rham and Hodge-Tate period morphisms:

$$\begin{array}{ccc} & \mathrm{Sht}(G,\mu,b)_\infty & \\ {\scriptstyle \pi_{dR}} \swarrow & & \searrow {\scriptstyle \pi_{HT}} \\ \mathrm{Gr}_\mu^a & & \mathrm{Gr}_{\mu^{-1}}^{Newt=[b]}. \end{array}$$

One can replace Gr_μ by $\mathrm{Gr}_{\leq\mu}$ in the above construction to get the diamond $\mathrm{Sht}(G,\leq\mu,b)_\infty$, which is exactly the version of moduli space of local G-Shtukas with one leg bounded by μ studied in [40].

If μ is minuscule, we have $\mathrm{Sht}(G,\leq\mu,b)_\infty = \mathrm{Sht}(G,\mu,b)_\infty$, and we will also use the notation $\mathcal{M}(G,\mu,b)_\infty$ for $\mathrm{Sht}(G,\mu,b)_\infty$. In this case $\mathrm{Gr}_\mu^a \simeq \mathscr{F}\ell(G,\mu,b)^\diamond$ and $\mathscr{F}\ell(G,\mu,b) \subset \mathscr{F}\ell(G,\mu)$ is the admissible locus introduced in [3] Definition 3.1. The reader can assume that μ is minuscule and work with the above under this condition until section 6.

3. Newton strata and Harder-Narasimhan strata on p-adic flag varieties

Let G be a connected reductive group over $\mathbb{Q}_p$ and $\{\mu\}$ be the conjugacy class of cocharacters $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$. Recall the Kottwitz set (cf. [25] section 6, here we use the notation of [3] 2.1)

$$B(G,\mu) = \{[b] \in B(G) \mid \nu([b]) \leq \mu^\diamond, \quad \kappa([b]) = \mu^\sharp\}.$$

[7] In this paper we only consider local Shtuka data with one conjugacy class $\{\mu\}$.

We have also (cf. [36] 2.2)

$$A(G,\mu) = \{[b] \in B(G) \mid \nu([b]) \leq \mu^{\diamond}\}.$$

Both $B(G,\mu)$ and $A(G,\mu)$ are finite subsets of $B(G)$, equipped with the induced partial order $\leq$.

In the rest of this section, we will mainly consider the induced conjugacy class $\{\mu^{-1}\}$ instead. Let $\mathscr{F}\ell(G,\mu^{-1})$ be the flag variety defined over $E = E(G,\{\mu^{-1}\})$ attached to $(G,\{\mu^{-1}\})$, which we consider as an *adic space*. We are interested in the geometry of the p-adic flag variety $\mathscr{F}\ell(G,\mu^{-1})$ from the point of view of p-adic Hodge theory. After reviewing the Newton stratification introduced in [2], we define and study the Harder-Narasimhan strata of the p-adic flag variety $\mathscr{F}\ell(G,\mu^{-1})$, following the lines in [6] chapter VI. As mentioned in the introduction, these strata can be studied by the theory of [6] Part 3. The direct approach here has the advantage that the stratification is defined over E. On the other hand, these strata generalize the Harder-Narasimhan strata in the case of GL_n studied by Fargues in [13]. Assume that μ is *minuscule* in this section.

3.1. **Newton strata.** We first consider the Newton strata. Let $C|E$ be an algebraically closed perfectoid field, and $\mathcal{E}$ be a G-bundle on the Fargues-Fontaine curve $X = X_{C^\flat}$. Then since μ is minuscule, by [2] 3.4.5, [10] 4.2 and [12] 3.20, for any $x \in \mathscr{F}\ell(G,\mu^{-1})(C,\mathcal{O}_C)$ we can associate to it a modification $\mathcal{E}_x$ of $\mathcal{E}$ at ∞. Consider the case $\mathcal{E} = \mathcal{E}_1$, the trivial G-bundle. The isomorphism class of $\mathcal{E}_{1,x}$ defines a point $b(\mathcal{E}_{1,x}) \in B(G)$. Letting C vary, we get a map

$$Newt : |\mathscr{F}\ell(G,\mu^{-1})| \longrightarrow B(G).$$

We can determine the image of the map $Newt$ as in the following theorem.

Theorem 3.1. (1) *We have the following decomposition of $\mathscr{F}\ell(G,\mu^{-1})$ into locally closed subsets over E:*

$$\mathscr{F}\ell(G,\mu^{-1}) = \coprod_{[b]\in B(G,\mu)} \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]},$$

such that for $x \in \mathscr{F}\ell(G,\mu^{-1})(C,\mathcal{O}_C)$, we have

$$x \in \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]}(C,\mathcal{O}_C) \Leftrightarrow \mathcal{E}_{1,x} \simeq \mathcal{E}_b.$$

The open stratum is associated to the unique basic element $[b_0] \in B(G,\mu)$. Each stratum $\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]}$ is stable under the $G(\mathbb{Q}_p)$-action on $\mathscr{F}\ell(G,\mu^{-1})$.

(2) *We have the following dimension formula: for $[b] \in B(G,\mu)$,*

$$\dim \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]} = \langle \mu - \nu([b]), 2\rho\rangle,$$

where ρ is the half of the sum of positive roots of G.

Proof. (1) follows from [2] Proposition 3.5.7, Corollary 3.5.9 and [34] Proposition A.9. The fact that each stratum is locally closed comes from the upper semi-continuity of the Newton map (cf. [22] and [40] subsection 22.5).

(2) follows from the theory of local Shimura varieties (see also [2] Proposition 4.2.23 for the PEL case). More precisely, consider the local Shimura datum $(G,\{\mu\},[b])$. Fix a representative $b \in G(\breve{\mathbb{Q}}_p)$ of $[b]$. We have the associated local Shimura variety at infinite level $\mathcal{M}(G,\mu,b)_\infty$, which fits into the following diagram

$$\begin{array}{ccc} & \mathcal{M}(G,\mu,b)_\infty & \\ \swarrow^{\pi_{dR}} & & \searrow^{\pi_{HT}} \\ \mathscr{F}\ell(G,\mu,b)^{a,\diamond} & & \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b],\diamond}, \end{array}$$

where π_{dR} is the Hodge-de Rham period map, which is a $\underline{G(\mathbb{Q}_p)}$-torsor, and π_{HT} is the Hodge-Tate period map, which is a $\widetilde{J}_b$-torsor. Here $\widetilde{J}_b = \underline{Aut}(\mathcal{E}_b)$ and we have dim $\widetilde{J}_b = \langle \nu([b]), 2\rho \rangle$ (cf. [12] for example). As μ is minuscule, dim $\mathscr{F}\ell(G, \mu) = \langle \mu, 2\rho \rangle$. As π_{dR} is pro-étale and $\mathscr{F}\ell(G, \mu, b)^a \subset \mathscr{F}\ell(G, \mu)$ is open, dim $\mathcal{M}(G, \mu, b)_\infty =$ dim $\mathscr{F}\ell(G, \mu, b)^a = \langle \mu, 2\rho \rangle$. Thus dim $\mathscr{F}\ell(G, \mu^{-1})^{Newt=[b]} = \langle \mu - \nu([b]), 2\rho \rangle$ (see also [3] Proposition 5.3). □

Remark 3.2. *We call the decomposition in the above theorem the Newton stratification. However, we don't know whether the closure relation holds. Thus the word "stratification" in this paper has only a weak sense.*

3.2. **Harder-Narasimhan strata.** Recall that we have the finite subsets $B(G, \mu) \subset A(G, \mu) \subset B(G)$. Then under the Newton map $\nu : B(G) \to \mathcal{N}(G)$ we have

$$\nu(A(G, \mu)) = \nu(B(G, \mu)).$$

Let

$$\mathcal{N}(G, \mu) \subset \mathcal{N}(G)$$

be the common image of $A(G, \mu)$ and $B(G, \mu)$ under the Newton map. In [3] Corollary 4.7 we gave an internal description of the set $\mathcal{N}(G, \mu)$ (using roots and weights). Here is an external (Tannakian) description which we will need.

Lemma 3.3. *Let $v \in \mathcal{N}(G)$. We have*

(1) *$v \in Im\nu$ if and only if for any representation $(V, \rho) \in \mathrm{Rep}\, G$ we have $\rho(v) \in Im\nu_{\mathrm{GL}(V)}$.*

(2) *$v \in \mathcal{N}(G, \mu)$ if and only if for any representation $(V, \rho) \in \mathrm{Rep}\, G$ we have $\rho(v) \in \mathcal{N}(\mathrm{GL}(V), \rho \circ \mu)$.*

Proof. (1) The only if part follows from the functoriality of the slope map $\nu : B(\cdot) \to \mathcal{N}(\cdot)$. The if part follows from the Tannakian definition of ν, cf. [23] 4.2.

(2) The only if part follows from the functorialities of the slope map $\nu : B(\cdot) \to \mathcal{N}(\cdot)$ and the Kottwitz map $\kappa : B(\cdot) \to \pi_1(\cdot)_\Gamma$ and the properties of the partial order on $B(G)$ and $\mathcal{N}(G)$. To show the if part, by (1) we have found $[b] \in B(G)$ such that $\nu([b]) = v$ and $v \leq \mu^\diamond$ by the properties of the partial order. Then by definition we have $[b] \in A(G, \mu)$. Thus $v = \nu([b]) \in \nu(A(G, \mu)) = \mathcal{N}(G, \mu)$. □

Now we consider Harder-Narasimhan stratifications. Let $C|E$ be an algebraically closed perfectoid field. Applying Theorem 2.4 to the admissible modification $(\mathcal{E}, \mathcal{E}', f)$ with $\mathcal{E} = \mathcal{E}_1$ and $\mathcal{E}' = \mathcal{E}_{1,x}$ for a point $x \in \mathscr{F}\ell(G, \mu^{-1})(C, \mathcal{O}_C)$, we get a well defined map

$$\begin{aligned} \mathscr{F}\ell(G, \mu^{-1})(C, \mathcal{O}_C) &\longrightarrow \mathcal{N}(G), \\ x &\longmapsto \nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f). \end{aligned}$$

We denote $\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f)^* = w_0(-\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f))$.

Proposition 3.4. *For any $x \in \mathscr{F}\ell(G, \mu^{-1})(C, \mathcal{O}_C)$, we have:*

(1) *The inequality of elements in $\mathcal{N}(G)$:*

$$\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f) \leq \nu(\mathcal{E}_{1,x}).$$

(2) *The Harder-Narasimhan vector $\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f)^*$ lies in $\mathcal{N}(G, \mu)$.*

Proof. (1) By Theorem 2.4 and [35] Lemma 2.2 (see also [6] Proposition 6.3.9), it suffices to show that for any $(V, \rho) \in \mathrm{Rep}\, G$,

$$\nu(\mathcal{E}_{1,V}, \mathcal{E}_{1,x,V}, f_V) \leq \nu(\mathcal{E}_{1,x,V}) \in \mathcal{N}(\mathrm{GL}(V)).$$

This is exactly [5] Proposition 44. See also [13] Proposition 14.

(2) By Lemma 3.3, it suffices to show for any $(V, \rho) \in \mathrm{Rep}\, G$, $\rho(\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f)^*) \in \mathcal{N}(\mathrm{GL}(V), \rho \circ \mu)$. By (1), we have $\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f)^* \leq \nu(\mathcal{E}_{1,x})^* := w_0(-\nu(\mathcal{E}_{1,x}))$. We conclude by the construction of $\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f)$. □

For any $x \in \mathscr{F}\ell(G,\mu^{-1})(C,\mathcal{O}_C)$ we write $HN(x)=\nu(\mathcal{E}_1,\mathcal{E}_{1,x},f)^*$. Letting C vary, we get the following map on topological spaces

$$HN : |\mathscr{F}\ell(G,\mu^{-1})| \longrightarrow \mathcal{N}(G,\mu).$$

Theorem 3.5. *The above map HN is upper semi-continuous, that is, for any $v \in \mathcal{N}(G,\mu)$, the subset*

$$\mathscr{F}\ell(G,\mu^{-1})^{HN\geq v} := \{x \in |\mathscr{F}\ell(G,\mu^{-1})| \, | HN(x) \geq v\}$$

is closed. In particular, the subset

$$\mathscr{F}\ell(G,\mu^{-1})^{HN=v} := \{x \in |\mathscr{F}\ell(G,\mu^{-1})| \, | HN(x) = v\}$$

is locally closed.

Proof. For any $x \in \mathscr{F}\ell(G,\mu^{-1})(C,\mathcal{O}_C)$, since by Theorem 2.4

$$\nu(\mathcal{E}_1,\mathcal{E}_{1,x},f)=\nu(\mathcal{F}_x)$$

with $\mathcal{F}_x \in \mathrm{Fil}_C(\omega^G)$ attached to x. Then the arguments in the proof of [6] Theorem 6.3.5 (see also the proof of the following Theorem 3.9) and Proposition 6.3.12 apply to the p-adic setting. □

In the following, we will identify $\mathcal{N}(G,\mu)$ with $B(G,\mu)$ by the Newton map, and for $x \in \mathscr{F}\ell(G,\mu^{-1})(C,\mathcal{O}_C)$ we will also write $HN(x)=b(\mathcal{E}_1,\mathcal{E}_{1,x},f)\in B(G,\mu)$. We have the following stratification over E:

$$\mathscr{F}\ell(G,\mu^{-1}) = \coprod_{[b]\in B(G,\mu)} \mathscr{F}\ell(G,\mu^{-1})^{HN=[b]}.$$

For any $[b]\in B(G,\mu)$, the stratum $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b]}$ is a locally closed subspace of $\mathscr{F}\ell(G,\mu^{-1})$, and it is stable under the action of $G(\mathbb{Q}_p)$ on $\mathscr{F}\ell(G,\mu^{-1})$.

Let $[b_0]\in B(G,\mu)$ be the basic element. Then the stratum

$$\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_0]}$$

is open, which is also called the semi-stable locus of $\mathscr{F}\ell(G,\mu^{-1})$. We have a description for $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_0]}$, which is similar to [3] Proposition 2.7, but here we don't need the assumption that G is quasi-split. To state it, we first need some more notations. Recall that after fixing a maximal torus inside a Borel subgroup $T\subset B\subset G_{\overline{\mathbb{Q}}_p}$, we view $\mu\in X_*(T)^+$. Let B^- be the Borel opposite to B. Then by construction $P_{\mu^{-1}}\supset B^-$. Let $P_0\subset G$ be a minimal parabolic subgroup over $\mathbb{Q}_p$ such that $P_{0,\overline{\mathbb{Q}}_p}\supset B^-$. Let M_0 be a Levi subgroup of P_0. For any standard parabolic $P\supset P_0$ with associated standard Levi $M\supset M_0$, we view

$$X^*(P/Z_G)\subset X^*(P)=X^*(M)=X^*(M_{ab})\subset X^*(Z_M),$$

where M_{ab} is the maximal abelian quotient of M and $Z_M\to M_{ab}$ is the natural isogeny. From the set $\Delta_P^\vee$, we get the following dominant set

$$X^*(P)^+=X^*(M)^+=\{\chi\in X^*(Z_M)|\, \langle\chi,\alpha^\vee\rangle\geq 0,\,\forall\,\alpha^\vee\in\Delta_P^\vee\}$$

and $X^*(P/Z_G)^+=X^*(P/Z_G)\cap X^*(P)^+$. Similarly, we have

$$X^*(P/Z_G)^\Gamma\subset X^*(P)^\Gamma=X^*(M)^\Gamma=X^*(A'_M)\subset X^*(A_M)$$

and $A_M\to A'_M$ is the natural isogeny. Using the set $\Delta_{0,P}^\vee$, we define similarly $X^*(P)^{\Gamma,+}$ and $X^*(P/Z_G)^{\Gamma,+}$.

Proposition 3.6. *Let $x\in\mathscr{F}\ell(G,\mu^{-1})(C,\mathcal{O}_C)$. Then $x\in\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_0]}(C,\mathcal{O}_C)$ if and only if for any standard parabolic P and any $\chi\in X^*(P/Z_G)^+$, we have*

$$\deg\chi_*(\mathcal{E}_{1,x})_P\leq 0,$$

where $(\mathcal{E}_{1,x})_P$ is the reduction of $\mathcal{E}_{1,x}$ to P induced by the reduction $\mathcal{E}_{1_P}$ of $\mathcal{E}_1$ to P.

Proof. This is essentially a reformulation of [6] Corollary 5.2.10. Indeed, consider the Schubert cell decomposition[8]

$$\mathscr{F}\ell(G,\mu^{-1})(C)=\coprod_{w\in W_P\backslash W/W_{P_{\mu^{-1}}}}\mathscr{F}\ell(G,\mu^{-1})(C)^w,$$

where

$$\mathscr{F}\ell(G,\mu^{-1})(C)^w=P(C)wP_{\mu^{-1}}(C)/P_{\mu^{-1}}(C)=P(C)/(P(C)\cap P_{\mu^{-1,w}}(C))=:\mathscr{F}\ell(P,\mu^{-1,w})(C).$$

Projection to the Levi quotient M of P induces an affine fibration:

$$\mathrm{pr}_w:\mathscr{F}\ell(P,\mu^{-1,w})(C)\to\mathscr{F}\ell(M,\mu^{-1,w})(C).$$

Now

$$(\mathcal{E}_{1,x})_P\times M\simeq\mathcal{E}_{1_M,\mathrm{pr}_w(x)},$$

and one can argue as in the proof of [3] Proposition 2.7 (see also the following Proposition 6.15), except in the last step we use [6] Corollary 5.2.10 instead. ☐

Remark 3.7. *We note that in the above proposition, for each P it suffices to consider the subset $\Delta_{0,P}\subset X^*(P/Z_G)^{\Gamma,+}\subset X^*(P/Z_G)^+$. In fact, it suffices to consider all maximal parabolic subgroups P, in which case each $\Delta_{0,P}$ consists of only one element.*

We have also the following GIT description for $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_0]}$.

Theorem 3.8 ([6] Theorem 6.2.8)**.** *Fix an invariant inner product on G and let $\mathcal{L}$ be the corresponding ample homogeneous $\mathbb{Q}$-line bundle on $\mathscr{F}\ell(G,\mu^{-1})$ (cf. [6] p. 146). Let K be a field extension of E and $x\in\mathscr{F}\ell(G,\mu^{-1})(K,\mathcal{O}_K)$. Then we have*

$$x\in\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_0]}(K,\mathcal{O}_K)\iff\forall\,\lambda:\mathbb{G}_m\to G_{der},\quad\mu^{\mathcal{L}}(x,\lambda)\geq 0.$$

The following theorem gives some basic properties of the Harder-Narasimhan stratification.

Theorem 3.9 ([13] Conjecture 2 (1))**.** *For any non basic $[b]\neq[b_0]$, the stratum $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b]}$ is a parabolic induction.*

Proof. We may assume that $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b]}\neq\emptyset$. We sketch the arguments following some ideas in [6]. Fix a minimal parabolic subgroup P_0 with Levi subgroup M_0 as above Proposition 3.6. Let T be a fixed maximal torus in M_0 defined over $\mathbb{Q}_p$. We introduce a finite set $\Theta(G,\mu)$ be the set of pairs (P,ν_P) with P a standard parabolic subgroup of G and $\nu_P\in X_*(T)_{\mathbb{Q}}/W_P$, satisfying the following two conditions:

(1) $\nu_P\equiv\mu^{-1}\mod W$,
(2) Let $\mu(\nu_P)\in X_*(A_P)_{\mathbb{Q}}$ be the image of ν_P under $X_*(T)_{\mathbb{Q}}\to X_*(A'_P)_{\mathbb{Q}}\xrightarrow{\sim}X_*(A_P)_{\mathbb{Q}}$. Then $\langle\mu(\nu_P),\alpha\rangle>0,\quad\forall\,\alpha\in\Delta_{0,P}$.

