STRATIFICATIONS IN GOOD REDUCTIONS OF SHIMURA VARIETIES OF ABELIAN TYPE

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Abstract. In this paper we study the geometry of good reductions of Shimura varieties of abelian type. More precisely, we construct the Newton stratification, Ekedahl-Oort stratification, and central leaves on the special fiber of a Shimura variety of abelian type at a good prime. We establish several basic properties of these stratifications, including the non-emptiness, closure relation and dimension formula, generalizing those previously known in the PEL and Hodge type cases. We also study the relations between these stratifications, both in general and in some special cases, such as those of fully Hodge-Newton decomposable type. We investigate the examples of quaternionic and orthogonal Shimura varieties in details.

Contents

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Introduction

Understanding the geometric properties of Shimura varieties in mixed characteristic has been a central problem in arithmetic algebraic geometry and Langlands program. In this paper we study the geometry of good reductions of Shimura varieties of abelian type, based on the works of Kisin [18] and Vasiu [45] where smooth integral canonical models for these Shimura varieties were already available, and following the genereal guideline proposed by He-Rapoport in [15] (see also [38]) where basic axioms were postulated to study various stratifications on the special fibers of certain integral models of Shimura varieties.

A Shimura datum (G, X) is said to have good reduction at a prime p, if $G_{\mathbb{Q}_p}$ extends to a reductive group $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$. Let E be the reflex field of (G, X). We will fix a place v of E over p, and write $O_{E,(v)}$ for the ring of integers. For $K = K_pK^p$ with $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$, Langlands and Milne conjectured (cf. [32] section 2) that the pro-variety

$$\mathrm{Sh}_{K_p}(G, X) := \varprojlim_{K^p} \mathrm{Sh}_{K_pK^p}(G, X),$$

where K^p runs through compact open subgroups of $G(\mathbb{A}_f^p)$, has an integral canonical model $\mathscr{S}_{K_p}(G, X)$ over $O_{E,(v)}$. The prime to p Hecke action of $G(\mathbb{A}_f^p)$ on $\mathrm{Sh}_{K_p}(G, X)$ should extend to $\mathscr{S}_{K_p}(G, X)$, and when K^p varies the inverse system of $\mathscr{S}_{K_pK^p}(G, X) := \mathscr{S}_{K_p}(G, X)/K^p$ should be a system of smooth models of $\mathrm{Sh}_{K_pK^p}(G, X)$ with étale transition morphisms. Thanks to the works of Kisin [18] and Vasiu [45], smooth integral canonical models are known to exist if the Shimura datum (G, X) is of abelian type and $p > 2$ (by recent work of Kim and Madapusi Pera [17], integral canonical models for Shimura varieties of abelian type at $p = 2$ are also known to exist; however we will restrict ourselves to the case $p > 2$ in this paper). Thus it is natural to investigate geometry of the (geometric) special fibers $\mathscr{S}_{K_pK^p,0}(G, X)$ over $\overline{\kappa}$[1] of these models, where κ is the residue field of $O_{E,(v)}$. In the following, (G, X) will always be a Shimura datum of abelian type with good reduction at $p > 2$.

It turns out the geometry of Shimura varieties in characteristic p is much finer than that in characteristic 0, in the sense that there are several invariants in characteristic p, which are stable under the prime to p Hecke action, leading to various natural stratifications of the special fiber $\mathscr{S}_{K_pK^p,0}(G, X)$. Following Oort (in the Siegel case, see [35] for example), Viehmann-Wedhorn (in the PEL type case, cf. [48]) and many others (see the references of [48, 47] for example), we mainly concentrate on the *Newton stratification*, the *Ekedahl-Oort stratification*, and the *central leaves* in this paper. In fact in this paper we will only be concerned with some basic properties of these stratifications, and the relations between these strata. Our study here can be put[2] in the general framework proposed by He-Rapoport in [15], where more group theoretic aspects are emphasized (compare also [38, 9, 10]).