A such pair (P,ν_P) is called a HN type. Let $\mathcal{H}(G,\mu)$ be the set of HN vectors which contribute in the HN stratification. Then we have an inclusion $\mathcal{H}(G,\mu)\hookrightarrow\mathcal{N}(G,\mu)$ by Proposition 3.4. We have also a natural surjective map

$$H:\Theta(G,\mu)\twoheadrightarrow\mathcal{H}(G,\mu)$$

sending a HN type to its HN vector. In the following we fix a finite extension $\widetilde{E}$ of E which splits G and base change everything to $\widetilde{E}$. We will denote by the same notations over $\widetilde{E}$. Similar to [6] p. 152 (and p. 280-281), we have a refinement of the Harder-Narasimhan stratification

$$\mathscr{F}\ell(G,\mu^{-1})=\coprod_{\theta\in\Theta(G,\mu)}\mathscr{F}\ell(G,\mu^{-1})^{\theta},$$

[8] Since the Schubert cell decomposition exists on the algebraic varieties level, we omit $\mathcal{O}_C$ here to simplify the notations.

which is $G(\mathbb{Q}_p)$-equivariant and such that

$$\mathscr{F}\ell(G,\mu^{-1})^{HN=v} = \coprod_{\theta\in\Theta(G,\mu),H(\theta)=v} \mathscr{F}\ell(G,\mu^{-1})^{\theta}.$$

Fix a HN type $\theta = (P,\nu_P) \in \Theta(G,\mu)$. Consider the P-orbits in the flag variety $\mathscr{F}\ell(G,\mu^{-1}) = G/P_{\mu^{-1}}$. Then ν_P determines a unique Schubert cell

$$\mathscr{F}\ell(P,\nu_P) = PwP_{\mu^{-1}}/P_{\mu^{-1}}$$

where $w \in W_P \setminus W/W_{P_{\mu^{-1}}}$ such that $\nu_P = \mu^{-1,w}$. By abuse of notation, we still denote w the minimal length representative in the corresponding coset $W_P w W_{P_{\mu^{-1}}}$. Let M be the standard Levi of P with induced ν_M. Then the natural projection

$$\mathscr{F}\ell(P,\nu_P) \to \mathscr{F}\ell(M,\nu_M)$$

is an affine bundle of rank $\ell(w)$. Set

$$\mathscr{F}\ell(P,\nu_P)^{\theta} = \mathscr{F}\ell(G,\mu^{-1})^{\theta} \cap \mathscr{F}\ell(P,\nu_P).$$

The $G(\mathbb{Q}_p)$-action restricts to an action of $P(\mathbb{Q}_p)$ on $\mathscr{F}\ell(P,\nu_P)^{\theta}$. Let $\mathscr{F}\ell(M,\nu_M)^{ss}$ be the open HN stratum for the flag variety $\mathscr{F}\ell(M,\nu_M)$. Then the above projection $\mathscr{F}\ell(P,\nu_P) \to \mathscr{F}\ell(M,\nu_M)$ restricts to an affine fibration of rank $\ell(w)$

$$\mathscr{F}\ell(P,\nu_P)^{\theta} \to \mathscr{F}\ell(M,\nu_M)^{ss}.$$

We have a homeomorphism

$$\mathscr{F}\ell(P,\nu_P)^{\theta} \times_{P(\mathbb{Q}_p)} G(\mathbb{Q}_p) \xrightarrow{\sim} \mathscr{F}\ell(G,\mu^{-1})^{\theta}.$$

Thus the stratum $\mathscr{F}\ell(G,\mu^{-1})^{\theta}$ is an affine bundle of rank $\ell(w)$ over $\mathscr{F}\ell(M,\nu_M)^{ss} \times_{P(\mathbb{Q}_p)} G(\mathbb{Q}_p)$. We deduce that for any $v \in \mathcal{N}(G,\mu)$, the stratum $\mathscr{F}\ell(G,\mu^{-1})^{HN=v}$ is a parabolic induction.

□

Remark 3.10. *We know the dimension formula for the basic stratum, since it is open. For any non basic $[b] \neq [b_0]$, if the stratum $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b]} \neq \emptyset$, then by the above proof we have*

$$\dim \mathscr{F}\ell(G,\mu^{-1})^{HN=[b]} = \max_{w} \langle \mu^{-1,w}, 2\rho_M\rangle + \ell(w),$$

where $M = M_v \subset P = P_v$ with $v = w_0(-\nu([b]))$ and w runs through the set $w \in {}^{P}W^{P_{\mu^{-1}}}$ such that $\langle \mu^{-1,w}, \alpha\rangle > 0$ for any $\alpha \in \Delta_{0,P}$. Here we view $\mu^{-1,w} \in X_*(A_P)_{\mathbb{Q}}$ under the above map $X_*(T)_{\mathbb{Q}} \to X_*(A_P)_{\mathbb{Q}}$. In fact, Conjecture 2 (2) of [13] *predicts that for any $[b] \in B(G,\mu)$ such that the stratum $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b]} \neq \emptyset$, we have*

$$\dim \mathscr{F}\ell(G,\mu^{-1})^{HN=[b]} = \langle \mu - \nu([b]), 2\rho\rangle.$$

This is verified in the case $G = \mathrm{GL}_n$ by Fargues in [13] *Proposition 23. The above theorem was also proved by Fargues in* [13] *Propositions 21 and 22 in the case $G = \mathrm{GL}_n$ by a different method.*

Remark 3.11. *By Theorems 3.9 and 3.8, we can calculate the $\ell(\neq p)$-adic cohomology of $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b]}$. Indeed, by 3.9 it suffices to consider the open stratum $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_0]}$. By the GIT description in 3.8, we can follow* [6] *chapter VII.2 to calculate the Euler-Poincaré characteristic, and* [27] *to calculate the individual cohomology groups (see also* [28, 30]*).*

Remark 3.12. *For any $[b] \in B(G,\mu)$ with the associated HN stratum $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b]}$, we neither know its non-emptiness*[9]*, nor the closure relation. If μ is non minuscule, then*

[9] If $[b]$ is basic, then the associated stratum is open and non-empty. Thus the non-emptiness is a problem on non-basic strata. For the case of GL_n, see [29] for a complete solution. For the general case, see [6] Remark 9.6.3 for some hints.

in [6] *the authors there gave a counter example for the closure relation, see loc. cit. Example 2.3.7. However the closure relation may hold for minuscule* μ *and in the general case for the* B_{dR}^{+}*-affine Schubert cells (see section 6).*

3.3. **Newton strata vs Harder-Narasimhan strata.** By [19] Theorem 0.1, there exists a unique maximal element $[b_1]$ in $B(G,\mu)$ for the partial order $\leq$. If G is quasi-split, then $[\nu_{b_1}] = \mu^\diamond$. In the general case, this is *not true*, see [20] Example 3.1. We call the stratum

$$\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_1]} \quad (\text{resp.} \quad \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b_1]})$$

the μ-ordinary Harder-Narasimhan (resp. Newton) stratum. Both of the μ-ordinary strata $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_1]}$ and $\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b_1]}$ are *closed* in $\mathscr{F}\ell(G,\mu^{-1})$, by the semi-continuity of the maps $Newt$ and HN. Proposition 3.4 implies that we have the inclusion

$$\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_1]} \subset \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b_1]}.$$

Proposition 3.13 ([13] Conjecture 1 (2))**.** *Assume that G is quasi-split. Then we have always*

$$\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_1]} = \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b_1]}.$$

In particular $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_1]} \neq \emptyset$ *in this case.*

Proof. Let $C|E$ be any algebraically closed perfectoid field. We have to show that for any point $x \in \mathscr{F}\ell(G,\mu^{-1})(C,\mathcal{O}_C)$ such that $\nu(\mathcal{E}_{1,x})^* = [\nu_{b_1}]$, then $\nu(\mathcal{E}_1,\mathcal{E}_{1,x},f)^* = [\nu_{b_1}]$. Since G is quasi-split, $[\nu_{b_1}] = \mu^\diamond$. Then this follows from [5] Proposition 48. □

Let $[b_0] \in B(G,\mu)$ be the unique *basic* element. By the last two subsections, we have the *open* subspaces $\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b_0]}$ and $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b_0]}$ of $\mathscr{F}\ell(G,\mu^{-1})$. Proposition 3.4 implies that we have the inclusion

$$\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b_0]} \subset \mathscr{F}\ell(G,\mu^{-1})^{HN=[b_0]}.$$

In section 5, we will classify the case when the following equality holds

$$\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b_0]} = \mathscr{F}\ell(G,\mu^{-1})^{HN=[b_0]}.$$

4. The twin towers principle and dualities for Newton and HN stratifications

Let G be a connected reductive group over $\mathbb{Q}_p$ and $[b] \in B(G)_{basic}$ be a *basic* element. Fix a representative $b \in G(\breve{\mathbb{Q}}_p)$ of $[b]$. We have the associated reductive group J_b over $\mathbb{Q}_p$, which is an inner form of G. Fix an isomorphism $J_{b,\breve{\mathbb{Q}}_p} \cong G_{\breve{\mathbb{Q}}_p}$. Let Bun_G be the groupoid of G-bundles on the Fargues-Fontaine curve, cf. [12] section 2, which is a *small* v*-stack* (over $\mathbb{F}_p$) in the sense of [38].

4.1. **The twin towers principle.** In [3] 5.1, we have introduced the so called "twin towers principle", which is the following isomorphism

$$\mathrm{Bun}_{J_b} \cong \mathrm{Bun}_G,$$

that is to say there is an equivalence of groupoids between G-bundles and J_b-bundles on the curve. In fact, $J_b \times X$ is the twisted pure inner form of $G \times X$ obtained by twisting by the G-torsor $\mathcal{E}_b$,

$$J_b \times X = \underline{\mathrm{Aut}}(\mathcal{E}_b)$$

as a group over the curve. If $\mathcal{E}$ is a G-bundle on X one associates to it the J_b-bundle

$$\underline{\mathrm{Isom}}(\mathcal{E}_b, \mathcal{E}).$$

At the level of points of the preceding small v-stacks this gives the well known bijection

$$B(J_b) \xrightarrow{\sim} B(G)$$

that sends [1] to $[b]$ and $[b^{-1}]$ to [1]. Here $[b^{-1}] \in B(J_b)$ is the class defined by

$$b^{-1} \in J_b(\breve{\mathbb{Q}}_p) = G(\breve{\mathbb{Q}}_p).$$

In fact, we have the following commutative diagrams on the compatibilities for Newton maps and Kottwitz maps:

$$\begin{array}{ccc} B(J_b) & \xrightarrow{\sim} & B(G) \\ \downarrow{\scriptstyle \nu_{J_b}} & & \downarrow{\scriptstyle \nu_G} \\ \mathcal{N}(H) & \xrightarrow{\cdot\nu([b])} & \mathcal{N}(H), \end{array} \qquad \begin{array}{ccc} B(J_b) & \xrightarrow{\sim} & B(G) \\ \downarrow{\scriptstyle \kappa_{J_b}} & & \downarrow{\scriptstyle \kappa_G} \\ \pi_1(H)_\Gamma & \xrightarrow{+\kappa([b])} & \pi_1(H)_\Gamma. \end{array}$$

Let us make a comment on the notations. Here we have identified $\mathcal{N}(G) = \mathcal{N}(J_b) = \mathcal{N}(H)$ and $\pi_1(G)_\Gamma = \pi_1(J_b)_\Gamma = \pi_1(H)_\Gamma$, where H is a fixed quasi-split inner form of G (and thus of J_b). Recall that $\pi_1(G)_\Gamma$ is an abelian group, for which we will write the group law additively and the identity as 0; on the other hand, $\mathcal{N}(G) \subset X_*(G)_{\mathbb{Q}}/G$, the later has a commutative ordered monoid structure, and we will write its semi-group low multiplicatively.

The equivalence $\mathrm{Bun}_{J_b} \cong \mathrm{Bun}_G$ respects modifications of a given type μ, that is to say it identifies the corresponding Hecke stacks of modifications. Let $\{\mu\}$ be a conjugacy class of cocharacter $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$. In the rest of this section, we assume that $[b] = [b_0] \in B(G, \mu)$ is the *basic* element (in $B(G)$). The isomorphism $J_{b,\overline{\mathbb{Q}}_p} \simeq G_{\overline{\mathbb{Q}}_p}$ induces a conjugacy class of cocharacter $\{\mu\}$ of J_b. Then

$$[b^{-1}] \in B(J_b, \mu^{-1})$$

is the basic element (in $B(J_b)$), and $[b^{-1}] \mapsto [1]$ via the above bijection $B(J_b) \xrightarrow{\sim} B(G)$. One thus has

$$J_{b^{-1}} = G.$$

Recall that in [3] 4.1 we have introduced the following generalized Kottwitz sets

$$B(G, 0, \nu_b\mu^{-1}) := \{[b'] \in B(G) \mid \kappa([b']) = 0, \quad \nu([b']) \leq \nu([b])\omega_0(-\mu^\diamond)\}$$

and

$$B(J_b, 0, \nu_{b^{-1}}\mu) := \{[b''] \in B(J_b) \mid \kappa([b'']) = 0, \quad \nu([b'']) \leq \nu([b^{-1}])\mu^\diamond\},$$

which are finite subsets of $B(G)$ and $B(J_b)$ respectively. They contain the trivial classes $[1] \in B(G)$ and $[1] \in B(J_b)$ respectively. One checks directly the following lemma:

Lemma 4.1. *The bijection $B(J_b) \xrightarrow{\sim} B(G)$ induces the following bijections:*

$$B(J_b, \mu^{-1}) \xrightarrow{\sim} B(G, 0, \nu_b\mu^{-1}), \quad B(J_b, 0, \nu_{b^{-1}}\mu) \xrightarrow{\sim} B(G, \mu).$$

Consider the following p-adic flag varieties (as adic spaces) over $\breve{E}$:

$$\mathscr{F}\ell(G, \mu), \quad \mathscr{F}\ell(G, \mu^{-1}), \quad \mathscr{F}\ell(J_b, \mu), \quad \text{and} \quad \mathscr{F}\ell(J_b, \mu^{-1}).$$

We have identifications:

$$\mathscr{F}\ell(G, \mu) = \mathscr{F}\ell(J_b, \mu), \quad \mathscr{F}\ell(G, \mu^{-1}) = \mathscr{F}\ell(J_b, \mu^{-1}).$$

To summarize, we have the following data:

- the triples $(G, \{\mu^{-1}\}, [1])$ and $(J_b, \{\mu\}, [1])$ (which we call the *Hodge-Tate side* for G and J_b respectively),
- the local Shtuka data $(G, \{\mu\}, [b])$ and $(J_b, \{\mu^{-1}\}, [b^{-1}])$ (which we call the *de Rham side* for G and J_b respectively).

In the rest of this section we will assume that μ is *minuscule*.

4.2. **Newton strata on the de Rham side.** In the last section, we studied the geometry of $\mathscr{F}\ell(G,\mu^{-1})$ by modifications of the trivial G-bundle $\mathcal{E}_1^G$. Now we continue to study the flag variety $\mathscr{F}\ell(G,\mu^{-1}) = \mathscr{F}\ell(J_b,\mu^{-1})$ by modifications of the J_b-bundle $\mathcal{E}_{b^{-1}}^{J_b}$. From the local Shimura datum $(J_b,\{\mu^{-1}\},[b^{-1}])$, in [3] subsection 5.3, we have constructed a stratification of $\mathscr{F}\ell(J_b,\mu^{-1})$ by locally closed subsets

$$\mathscr{F}\ell(J_b,\mu^{-1}) = \coprod_{[b']\in B(J_b,0,\nu_{b^{-1}}\mu)} \mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^{Newt=[b']},$$

which we call the Newton[10] stratification. Let $C|\breve{E}$ be an algebraically closed perfectoid field. For any point $x\in\mathscr{F}\ell(J_b,\mu^{-1})(C,\mathcal{O}_C)$, we get a modification $\mathcal{E}_{b^{-1},x}^{J_b}$ of the J_b-bundle $\mathcal{E}_{b^{-1}}^{J_b}$ on the Fargues-Fontaine $X = X_\flat$. Then by definition

$$x\in\mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^{Newt=[b']}(C,\mathcal{O}_C) \quad\Leftrightarrow\quad b(\mathcal{E}_{b^{-1},x}^{J_b}) = [b'].$$

We have the associated p-adic period domain

$$\mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^a := \mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^{Newt=[1]},$$

which is the maximal open stratum.

Similarly, starting from the local Shimura datum $(G,\{\mu\},[b])$ we can study the geometry of $\mathscr{F}\ell(G,\mu)$ by modifications of the G-bundle $\mathcal{E}_b^G$. More precisely, we have the Newton stratification

$$\mathscr{F}\ell(G,\mu) = \coprod_{[b']\in B(G,0,\nu_b\mu^{-1})} \mathscr{F}\ell(G,\mu,b)^{Newt=[b']},$$

and the associated p-adic period domain

$$\mathscr{F}\ell(G,\mu,b)^a := \mathscr{F}\ell(G,\mu,b)^{Newt=[1]}.$$

Recall that inside $\mathscr{F}\ell(G,\mu^{-1})$ and $\mathscr{F}\ell(J_b,\mu)$, we have respectively the open Newton strata $\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]}$ and $\mathscr{F}\ell(J_b,\mu)^{Newt=[b^{-1}]}$ introduced in subsection 3.1.

Lemma 4.2. *Under the identification $\mathscr{F}\ell(G,\mu^{-1}) = \mathscr{F}\ell(J_b,\mu^{-1})$, we have*

$$\mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^a = \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]}.$$

Similarly, under the identification $\mathscr{F}\ell(G,\mu) = \mathscr{F}\ell(J_b,\mu)$, we have

$$\mathscr{F}\ell(G,\mu,b)^a = \mathscr{F}\ell(J_b,\mu)^{Newt=[b^{-1}]}.$$

Proof. We only check the identity $\mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^a = \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]}$. Let C be any algebraically closed complete extension of $\breve{E}$ and let $x\in\mathscr{F}\ell(G,\mu^{-1})(C,\mathcal{O}_C) = \mathscr{F}\ell(J_b,\mu^{-1})(C,\mathcal{O}_C)$. Then we have

$$\begin{aligned} x\in\mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^a(C,\mathcal{O}_C) &\Leftrightarrow \mathcal{E}_{b^{-1},x}^{J_b} = \mathcal{E}_1^{J_b} \\ &\Leftrightarrow \mathcal{E}_{1,x}^G = \mathcal{E}_b^G \\ &\Leftrightarrow x\in\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]}(C,\mathcal{O}_C). \end{aligned}$$

□

[10]In [3] this is called the Harder-Narasimhan stratification. Here we change the terminology and modify the notation, since later we will introduce another stratification with the same index set, which we will call the Harder-Narasimhan stratification following [6], as an analogy of that introduced in last section.