We mention that if (G, X) is of PEL type, then we can use the explicit moduli interpretation to treat the geometry of the special fibers. In the general Hodge type case, at the current stage we do not know whether there exists moduli interpretation in mixed characteristic. But there still exists an abelian scheme together with certain tensors over the special fiber of a Hodge type Shimura variety, and we can use of it to study the geometry modulo p, cf. [13, 51, 53, 54] for example. If now (G, X) is a general abelian type Shimura datum, which is the case we want to treat in this paper, then there is no abelian scheme nor

[1] Here in the introduction we work uniformly over $\overline{\kappa}$ for simplicity. We remind the reader that in the body part of this paper, we denote by $\mathscr{S}_{K_pK^p,0}(G, X)$ the special fiber over κ and by $\mathscr{S}_{K_pK^p,\overline{\kappa}}(G, X)$ the geometric special fiber over $\overline{\kappa}$.

[2] In fact the main part of [15] is to work with all parahoric levels at p. Here we restrict to the hyperspecial levels, as a first step toward the verification of the axioms in [15] in the abelian type case.

p-divisible groups over the associated Shimura varieties at all. Nevertheless, we can study them by choosing some related Hodge type Shimura varieties. This usually requires the study of some finer geometric structures on these Hodge type Shimura varieties. Along the way, we will also see some close relations between the strata of different Shimura varieties. To a certain extent, many of our following main results were previously known in the PEL type and Hodge type cases. Our modest goal here is to extend them to the abelian type case and hence in the full generality as in the work of Kisin [18], and to provide a useful documentary literature with a point of view toward possible applications to Langlands program.

Now we state our main results. Let $\{\mu\}$ be the Hodge cocharacter attached to the Shimura datum (G, X). The parametrizing set of the Newton stratification is the finite Kottwitz set $B(G, \mu)$ (cf. [23] section 6), which may be viewed as the set of isomorphism classes of F-isocrystals with G-structure associated to points in $\mathscr{S}_0 := \mathscr{S}_{K_pK^p,0}(G, X)$. Recall that there is a partial order $\leq$ on $B(G, \mu)$, cf. 2.1. In the classical Siegel case, one can realize $B(G, \mu)$ as the set of Newton polygons of the polarized p-divisible groups attached to points on the special fiber. The basic properties of the Newton stratification are as follows[3] (cf. Theorem 2.3.6).

Theorem A. *Each Newton stratum $\mathscr{S}_0^b$ is non-empty, and it is an equi-dimensional locally closed subscheme of $\mathscr{S}_0$ of dimension*

$$\langle \rho, \mu + \nu_G(b) \rangle - \frac{1}{2}\mathrm{def}_G(b).$$

Here ρ is the half-sum of positive roots of G, $\nu_G(b)$ is the Newton point associated to $[b] \in B(G, \mu)$, and $\mathrm{def}_G(b)$ is the number defined in Definition 2.1.4. Moreover, $\overline{\mathscr{S}_0^b}$, the closure of $\mathscr{S}_0^b$, is the union of strata $\mathscr{S}_0^{b'}$ with $[b'] \leq [b]$, and $\overline{\mathscr{S}_0^b} - \mathscr{S}_0^b$ is either empty or pure of codimension 1 in $\overline{\mathscr{S}_0^b}$.

We remark that the non-emptiness was conjectured by Rapoport (cf. [38] Conjecture 7.1) and by Fargues (cf. [8] page 55), and it has been proved by Viehmann-Wedhorn in the PEL type case (cf. [48]), and Dong-Uk Lee, Kisin-Madapusi Pera and Chia-Fu Yu respectively in the Hodge type case, see [25, 52] for example. The other statements are due to Hamacher in the PEL type and Hodge type cases, cf. [13, 12]. The dimension formula in the Hodge type case was proved independently by the second author in [54].

Let W be the (absolute) Weyl group of G, and we have a certain subset ${}^JW \subset W$ defined by $\{\mu\}$ equipped with a partial order $\preceq$, cf. 3.2. The parametrizing set of the Ekedahl-Oort stratification is the set ${}^JW$, which classifies isomorphism classes of G-zips (or "F-zips with G-structure") associated to points in $\mathscr{S}_0 = \mathscr{S}_{K_pK^p,0}(G, X)$. In the classical Siegel case, ${}^JW$ classifies the p-torsions of the polarized abelian varieties attached to points on the special fiber. The basic properties of the Ekedahl-Oort stratification are as follows (cf. Theorem 3.4.7).

Theorem B. (1) *Each Ekedahl-Oort stratum $\mathscr{S}_0^w$ is a non-empty, equi-dimensional locally closed subscheme of $\mathscr{S}_0$. Moreover, $\overline{\mathscr{S}_0^w}$, the closure of $\mathscr{S}_0^w$, is the union of strata $\mathscr{S}_0^{w'}$ with $w' \preceq w$.*
(2) *For $w \in {}^JW$, the dimension of $\mathscr{S}_0^w$ is $l(w)$, the length of w.*
(3) *Each stratum $\mathscr{S}_0^w$ is smooth and quasi-affine.*

We remark that the non-emptiness is due to Viehmann-Wedhorn in the PEL type case (cf. [48]), and Chia-Fu Yu in the Hodge type case (cf. [52]). In the projective Hodge type

[3] In fact the Newton stratification is defined over κ, and these properties are also true over κ, see subsections 2.2 and 2.3.

case, Koskivirta proved the non-emptiness independently, cf. [21]. The other statements in the PEL type case are due to Viehmann-Wedhorn (cf. [48]). In the Hodge type case, the quasi-affiness is due to Goldring-Koskivirta (cf. [11]), and the closure relation and dimension formula are due to the second author (cf. [53]).

Attached to the Shimura datum (G, X) we have an infinite set[4] $C(G^{\mathrm{ad}}, \mu)$, which may be viewed as the set of isomorphism classes of F-crystals with G^{ad}-structure associated to points in $\mathscr{S}_{K_pK^p,0}(G, X)$. Here G^{ad} is the adjoint group associated to G. We have surjections $C(G^{\mathrm{ad}}, \mu) \twoheadrightarrow B(G^{\mathrm{ad}}, \mu) \simeq B(G, \mu)$ and $C(G^{\mathrm{ad}}, \mu) \twoheadrightarrow {}^JW_{G^{\mathrm{ad}}} \simeq {}^JW_G$ which, roughly speaking, send F-crystals with G^{ad}-structure to the associated F-isocrystals with G^{ad}-structure and G^{ad}-zips respectively. Associated to an element $c \in C(G^{\mathrm{ad}}, \mu)$, we can define a central leaf, which is a finer structure than the above Newton and Ekedahl-Oort strata. In the Siegel case, a central leaf is the locus where one fixes an isomorphism class of the polarized p-divisible groups. The basic properties of central leaves are as follows (cf. Theorem 4.2.5).

Theorem C. *Each central leaf is a non-empty, smooth, equi-dimensional locally closed subscheme of $\mathscr{S}_0$. It is closed in the Newton stratum containing it. Any central leaf in a Newton stratum $\mathscr{S}_0^b$ is of dimension $\langle 2\rho, \nu_G(b)\rangle$. Here ρ is the half sum of positive roots of G.*

The non-emptiness in the abelian type case follows that in the Hodge type case, which is in turn a consequence of the non-emptiness of the Newton strata. In the PEL type case, see [48] Theorem 10.2. The other statements in the Hodge type case are due to Hamacher (cf. [13]; see also [12] in the PEL type case) and the second author (cf. [54]) respectively.

The ideas to prove the above theorems are as follows. We work first in the Hodge type case, where most of the above are known, see the above remarks after each theorems. To extend to the abelian type case, we first work with a Shimura datum of abelian type such that the group G is *adjoint*. By using a lemma of Kisin (cf. Lemma 2.3.2), we can find a Hodge type Shimura datum (G_1, X_1) such that

(1) $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \xrightarrow{\sim} (G, X)$ and Z_{G_1} is a torus;
(2) if (G, X) has good reduction at p, then (G_1, X_1) in (1) can be chosen to have good reduction at p, and such that $E(G, X)_p = E(G_1, X_1)_p$.

Then the integral canonical model for (G, X) is given by
$$\begin{aligned}\mathscr{S}_{K_p}(G, X) &= [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p}}(G_1, X_1)^+]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ \\ &= [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_p}(G, X)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ,\end{aligned}$$
where on geometric connected components we have
$$\mathscr{S}_{K_p}(G, X)^+ = \mathscr{S}_{K_{1,p}}(G_1, X_1)^+/\Delta$$
with
$$\Delta = \mathrm{Ker}(\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ \to \mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ).$$
To show the induced Newton stratification, Ekedahl-Oort stratification, central leaves on $\mathscr{S}_{K_{1,p},0}(G_1, X_1)^+$ desecend to $\mathscr{S}_{K_p,0}(G, X)$, the key point is to show the Newton strata, Ekedahl-Oort strata, and central leaves of $\mathscr{S}_{K_{1,p},0}(G_1, X_1)^+$ are stable under the action of Δ, and their quotients by Δ are well defined. By [20] 4.4 the action of Δ can be described by certain construction of twisting of abelian varieties. This leads us to study the effect to p-divisible groups with additional structures under the construction of twisting abelian varieties in [20]. Using the fact that Z_{G_1} is a torus, we can show this twisting does not