4.3. **Twin tower local Shimura varieties.** ([8, 9], [39] section 7, [40] Corollary 23.2.3.) Consider the local Shimura variety with infinite level

$$\mathcal{M}(G,\mu,b)_\infty,$$

which is the moduli space classifying

- either modifications of type μ between $\mathcal{E}_b^G$ and $\mathcal{E}_1^G$ over $\mathscr{F}\ell(G,\mu,b)^a$,
- or modifications of type μ^{-1} between $\mathcal{E}_1^G$ and $\mathcal{E}_b^G$ over $\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b]} = \mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^a$.

Similarly, we have the local Shimura variety with infinite level

$$\mathcal{M}(J_b,\mu^{-1},b^{-1})_\infty,$$

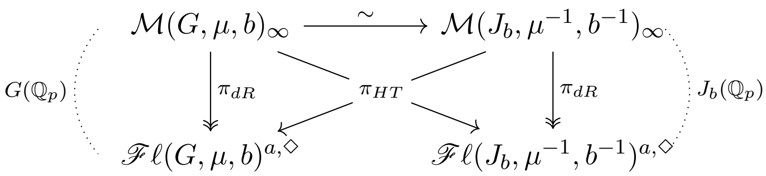
which is the moduli space classifying

- either modifications of type μ^{-1} between $\mathcal{E}_{b^{-1}}^{J_b}$ and $\mathcal{E}_1^{J_b}$ over $\mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^a$,
- or modifications of type μ between $\mathcal{E}_1^{J_b}$ and $\mathcal{E}_{b^{-1}}^{J_b}$ over $\mathscr{F}\ell(J_b,\mu)^{Newt=[b^{-1}]} = \mathscr{F}\ell(G,\mu,b)^a$.

At the end, the twin tower principle induces a $J_b(\mathbb{Q}_p)\times G(\mathbb{Q}_p)$-isomorphism of local Shimura varieties with infinite level

$$\mathcal{M}(G,\mu,b)_\infty \xrightarrow{\sim} \mathcal{M}(J_b,\mu^{-1},b^{-1})_\infty$$

as diamonds on $\mathrm{Spa}(\breve{E})^\diamond$. This fits into a twin towers diagram using the de Rham and Hodge-Tate period morphisms that allow us to collapse each tower on its base



where:

- $\mathcal{M}(G,\mu,b)_\infty$ classifies modifications of type μ between $\mathcal{E}_b^G$ and $\mathcal{E}_1^G$ over $\mathscr{F}\ell(G,\mu,b)^a$.
- For such a modification its image by π_{dR} is x if $\mathcal{E}_1^G = \mathcal{E}_{b,x}^G$. Its image by π_{HT} is y if $\mathcal{E}_b^G = \mathcal{E}_{1,y}^G$.
- $\mathcal{M}(J_b,\mu^{-1},b^{-1})_\infty$ classifies modifications of type μ^{-1} between $\mathcal{E}_{b^{-1}}^{J_b}$ and $\mathcal{E}_1^{J_b}$ over $\mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^a$.
- For such a modification its image by π_{dR} is x if $\mathcal{E}_1^{J_b} = \mathcal{E}_{b^{-1},x}^{J_b}$. Its image by π_{HT} is y if $\mathcal{E}_{b^{-1}}^{J_b} = \mathcal{E}_{1,y}^{J_b}$.

4.4. **Harder-Narasimhan strata on the de Rham side.** Now we continue to look at the p-adic flag variety $\mathscr{F}\ell(J_b,\mu^{-1})$. In [6] chapter IX.6, Dat-Orlik-Rapoport introduced a stratification of $\mathscr{F}\ell(J_b,\mu^{-1})$ by locally closed subsets (indexed by Harder-Narasimhan vectors)

$$\mathscr{F}\ell(J_b,\mu^{-1}) = \coprod_{v\in\mathcal{H}(\mathbb{J},\mu^{-1})} \mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^{HN=v},$$

which they called the Harder-Narasimhan stratification. Here $\mathbb{J}$ is the augmented group attached to J_b and b^{-1} as in [6] Example 9.1.22. Let $C|\breve{E}$ be an algebraically closed perfectoid field. For any $x\in\mathscr{F}\ell(J_b,\mu^{-1})(C,\mathcal{O}_C)$, we have the modification triple $(\mathcal{E}_{b^{-1},x}^{J_b},\mathcal{E}_{b^{-1}}^{J_b},f)$ of J_b-bundles on $X=X_{C^\flat}$. We write

$$\nu(\mathcal{E}_{b^{-1},x}^{J_b},\mathcal{E}_{b^{-1}}^{J_b},f) = \nu(N_{b^{-1}},\mathcal{F}_x)$$

for the filtered F-isocrystal with J_b-structure $(N_{b^{-1}},\mathcal{F}_x)$ attached to $(\mathcal{E}_{b^{-1},x}^{J_b},\mathcal{E}_{b^{-1}}^{J_b},f)$ constructed in subsection 2.2.

Proposition 4.3. *For any $x\in\mathscr{F}\ell(J_b,\mu^{-1})(C,\mathcal{O}_C)$,*

(1) *we have the following inequality in $\mathcal{N}(J_b)$*

$$\nu(\mathcal{E}^{J_b}_{b^{-1},x}, \mathcal{E}^{J_b}_{b^{-1}}, f) \leq \nu(\mathcal{E}^{J_b}_{b^{-1},x}).$$

(2) *The Newton map for J_b induces an injection*

$$\mathcal{H}(\mathbb{J}, \mu^{-1}) \hookrightarrow B(J_b, 0, \nu_{b^{-1}}\mu).$$

Proof. Under the bijection $B(J_b) \xrightarrow{\sim} B(G), [b^{-1}] \mapsto [1]$ and the identification $\mathscr{F}\ell(J_b, \mu^{-1}) = \mathscr{F}\ell(G, \mu^{-1})$, we have

$$\nu(\mathcal{E}^{J_b}_{b^{-1},x}, \mathcal{E}^{J_b}_{b^{-1}}, f) = \nu_b \nu(\mathcal{E}_{1,x}, \mathcal{E}_1, f) = \nu_b \nu(\mathcal{E}_1, \mathcal{E}_{1,x}, \mathcal{E}_1, f)$$

(for the second "=", see subsection 2.3) and $\nu(\mathcal{E}^{J_b}_{b^{-1},x}) = \nu_b \nu(\mathcal{E}_{1,x})$. Since $[b]$ is basic, we have

$$\nu_b \nu(\mathcal{E}_1, \mathcal{E}_{1,x}, \mathcal{E}_1, f) \leq \nu_b \nu(\mathcal{E}_{1,x}) \Leftrightarrow \nu(\mathcal{E}_1, \mathcal{E}_{1,x}, \mathcal{E}_1, f) \leq \nu(\mathcal{E}_{1,x}).$$

Therefore (1) is equivalent to Proposition 3.4 (1). The proof of (2) is similar, which is equivalent to Proposition 3.4 (2) (using Lemma 4.1). □

We get the composition

$$|\mathscr{F}\ell(J_b, \mu^{-1})| \to \mathcal{H}(\mathbb{J}, \mu^{-1}) \hookrightarrow B(J_b, 0, \nu_{b^{-1}}\mu)$$

and we write $b(\mathcal{E}^{J_b}_{b^{-1},x}, \mathcal{E}^{J_b}_{b^{-1}}, f) \in B(J_b, 0, \nu_{b^{-1}}\mu)$. Therefore, starting from the local Shimura datum $(J_b, \{\mu^{-1}\}, [b^{-1}])$, for the flag variety $\mathscr{F}\ell(J_b, \mu^{-1})$, we have the Harder-Narasimhan stratification:

$$\mathscr{F}\ell(J_b, \mu^{-1}) = \coprod_{[b'] \in B(J_b, 0, \nu_{b^{-1}}\mu)} \mathscr{F}\ell(J_b, \mu^{-1}, b^{-1})^{HN=[b']}.$$

Similarly, starting from the local Shimura datum $(G, \{\mu\}, [b])$, for the flag variety $\mathscr{F}\ell(G, \mu)$, we have the Harder-Narasimhan stratification:

$$\mathscr{F}\ell(G, \mu) = \coprod_{[b'] \in B(G, 0, \nu_b \mu^{-1})} \mathscr{F}\ell(G, \mu, b)^{HN=[b']}.$$

The open Harder-Narasimhan stratum $\mathscr{F}\ell(G, \mu, b)^{HN=[1]}$ corresponds to the trivial element $[1] \in B(G, 0, \nu_b \mu^{-1})$, which is also denoted by (cf. [37] chapter 1)

$$\mathscr{F}\ell(G, \mu, b)^{wa} := \mathscr{F}\ell(G, \mu, b)^{HN=[1]}.$$

Moreover, by Proposition 4.3 (1) (applied to $(G, \{\mu\}, [b])$), we have

$$\mathscr{F}\ell(G, \mu, b)^{a} \subset \mathscr{F}\ell(G, \mu, b)^{wa}.$$

Alternatively, the above inclusion also follows from the theorem of Colmez-Fontaine (cf. [14] chapter 10). Our argument above shows that it is equivalent to the inclusion $\mathscr{F}\ell(G, \mu^{-1})^{Newt=[b]} \subset \mathscr{F}\ell(G, \mu^{-1})^{HN=[b]}$, see subsection 3.3.

4.5. **Dualities for Newton and Harder-Narasimhan stratifications.** Consider the p-adic flag variety $\mathscr{F}\ell(J_b, \mu^{-1})$. Starting from the datum $(J_b, \{\mu^{-1}\}, [b^{-1}])$ (*de Rham side for the group J_b*), by subsection 4.4 we have the Harder-Narasimhan stratification:

$$\mathscr{F}\ell(J_b, \mu^{-1}) = \coprod_{[b'] \in B(J_b, 0, \nu_{b^{-1}}\mu)} \mathscr{F}\ell(J_b, \mu^{-1}, b^{-1})^{HN=[b']}.$$

By subsection 4.2, we have also the Newton stratification:

$$\mathscr{F}\ell(J_b, \mu^{-1}) = \coprod_{[b'] \in B(J_b, 0, \nu_{b^{-1}}\mu)} \mathscr{F}\ell(J_b, \mu^{-1}, b^{-1})^{Newt=[b']}.$$

Recall the Harder-Narasimhan and Newton stratifications for $\mathscr{F}\ell(G, \mu^{-1})$ introduced in section 3 starting from the datum $(G, \{\mu^{-1}\}, [1])$ (*Hodge-Tate side for the group G*):

$$\mathscr{F}\ell(G, \mu^{-1}) = \coprod_{[b'] \in B(G, \mu)} \mathscr{F}\ell(G, \mu^{-1})^{HN=[b']}, \quad \mathscr{F}\ell(G, \mu^{-1}) = \coprod_{[b'] \in B(G, \mu)} \mathscr{F}\ell(G, \mu^{-1})^{Newt=[b']}.$$

We have the following generalization of Lemma 4.2, which says that under the twin towers principle, the corresponding Harder-Narasimhan and Newton stratifications introduced in section 3 and here are identical.

Theorem 4.4. *Under the identification*

$$\mathscr{F}\ell(G,\mu^{-1}) = \mathscr{F}\ell(J_b,\mu^{-1}),$$

for any $[b'] \in B(G,\mu)$ *corresponding to* $[b''] \in B(J_b, 0, \nu_{b^{-1}}\mu)$ *under the bijection (cf. Lemma 4.1)*

$$B(G,\mu) \xrightarrow{\sim} B(J_b, 0, \nu_{b^{-1}}\mu),$$

we have

(1) $\mathscr{F}\ell(G,\mu^{-1})^{HN=[b']} = \mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^{HN=[b'']}$.
(2) $\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b']} = \mathscr{F}\ell(J_b,\mu^{-1},b^{-1})^{Newt=[b'']}$.

Proof. The proof for (2) is identical with the proof for Lemma 4.2, which is in fact also [3] Proposition 5.3.

The proof for (1) is in fact similar, which follows from the functoriality of the Harder-Narasimhan filtrations and the morphisms HN: let C be any algebraically closed complete extension of $\breve{E}$, we have the following commutative diagram

$$\begin{array}{ccc} \mathscr{F}\ell(J_b,\mu^{-1})(C,\mathcal{O}_C) & \xrightarrow{\sim} & \mathscr{F}\ell(G,\mu^{-1})(C,\mathcal{O}_C) \\ \downarrow{\scriptstyle HN_{J_b}} & & \downarrow{\scriptstyle HN_G} \\ \mathcal{N}(H) & \xrightarrow{\cdot\nu([b])} & \mathcal{N}(H), \end{array}$$

see the similar diagrams for ν and κ in subsection 4.1.

In fact, (1) also follows from [6] p. 252 (3.3), Proposition 9.5.3 (iii) and Remarks 9.6.18 (ii). □

Similarly, starting from $(G,\{\mu\},[b])$ (*de Rham side for the group* G), for the flag variety $\mathscr{F}\ell(G,\mu)$, we have the Harder-Narasimhan stratification (see subsection 4.4)

$$\mathscr{F}\ell(G,\mu) = \coprod_{[b']\in B(G,0,\nu_b\mu^{-1})} \mathscr{F}\ell(G,\mu,b)^{HN=[b']}$$

and the Newton stratification (see subsection 4.2)

$$\mathscr{F}\ell(G,\mu) = \coprod_{[b']\in B(G,0,\nu_b\mu^{-1})} \mathscr{F}\ell(G,\mu,b)^{Newt=[b']}$$

introduced in this section. Recall also the Harder-Narasimhan and Newton stratifications for $\mathscr{F}\ell(J_b,\mu)$ in section 3 starting from the datum $(J_b,\{\mu\},[1])$ (*Hodge-Tate side for the group* J_b):

$$\mathscr{F}\ell(J_b,\mu) = \coprod_{[b']\in B(J_b,\mu^{-1})} \mathscr{F}\ell(J_b,\mu)^{HN=[b']}, \quad \mathscr{F}\ell(J_b,\mu) = \coprod_{[b']\in B(J_b,\mu^{-1})} \mathscr{F}\ell(J_b,\mu)^{Newt=[b']}.$$

The following corollary is clear now.

Corollary 4.5. *Under the identification*

$$\mathscr{F}\ell(J_b,\mu) = \mathscr{F}\ell(G,\mu),$$

for any $[b'] \in B(G,\mu)$ *corresponding to* $[b''] \in B(J_b, 0, \nu_{b^{-1}}\mu)$ *under the bijection (cf. Lemma 4.1)*

$$B(J_b,\mu^{-1}) \xrightarrow{\sim} B(G,0,\nu_b\mu^{-1}),$$

we have

(1) $\mathscr{F}\ell(J_b,\mu)^{HN=[b']} = \mathscr{F}\ell(G,\mu,b)^{HN=[b'']}$.
(2) $\mathscr{F}\ell(J_b,\mu)^{Newt=[b']} = \mathscr{F}\ell(G,\mu,b)^{Newt=[b'']}$.

5. Fully Hodge-Newton decomposable case

We keep the notations of the last section. Let $(G, \{\mu\}, [b])$ be a local Shimura datum such that $[b] \in B(G, \mu)$ is *basic*. In particular μ is *minuscule*. We get the dual local Shimura datum $(J_b, \{\mu^{-1}\}, [b^{-1}])$.

Recall that (cf. [17] Definition 2.1 and [3] 4.3) we have the notion of *fully Hodge-Newton decomposability* for the Kottwitz set $B(G, \mu)$ (or the pair $(G, \{\mu\})$). This notion can be generalized to the sets $B(G, 0, \nu_b\mu^{-1})$ and $B(J_b, 0, \nu_{b^{-1}}\mu)$.

Now we can summarize the various equivalent conditions for fully Hodge-Newton decomposability studied in [3] and here.

Theorem 5.1. *The following are equivalent:*

1. $B(G, \mu)$ *is fully Hodge-Newton decomposable.*
2. $B(J_b, \mu^{-1})$ *is fully Hodge-Newton decomposable.*
3. $B(G, 0, \nu_b\mu^{-1})$ *is fully Hodge-Newton decomposable.*
4. $B(J_b, 0, \nu_{b^{-1}}\mu)$ *is fully Hodge-Newton decomposable.*
5. $\mathscr{F}\ell(G, \mu, b)^a = \mathscr{F}\ell(G, \mu, b)^{wa}$.
6. $\mathscr{F}\ell(J_b, \mu)^{Newt=[b^{-1}]} = \mathscr{F}\ell(J_b, \mu)^{HN=[b^{-1}]}$.
7. $\mathscr{F}\ell(J_b, \mu^{-1}, b^{-1})^a = \mathscr{F}\ell(J_b, \mu^{-1}, b^{-1})^{wa}$.
8. $\mathscr{F}\ell(G, \mu^{-1})^{Newt=[b]} = \mathscr{F}\ell(G, \mu^{-1})^{HN=[b]}$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from [3] Corollary 4.15. Replacing $(G, \{\mu\}, [b])$ by $(J_b, \{\mu^{-1}\}, [b^{-1}])$, we get the equivalences (2) $\Leftrightarrow$ (1) $\Leftrightarrow$ (4).

The equivalence (1) $\Leftrightarrow$ (5) was proved in [3] Theorem 6.1, thus we get also the equivalence (2) $\Leftrightarrow$ (7).

The equivalences (5) $\Leftrightarrow$ (6) and (7) $\Leftrightarrow$ (8) follow from Theorem 4.4 and Corollary 4.5 respectively. Therefore all the above statements are equivalent.

□

Remark 5.2. *In the above theorem, the equivalences (1)-(4) are taken from* [3]*, which are purely group theoretical statements. To show the equivalences with the remaining (5)-(8), we have taken* [3] *Theorem 6.1 as one of the key ingredients. On the other hand, one can show*[11] *the equivalence* (1) $\Leftrightarrow$ (8) *directly, by using similar (and in fact easier) arguments as in the proof of* [3] *Theorem 6.1. Then using Theorem 4.4 and Corollary 4.5, we get another proof of* [3] *Theorem 6.1, although essentially the two proofs are the same. We leave the details to the interested reader.*

Remark 5.3. *There are some further (conjectural) equivalences for the fully Hodge-Newton decomposable condition (1). For example, we refer the reader to*

1. [3] *Conjecture 7.2 (in terms of fundamental domains of p-adic period domains and local Shimura varieties),*
2. [17] *Theorem 2.3 (in terms of the geometry of affine Deligne-Lusztig varieties).*

6. Non minuscule cocharacters

In this section, we indicate how to generalize the constructions and results in previous sections to a general (not necessarily minuscule) cocharacter μ. Roughly, we need to replace flag varieties and local Shimura varieties by the corresponding B_{dR}^+-affine Schubert cells and moduli of local G-Shtukas respectively. We have Newton and Harder-Narasimhan stratifications on both sides (Gr_μ and $\mathrm{Gr}_{\mu^{-1}}$), generalizing the previous constructions in sections 3 and 4. It turns out that the HN stratifications on both sides are pullbacks of the corresponding HN stratifications on flag varieties via the Bialynicki-Birula map. Then we analyze the geometry of Gr_μ using affine Schubert cells of the Levi subgroups, cf. 6.6, which is in some sense a theory of (generalized) semi-infinite orbits for B_{dR}^+-affine Grassmannians. This is the key last step to prove the generalization of [3] Theorem 6.1.

[11] This is exactly what the author did at the beginning when preparing this article.