[4]Here the more natural set should be $C(G, \mu)$; however, there will be no difference if the center Z_G of G is connected, cf. Lemma 4.2.1.

change the associated p-divisible groups with additional structures, and thus the Newton strata, Ekedahl-Oort strata, and central leaves of $\mathscr{S}_{K_{1,p},0}(G_1, X_1)^+$ are stable under the action of Δ, and their quotients by Δ are well defined. For a general Shimura datum of abelian type (G, X), we first pass to the associated adjoint Shimura datum $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ and apply the above construction to $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Then we define the Newton stratification, Ekedahl-Oort stratification, and central leaves on $\mathscr{S}_{K_p,0}(G, X)$ by pullling back those on $\mathscr{S}_{K_p^{\mathrm{ad}},0}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ under the natural morphism $\mathscr{S}_{K_p,0}(G, X) \to \mathscr{S}_{K_p^{\mathrm{ad}},0}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$.

In fact, there is an alternative way to define the Newton stratification, Ekedahl-Oort stratification, and central leaves on $\mathscr{S}_{K_p,0}(G, X)$, by using the filtered F-crystal with G^c-structure

$$\omega_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Z}_p}(G^c) \to \mathrm{FFCrys}_{\widehat{\mathscr{S}_{K_p}}(G,X)}$$

on $\mathscr{S}_{K_p}(G, X)$ constructed by Lovering in [28], which may be viewed as a crystalline model of the universal de Rham bundle $\omega_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}(G^c) \to \mathrm{Fil}^{\nabla}_{\widehat{\mathscr{S}_{K_p}}(G,X)^{rig}}$, see [26]. Here $G^c = G/Z_G^{nc}$ and $Z_G^{nc} \subset Z_G$ is the largest subtorus of Z_G that is split over $\mathbb{R}$ but anisotropic over $\mathbb{Q}$, $\widehat{\mathscr{S}_{K_p}}(G, X)$ is the p-adic completion of $\mathscr{S}_{K_p}(G, X)$ along its special fiber, $\widehat{\mathscr{S}_{K_p}}(G, X)^{rig}$ is the associated adic space, and $\mathrm{FFCrys}_{\widehat{\mathscr{S}_{K_p}}(G,X)}$ (resp. $\mathrm{Fil}^{\nabla}_{\widehat{\mathscr{S}_{K_p}}(G,X)^{rig}}$) is the category of filtered F-crystals (resp. filtered isocrystals) on $\widehat{\mathscr{S}_{K_p}}(G, X)$ (resp. $\widehat{\mathscr{S}_{K_p}}(G, X)^{rig}$), cf. 5.1.This construction in turn uses ideas from [27] where one constructs an auxiliary Shimura datum of abelian type $(\mathcal{B}, X')$, such that there is a commutative diagram of Shimura data

$$\begin{tikzcd} (\mathcal{B}, X') \arrow[r] \arrow[d] & (G_1, X_1) \arrow[d] \\ (G, X) \arrow[r] & (G^{\mathrm{ad}}, X^{\mathrm{ad}}) \end{tikzcd}$$

inducing a commutative diagram of (integral models of) Shimura varieties

$$\begin{tikzcd} \mathscr{S}_{K_{\mathcal{B},p}}(\mathcal{B}, X') \arrow[r] \arrow[d] & \mathscr{S}_{K_{1,p}}(G_1, X_1) \arrow[d] \\ \mathscr{S}_{K_p}(G, X) \arrow[r] & \mathscr{S}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}). \end{tikzcd}$$

Using the auxiliary Shimura datum of abelian type $(\mathcal{B}, X')$, one can then construct the filtered F-crystal with G^c-structure on $\mathscr{S}_{K_p,0}(G, X)$ from that on $\mathscr{S}_{K_{1,p},0}(G_1, X_1)$. If (G, X) is of Hodge type, it is easy to see the construction of the Newton stratification, Ekedahl-Oort stratification, and central leaves using the filtered F-crystal with G^c-structure coincides with the construction above. From this we can deduce that the two constructions (using the *right lower triangle* and the *left upper triangle* respectively) of Newton and Ekedahl-Oort stratifications coincide for a general abelian type Shimura datum (G, X), cf. 5.4.3, 5.4.4. If the center Z_G is connected we can show the two constructions of central leaves also coincide, cf. 5.4.5.

We also study the relations between the Newton stratification, Ekedahl-Oort stratification, and central leaves using the group theoretic methods in [34, 9, 10]. The main results are summarized as follows, cf. Proposition 6.2.3, Corollary 3.4.8, Example 6.2.4, Propositions 6.2.5 and 6.2.7. As the above, after fixing a prime to p level $K^p \subset G(\mathbb{A}_f^p)$, we simply write $\mathscr{S}_0 = \mathscr{S}_{K_pK^p,0}(G, X)$.

Theorem D. (1) *Each Newton stratum contains a minimal Ekedahl-Oort stratum (i.e. an Ekedahl-Oort stratum which is a central leaf). Moreover, if G splits, then each Newton stratum contains a unique minimal Ekedahl-Oort stratum.*

(2) *The ordinary Ekedahl-Oort stratum (i.e. the open Ekedahl-Oort stratum) coincides with the μ-ordinary locus (i.e. the open Newton stratum), which is a central leaf.*

(3) *For any $b \in B(G, \mu)$ and $w \in {}^J W$, we have*

$$\mathscr{S}_0^b \cap \mathscr{S}_0^w \neq \emptyset \Longleftrightarrow X_w(b) \neq \emptyset,$$

where $X_w(b) := \{gK \mid g^{-1}b\sigma(g) \in K \cdot_\sigma \mathcal{I}w\mathcal{I}\} \subseteq G(L)/K$, with $L = W(\overline{\kappa})_{\mathbb{Q}}, W = W(\overline{\kappa}), K = G(W)$, σ is the Frobenius on L and W, $\mathcal{I} \subset G(\mathbb{Z}_p)$ is the Iwahori subgroup, and $K \cdot_\sigma \mathcal{I}w\mathcal{I}$ is as in 6.1.6.

(4) *Let (G, X) be a Shimura datum of abelian type with good reduction at p whose attached pair $(G_{\mathbb{Q}_p}, \mu)$ is fully Hodge-Newton decomposable (cf. Definition 6.1.10 and* [10]*), then*

 (a) *each Newton stratum of $\mathscr{S}_0$ is a union of Ekedahl-Oort strata;*

 (b) *each Ekedahl-Oort stratum in a non-basic Newton stratum is a central leaf, and it is open and closed in the Newton stratum, in particular, non-basic Newton strata are smooth;*

 (c) *if $(G_{\mathbb{Q}_p}, \mu)$ is of Coxeter type (cf. 6.1.7 and* [9]*), then for two Ekedahl-Oort stratum $\mathscr{S}_0^1$ and $\mathscr{S}_0^2$, $\mathscr{S}_0^1$ is in the closure of $\mathscr{S}_0^2$ if and only if $\dim(\mathscr{S}_0^2) > \dim(\mathscr{S}_0^1)$.*

Here the first statement was proved in the PEL case by Viehmann-Wedhorn ([48]) under certain condition and by Nie in [34]. The proof of [34] is based on some group theoretic results, thus it also applies to our situation. In the Hodge type case the statement (2) is due to Wortmann, see [51]. The statements in (4) are first due to Görtz-He-Nie ([10], see also [9]) under the assumption that the axioms of [15] are verified. Here we do not use any unproved hypothesis or axioms.

In this paper we discuss our constructions for the examples of quaternionic and orthogonal Shimura varieties in details. These are of abelian type but mostly not of Hodge type Shimura varieties, and we do hope that our constructions (for these interesting and important Shimura varieties) will find interesting applications to number theory.

We now briefly describe the structure of this article. In the first section, we first review the construction of integral canonical models for Shimura varieties of abelian type following [18]. We then study twisting of p-divisible groups in a general setting which will be used later. In sections 2-4, we construct and study the Newton stratification, Ekedahl-Oort stratification, and central leaves respectively by using the approach of passing to adjoint. We discuss the example of quaternionic Shimura varieties in each section. In section 5, we revisit our constructions of stratifications using the filtered F-crystal with G^c-structure of [28]. In section 6, we study the relations between the Newton stratification, the Ekedahl-Oort stratification, and the central leaves both in the general and special setting. Finally, in section 7 we discuss the examples of GSpin and SO Shimura varieties in details.