6.1. **B_{dR}^+-affine Grassmannians and B_{dR}^+-affine Schubert varieties.** Let G be a connected reductive group over $\mathbb{Q}_p$. Recall the B_{dR}^+-affine Grassmannian Gr_G is the *small v-sheaf* (cf. [40] 17.2 and [38]) over $\mathrm{Spd}\,\mathbb{Q}_p := (\mathrm{Spa}\,\mathbb{Q}_p)^\diamond$ such that for any affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over $\mathbb{Q}_p$,

$$\mathrm{Gr}_G(S) = \{(\mathcal{E}, \beta)\}/\simeq$$

where

- $\mathcal{E}$ is a G-torsor over $\mathrm{Spec} B_{dR}^+(R)$,
- $\beta : \mathcal{E} \to \mathcal{E}^0$ is a trivialization over $\mathrm{Spec} B_{dR}(R)$; here $\mathcal{E}^0$ is the trivial G-torsor,

cf. [12] 3.1 and [2] Definition 3.4.1. Equivalently,

$$\mathrm{Gr}_G = LG/L^+G,$$

where LG and L^+G are the loop groups such that

$$LG(\mathrm{Spa}(R, R^+)) = G(B_{dR}(R)), \quad \text{and} \quad L^+G(\mathrm{Spa}(R, R^+)) = G(B_{dR}^+(R)).$$

See also [40] Definition 20.2.1 and Proposition 20.2.2 (where it is called the Beilinson-Drinfeld Grassmannian over $\mathrm{Spd}\,\mathbb{Q}_p$). Let $C|\mathbb{Q}_p$ be an algebraically closed perfectoid field and $\mathrm{Spd}\,C := (\mathrm{Spa}\,C)^\diamond$. The base change $\mathrm{Gr}_{G,\mathrm{Spd}\,C}$ of Gr_G to $\mathrm{Spd}\,C$ is given by Definition 19.1.1 of [40].

Let $T \subset B \subset G_{\overline{\mathbb{Q}}_p}$ be a maximal torus inside a Borel subgroup of $G_{\overline{\mathbb{Q}}_p}$. We have the set of dominant cocharacters $X_*(T)^+$ of T with respective to B, which is a set of representatives for $X_*(T)/W$ where W is the absolute Weyl group of G. Recall that we have the Cartan decomposition[12]

$$G(B_{dR}(C)) = \coprod_{\mu \in X_*(T)^+} G(B_{dR}^+(C))\mu(\xi)^{-1}G(B_{dR}^+(C)),$$

where $\xi \in B_{dR}^+(C)$ is a fixed uniformizer. Any $\mu \in X_*(T)^+$ defines a closed subfunctor

$$\mathrm{Gr}_{\leq\mu}$$

of $\mathrm{Gr}_{G,\mathrm{Spd}\,E}$, with an open subfunctor $\mathrm{Gr}_\mu \subset \mathrm{Gr}_{\leq\mu}$, where $E = E(G, \{\mu\})$ is the field of definition of $\{\mu\}$. By definition (cf. [40] Definition 19.2.2), $\mathrm{Gr}_{\leq\mu}$ (resp. Gr_μ) parametrizes those $(\mathcal{E}, \beta)$ such that over any geometric points x, the relative position $Inv(\beta_x)$ is bounded (resp. exactly given) by μ. One of the main results of [40] is the following theorem.

Theorem 6.1 ([40] Theorem 19.2.4, Corollary 19.3.4 and Proposition 20.2.3)**.** *$\mathrm{Gr}_{\leq\mu}$ is a spatial diamond, and it is proper over $\mathrm{Spd}\,E$. Gr_μ is then a locally spatial diamond.*

By definition we have a stratification of diamonds

$$\mathrm{Gr}_{\leq\mu} = \coprod_{\mu' \leq \mu} \mathrm{Gr}_{\mu'}.$$

In particular if μ is *minuscule*, we have $\mathrm{Gr}_\mu = \mathrm{Gr}_{\leq\mu}$.

The inclusion $L^+G \subset LG$ induces a natural action of L^+G on Gr_G. For any perfectoid affinoid $\mathbb{Q}_p$-algebra (R, R^+), let $\xi \in B_{dR}^+(R)$ denote a uniformizer. For any $\mu \in X_*(T)$, we write $\xi^\mu = \mu(\xi)$ and $t^\mu = \mu(\xi)^{-1}$ for the corresponding elements in LG. By abuse of notation we also denote ξ^μ and t^μ the associated points in Gr_G. The diamond Gr_μ can be described as usual the orbit L^+Gt^μ, and we have

$$\mathrm{Gr}_\mu \simeq \frac{L^+G}{L^+G \cap t^\mu L^+G t^{-\mu}}.$$

Recall that for a diamond $\mathcal{D}$ we have its underlying topological space $|\mathcal{D}|$. We call a sub diamond $\mathcal{D}'$ is dense in $\mathcal{D}$, if $|\mathcal{D}'| \subset |\mathcal{D}|$ is dense.

Proposition 6.2. (1) *The open sub diamond $\mathrm{Gr}_\mu \subset \mathrm{Gr}_{\leq\mu}$ is dense in $\mathrm{Gr}_{\leq\mu}$.*

[12] Here we follow [2] and [3] to normalize the sign.

(2) *The dimension*[13] *of* Gr_μ *(and thus* $\mathrm{Gr}_{\leq\mu}$*) is* $\langle 2\rho, \mu\rangle$*, where* ρ *is as usual the half sum of positive (absolute) roots of* G*.*

Proof. For both statements we may assume that the base field is $\widehat{\overline{E}}$.

(1) We imitate the proof of [44] Proposition 2.1.5 (2) in the equal characteristic setting. If $\lambda \leq \mu$, then there exists a positive coroot α such that $\mu - \alpha$ is dominant and $\lambda \leq \mu - \alpha \leq \mu$. Thus it suffices to show that $t^{\mu-\alpha}$ is contained in the closure of Gr_μ. To prove this, we will construct a curve $C \simeq \mathbb{P}^{1,\diamond}$ in $\mathrm{Gr}_{\leq\mu}$ such that $t^{\mu-\alpha} \in C$ and $C \setminus \{t^{\mu-\alpha}\} \subset \mathrm{Gr}_\mu$.

For any integer m, let $t^{\lambda_m} := \begin{pmatrix} t^m & 0 \\ 0 & 1 \end{pmatrix}$, regarded as an element in $\mathrm{PGL}_2(B_{dR})$. Let $K_m = \mathrm{Ad}_{t^{\lambda}_m}(L^+\mathrm{SL}_2) \subset L\mathrm{SL}_2$. Then

$$\sigma_m := \begin{pmatrix} 0 & -t^m \\ t^{-m} & 0 \end{pmatrix} \in K_m.$$

Consider the map $L^+\mathrm{SL}_2 \to \mathrm{SL}_2$ induced by the natural map $\theta : B^+_{dR}(R) \to R$ for any perfectoid algebra R over $\mathbb{Q}_p$. Let $L^{>0}\mathrm{SL}_2$ be its kernel and set $K_m^{(1)} = \mathrm{Ad}_{t^{\lambda}_m}(L^{>0}\mathrm{SL}_2)$. Then $K_m/K_m^{(1)} \simeq \mathrm{SL}_2$. Let $i_\alpha : \mathrm{SL}_2 \to G$ be the canonical homomorphism associated to α. We get the induced map $Li_\alpha : L\mathrm{SL}_2 \to LG$. Let $m = \langle \mu, \alpha\rangle - 1$ and consider

$$C_{\mu,\alpha} := Li_\alpha(K_m)t^\mu.$$

Since $Li_\alpha(K_m^{(1)}) \subset L^+G \cap t^\mu L^+G t^{-\mu}$, $C_{\mu,\alpha}$ is a homogenous space under $K_m/K_m^{(1)} = \mathrm{SL}_2$. One gets then

$$C_{\mu,\alpha} \simeq \mathbb{P}^{1,\diamond}, \quad \text{and} \quad (L^+G \cap Li_\alpha(K_m))t^\mu \simeq \mathbb{A}^{1,\diamond} \subset \mathbb{P}^{1,\diamond}.$$

In addition,

$$C_{\mu,\alpha} \setminus (L^+G \cap Li_\alpha(K_m))t^\mu = i_\alpha(\sigma_m)t^\mu = t^{\mu-\alpha}L^+G.$$

Thus $C_{\mu,\alpha}$ is the desired curve.

(2) Since $\langle 2\rho, \mu\rangle = \langle 2\rho, -w_0\mu\rangle$ and $\dim \mathrm{Gr}_\mu = \dim \mathrm{Gr}_{\mu^{-1}}$ (note that $LG \to LG, \quad g \mapsto g^{-1}$ induces an isomorphism $\mathrm{Gr}_\mu \simeq \mathrm{Gr}_{\mu^{-1}}$), we consider $\mathrm{Gr}_{\mu^{-1}} = L^+G\xi^\mu$. Let Φ^+ be the set of positive (absolute) roots of G for the choice of the above Borel subgroup $B \subset G_{\overline{\mathbb{Q}}_p}$. Consider the parabolic subgroups P_μ and $P_{\mu^{-1}}$ defined by the roots α such that $\langle \alpha, \mu\rangle \geq 0$ and $\langle \alpha, \mu\rangle \leq 0$ respectively. Then $P_{\mu^{-1}}$ is the opposite parabolic of P_μ. Let $U = U_{P_\mu}$ be the unipotent radical of P_μ. Then $U \times P_{\mu^{-1}} \subset G$ defines an open subspace. Consider the associated open functor $L^+(U \times P_{\mu^{-1}}) = L^+U \times L^+P_{\mu^{-1}} \subset L^+G$. Then since $L^+P_{\mu^{-1}} \subset L^+G \cap \xi^\mu L^+G\xi^{-\mu}$ acts trivially on ξ^μ, we have open functor $L^+U\xi^\mu \subset L^+G\xi^\mu = \mathrm{Gr}_{\mu^{-1}}$. By definition, $U = \prod_{\alpha\in\Phi^+, \langle\alpha,\mu\rangle>0} U_\alpha$ with U_α the subgroup of G corresponding to the root α. Then

$$\begin{aligned} L^+U\xi^\mu &= (\prod_{\alpha\in\Phi^+,\langle\alpha,\mu\rangle>0} L^+U_\alpha)\xi^\mu \\ &= \prod_{\alpha\in\Phi^+,\langle\alpha,\mu\rangle>0} (L^+U_\alpha\xi^\mu) \\ &= \prod_{\alpha\in\Phi^+,\langle\alpha,\mu\rangle>0} \mathbb{B}^+_{dR}/\xi^{\langle\alpha,\mu\rangle}\xi^\mu, \end{aligned}$$

where $\mathbb{B}^+_{dR}$ is the functor which sends a perfectoid affinoid $\mathbb{Q}_p$-algebra (R, R^+) to $B^+_{dR}(R)$, and the last "=" comes from the fact that $L^+U_\alpha \simeq \mathbb{B}^+_{dR}$ which acts on ξ^μ through $\mathbb{B}^+_{dR}/\xi^{\langle\alpha,\mu\rangle}$. Moreover the action of $\mathbb{B}^+_{dR}/\xi^{\langle\alpha,\mu\rangle}$ on ξ^μ is free, thus

$$L^+U\xi^\mu \simeq \prod_{\alpha\in\Phi^+,\langle\alpha,\mu\rangle>0} \mathbb{B}^+_{dR}/\xi^{\langle\alpha,\mu\rangle}.$$

[13]We refer the reader to [38] section 21 for the definition and discussions for dimensions of diamonds.

By [40] subsection 15.2, for each α as above, the Banach-Colmez space $\mathbb{B}_{dR}^+/\xi^{\langle\alpha,\mu\rangle}$ is a diamond, which is a successive extension of $\mathbb{A}^{1,\diamond}$ and $\dim \mathbb{B}_{dR}^+/\xi^{\langle\alpha,\mu\rangle} = \langle\alpha,\mu\rangle$. Therefore,

$$\dim \mathrm{Gr}_\mu = \dim L^+U\xi^\mu = \sum_{\alpha\in\Phi^+,\langle\alpha,\mu\rangle>0} \langle\alpha,\mu\rangle = \langle 2\rho,\mu\rangle.$$

□

For $\mu \in X_*(T)^+$, as in the above proof let P_μ be the parabolic subgroup of $G_{\overline{\mathbb{Q}}_p}$ associated to the roots α such that $\langle\alpha,\mu\rangle \geq 0$. In other words P_μ is given by the formula

$$P_\mu = \{g \in G \,|\, \lim_{t\to 0} \mu(t)g\mu(t)^{-1} \,\text{exists}\}.$$

Then $P_\mu \supset B$. Consider the flag variety $\mathscr{F}\ell(G,\mu) = G_C/P_\mu$, which is defined over E. By [2] Proposition 3.4.3,Theorem 3.4.5 and [40] Proposition 19.4.2, there is a natural Bialynicki-Birula map[14] for diamonds over E

$$\pi_\mu : \mathrm{Gr}_\mu \to \mathscr{F}\ell(G,\mu)^\diamond,$$

which is an isomorphism if μ is minuscule. Let us recall the definition of π_μ. Group theoretically, over C it is the projection

$$\pi_\mu : \mathrm{Gr}_\mu \simeq \frac{L^+G}{L^+G \cap t^\mu L^+G t^{-\mu}} \longrightarrow G_C/P_\mu$$

induced by the projection

$$\theta : L^+G(R) = G(B_{dR}^+(R)) \to G(R)$$

for any C-perfectoid algebra R. Alternatively, we can give the moduli interpretation as follows. By Tannakian formalism, it is enough to define it for GL_n. In this case, μ is given by a tuple of integers $(m_1,\dots,m_n)$ with $m_1 \geq \cdots \geq m_n$. Then Gr_μ parametrizes lattices $\Xi \subset B_{dR}(R)^n$ of relative position $(m_1,\dots,m_n)$. For any such lattice, we can define a descending filtration $\mathrm{Fil}_\Xi^\bullet$ on the residue $R^n = B_{dR}^+(R)^n/\xi B_{dR}^+(R)^n$ with

$$\mathrm{Fil}_\Xi^i = \frac{\xi^i\Xi \cap B_{dR}^+(R)^n}{\xi^i\Xi \cap \xi B_{dR}^+(R)^n}.$$

The stabilizer of this filtration defines a parabolic which is conjugate to P_μ. This gives the desired $\pi_\mu : \mathrm{Gr}_\mu \to \mathscr{F}\ell(G,\mu)^\diamond$. From the construction we see that in general, π_μ is surjective, and in fact it is a fibration in diamonds associated to affine spaces.

For C-points, recall (cf. subsection 2.6) $\mathscr{F}\ell(G,\mu)(C) = \{\mathcal{F} \in \mathrm{Fil}_C(\omega^G) \,|\, \mathcal{F}\,\text{has type}\,\mu\}$, where $\mathrm{Fil}_C(\omega^G)$ is the set of $\mathbb{Q}$-filtrations over C of the standard fiber functor ω^G. The map $\mathrm{Gr}_\mu(C,\mathcal{O}_C) \to \mathscr{F}\ell(G,\mu)(C)$ sends a G-torsor to a "G-filtration". We can define similarly

$$\pi : \mathrm{Gr}_G(C,\mathcal{O}_C) \to \mathrm{Fil}_C(\omega^G),$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Gr}_G(C,\mathcal{O}_C) & \xrightarrow{\pi} & \mathrm{Fil}_C(\omega^G) \\ \downarrow & & \downarrow \\ X_*(T)^+ & \longrightarrow & X_*(G)_{\mathbb{Q}}/G, \end{array}$$

where the left vertical arrow is given by the Cartan decomposition, the right vertical arrow is given by taking a splitting modulo conjugacy, and the bottom arrow is given by the identifications $X_*(T)^+ = X_*(T)/W = X_*(G)/G$ and the inclusion $X_*(G)/G \hookrightarrow X_*(G)_{\mathbb{Q}}/G$.

[14]Note that according to our convention, here π_μ agrees with that in [2] Proposition 3.4.3, and it is the $\pi_{\mu^{-1}}$ of that in [40] Proposition 19.4.2.

Now let H be an arbitrary linear algebraic group over $\mathbb{Q}_p$. Then we define the B_{dR}^+-affine Grassmannian $\mathrm{Gr}_H = LH/L^+H$ similarly as above.

Proposition 6.3. Gr_H *is representable by an ind-diamond, which is ind-proper if H is reductive.*

Proof. As in the proof of [31] Theorem 1.4, we can take a faithful representation $H \hookrightarrow \mathrm{GL}_n$ such that GL_n/H is quasi-affine. Then the arguments in the proof of [40] Lemma 19.1.5 show that the induced map $\mathrm{Gr}_H \to \mathrm{Gr}_{\mathrm{GL}_n}$ is a locally closed embedding. Since $\mathrm{Gr}_{\mathrm{GL}_n}$ is representable by an ind-diamond by [40] 19.3, we conclude that Gr_H is also representable by an ind-diamond. In case H is reductive, Theorem 19.2.4 of [40] implies that it is ind-proper. □

6.2. **Hecke stacks and B_{dR}^+-affine Schubert cells.** Fix a dominant cocharacter $\mu \in X_*(T)^+$. We have the Hecke stack $Hecke^{\mu}$ over $\overline{\mathbb{F}}_p$ (here we slightly modify the definition in [12] 3.4): for any $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$, $Hecke^{\mu}(\mathrm{Spa}(R, R^+))$ is the groupoid of quadruples $(\mathcal{E}_1, \mathcal{E}_2, D, f)$, where

- $\mathcal{E}_1$ and $\mathcal{E}_2$ are G-bundles on X_R,
- D is an effective Cartier divisor of degree 1 on X_R,
- $f : \mathcal{E}_1|_{X_R\setminus D} \stackrel{\sim}{\longrightarrow} \mathcal{E}_2|_{X_R\setminus D}$ is a modification of G-bundles, such that the type of f_x is μ for any geometric point $x = \mathrm{Spa}(C(x), C(x)^+) \to \mathrm{Spa}(R, R^+)$.

This Hecke stack fits into the following diagram

$$\begin{array}{ccc} & Hecke^{\mu} & \\ \swarrow^{\overleftarrow{h}} & & \searrow^{\overrightarrow{h}} \\ \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} & & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times Div^1_X, \end{array}$$

where $Div^1_X = X^{\diamond} \times \mathrm{Spa}(\breve{\mathbb{Q}}_p)^{\diamond}/\varphi^{\mathbb{Z}}$ is the diamond of degree one divisors on X and

$$\overleftarrow{h}(\mathcal{E}_1, \mathcal{E}_2, f, D) = \mathcal{E}_2, \quad \overrightarrow{h}(\mathcal{E}_1, \mathcal{E}_2, f, D) = (\mathcal{E}_1, D).$$

The above diagram is the stack version of the diagram in subsection 2.1.