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1. Good reductions of Shimura varieties of abelian type

In this section, we will recall Kisin's construction of integral canonical models for Shimura varieties of abelian type in [18]. We will start with the construction for those of Hodge type, and then pass to abelian type as in [18]. We will assume that $p>2$ through out this paper. But we would like to mention that by recent work of W. Kim and K. Madapusi Pera [17], integral canonical models for Shimura varieties of abelian type at $p=2$ are known to exist. We expect that our constructions also work in those case.

1.1. **Integral models for Shimura varieties of Hodge type.** Let (G,X) be a Shimura datum of Hodge type with good reduction at p. We recall the construction and basic results for $\mathscr{S}_{K_p}(G,X)$.

For a symplectic embedding $i:(G,X)\hookrightarrow(\mathrm{GSp}(V,\psi),X')$, by [18] Lemma 2.3.1, there exists a $\mathbb{Z}_p$-lattice $V_{\mathbb{Z}_p}\subseteq V_{\mathbb{Q}_p}$, such that $i_{\mathbb{Q}_p}:G_{\mathbb{Q}_p}\subseteq \mathrm{GL}(V_{\mathbb{Q}_p})$ extends uniquely to a closed embedding $G_{\mathbb{Z}_p}\hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_p})$. So there is a $\mathbb{Z}$-lattice $V_{\mathbb{Z}}\subseteq V$ such that $G_{\mathbb{Z}_{(p)}}$, the Zariski closure of G in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$, is reductive, as the base change to $\mathbb{Z}_p$ of $G_{\mathbb{Z}_{(p)}}$ is $G_{\mathbb{Z}_p}$. Moreover, by Zarhin's trick, we can assume that ψ is perfect on $V_{\mathbb{Z}}$. The integral canonical model $\mathscr{S}_K(G,X)$ of $\mathrm{Sh}_K(G,X)$ is constructed as follows. Let $K'_p=\mathrm{GSp}(V_{\mathbb{Z}_{(p)}},\psi)(\mathbb{Z}_p)$, we can choose $K'=K'_pK'^p\subseteq \mathrm{GSp}(V,\psi)(\mathbb{A}_f)$ with K'^p small enough and containing K^p, such that $\mathrm{Sh}_{K'}(\mathrm{GSp}(V,\psi),X)$ affords a moduli interpretation, and that the natural morphism

$$f:\mathrm{Sh}_K(G,X)\to \mathrm{Sh}_{K'}(\mathrm{GSp}(V,\psi),X)_E$$

is a closed embedding. Let $g=\frac{1}{2}\mathrm{dim}(V)$, and $\mathscr{A}_{g,1,K'}$ be the moduli scheme of principally polarized abelian schemes over $\mathbb{Z}_{(p)}$-schemes with level K'^p structure. Then $\mathrm{Sh}_{K'}(\mathrm{GSp}(V,\psi),X)$ is the generic fiber of $\mathscr{A}_{g,1,K'}$, and the integral canonical model $\mathscr{S}_K(G,X)$ is defined to be the normalization of the Zariski closure of $\mathrm{Sh}_K(G,X)$ in $\mathscr{A}_{g,1,K'}\otimes O_{E,(v)}$.

Theorem 1.1.1 ([18] Theorem 2.3.8)**.** *The $O_{E,(v)}$-scheme $\mathscr{S}_K(G,X)$ is smooth, and morphisms in the inverse system $\varprojlim_{K^p}\mathscr{S}_K(G,X)$ are étale.*

The scheme $\mathscr{S}_K(G,X)$ is uniquely determined by the Shimura datum and the group K in the sense that $\varprojlim_{K^p}\mathscr{S}_K(G,X)$ satisfies a certain extension property (see [18] 2.3.7 for the precise statement). This implies that the $G(\mathbb{A}_f^p)$-action on $\varprojlim_{K^p}\mathrm{Sh}_K(G,X)$ extends to $\varprojlim_{K^p}\mathscr{S}_K(G,X)$.

Let $\mathcal{A}\to\mathscr{S}_K(G,X)$ be the pull back to $\mathscr{S}_K(G,X)$ of the universal abelian scheme on $\mathscr{A}_{g,1,K'/\mathbb{Z}_{(p)}}$. Consider the vector bundle $\mathcal{V}:=\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathscr{S}_K(G,K))$ over $\mathcal{A}/\mathscr{S}_K(G,K)$. Kisin also constructed in [18] certain sections of $\mathcal{V}^{\otimes}$ which will play an important role in this paper. Let $\mathcal{V}_{\mathrm{Sh}_K(G,X)}$ be the base change of $\mathcal{V}$ to $\mathrm{Sh}_K(G,X)$, which is $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathrm{Sh}_K(G,K))$ by base change of de Rham cohomology. By [18] Proposition 1.3.2, there is a tensor $s\in V^{\otimes}_{\mathbb{Z}_{(p)}}$ defining $G_{\mathbb{Z}_{(p)}}\subseteq\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$. This tensor gives a section $s_{\mathrm{dR}/E}$ of $\mathcal{V}^{\otimes}_{\mathrm{Sh}_K(G,X)}$, which is actually defined over $O_{E,(v)}$. More precisely, we have the following result.

Proposition 1.1.2 ([18] Corollary 2.3.9)**.** *The section $s_{\mathrm{dR}/E}$ of $\mathcal{V}^{\otimes}_{\mathrm{Sh}_K(G,X)}$ extends to a section s_{dR} of $\mathcal{V}^{\otimes}$.*

Let $\mathbb{D}(\mathcal{A})$ be the Dieudonné crystal of $\mathcal{A}[p^\infty]$, then s_{dR} (and hence s) induces an injection of crystals $s_{\mathrm{cris}}:\mathbf{1}\to\mathbb{D}(\mathcal{A})^{\otimes}$, such that $s_{\mathrm{cris}}[\frac{1}{p}]:\mathbf{1}[\frac{1}{p}]\to\mathbb{D}(\mathcal{A})^{\otimes}[\frac{1}{p}]$ is Frobenius equivariant. We will simply call s_{cris} a tensor of $\mathbb{D}(\mathcal{A})^{\otimes}$.

1.1.3. We need to work with geometrically connected components. Fix a connected component $X^+\subseteq X$. For a compact open subgroup $K\subseteq G(\mathbb{A}_f)$ as before, i.e. $K=K_pK^p$ with $K_p=G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ and $K^p\subseteq G(\mathbb{A}_f^p)$ open compact and small enough, we denote by $\mathrm{Sh}_K(G,X)^+ \subseteq \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ the geometrically connected component which is the image of $X^+ \times 1$. Then by [18] 2.2.4, $\mathrm{Sh}_K(G,X)^+$ is defined over E^p, the maximal unramified extension of E. Let $O_{(p)}$ be the localization at (p) of the ring of integers of E^p, we write $\mathscr{S}_K(G,X)^+$ for the closure of $\mathrm{Sh}_K(G,X)^+$ in $\mathscr{S}_K(G,X) \otimes O_{(p)}$, and set $\mathscr{S}_{K_p}(G,X)^+ := \varprojlim_{K^p} \mathscr{S}_K(G,X)^+$.

Recall that by [18] 3.2 there exists an adjoint action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on $\mathrm{Sh}_{K_p}(G,X)$ induced by conjugation of G. The adjoint action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ on $\mathrm{Sh}_{K_p}(G,X)$ extends to $\mathscr{S}_{K_p}(G,X)$. It leaves $\mathrm{Sh}_{K_p}(G,X)^+$ stable, and hence induces an action on $\mathscr{S}_{K_p}(G,X)^+$. We will describe this action following [20] in the next subsection.

We remark that $\mathscr{S}_{K_p}(G,X)^+$ has connected special fiber. Indeed, by [29], it has a smooth compactification $\mathscr{S}_{K_p}(G,X)^+_{\mathrm{tor}}$ such that the boundary is either empty or a relative divisor. Let H^0 be the ring of regular functions on $\mathscr{S}_{K_p}(G,X)^+_{\mathrm{tor}}$. It is a finite $O_{(p)}$-algebra in E^p. Noting that H^0 is normal, we have $H^0 = O_{(p)}$. By Zariski's connectedness theorem, the special fiber of $\mathscr{S}_{K_p}(G,X)^+_{\mathrm{tor}}$ is connected, and hence that of $\mathscr{S}_{K_p}(G,X)^+$ is connected.