Let $[b] \in B(G, \mu)$ be the basic element. Fix a representative $b \in G(\breve{\mathbb{Q}}_p)$ of $[b]$ and we have the reductive group J_b. Let

$$x_1 : \mathrm{Spa}(\overline{\mathbb{F}}_p) \to [\mathrm{Spa}(\overline{\mathbb{F}}_p)/\underline{G(\mathbb{Q}_p)}] \to \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$$

and

$$x_b : \mathrm{Spa}(\overline{\mathbb{F}}_p) \to [\mathrm{Spa}(\overline{\mathbb{F}}_p)/\underline{J_b(\mathbb{Q}_p)}] \to \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$$

be the points associated to the classes [1] and $[b]$. Consider the diamonds Gr_{μ} and $\mathrm{Gr}_{\mu^{-1}}$ over $\breve{E}$. Then we have the following enlarged diagram[15] where Gr_{μ} and $\mathrm{Gr}_{\mu^{-1}}$ appear:

$$\begin{array}{ccccccc} & \mathrm{Gr}_{\mu} & & & & \mathrm{Gr}_{\mu^{-1}} & \\ \swarrow & & \searrow^{i_b} & & \swarrow^{i_1} & & \searrow \\ \mathrm{Spa}(\overline{\mathbb{F}}_p) & & & Hecke^{\mu} & & & \mathrm{Spa}(\overline{\mathbb{F}}_p) \\ & \searrow^{x_b} & \swarrow^{\overleftarrow{h}} & & \searrow^{\overrightarrow{h}} & \swarrow^{(x_1 id)} & \\ & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} & & & & \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times Div^1_X, & \end{array}$$

where both the squares are cartesian. In particular, we get

$$\mathrm{Gr}_{\mu^{-1}} \longrightarrow \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$$

[15] We can add $\mathrm{Sht}(G, \mu, b)_{\infty}$ on the top together with the period maps π_{dR} and π_{HT} to get a further cartesian square and thus a even larger diagram, cf. [12] 8.2.

which is the composition $\overleftarrow{h} \circ i_1$, and

$$\mathrm{Gr}_\mu \longrightarrow \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$$

which is the composition $\mathrm{pr} \circ \overrightarrow{h} \circ i_b$, where $\mathrm{pr} : \mathrm{Bun}_{G,\overline{\mathbb{F}}_p} \times Div^1_X \to \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$ is the natural projection.

We have also the version of Hecke stack $Hecke^{\leq\mu}$, which can be defined similarly and it is related to $\mathrm{Gr}_{\leq\mu}$ and $\mathrm{Gr}_{\leq\mu^{-1}}$ as above.

6.3. **Newton and Harder-Narasimhan stratifications on** $\mathrm{Gr}_{\mu^{-1}}$**.** Fix a dominant cocharacter $\mu \in X_*(T)^+$. Consider the affine Schubert cells Gr_μ and $\mathrm{Gr}_{\mu^{-1}}$.

We first study the geometry of $\mathrm{Gr}_{\mu^{-1}}$ using modifications of the trivial G-bundle $\mathcal{E}_1$. Consider the morphism $\mathrm{Gr}_{\mu^{-1}} \longrightarrow \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$ constructed above. The induced map on the sets of C-valued points can be described in more concrete terms. Let $C|\breve{E}$ be an algebraically closed perfectoid field. For any $x \in \mathrm{Gr}_{\mu^{-1}}(C, \mathcal{O}_C)$, we have modification $\mathcal{E}_{1,x}$ of $\mathcal{E}_1$. The isomorphism class of $\mathcal{E}_{1,x}$ defines a point $b(\mathcal{E}_{1,x}) \in B(G)$. We write $Newt : \mathrm{Gr}_{\mu^{-1}}(C, \mathcal{O}_C) \to B(G)$ for the map.

Proposition 6.4. *The image of the induced map*

$$Newt : \mathrm{Gr}_{\mu^{-1}}(C, \mathcal{O}_C) \to B(G)$$

is $B(G, \mu)$*.*

Proof. The fact that the image of the above map is included in $B(G, \mu)$ follows from [2] Proposition 3.5.3. To show the surjectivity, if μ is minuscule, then it follows from [34] Proposition A.9. The arguments in loc. cit. in fact apply to the general case. For the reader's convenience, we sketch the arguments. Consider Gr_μ and let $[b] \in B(G, \mu)$ be any element. Fix a representative $b \in G(\breve{\mathbb{Q}}_p)$ of $[b]$ and let $\mathrm{Gr}^a_\mu \subset \mathrm{Gr}_\mu$ be the associated admissible locus (here $\mathrm{Gr}^a_\mu = \mathrm{Gr}_\mu \cap \mathrm{Gr}^a_{\leq\mu}$ and $\mathrm{Gr}^a_{\leq\mu}$ is the admissible locus introduced in the proof of [40] Proposition 23.3.3), and $\mathscr{F}\ell(G, \mu, b)^{wa} \subset \mathscr{F}\ell(G, \mu)$ be the associated weakly admissible locus (cf. [37, 6]). Then the Bialynicki-Birula map induces a morphism of diamonds

$$\pi_\mu : \mathrm{Gr}^a_\mu \to \mathscr{F}\ell(G, \mu, b)^{wa,\diamond}.$$

By the theorem of Colmez-Fontaine (cf. [14] chapter 10), we have $\mathrm{Gr}^a_\mu(K, \mathcal{O}_K) = \mathscr{F}\ell(G, \mu, b)^{wa}(K, \mathcal{O}_K)$ for any finite extension $K|\breve{E}$. Thus $\mathscr{F}\ell(G, \mu, b)^{wa} \neq \emptyset$ if and only if $\mathrm{Gr}^a_\mu \neq \emptyset$. Since $[b] \in B(G, \mu)$, by [36] Proposition 3.1, $\mathscr{F}\ell(G, \mu, b)^{wa} \neq \emptyset$ and thus $\mathrm{Gr}^a_\mu \neq \emptyset$. Take a point $x \in \mathrm{Gr}_{\mu^{-1}}(C, \mathcal{O}_C)$. By definition,

$$x \in \mathrm{Gr}^{Newt=[b]}_{\mu^{-1}}(C, \mathcal{O}_C) \quad \Leftrightarrow \quad \mathcal{E}_{1,x} \simeq \mathcal{E}_b \quad \Leftrightarrow \quad \mathcal{E}_1 = \mathcal{E}_{b,x^*}$$

for some $x^* \in \mathrm{Gr}_\mu(C, \mathcal{O}_C)$. This is equivalent to $x^* \in \mathrm{Gr}^a_\mu(C, \mathcal{O}_C)$. Thus we get for any $[b] \in B(G, \mu)$, $\mathrm{Gr}^{Newt=[b]}_{\mu^{-1}}(C, \mathcal{O}_C) \neq \emptyset$. □

Letting C vary, we get a map

$$Newt : |\mathrm{Gr}_{\mu^{-1}}| \longrightarrow B(G, \mu).$$

By [22] (in the case $G = \mathrm{GL}_n$) and [40] Corollary 22.5.1, this map is upper semi-continuous. The Newton stratification of $\mathrm{Gr}_{\mu^{-1}}$ is the following stratification in diamonds over E:

$$\mathrm{Gr}_{\mu^{-1}} = \coprod_{[b'] \in B(G,\mu)} \mathrm{Gr}^{Newt=[b']}_{\mu^{-1}}.$$

The open Newton stratum $\mathrm{Gr}^{Newt=[b]}_{\mu^{-1}}$ is associated to the basic element $[b] \in B(G, \mu)$. We have a natural action of $G(\mathbb{Q}_p)$ on $\mathrm{Gr}_{\mu^{-1}}$ and for any $[b'] \in B(G, \mu)$, the stratum $\mathrm{Gr}^{Newt=[b']}_{\mu^{-1}}$ is stable under this action.

Proposition 6.5. *We have the following dimension formula: for $[b'] \in B(G,\mu)$,*

$$\dim \mathrm{Gr}_{\mu^{-1}}^{Newt=[b']} = \langle \mu - \nu([b']), 2\rho \rangle.$$

Proof. This is essentially the same as the proof of Theorem 3.1 (2), using the diagram in subsection 2.6 and the dimension formula $\dim \mathrm{Gr}_\mu = \langle \mu, 2\rho \rangle$ of Proposition 6.2 (2). □

For any point $x \in \mathrm{Gr}_{\mu^{-1}}(C, \mathcal{O}_C)$, consider the admissible modification $(\mathcal{E}_1, \mathcal{E}_{1,x}, f)$ and $\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f) \in \mathcal{N}(G)$. Then Proposition 3.4 still holds in this setting, since in the proof we don't need the minuscule condition. In other words, we have

$$\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f) \leq \nu(\mathcal{E}_{1,x})$$

and

$$\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f)^* = w_0(-\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f)) \in \mathcal{N}(G,\mu).$$

Letting C vary, we get a map

$$HN : |\mathrm{Gr}_{\mu^{-1}}| \longrightarrow \mathcal{N}(G,\mu).$$

Theorem 6.6. *The above map HN is upper semi-continuous, that is, for any $v \in \mathcal{N}(G,\mu)$, the subset*

$$\mathrm{Gr}_{\mu^{-1}}^{HN \geq v} := \{x \in |\mathrm{Gr}_{\mu^{-1}}| \, | HN(x) \geq v\}$$

is closed. In particular, the subset

$$\mathrm{Gr}_{\mu^{-1}}^{HN = v} := \{x \in |\mathrm{Gr}_{\mu^{-1}}| \, | HN(x) = v\}$$

is locally closed, thus it defines a sub diamond of $\mathrm{Gr}_{\mu^{-1}}$.

Proof. This is similar to the proof of Theorem 3.5: for any $x \in \mathrm{Gr}_{\mu^{-1}}(C, \mathcal{O}_C)$, by Theorem 2.4

$$\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f) = \nu(\mathcal{F}_x)$$

with $\mathcal{F}_x \in \mathrm{Fil}_C(\omega^G)$ attached to $\pi_{\mu^{-1}}(x) \in \mathscr{F}\ell(G,\mu^{-1})(C, \mathcal{O}_C)$. Thus the map $HN : |\mathrm{Gr}_{\mu^{-1}}| \longrightarrow \mathcal{N}(G,\mu)$ factors through $HN : |\mathscr{F}\ell(G,\mu^{-1})| \longrightarrow \mathcal{N}(G,\mu)$, via the Bialynicki-Birula map. For the flag variety $\mathscr{F}\ell(G,\mu^{-1})$, this follows from (the proof of) [6] Theorem 6.3.5 and Proposition 6.3.12.

□

In the following, we will identify $\mathcal{N}(G,\mu)$ with $B(G,\mu)$ by the Newton map. We have the following stratification of diamonds over E:

$$\mathrm{Gr}_{\mu^{-1}} = \coprod_{[b'] \in B(G,\mu)} \mathrm{Gr}_{\mu^{-1}}^{HN=[b']}.$$

For any $[b'] \in B(G,\mu)$, the stratum $\mathrm{Gr}_{\mu^{-1}}^{HN=[b']}$ is stable under the action of $G(\mathbb{Q}_p)$ on $\mathrm{Gr}_{\mu^{-1}}$. By the proof of Theorem 6.6, this stratification is the pullback of

$$\mathscr{F}\ell(G,\mu^{-1}) = \coprod_{[b'] \in B(G,\mu)} \mathscr{F}\ell(G,\mu^{-1})^{HN=[b']}$$

via the Bialynicki-Birula map

$$\pi_{\mu^{-1}} : \mathrm{Gr}_{\mu^{-1}} \to \mathscr{F}\ell(G,\mu^{-1})^\diamond.$$

The open Harder-Narasimhan stratum $\mathrm{Gr}_{\mu^{-1}}^{HN=[b]}$ is associated to the basic element $[b] \in B(G,\mu)$. Note that theorem 3.9 still holds for the flag variety $\mathscr{F}\ell(G,\mu^{-1})$ (which is reduced to [6] Theorem 6.3.5). Pulling back under $\pi_{\mu^{-1}} : \mathrm{Gr}_{\mu^{-1}} \to \mathscr{F}\ell(G,\mu^{-1})^\diamond$, we get

Corollary 6.7. *For any non basic $[b'] \in B(G,\mu)$, the stratum $\mathrm{Gr}_{\mu^{-1}}^{HN=[b']}$ is a parabolic induction.*

6.4. **Newton and Harder-Narasimhan stratifications on Gr_μ.** Let $[b] \in B(G,\mu)$ be the *basic* element. Now we study the geometry of Gr_μ using modifications of the G-bundle $\mathcal{E}_b$. Consider the map $\mathrm{Gr}_\mu \to \mathrm{Bun}_{G,\overline{\mathbb{F}}_p}$ constructed in subsection 6.2. Let $C|\breve{E}$ be an algebraically closed perfectoid field. The induced map on the sets of C-valued points can be described In more concrete terms. For any $x \in \mathrm{Gr}_\mu(C, \mathcal{O}_C)$, we have modification $\mathcal{E}_{b,x}$ of $\mathcal{E}_b$. The isomorphism class of $\mathcal{E}_{b,x}$ defines a point $b(\mathcal{E}_{b,x}) \in B(G)$. We write $Newt : \mathrm{Gr}_\mu(C, \mathcal{O}_C) \to B(G)$ for the map.

Proposition 6.8. *The image of the induced map $Newt : \mathrm{Gr}_\mu(C, \mathcal{O}_C) \to B(G)$ is $B(G, 0, \nu_b\mu^{-1})$.*

Proof. For μ minuscule, this has been studied in [3] section 5 (see also [34] A.10). The arguments in [3] section 5 work in the general case. See also the proof of Proposition 6.4. □

Letting C vary, we get a map

$$Newt : |\mathrm{Gr}_\mu| \longrightarrow B(G, 0, \nu_b\mu^{-1}),$$

which is upper semi-continuous by [22, 40]. Thus we have the Newton stratification[16] of diamonds over $\breve{E}$:

$$\mathrm{Gr}_\mu = \coprod_{[b'] \in B(G,0,\nu_b\mu^{-1})} \mathrm{Gr}_\mu^{Newt=[b']}.$$

For any $[b'] \in B(G, 0, \nu_b\mu^{-1})$, the stratum $\mathrm{Gr}_\mu^{Newt=[b']}$ is stable under the action of $J_b(\mathbb{Q}_p)$ on Gr_μ. The open Newton stratum $\mathrm{Gr}_\mu^{Newt=[1]}$ corresponds to the trivial element $[1] \in B(G, 0, \nu_b\mu^{-1})$, which we will also denote by Gr_μ^a (the admissible locus inside Gr_μ with respective to $(G, \{\mu\}, [b])$, which we already introduced in the proof of Proposition 6.4).

Now we want to define a Harder-Narasimhan stratification on Gr_μ. We first come back to the p-adic flag variety $\mathscr{F}\ell(G,\mu)$. Starting from the local Shtuka datum $(G, \{\mu\}, [b])$, Dat-Orlik-Rapoport introduced a stratification on $\mathscr{F}\ell(G,\mu)$ indexed by HN vectors, cf. [6] IX.6 (and the previous subsection 4.4). The index set of this stratification is denoted by $\mathcal{H}(\mathbb{G}, \mu)$ in loc. cit., which is a finite subset of $\mathcal{N}(G)$, where $\mathbb{G}$ is the augmented group attached to G and b in [6] Example 9.1.22. Similar to Proposition 4.3, we can prove that the Newton map ν induces an injection $\mathcal{H}(\mathbb{G}, \mu) \hookrightarrow B(G, 0, \nu_b\mu^{-1})$. Therefore, we have

$$\mathscr{F}\ell(G,\mu) = \coprod_{[b'] \in B(G,0,\nu_b\mu^{-1})} \mathscr{F}\ell(G,\mu,b)^{HN=[b']}.$$

There is a unique minimal element v_0 in $\mathcal{H}(\mathbb{G}, \mu)$, which corresponds to $[1] \in B(G, 0, \nu_b\mu^{-1})$. The corresponding stratum (called semi-stable locus in [6]) is the weakly admissible locus

$$\mathscr{F}\ell(G,\mu,b)^{wa} \subset \mathscr{F}\ell(G,\mu)$$

previously studied in [37] chapter 1. Via the Bialynicki-Birula map

$$\pi_\mu : \mathrm{Gr}_\mu \to \mathscr{F}\ell(G,\mu)^\diamond,$$

the above stratification on $\mathscr{F}\ell(G,\mu)$ induces a stratification[17] on Gr_μ:

$$\mathrm{Gr}_\mu = \coprod_{[b'] \in B(G,0,\nu_b\mu^{-1})} \mathrm{Gr}_\mu^{HN=[b']},$$

which we call the Harder-Narasimhan stratification. The open Harder-Narasimhan stratum $\mathrm{Gr}_\mu^{HN=[1]}$ corresponds to the trivial element $[1] \in B(G, 0, \nu_b\mu^{-1})$, which we will also denote by Gr_μ^{wa} (the weakly admissible locus inside Gr_μ with respective to $(G, \{\mu\}, [b])$).

[16] Note here we have used simplified notations compared with the minuscule case: we have omitted the subscript b on each Newton stratum.

[17] Similar as Newton stratification case here, we have used simplified notations.

Remark 6.9. *In this subsection, to define the Newton and Harder-Narasimhan stratification, in fact we don't need the assumption that $[b]$ is basic. However, we don't know a description of the index set for a non basic $[b]$. Nevertheless, the open strata Gr_μ^a and Gr_μ^{wa} are always well defined.*

Consider the dual local Shtuka datum $(J_b, \{\mu^{-1}\}, [b^{-1}])$ and the triple $(J_b, \{\mu\}, [1])$. Then we can consider the Newton and Harder-Narasimhan stratifications on $\mathrm{Gr}_{J_b,\mu^{-1}}$ and $\mathrm{Gr}_{J_b,\mu}$ as before. Since $[b]$ is basic, the isomorphism $J_{b,\breve{\mathbb{Q}}_p} \overset{\sim}{\to} G_{\breve{\mathbb{Q}}_p}$ induces identifications $\mathrm{Gr}_{G,\mu} = \mathrm{Gr}_{J_b,\mu}$ and $\mathrm{Gr}_{G,\mu^{-1}} = \mathrm{Gr}_{J_b,\mu^{-1}}$ as diamonds over $\mathrm{Spd}\,\breve{E}$. The results of subsection 4.3 still hold (cf. [40] subsection 23.3). Now the following generalization of Theorem 4.4 and Corollary 4.5 is clear:

Theorem 6.10. (1) *Under the identification $\mathrm{Gr}_{G,\mu^{-1}} = \mathrm{Gr}_{J_b,\mu^{-1}}$, for any $[b'] \in B(G,\mu)$ corresponding to $[b''] \in B(J_b, 0, \nu_{b^{-1}}\mu)$ under the bijection (cf. Lemma 4.1)*

$$B(G,\mu) \overset{\sim}{\longrightarrow} B(J_b, 0, \nu_{b^{-1}}\mu),$$

we have
(a) $\mathrm{Gr}_{G,\mu^{-1}}^{HN=[b']} = \mathrm{Gr}_{J_b,\mu^{-1}}^{HN=[b'']}$.
(b) $\mathrm{Gr}_{G,\mu^{-1}}^{Newt=[b']} = \mathrm{Gr}_{J_b,\mu^{-1}}^{Newt=[b'']}$.

(2) *Under the identification $\mathrm{Gr}_{J_b,\mu} = \mathrm{Gr}_{G,\mu}$, for any $[b'] \in B(G,\mu)$ corresponding to $[b''] \in B(J_b, 0, \nu_{b^{-1}}\mu)$ under the bijection (cf. Lemma 4.1)*

$$B(J_b,\mu^{-1}) \overset{\sim}{\longrightarrow} B(G, 0, \nu_b\mu^{-1}),$$

we have
(a) $\mathrm{Gr}_{J_b,\mu}^{HN=[b']} = \mathrm{Gr}_{G,\mu}^{HN=[b'']}$.
(b) $\mathrm{Gr}_{J_b,\mu}^{Newt=[b']} = \mathrm{Gr}_{G,\mu}^{Newt=[b'']}$.

6.5. **Extensions to $\mathrm{Gr}_{\leq\mu}$ and $\mathrm{Gr}_{\leq\mu^{-1}}$.** We can extend the above constructions to $\mathrm{Gr}_{\leq\mu}$ and $\mathrm{Gr}_{\leq\mu^{-1}}$. First, we note the following lemma.

Lemma 6.11. *For $\mu_1, \mu_2 \in X_*(T)^+$ with $w_0(-\mu_1) \leq w_0(-\mu_2)$, we have a natural injection $B(G,\mu_1) \hookrightarrow B(G,\mu_2)$.*

Proof. The assumption $w_0(-\mu_1) \leq w_0(-\mu_2)$ implies that $\mu_1 \leq \mu_2$ and thus $\mu_1^\diamond \leq \mu_2^\diamond$. Recall that by [25] 4.13,

$$(\kappa,\nu) : B(G) \to \pi_1(G)_\Gamma \times \mathcal{N}(G)$$

is injective. For $[b] \in B(G,\mu_1)$, consider the pair $(\mu_2^\sharp, \nu([b]) \in \pi_1(G)_\Gamma \times \mathcal{N}(G)$. It comes from a unique element $[b'] \in B(G)$ under the injection $(\kappa,\nu) : B(G) \hookrightarrow \pi_1(G)_\Gamma \times \mathcal{N}(G)$, since κ is surjective and $\mu_2^\sharp \equiv \nu([b])$ in $\pi_1(G)_\Gamma$. Then since $\nu([b']) = \nu([b]) \leq \mu_1^\diamond \leq \mu_2^\diamond$, by definition $[b'] \in B(G,\mu_2)$. In this way we get an injection $B(G,\mu_1) \hookrightarrow B(G,\mu_2)$. □

By the above lemma, we can define Newton and Harder-Narasimhan stratifications on $\mathrm{Gr}_{\leq\mu^{-1}}$ by modifications of the trivial G-bundle $\mathcal{E}_1$, with both of the index sets as $B(G,\mu)$. These strata will be the union over all $(\mu')^{-1} \leq \mu^{-1}$ of the corresponding strata (could be empty) inside $\mathrm{Gr}_{(\mu')^{-1}}$. Similar, for $[b] \in B(G,\mu)$ basic, we can define Newton and Harder-Narasimhan stratifications on $\mathrm{Gr}_{\leq\mu}$ by modifications of the G-bundle $\mathcal{E}_b$, with both of the index sets as $B(G, 0, \nu_b\mu^{-1})$. The strata will be the union over all $\mu' \leq \mu$ of the corresponding strata (which could be empty) inside $\mathrm{Gr}_{\mu'}$. We will use the version of moduli of local G-Shtukas $\mathrm{Sht}(G, \leq \mu, b)_\infty$ in this setting. We can consider the dual local Shtuka datum $(J_b, \{\mu^{-1}\}, [b^{-1}])$. Then the constructions and results in subsections 6.3 and 6.4, in particular Theorem 6.10, can be generalized to the current setting. We leave the details to the interested reader.

6.6. **Fargues-Rapoport conjecture for general μ.** Let G be a reductive group over $\mathbb{Q}_p$, $P \subset G$ a parabolic subgroup over $\mathbb{Q}_p$, M a Levi subgroup contained in P, which is identified with the reductive quotient of P. Take a maximal torus inside a Borel $T \subset B \subset G_{\overline{\mathbb{Q}}_p}$ and assume $B \subset P_{\overline{\mathbb{Q}}_p}$ and thus $T \subset M_{\overline{\mathbb{Q}}_p}$. We have the set of dominant cocharacters $X_*(T)^+$. Let $B \cap M_{\overline{\mathbb{Q}}_p}$ be the induced Borel of $M_{\overline{\mathbb{Q}}_p}$. Then we get the set of M-dominant cocharacters $X_*(T)^+_M$. We have the inclusion $X_*(T)^+ \subset X_*(T)^+_M$.

To simplify notations, the base field in this subsection will be C, an algebraically closed perfectoid field of characteristic 0 (in fact an extension $F|\mathbb{Q}_p$ which splits G will be enough). In the following we will write $B_{dR} = B_{dR}(C)$. Consider B^+_{dR}-affine Grassmannians $\mathrm{Gr}_M, \mathrm{Gr}_G$ and Gr_P (cf. Proposition 6.3) over C. The inclusion $P \subset G$ and the projection $P \to M$ induce the following diagram of B^+_{dR}-affine Grassmannians:

$$\mathrm{Gr}_M \xleftarrow{\ \mathrm{pr}\ } \mathrm{Gr}_P \xrightarrow{\ i\ } \mathrm{Gr}_G.$$

We have the following fact: the Iwasawa decomposition

$$G(B_{dR}) = P(B_{dR})G(B^+_{dR})$$

induces a bijection

$$i : \mathrm{Gr}_P(C, \mathcal{O}_C) = P(B_{dR})/P(B^+_{dR}) \xrightarrow{\sim} \mathrm{Gr}_G(C, \mathcal{O}_C) = G(B_{dR})/G(B^+_{dR}).$$

Let $U_P \subset P$ be the unipotent radical of P. Since G/M (resp. G/U_P) is affine (resp. quasi-affine), the natural inclusion $M \subset G$ (resp. $U_P \subset G$) induces a closed embedding $\mathrm{Gr}_M \hookrightarrow \mathrm{Gr}_G$ by [40] Lemma 19.1.5 (resp. a locally closed embedding $\mathrm{Gr}_{U_P} \hookrightarrow \mathrm{Gr}_G$ by the proof of Proposition 6.3). For any $\lambda \in X_*(T)^+_M$, we have the locally spatial diamond $\mathrm{Gr}_{M,\lambda} \subset \mathrm{Gr}_M$. Consider the locally closed sub ind-diamond

$$S_\lambda := i\big(\mathrm{pr}^{-1}(\mathrm{Gr}_{M,\lambda})\big) \subset \mathrm{Gr}_G.$$

This is identified with the orbit $LU_P\mathrm{Gr}_{M,\lambda}$ for the natural action LU_P on Gr_G induced by $LU_P \subset LG$. The natural product defines a map $LU_P \times LM \to LG$ which induces a map $\mathrm{Gr}_{U_P} \times \mathrm{Gr}_M \to \mathrm{Gr}_G$. Then we have

$$S_\lambda = \mathrm{Gr}_{U_P}\mathrm{Gr}_{M,\lambda} \subset \mathrm{Gr}_G,$$

where $\mathrm{Gr}_{U_P}\mathrm{Gr}_{M,\lambda}$ denotes the image of $\mathrm{Gr}_{U_P} \times \mathrm{Gr}_{M,\lambda}$ under $\mathrm{Gr}_{U_P} \times \mathrm{Gr}_M \to \mathrm{Gr}_G$. The Iwasawa decomposition above implies that

$$\mathrm{Gr}_G = \coprod_{\lambda \in X_*(T)^+_M} S_\lambda.$$

In the following we consider the partial order $\leq_P$[18] on $X_*(T)$ (and the restriction to $X_*(T)^+_M$) with respective to the coroots appearing in $\mathrm{Lie}\, U_P$. When the setting is clear, we simply write $\lambda_1 \leq \lambda_2$ for $\lambda_1, \lambda_2 \in X_*(T)^+_M$ and $\lambda_1 \leq_P \lambda_2$. For any $\lambda \in X_*(T)^+_M$, like in the classical setting, S_λ is of infinite dimensional. Nevertheless, we have

Proposition 6.12. *The closure $\overline{S_\lambda}$ of S_λ is given by $S_{\leq\lambda} := \coprod_{\lambda' \leq \lambda} S_{\lambda'}$. More precisely, for any $\mu \in X_*(T)^+$, we have*

$$\overline{S_\lambda \cap \mathrm{Gr}_{\leq\mu}} = \coprod_{\lambda' \leq \lambda} S_{\lambda'} \cap \mathrm{Gr}_{\leq\mu}.$$

[18] Note that this is different from the partial order $\leq_M$ used in some literatures, e.g. [16] 5.1, where one uses simple coroots of M.

Proof. We follow the argument of [44] Proposition 5.3.6. We show firstly that $S_{\leq\lambda}$ is closed. First, assume that G_{der} is simply connected. For any highest weight representation V_χ of G, let ℓ_χ be the corresponding highest weight line. Then we have the following description

$$S_{\leq\lambda} = \bigcap_{V_\chi}\{(\mathcal{E},\beta) \in \mathrm{Gr}_G |\, \beta^{-1}(\ell_\chi) \subset t^{-\langle\chi,\lambda\rangle}(\mathcal{E}_{V_\chi})\},$$

where the intersection runs through all highest weight representations V_χ of G, and $\mathcal{E}_{V_\chi} = \mathcal{E} \times^G V_\chi$ is the induced vector bundle. It suffices to prove the locus

$$\{(\mathcal{E},\beta) \in \mathrm{Gr}_G |\, \beta^{-1}(\ell_\chi) \subset t^{-\langle\chi,\lambda\rangle}(\mathcal{E}_{V_\chi})\} \subset \mathrm{Gr}_G$$

is closed. This follows from the proof of [40] Lemma 19.1.4. For general G, one can pass to a z-extension to reduce to the case when G_{der} is simply connected.

Now we show $\overline{S_\lambda} = S_{\leq\lambda}$. For $\lambda' \leq \lambda$, there exists a positive coroot α appearing in $\mathrm{Lie}\, U_P$ such that $\lambda - \alpha$ is M-dominant and $\lambda' \leq \lambda - \alpha \leq \lambda$. Then the arguments in the proof of Proposition 6.2 (1) apply. □

Let $\mu \in X_*(T)^+$ be fixed and consider $\mathrm{Gr}_{G,\mu}$. For any $\lambda \in X_*(T)^+_M$, note that

$$S_\lambda \cap \mathrm{Gr}_{G,\mu} \neq \emptyset \quad \Longleftrightarrow \quad LU_P t^\lambda \cap \mathrm{Gr}_{G,\mu} \neq \emptyset.$$

Indeed, to prove the direction "$\Rightarrow$", it suffices to work with an algebraically closed field C and then use the normality of U_P. Set

$$S_M(\mu) := \{\lambda \in X_*(T)^+_M |\, S_\lambda \cap \mathrm{Gr}_{G,\mu} \neq \emptyset\}.$$

The stratification $\mathrm{Gr}_G = \coprod_{\lambda \in X_*(T)^+_M} S_\lambda$ induces a stratification

$$\mathrm{Gr}_{G,\mu} = \coprod_{\lambda \in S_M(\mu)} S_\lambda \cap \mathrm{Gr}_{G,\mu}.$$

For each $\lambda \in S_M(\mu)$, for simplicity we denote $\mathrm{Gr}_{G,\lambda} = S_\lambda \cap \mathrm{Gr}_{G,\mu}$, so that

$$\mathrm{Gr}_{G,\mu} = \coprod_{\lambda \in S_M(\mu)} \mathrm{Gr}_{G,\lambda}.$$

To describe the index set $S_M(\mu)$, first note by [16] Lemma 5.4.1

$$S_M(\mu) \subset \Sigma(\mu)_{M-dom},$$

where $\Sigma(\mu)_{M-dom} \subset X_*(T)^+_M$ is the set of M-dominant elements in $\{\mu' \in X_*(T) |\, \mu'_{dom} \leq \mu\}$. Indeed, to describe $S_M(\mu)$ we may choose any algebraically closed perfectoid field $C|\mathbb{Q}_p$ and consider the C-points of $\mathrm{Gr}_{G,\mu}(C, \mathcal{O}_C)$. Then $\lambda \in S_M(\mu)$ if and only if $\lambda \in X_*(T)^+_M$ and $U_P(B_{dR}(C))t^\lambda \cap G(B^+_{dR}(C))t^\mu G(B^+_{dR}(C)) \neq \emptyset$ (both as subsets of $G(B_{dR}(C))$). Fixing an isomorphism $B_{dR}(C) \simeq C((t))$, we translate these to subsets of $G\Big(C((t))\Big)$. As in the proof of [16] Lemma 5.4.1 (which is purely group theoretical and applies to general base fields), $\lambda \in \Sigma(\mu)_{M-dom}$ if and only if $\lambda \in X_*(T)^+_M$ and $U_B\Big(C((t))\Big)t^\lambda \cap G(C[[t]])t^\mu G(C[[t]]) \neq \emptyset$, where U_B is the unipotent radical of B.

Recall that attached to μ we have the parabolic subgroup $P_\mu \subset G_{\overline{\mathbb{Q}}_p}$. Let W (resp. W_P, W_{P_μ}) be the absolute Weyl group of G (resp. P, P_μ). We have the following inclusion:

$$W\mu \cap X_*(T)^+_M \subset S_M(\mu).$$

The set $W\mu \cap X_*(T)^+_M$ can be described as

$$W\mu \cap X_*(T)^+_M = {}^P W^{P_\mu}\mu,$$

where ${}^P W^{P_\mu} \subset W$ is the set of minimal length representatives in the corresponding coset in $W_P \setminus W / W_{P_\mu}$. Then the element

$$\lambda_0 = \mu \in S_M(\mu)$$

is the unique maximal element with respective to the partial order $\leq_P$. When μ is minuscule, we have $W\mu \cap X_*(T)^+_M = {}^PW^{P_\mu}\mu = S_M(\mu)$. In this case, under the isomorphism $\mathrm{Gr}_{G,\mu} \xrightarrow{\sim} \mathscr{F}\ell(G,\mu)^\diamond$, for $\lambda = w\mu$ with $w \in {}^PW^{P_\mu}$, we have

$$S_\lambda \cap \mathrm{Gr}_{G,\mu} \simeq (U_P w w_0 P_\mu / P_\mu)^\diamond,$$

where $w_0 \in W$ is the element of maximal length.

Remark 6.13. *The reader who believes the geometric Satake equivalence for B^+_{dR}-affine Grassmannians (cf. [15]) can have the following descriptions of $S_M(\mu)$:*

Let $\widehat{G}$ be the dual reductive group of G (over some characteristic zero algebraically closed field) and $\widehat{M} \subset \widehat{G}$ be Levi subgroup defined by the dual root datum of M. Similarly let $\widehat{T} \subset \widehat{B} \subset \widehat{G}$ be the maximal torus dual to T inside the Borel subgroup of $\widehat{G}$ dual to B. Then we may view $\mu \in X^*(\widehat{T})^+ = X_*(T)^+$. Consider the irreducible representation V_μ of highest weight μ of $\widehat{G}$. The geometric Satake equivalence in the current setting implies that $S_M(\mu)$ is the set of $\widehat{M}$-dominant weights of $\widehat{T}$ such that the associated highest weight representations of $\widehat{M}$ appear in the restricted representation $V_\mu|_{\widehat{M}}$:

$$S_M(\mu) = \{\lambda \in X^*(T)^+_{\widehat{M}} | \, 0 \neq V_\lambda \subset V_\mu|_{\widehat{M}}\},$$

where for any $\lambda \in X^*(T)^+_{\widehat{M}}$, V_λ is the irreducible representation of $\widehat{M}$ of highest weight λ.

We identify $W = W(\widehat{G})$ and $X^*(\widehat{T})^+_{\widehat{M}} = X_*(T)^+_M$. The set $W\mu \cap X^*(\widehat{T})^+_{\widehat{M}} = {}^PW^{P_\mu}\mu$ appears naturally when considering the decomposition of $V_\mu|_{\widehat{M}}$ into irreducible representations of $\widehat{M}$: we view $\mu \in X^*(\widehat{T})^+_{\widehat{M}}$, then the associated irreducible representation $V^{\widehat{M}}_\mu$ of $\widehat{M}$ appears in $V_\mu|_{\widehat{M}}$. Consider the adjoint action of W on $V_\mu = V_\mu|_{\widehat{M}}$. For any $w \in {}^PW^{P_\mu}$, we have

$$wV^{\widehat{M}}_\mu = V_{w\mu} \subset V_\mu|_{\widehat{M}}.$$

Any $\lambda \in S_M(\mu)$ is of the form

$$\lambda = \mu - \sum_{\alpha \in \Delta \setminus \Delta_{\widehat{M}}} n_\alpha \alpha, \quad n_\alpha \in \mathbb{N}, \forall\, \alpha,$$

where $\Delta = \Delta_{\widehat{G}}$ (resp. $\Delta_{\widehat{M}}$) is the set of simple roots of $\widehat{G}$ (resp. $\widehat{M}$). Therefore,

$$W\mu \cap X^*(\widehat{T})^+_{\widehat{M}} = W\mu \cap X_*(T)^+_M \subset S_M(\mu)$$

and $\mu \in S_M(\mu)$ is the unique maximal element.

In the following we sketch how to generalize the arguments in the proof of [3] Theorem 6.1 to the non minuscule case.

Fix a $\mu \in X_*(T)^+$ and consider $\mathrm{Gr}_\mu = \mathrm{Gr}_{G,\mu}$, which is defined over $\mathrm{Spd}\, E$ with $E = E(G, \{\mu\})$. As usual, let $\breve{E} = \widehat{E^{ur}}$ be the completion of the maximal unramified extension of E. We will study $\mathrm{Gr}_{G,\mu}$ over $\mathrm{Spd}\, \breve{E}$. First of all, we explain that the set $S_M(\mu)$ and the above diagram of B^+_{dR}-affine Grassmannians naturally arise when considering reductions of modifications of G-bundles to P-bundles (resp. M-bundles), cf. Lemma 6.14.

For $C|\breve{E}$ any algebraically closed perfectoid field, let $X = X_{C^\flat}$ be the Fargues-Fontaine curve over $\mathbb{Q}_p$ attached to $C^\flat$. Let $b \in G(\breve{\mathbb{Q}}_p)$ be an element with associated class $[b] \in B(G)$ and the G-bundle $\mathcal{E}_b$ on X (cf. [11]). For a Levi subgroup M of G, recall that (cf. [3] Definition 2.5) we have the notion of reductions of b to M. Such a reduction is given by an element $b_M \in M(\breve{\mathbb{Q}}_p)$ together with an element $g \in G(\breve{\mathbb{Q}}_p)$ such that $b = g b_M \sigma(g)^{-1}$. Then the M-bundle $\mathcal{E}_{b_M}$ is a reduction of $\mathcal{E}_b$. If $M \subset P$ for some parabolic subgroup P of G, let $b_P \in P(\breve{\mathbb{Q}}_p)$ be the image of b_M. This defines a reduction of b to P, and thus a reduction of the G-bundle $\mathcal{E}_b$ to a P-bundle $\mathcal{E}_{b_P}$. By construction, $\mathcal{E}_{b_P} = \mathcal{E}_{b_M} \times_M P$.

For any $x \in \mathrm{Gr}_G(C, \mathcal{O}_C)$, we can define a modification $\mathcal{E}_{b,x}$ of $\mathcal{E}$, thus a map

$$\mathrm{Gr}_G(C, \mathcal{O}_C) \to H^1_{\text{ét}}(X, G).$$

It is functorial in the following sense: we have similar maps

$$\mathrm{Gr}_P(C, \mathcal{O}_C) \to H^1_{\text{ét}}(X, P), \quad y \mapsto \mathcal{E}_{b_P,y},$$

$$\mathrm{Gr}_M(C, \mathcal{O}_C) \to H^1_{\text{ét}}(X, M), \quad z \mapsto \mathcal{E}_{b_M,z},$$

by considering modifications of the P-bundle $\mathcal{E}_{b_P}$ and the M-bundle $\mathcal{E}_{b_M}$ respectively. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Gr}_G(C, \mathcal{O}_C) & \longrightarrow & H^1_{\text{ét}}(X, G) \\ \uparrow & & \uparrow \\ \mathrm{Gr}_P(C, \mathcal{O}_C) & \longrightarrow & H^1_{\text{ét}}(X, P) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_M(C, \mathcal{O}_C) & \longrightarrow & H^1_{\text{ét}}(X, M), \end{array}$$

where the arrows on the right hand side are $\mathcal{E} \mapsto \mathcal{E} \times_P G, \quad \mathcal{E} \mapsto \mathcal{E} \times_P M$, the push forwards of P-bundles along $P \subset G$ and $P \to M$ respectively. By Iwasawa decomposition, the map $\mathrm{Gr}_P(C, \mathcal{O}_C) = P(B_{dR})/P(B^+_{dR}) \xrightarrow{\sim} \mathrm{Gr}_G(C, \mathcal{O}_C) = G(B_{dR})/G(B^+_{dR})$ is a bijection. For $x \in \mathrm{Gr}_G(C, \mathcal{O}_C)$, let $y \in \mathrm{Gr}_P(C, \mathcal{O}_C)$ be its inverse image under this bijection. Then

$$\mathcal{E}_{b_P,y} \times_P G = \mathcal{E}_{b,x},$$

i.e. $\mathcal{E}_{b_P,y}$ is a reduction to P of $\mathcal{E}_{b,x}$. By [3] Lemma 2.5, $\mathcal{E}_{b_P,y}$ is the reduction to P of $\mathcal{E}_{b,x}$ induced by the reduction $\mathcal{E}_{b_P}$ of $\mathcal{E}_b$. We will also write

$$\mathcal{E}_{b_P,y} = (\mathcal{E}_{b,x})_P$$

for this reduction. Recall that we have the decomposition

$$\mathrm{Gr}_{G,\mu}(C, \mathcal{O}_C) = \coprod_{\lambda \in S_M(\mu)} \mathrm{Gr}_{G,\lambda}(C, \mathcal{O}_C).$$

For $\lambda \in S_M(\mu)$, let $\mathrm{pr}_\lambda : \mathrm{Gr}_{G,\lambda}(C, \mathcal{O}_C) \to \mathrm{Gr}_{M,\lambda}(C, \mathcal{O}_C)$ be the projection. The following generalization of [3] Lemma 2.6 is clear now.

Lemma 6.14. *For any $x \in \mathrm{Gr}_{G,\mu}(C, \mathcal{O}_C)$, let $\lambda \in S_M(\mu)$ be such that $x \in \mathrm{Gr}_{G,\lambda}(C, \mathcal{O}_C)$. Then there is an isomorphism of M-bundles*

$$(\mathcal{E}_{b,x})_P \times_P M \simeq \mathcal{E}_{b_M, \mathrm{pr}_\lambda(x)},$$

where $(\mathcal{E}_{b,x})_P$ is the reduction of $\mathcal{E}_{b,x}$ induced by the reduction $\mathcal{E}_{b_P}$ of $\mathcal{E}_b$ as above.

For any $b \in G(\breve{\mathbb{Q}}_p)$ with associated class $[b] \in B(G)$, in [37] Rapoport-Zink introduced the weakly admissible locus $\mathscr{F}\ell(G, \mu, b)^{wa} \subset \mathscr{F}\ell(G, \mu)$, which is an open subspace of the adic space $\mathscr{F}\ell(G, \mu)$ over $\breve{E}$. Up to isomorphism, it depends only on $[b]$. Recall that we have the finite set $A(G, \mu) \subset B(G)$ (see the beginning of section 3). By [36], we have

$$\mathscr{F}\ell(G, \mu, b)^{wa} \neq \emptyset \Leftrightarrow [b] \in A(G, \mu).$$

Consider the diamond $\mathrm{Gr}_{G,\mu}$ over $\breve{E}$. Assume that $[b] \in A(G, \mu)$ and we define (see subsection 6.4 and in particular Remark 6.9, where we don't require $[b]$ to be basic)

$$\mathrm{Gr}^{wa}_\mu := \mathrm{Gr}^{wa}_{G,\mu,b} \subset \mathrm{Gr}_\mu = \mathrm{Gr}_{G,\mu}$$

as the inverse image of $\mathscr{F}\ell(G, \mu, b)^{wa,\diamond}$ under the map $\pi_\mu : \mathrm{Gr}_\mu \to \mathscr{F}\ell(G, \mu)^\diamond$ over $\mathrm{Spd}\, \breve{E}$. The above Lemma 6.14 implies the following generalization of [3] Proposition 2.7 in the non minuscule case.

Proposition 6.15. *Assume that G is quasi-split and $[b] \in A(G,\mu)$. Then $x \in \mathrm{Gr}_{G,\mu}(C,\mathcal{O}_C)$ is weakly admissible if and only if for any standard parabolic P with associated standard Levi M, any reduction b_M of b to M, and any $\chi \in X^*(P/Z_G)^+$, we have*

$$\deg \chi_*(\mathcal{E}_{b,x})_P \leq 0,$$

where $(\mathcal{E}_{b,x})_P$ is the reduction to P of $\mathcal{E}_{b,x}$ induced by the reduction $\mathcal{E}_{b_P}$ of $\mathcal{E}_b$ as above.

Proof. By definition, $x \in \mathrm{Gr}_\mu^{wa}(C,\mathcal{O}_C) \Leftrightarrow \pi_\mu(x) \in \mathscr{F}\ell(G,\mu,b)^{wa}(C,\mathcal{O}_C)$, where $\pi_\mu : \mathrm{Gr}_\mu(C,\mathcal{O}_C) \to \mathscr{F}\ell(G,\mu)(C,\mathcal{O}_C)$ is the Bialynicki-Birula map. Recall that we have similarly $\pi : \mathrm{Gr}_G(C,\mathcal{O}_C) \to \mathrm{Fil}_C(\omega^G)$. The map π is functorial. In particular, for P and M as above, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Gr}_G(C,\mathcal{O}_C) & \longrightarrow & \mathrm{Gr}_P(C,\mathcal{O}_C) & \longrightarrow & \mathrm{Gr}_M(C,\mathcal{O}_C) \\ \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_P} & & \downarrow{\scriptstyle \pi_M} \\ \mathrm{Fil}_C(\omega^G) & \longrightarrow & \mathrm{Fil}_C(\omega^P) & \longrightarrow & \mathrm{Fil}_C(\omega^M), \end{array}$$

where $Gr_G(C,\mathcal{O}_C) \to \mathrm{Gr}_P(C,\mathcal{O}_C)$ is the inverse of the natural bijection $\mathrm{Gr}_P(C,\mathcal{O}_C) \to \mathrm{Gr}_G(C,\mathcal{O}_C)$ induced by $P \subset G$ and the Iwasawa decomposition as above, $\mathrm{Fil}_C(\omega^G) \to \mathrm{Fil}_C(\omega^P)$ is the map defined by [6] Proposition 4.2.17, and the other arrows are naturally defined by $P \to M$.

Let $\lambda \in S_M(\mu)$ be such that $x \in \mathrm{Gr}_{G,\lambda}(C,\mathcal{O}_C)$. By Lemma 6.14, $(\mathcal{E}_{b,x})_P \times_P M \simeq \mathcal{E}_{b_M,\mathrm{pr}_\lambda(x)}$. For any $\chi \in X^*(P/Z_G)^+$, since it factorizes through M, we have $\chi_*(\mathcal{E}_{b,x})_P = \chi_*\mathcal{E}_{b_M,\mathrm{pr}_\lambda(x)}$. By [2] Lemma 3.5.5,

$$c_1^M(\mathcal{E}_{b_M,\mathrm{pr}_\lambda(x)}) = \lambda^\sharp - \kappa_M(b_M) \in \pi_1(M)_\Gamma.$$

Therefore,

$$\begin{aligned} \deg \chi_*(\mathcal{E}_{b,x})_P &= \deg \chi_*\mathcal{E}_{b_M,\mathrm{pr}_\lambda(x)} \\ &= \deg\Big(\chi(b_M), \chi \circ \pi_M\big(\mathrm{pr}_\lambda(x)\big)\Big) \\ &= \deg\Big(\chi(b_M), \chi \circ \lambda\Big), \end{aligned}$$

where the last two terms are the degrees of rank one filtered isocrystals. Thus we are reduced to [6] Corollary 9.2.30. □

Before proceeding further, let us fix some conventions. Recall that we have the following commutative diagram for the Kottwitz and Newton maps (see [35] p. 162):

$$\begin{array}{ccc} B(G) & \xrightarrow{\ \nu\ } & \mathcal{N}(G) \\ \downarrow{\scriptstyle \kappa} & & \downarrow \\ \pi_1(G)_\Gamma & \longrightarrow & \pi_1(G)_{\Gamma,\mathbb{Q}}, \end{array}$$

where we identify

$$\pi_1(G)_{\Gamma,\mathbb{Q}} = \pi_1(G)^\Gamma_\mathbb{Q} = X_*(Z_G)^\Gamma_\mathbb{Q} = X_*(A_G)_\mathbb{Q},$$

where A_G is the maximal split torus inside the center Z_G of G. For an element $v \in \mathcal{N}(G)$, in the following we will denote its image in $\pi_1(G)_{\Gamma,\mathbb{Q}}$, by the same notation v for simplicity. For a Levi subgroup $M \subset G$, we have the corresponding commutative diagram as above for G, which maps to that for G, since all the maps in the diagram is functorial.

Let $b \in G(\breve{\mathbb{Q}}_p)$ be such that $[b] \in B(G,\mu) \subset A(G,\mu)$ is basic. Recall that we have the weakly admissible locus $\mathrm{Gr}_\mu^{wa} \subset \mathrm{Gr}_\mu$, which is defined as the inverse image of $\mathscr{F}\ell(G,\mu,b)^{wa,\diamond}$ under the map $\pi_\mu : \mathrm{Gr}_\mu \to \mathscr{F}\ell(G,\mu)^\diamond$. Recall that we have also the admissible locus

$$\mathrm{Gr}_\mu^a := \mathrm{Gr}_{G,\mu,b}^a \subset \mathrm{Gr}_\mu = \mathrm{Gr}_{G,\mu},$$

which is defined as the open Newton stratum $\mathrm{Gr}_\mu^{Newt=[1]}$ inside the Newton stratification (see subsection 6.4)

$$\mathrm{Gr}_\mu = \coprod_{[b']\in B(G,0,\nu_b\mu^{-1})} \mathrm{Gr}_\mu^{Newt=[b']}.$$

The theorem of Colmez-Fontaine (see [14] chapter 10) implies that under $\pi_\mu : \mathrm{Gr}_\mu \to \mathscr{F}\ell(G,\mu)^\diamond$, we have

$$\pi_\mu(\mathrm{Gr}_\mu^a) \subset \mathscr{F}\ell(G,\mu,b)^{wa,\diamond},$$

see also the proof of Proposition 6.4. Thus we have the inclusion of locally spatial diamonds over $\mathrm{Spd}\,\breve{E}$:

$$\mathrm{Gr}_\mu^a \subset \mathrm{Gr}_\mu^{wa}.$$

Theorem 6.16. *Assume that $[b] \in B(G,\mu)$ is basic. Then the following statements are equivalent:*
$B(G,\mu)$ is fully Hodge-Newton decomposable $\Longleftrightarrow \mathrm{Gr}_\mu^a = \mathrm{Gr}_\mu^{wa}$.

Proof. With all the ingredients at hand, the arguments in the proof of [3] Theorem 6.1 apply here. We first assume that G is quasi-split.

The direction "⇒": The arguments is identical to the direction (1) ⇒ (2) in [3] Theorem 6.1, using

- the Newton stratification $\mathrm{Gr}_\mu = \coprod_{[b']\in B(G,0,\nu_b\mu^{-1})} \mathrm{Gr}_\mu^{Newt=[b']}$,
- [3] Corollary 4.15 and Lemma 4.11,
- the above Lemma 6.14,
- [3] Lemmas 6.2 and 6.3,
- the above Proposition 6.15.

We leave the details to the readers.

The direction "⇐": We follow the arguments in the direction (2) ⇒ (1) of [3] Theorem 6.1, except in the last step Proposition 6.12 will be used. For the reader's convenience, and to clarify the ideas, we sketch the arguments as follows. We use the notations of [3].

We prove that if $B(G,\mu)$ is not fully Hodge-Newton decomposable, then $\mathrm{Gr}_\mu^{wa} \supsetneq \mathrm{Gr}_\mu^a$, i.e. there exists a point $x \in \mathrm{Gr}_\mu^{wa}(C,\mathcal{O}_C) \setminus \mathrm{Gr}_\mu^a(C,\mathcal{O}_C)$, for any algebraically closed perfectoid field $C|\breve{E}$.

By [3] Corollary 4.15, $B(G,0,\nu_b\mu^{-1})$ is not fully Hodge-Newton decomposable, and thus by [3] Proposition 4.13 (and its proof), there exists $\alpha \in \Delta_0$ such that

$$\langle -w_0\mu, \tilde{\omega}_\alpha\rangle > 1,$$

where $\tilde{\omega}_\alpha = \sum_{\gamma\in\Phi,\gamma|_A=\alpha} \omega_\gamma$. Let $\beta \in \Delta$ such that $\beta|_A = \alpha$ with corresponding coroot $\beta^\vee \in \Delta^\vee$. Then $\langle \beta^\vee, \tilde{\omega}_\alpha\rangle = \langle (\beta^\vee)^\diamond, \tilde{\omega}_\alpha\rangle = 1$ and thus

$$\langle -w_0\mu - \beta^\vee, \tilde{\omega}_\alpha\rangle > 0.$$

Let M be the standard Levi subgroup such that $\Delta_{0,M} = \Delta_0\backslash\{\alpha\}$. Write P the associated standard parabolic subgroup. Then the element $(\beta^\vee)^\sharp \in \pi_1(G)_\Gamma$ admits to a lift to $\pi_1(M)_\Gamma$, which we still denote by

$$(\beta^\vee)^\sharp \in \pi_1(M)_\Gamma = \Big(X_*(T)/\langle\Phi_M^\vee\rangle\Big)_\Gamma.$$

Let $[b'_M] \in B(M)_{basic}$ be the basic element in $B(M)$ such that it is mapped to $(\beta^\vee)^\sharp$ under the bijection $\kappa_M : B(M)_{basic} \xrightarrow{\sim} \pi_1(M)_\Gamma$. Let $[b'] \in B(G)$ be the image of $[b'_M]$ under the natural map $B(M) \to B(G)$. Then by construction

$$M_{b'} = M, \quad [b'] \in B(G,0,\nu_b\mu^{-1})$$

and $([b'], \nu_b\mu^{-1})$ is not Hodge-Newton decomposable.

Consider the Newton stratum $Z := \mathrm{Gr}_{\mu}^{Newt=[b']}$ attached to $[b']$. Then naturally $Z(C, \mathcal{O}_C) \bigcap \mathrm{Gr}_{\mu}^{wa}(C, \mathcal{O}_C) \subset \mathrm{Gr}_{\mu}^{wa}(C, \mathcal{O}_C) \setminus \mathrm{Gr}_{\mu}^{a}(C, \mathcal{O}_C)$. We claim that

$$Z(C, \mathcal{O}_C) \bigcap \mathrm{Gr}_{\mu}^{wa}(C, \mathcal{O}_C) \neq \emptyset.$$

This will conclude the proof of the direction "$\Leftarrow$".

Suppose the claim was not true, i.e. for any $x \in Z(C, \mathcal{O}_C)$, x is non weakly admissible. By the definition of Z, we have

$$\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}.$$

By Proposition 6.15, there exists a standard maximal parabolic Q with the corresponding Levi M_Q, a reduction $b_{M_{\mathbb{Q}}}$ of b to M_Q, a character $\chi \in X_*(Q/Z_G)^+$ such that $\deg\chi_*(\mathcal{E}_{b,x})_Q > 0$. Consider the map

$$v : X^*(Q/Z_G) \to \mathbb{Z}, \quad \chi' \mapsto \deg\chi'_*(\mathcal{E}_{b,x})_Q.$$

It defines an element $v \in \mathcal{N}(G)$ and we have $v \leq \nu(\mathcal{E}_{b,x})$ by [3] Theorem 1.8. From this inequality, we get $Q = P$ and $(\mathcal{E}_{b,x})_Q$ is the canonical reduction of $\mathcal{E}_{b,x}$ to the maximal parabolic $Q = P$.

Consider the decomposition $\mathrm{Gr}_{G,\mu}(C, \mathcal{O}_C) = \coprod_{\lambda \in S_M(\mu)} \mathrm{Gr}_{G,\lambda}(C, \mathcal{O}_C)$. Let $\lambda \in S_M(\mu)$ such that $x \in \mathrm{Gr}_{G,\lambda}(C, \mathcal{O}_C)$. By Lemma 6.14 we have

$$\mathcal{E}_{b'_M} = (\mathcal{E}_{b'})_P \times_P M = (\mathcal{E}_{b,x})_P \times_P M = \mathcal{E}_{b_M, \mathrm{pr}_\lambda(x)},$$

where $b_M = b_{M_Q}$ is the above reduction of b to $M_Q = M$. Therefore, by taking $-c_1^M(\cdot)$, we get

$$\kappa_M(b'_M) = \kappa_M(b_M) - \lambda^\sharp \in \pi_1(M)_\Gamma,$$

which implies

$$\nu_{b'_M} = \nu_{b_M} - \lambda^\sharp \otimes 1 \in \pi_1(M)_{\Gamma,\mathbb{Q}}$$

by our previous convention. As $\kappa_M(b'_M) = (\beta^\vee)^\sharp$ by construction, we get

$$\lambda^\sharp \otimes 1 = \nu_{b_M} - (\beta^\vee)^\sharp \otimes 1 \in \pi_1(M)_{\Gamma,\mathbb{Q}}. \tag{1}$$

Next we pass to the dual side. Consider the inner form J_b of G. Let $[b''] \in B(J_b)$ be the element which is mapped to $[b'] \in B(G)$ under the bijection $B(J_b) \xrightarrow{\sim} B(G)$. Since G is quasi-split and b' admits reductions to P and M (by construction), the groups P and M transfer to parabolic and Levi subgroups respectively of J_b, which we still denote by P and M by abuse of notation. Moreover, there exist corresponding reductions b''_M and b''_P of b'' to M and P respectively. The isomorphism $J_{b,\breve{\mathbb{Q}}_p} \simeq G_{\breve{\mathbb{Q}}_p}$ induces an identification $\mathrm{Gr}_{J_b,\mu} = \mathrm{Gr}_{G,\mu}$, and by Theorem 6.10 we have

$$\mathrm{Gr}_{J_b,\mu}^{Newt=[b'']} = \mathrm{Gr}_{G,\mu}^{Newt=[b']} = Z.$$

By Lemma 4.1, the bijection $B(J_b) \xrightarrow{\sim} B(G)$ restricts to a bijection $B(J_b, \mu^{-1}) \xrightarrow{\sim} B(G, 0, \nu_b\mu^{-1})$. As $[b'] \in B(G, 0, \nu_b\mu^{-1})$, we get $[b''] \in B(J_b, \mu^{-1})$. Consider the dual local Shtuka datum $(J_b, \{\mu^{-1}\}, [b''])$. We have the following diagram

$$\begin{array}{ccc} & \mathrm{Sht}(J_b, \mu^{-1}, b'')_\infty & \\ \swarrow^{\pi_{dR}} & & \searrow^{\pi_{HT}} \\ \mathrm{Gr}^{a}_{J_b,\mu^{-1}} & & \mathrm{Gr}_{J_b,\mu}^{Newt=[b'']}. \end{array}$$

Recall that we have our point $x \in Z(C, \mathcal{O}_C) = \mathrm{Gr}_{J_b,\mu}^{Newt=[b'']}(C, \mathcal{O}_C)$. Consider the subset

$$\pi_{dR}(\pi_{HT}^{-1}(x)) \subset \mathrm{Gr}_{J_b,\mu^{-1}}(C, \mathcal{O}_C).$$

For the parabolic P and Levi M of J_b, we consider the digram of the corresponding B_{dR}^+-affine Grassmannians. Under the identifications $\mathrm{Gr}_{G,\mu} = \mathrm{Gr}_{J_b,\mu}$ and $S_M^G(\mu) = S_M^{J_b}(\mu)$, the decompositions $\mathrm{Gr}_{G,\mu} = \coprod_{\lambda \in S_M^G(\mu)} \mathrm{Gr}_{G,\lambda}$ and $\mathrm{Gr}_{J_b,\mu} = \coprod_{\lambda \in S_M^{J_b}(\mu)} \mathrm{Gr}_{J_b,\lambda}$ coincide. We consider the side $\mathrm{Gr}_{J_b,\mu^{-1}}$. Let $z \in \pi_{dR}(\pi_{HT}^{-1}(x)) \subset \mathrm{Gr}_{J_b,\mu^{-1}}(C, \mathcal{O}_C)$ be a point. Consider the decomposition of $\mathrm{Gr}_{J_b,\mu^{-1}}(C, \mathcal{O}_C)$ indexed by $S_M(\mu^{-1}) := S_M^{J_b}(\mu^{-1})$. Let $\lambda' \in S_M(\mu^{-1})$ be such that

$$z \in \mathrm{Gr}_{J_b,\lambda'}(C, \mathcal{O}_C).$$

By Lemma 6.14 again, we have

$$(\mathcal{E}_{b'',z}) \times_P M \simeq \mathcal{E}_{b''_M, \mathrm{pr}_{\lambda'}(z)}.$$

Let $\lambda_0 := -w_0\mu \in S_M(\mu^{-1})$ be the maximal element. If $\lambda' = \lambda_0$, that is

$$\lambda' = -w_0\mu \in X_*(T)_M^+ \subset X_*(T),$$

then we have

$$\lambda' \otimes 1 = (-w_0\mu) \otimes 1 \in \pi_1(M)_{\mathbb{Q}} = \Big(X_*(T)/\langle \Phi_M^\vee \rangle\Big)_{\mathbb{Q}}.$$

Now we come back to the group G and consider M as a Levi subgroup of G. We have our previous notation $\pi_1(M)_{\Gamma,\mathbb{Q}}$, taking into account the Galois action on $G_{\breve{\mathbb{Q}}_p}$ defined by G over $\mathbb{Q}_p$. Then

$$(\lambda')^\sharp \otimes 1 = (-w_0\mu)^\sharp \otimes 1 \in \pi_1(M)_{\Gamma,\mathbb{Q}}. \tag{2}$$

Recall that we have the corresponding element $\lambda \in S_M(\mu)$ such that $x \in \mathrm{Gr}_{J_b,\lambda}(C, \mathcal{O}_C)$. Then

$$\begin{aligned}(\lambda')^\sharp \otimes 1 &= -\lambda^\sharp \otimes 1 \\ &= -\nu_{b_M} + (\beta^\vee)^\sharp \otimes 1 \in \pi_1(M)_{\Gamma,\mathbb{Q}},\end{aligned}$$

where the second "=" comes from equation 1. Combined with equation 2, we get

$$-\nu_{b_M} = (-w_0\mu)^\sharp \otimes 1 - (\beta^\vee)^\sharp \otimes 1 \in \pi_1(M)_{\Gamma,\mathbb{Q}}. \tag{3}$$

Pushing forward equation 3 to $\pi_1(G)_{\Gamma,\mathbb{Q}}$ and taking $\langle \cdot, \tilde{\omega}_\alpha \rangle$, we get

$$\langle -\nu_b, \tilde{\omega}_\alpha \rangle = \langle -w_0\mu - \beta^\vee, \tilde{\omega}_\alpha \rangle > 0.$$

This is a contradiction, since b is basic in G and thus $\langle -\nu_b, \tilde{\omega}_\alpha \rangle = 0$. Therefore, for any $z \in \pi_{dR}(\pi_{HT}^{-1}(x))$ such that $z \in \mathrm{Gr}_{J_b,\lambda'}(C, \mathcal{O}_C)$, we have

$$\lambda' \neq \lambda_0.$$

Now let $x \in Z(C, \mathcal{O}_C)$ vary. Since $\mathrm{Gr}^a_{J_b,\mu^{-1}}(C, \mathcal{O}_C) = \coprod_{x \in Z(C,\mathcal{O}_C)} \pi_{dR}(\pi_{HT}^{-1}(x))$, by the above discussion, we get

$$\mathrm{Gr}^a_{J_b,\mu^{-1}}(C, \mathcal{O}_C) \bigcap \mathrm{Gr}_{J_b,\lambda_0}(C, \mathcal{O}_C) = \emptyset.$$

As $\mathrm{Gr}^a_{J_b,\mu^{-1}} \subset \mathrm{Gr}_{J_b,\mu^{-1}}$ is open and $\mathrm{Gr}_{J_b,\lambda_0} \subset \mathrm{Gr}_{J_b,\mu^{-1}}$ is dense by Proposition 6.12, we must have

$$\mathrm{Gr}^a_{J_b,\mu^{-1}}(C, \mathcal{O}_C) \bigcap \mathrm{Gr}_{J_b,\lambda_0}(C, \mathcal{O}_C) \neq \emptyset.$$

This contradiction implies that the claim is true: $Z(C, \mathcal{O}_C) \bigcap \mathrm{Gr}^{wa}_\mu(C, \mathcal{O}_C) \neq \emptyset$. Thus we have proved the direction "$\Leftarrow$".

The general case: now consider the case G non necessarily quasi-split. Let G_{ad} be the adjoint group attached to G. Then we get a natural surjective morphism $\phi : \mathrm{Gr}_{G,\mu} \to \mathrm{Gr}_{G_{ad},\mu_{ad}}$. Let $[b_{ad}] \in B(G_{ad}, \mu_{ad})$ be the corresponding element under the bijection $B(G, \mu) \xrightarrow{\sim} B(G_{ad}, \mu_{ad})$. We consider the admissible locus and weakly admissible locus of $\mathrm{Gr}_{G_{ad},\mu_{ad}}$ with respective to b_{ad}. One checks easily that

$$\mathrm{Gr}^a_{G,\mu} = \phi^{-1}(\mathrm{Gr}^a_{G_{ad},\mu_{ad}}) \quad \text{and} \quad \mathrm{Gr}^{wa}_{G,\mu} = \phi^{-1}(\mathrm{Gr}^{wa}_{G_{ad},\mu_{ad}}).$$

Thus we are reduced to the case G is adjoint. Let H be a quasi-split inner form of G. Then H is adjoint and $G = J_{b^*}$ for some $[b^*] \in B(H)_{basic} = H^1(\mathbb{Q}_p, H)$. Let $[b^H] \in B(H)$ be the image of $[b]$ under the bijection $B(G) \xrightarrow{\sim} B(H)$. We can consider the admissible locus and weakly admissible locus inside $\mathrm{Gr}_{H,\mu}$ with respective to b^H. Under the identification $\mathrm{Gr}_{G,\mu} = \mathrm{Gr}_{H,\mu}$, we have

$$\mathrm{Gr}_{G,\mu}^{a} = \mathrm{Gr}_{H,\mu}^{a} \quad \text{and} \quad \mathrm{Gr}_{G,\mu}^{wa} = \mathrm{Gr}_{H,\mu}^{wa}.$$

Thus we are reduced to the quasi-split case as the last paragraph of the proof of [3] Theorem 6.1.

□

Come back to the Hodge-Tate side $\mathrm{Gr}_{\mu^{-1}}$. For any algebraically perfectoid field $C|\breve{E}$ and any $x \in \mathrm{Gr}_{\mu^{-1}}(C, \mathcal{O}_C)$, the inequality $\nu(\mathcal{E}_1, \mathcal{E}_{1,x}, f) \leq \nu(\mathcal{E}_{1,x})$ (see subsection 6.3 and Proposition 3.4 (1)) implies that we have always the inclusion for open Newton and Harder-Narasimhan strata:

$$\mathrm{Gr}_{\mu^{-1}}^{Newt=[b]} \subset \mathrm{Gr}_{\mu^{-1}}^{HN=[b]}.$$

Our previous efforts (cf. Theorems 5.1 and 6.10) imply the following enlarged version of Theorem 6.16:

Corollary 6.17. *Let $[b] \in B(G, \mu)$ be basic. The following statements are equivalent:*

(1) *$B(G, \mu)$ is fully Hodge-Newton decomposable,*
(2) *$\mathrm{Gr}_{\mu}^{a} = \mathrm{Gr}_{\mu}^{wa}$,*
(3) *$\mathrm{Gr}_{\mu^{-1}}^{Newt=[b]} = \mathrm{Gr}_{\mu^{-1}}^{HN=[b]}$.*

Of course, one can make the above corollary into a similar version as Theorem 5.1, by including the corresponding information for the dual local Shtuka datum $(J_b, \{\mu^{-1}\}, [b^{-1}])$. One can also generalize the results further to $\mathrm{Gr}_{\leq\mu}$ and $\mathrm{Gr}_{\leq\mu^{-1}}$. We leave these tasks to the reader.

7. Application to moduli of local G-Shtukas

Let $(G, \{\mu\}, [b])$ be a local Shtuka datum. Fix a representative $b \in G(\breve{\mathbb{Q}}_p)$ of $[b]$, and let $\mathrm{Sht}(G, \mu, b)_\infty$ be the associated moduli space of local G-Shtukas of type $\{\mu\}$ with infinite level.

Consider the Hodge-Tate period map of diamonds over $\breve{E}$

$$\pi_{HT} : \mathrm{Sht}(G, \mu, b)_\infty \longrightarrow \mathrm{Gr}_{\mu^{-1}}^{[b]},$$

where we write $\mathrm{Gr}_{\mu^{-1}}^{[b]} = \mathrm{Gr}_{\mu^{-1}}^{Newt=[b]}$ for the associated Newton stratum inside $\mathrm{Gr}_{\mu^{-1}}$ for simplicity. By subsection 6.3, the Harder-Narasimhan stratification on $\mathrm{Gr}_{\mu^{-1}}$ induces a Harder-Narasimhan stratification on $\mathrm{Gr}_{\mu^{-1}}^{[b]}$:

$$\mathrm{Gr}_{\mu^{-1}}^{[b]} = \coprod_{[b'] \in B(G,\mu), [b'] \leq [b]} \mathrm{Gr}_{\mu^{-1}}^{[b], HN=[b']}$$

where each $\mathrm{Gr}_{\mu^{-1}}^{[b],HN=[b']} \subset \mathrm{Gr}_{\mu^{-1}}^{[b]}$ is the pullback of $\mathrm{Gr}_{\mu^{-1}}^{HN=[b']} \subset \mathrm{Gr}_{\mu^{-1}}$ under the inclusion $\mathrm{Gr}_{\mu^{-1}}^{[b]} \subset \mathrm{Gr}_{\mu^{-1}}$, which is empty if $[b'] \geq [b]$ and $[b'] \neq [b]$ (see subsection 6.3). The above stratification in turn induces a Harder-Narasimhan stratification on $\mathrm{Sht}(G, \mu, b)_\infty$ by diamonds

$$\mathrm{Sht}(G, \mu, b)_\infty = \coprod_{[b'] \in B(G,\mu), [b'] \leq [b]} \mathrm{Sht}(G, \mu, b)_\infty^{HN=[b']},$$

where

$$\mathrm{Sht}(G, \mu, b)_\infty^{HN=[b']} = \pi_{HT}^{-1}(\mathrm{Gr}_{\mu^{-1}}^{[b], HN=[b']}).$$

By Corollary 6.7, we have

Corollary 7.1. *For any non basic $[b'] \in B(G,\mu)$ such that $[b'] \leq [b]$, the stratum $\mathrm{Sht}(G,\mu,b)_\infty^{HN=[b']}$ is a parabolic induction.*

Of course, when $[b]=[b_0]$ is basic, the above Harder-Narasimhan stratification on $\mathrm{Sht}(G,\mu,b)_\infty$ is trivial and thus Corollary 7.1 says nothing in this case.

We may also consider the Hodge-Tate period map of diamonds over $\breve{E}$

$$\pi_{HT}: \mathrm{Sht}(G,\leq\mu,b)_\infty \longrightarrow \mathrm{Gr}^{[b]}_{\leq\mu^{-1}}.$$

Then we have similar conclusion for $\mathrm{Sht}(G,\leq\mu,b)_\infty$ as above.

8. Application to Shimura varieties

Let $(\mathbf{G},\mathbf{X})$ be *an arbitrary Shimura datum.* Let p be a prime number. Consider the conjugacy class of Hodge cocharacters $\{\mu\}$ attached to $\mathbf{X}$, which we view a conjugacy class of cocharacters over $\overline{\mathbb{Q}}_p$. Set $G=\mathbf{G}_{\mathbb{Q}_p}$.

Let $v|p$ be a place of the reflex field $\mathbf{E}=\mathbf{E}(\mathbf{G},\mathbf{X})$ above p and $E=\mathbf{E}_v$. Let $\mathbf{K}\subset\mathbf{G}(A_f)$ be a sufficiently small open compact subgroup. Attached to $(\mathbf{G},\mathbf{X},\mathbf{K})$, we have the Shimura variety $\mathrm{Sh}_{\mathbf{K}}$ over the local reflex field E, which we view as an adic space. Assume that $\mathbf{K}$ is of the form $\mathbf{K}=KK^p$ with $K\subset G(\mathbb{Q}_p)$ and $K^p\subset\mathbf{G}(\mathbb{A}_f^p)$. Consider the p-adic flag variety $\mathscr{F}\ell(G,\mu^{-1})$ over E, on which we have an action of $G(\mathbb{Q}_p)$. Let G^c denote the quotient of G by the maximal $\mathbb{Q}$- anisotropic $\mathbb{R}$-split subtorus in the center Z_G of G. Then we have an induced action $G^c(\mathbb{Q}_p)$ on $\mathscr{F}\ell(G,\mu^{-1})$. Let $K^c\subset G^c(\mathbb{Q}_p)$ be the induced open compact subgroup. The quotient space

$$[\underline{K^c}\setminus\mathscr{F}\ell(G,\mu^{-1})^\diamond]$$

exists as *a small v-stack* in the sense of [38]. The main results of [26] and [7] imply that we have the Hodge-Tate period map

$$\pi_{HT}: \mathrm{Sh}_{\mathbf{K}}^\diamond \longrightarrow [\underline{K^c}\setminus\mathscr{F}\ell(G,\mu^{-1})^\diamond],$$

which is a morphism of small v-stacks over E. More precisely, by [26] Theorem 1.2 the universal p-adic local system over $\mathrm{Sh}_{\mathbf{K}}$ is de Rham, thus we get a relative Hodge-Tate filtration on it; by [7] Theorem 1.3, the type of this Hodge-Tate filtration is exactly given by $\{\mu^{-1}\}$.

Noting that the $\underline{K^c}$-action on $\mathscr{F}\ell(G,\mu^{-1})^\diamond$ preserves the Harder-Narasimhan stratification

$$\mathscr{F}\ell(G,\mu^{-1})^\diamond=\coprod_{[b]\in B(G,\mu)}\mathscr{F}\ell(G,\mu^{-1})^{HN=[b],\diamond},$$

we get a stratification on $\mathrm{Sh}_{\mathbf{K}}^\diamond$ via π_{HT}:

$$\mathrm{Sh}_{\mathbf{K}}^\diamond=\coprod_{[b]\in B(G,\mu)}\mathrm{Sh}_{\mathbf{K}}^{HN=[b]},$$

where

$$\mathrm{Sh}_{\mathbf{K}}^{HN=[b]}=\pi_{HT}^{-1}\Big(\big[\underline{K^c}\setminus\mathscr{F}\ell(G,\mu^{-1})^{HN=[b],\diamond}\big]\Big).$$

By Theorem 3.9, we have

Corollary 8.1. *For any non basic $[b]\neq[b_0]$, the stratum $\mathrm{Sh}_{\mathbf{K}}^{HN=[b]}$ is a parabolic induction.*

Similarly, the $\underline{K^c}$-invariant Newton stratification

$$\mathscr{F}\ell(G,\mu^{-1})^\diamond=\coprod_{[b]\in B(G,\mu)}\mathscr{F}\ell(G,\mu^{-1})^{Newt=[b],\diamond}$$

also induces a stratification[19] on $\mathrm{Sh}_{\mathbf{K}}^{\diamond}$:

$$\mathrm{Sh}_{\mathbf{K}}^{\diamond} = \coprod_{[b]\in B(G,\mu)} \mathrm{Sh}_{\mathbf{K}}^{Newt=[b]},$$

where

$$\mathrm{Sh}_{\mathbf{K}}^{Newt=[b]} = \pi_{HT}^{-1}\Big(\big[\underline{K^c} \setminus \mathscr{F}\ell(G,\mu^{-1})^{Newt=[b],\diamond}\big]\Big).$$

Then we have an inclusion of open strata

$$\mathrm{Sh}_{\mathbf{K}}^{Newt=[b_0]} \subset \mathrm{Sh}_{\mathbf{K}}^{HN=[b_0]}.$$

Theorem 5.1 implies

Corollary 8.2. *If the associated pair $(G,\{\mu\})$ is fully Hodge-Newton decomposable, then* $\mathrm{Sh}_{\mathbf{K}}^{Newt=[b_0]} = \mathrm{Sh}_{\mathbf{K}}^{HN=[b_0]}$.

We refer the reader to [13] 9.7.2 for some speculations on possible arithmetic applications related to the results above.

Remark 8.3. *The readers who prefer diamonds can replace the above by the following considerations. Let*

$$\mathrm{Sh}_{K^p} = \varprojlim_{K_p} \mathrm{Sh}_{\mathbf{K}}^{\diamond}$$

be the diamond of Shimura variety with infinite level at p and prime-to-p level K^p, on which we have a natural action of $G(\mathbb{Q}_p)$. Then we get the Hodge-Tate period map of diamonds[20] *over E*

$$\pi_{HT} : \mathrm{Sh}_{K^p} \longrightarrow \mathscr{F}\ell(G,\mu^{-1})^{\diamond},$$

which is $G(\mathbb{Q}_p)$-equivariant. We can define Harder-Narasimhan strata and Newton strata on the diamond Sh_{K^p}, which are inverse limits of the corresponding strata at finite levels. Corollaries 8.1 and 8.2 admit the corresponding diamond versions.

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[19] Let us call it the Newton stratification of $\mathrm{Sh}_{\mathbf{K}}^{\diamond}$. If there exists some nice integral model of $\mathrm{Sh}_{\mathbf{K}}$, then we have the Newton stratification on the special fiber and we can pullback it to the tube (the good reduction locus) over the special fiber. There exist some subtleties between the two Newton stratifications, cf. [2].

[20] If $(\mathbf{G},\mathbf{X})$ is of abelian type, then Sh_{K^p} is representable by a perfectoid space and π_{HT} comes from a morphism of adic spaces over E, cf. [41]. In the general case, see also [18] for the morphisms of diamonds π_{HT}.

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