1.2. **Integral models for Shimura varieties of abelian type.** We recall Kisin's construction of integral canonical models for Shimura varieties of abelian type. Recall a Shimura datum (G,X) is said to be of abelian type, if there is a Shimura datum of Hodge type (G_1,X_1) and a central isogeny $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ which induces an isomorphism of ajoint Shimura data $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \xrightarrow{\sim} (G^{\mathrm{ad}}, X^{\mathrm{ad}})$.

1.2.1. In order to explain the construction of integral canonical models for Shimura varieties of abelian type, and also for the convenience of the next subsection, we recall briefly Kisin's construction of twisting abelian varieties. The main reference is [20] 4.4.

Let R be a commutative ring, Z be a flat affine group scheme over $\operatorname{Spec} R$, and $\mathcal{P}$ be a Z-torsor. Then $\mathcal{P}$ is flat and affine. We write O_Z and $O_{\mathcal{P}}$ for the ring of regular functions on Z and $\mathcal{P}$ respectively. Let M be a R-module with Z-action, i.e. a homomorphism of fppf sheaves of groups $Z \to \mathbf{Aut}(M)$, then the subsheaf M^Z is a R-submodule of M. By [20] Lemma 4.4.3, the natural homomorphism

$$(M \otimes_R O_{\mathcal{P}})^Z \otimes_R O_{\mathcal{P}} \to M \otimes_R O_{\mathcal{P}} \tag{1.2.2}$$

is an isomorphism.

1.2.3. Let $R \subseteq \mathbb{Q}$ be a normal subring. For a scheme S, we define the R-isogeny category of abelian schemes over S to be the category of abelian schemes over S by tensoring the Hom groups by $\otimes_{\mathbb{Z}} R$. An object $\mathcal{A}$ in this category is called an abelian scheme up to R-isogeny over S. For T an S-scheme, we set $\mathcal{A}(T) = \mathrm{Mor}_S(T,\mathcal{A}) \otimes_{\mathbb{Z}} R$. We will write $\underline{\mathrm{Aut}}_R(\mathcal{A})$ for the R-group whose points in an R-algebra A are given by

$$\underline{\mathrm{Aut}}_R(\mathcal{A})(A) = ((\mathrm{End}_S \mathcal{A}) \otimes_R A)^\times.$$

Now we assume that Z is of finite type over $R \subseteq \mathbb{Q}$. Suppose that we are given a homomorphism of R-groups $Z \to \underline{\mathrm{Aut}}_R(\mathcal{A})$, we define a pre-sheaf $\mathcal{A}^{\mathcal{P}}$ by setting

$$\mathcal{A}^{\mathcal{P}}(T) = (\mathcal{A}(T) \otimes_R O_{\mathcal{P}})^Z.$$

By [20] Lemma 4.4.6, $\mathcal{A}^{\mathcal{P}}$ is a sheaf, represented by an abelian scheme up to R-isogeny.

1.2.4. Before describing the construction of integral canonical models for Shimura varieties of abelian type, we need to fix some notations. Let $H/\mathbb{Z}_{(p)}$ be a reductive group. For a subgroup $A \subseteq H(\mathbb{Z}_{(p)})$, we write A_+ for the pre-image in A of $H^{\mathrm{ad}}(\mathbb{R})^+$, the connected component of identity in $H^{\mathrm{ad}}(\mathbb{R})$; and A^+ for $A \cap H(\mathbb{R})^+$. We write $H(\mathbb{Z}_{(p)})^-$ (resp.

$H(\mathbb{Z}_{(p)})_+^-$) for the closure of $H(\mathbb{Z}_{(p)})^-$ (resp. $H(\mathbb{Z}_{(p)})_+^-$) in $H(\mathbb{A}_f^p)$. Let Z be the center of H, we set

$$\mathscr{A}(H) = H(\mathbb{A}_f^p)/Z(\mathbb{Z}_{(p)})^- *_{H(\mathbb{Z}_{(p)})+/Z(\mathbb{Z}_{(p)})} H^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$$

and

$$\mathscr{A}(H)^\circ = H(\mathbb{Z}_{(p)})_+^-/Z(\mathbb{Z}_{(p)})^- *_{H(\mathbb{Z}_{(p)})+/Z(\mathbb{Z}_{(p)})} H^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+,$$

where $X *_Y Z$ is the quotient of $X \rtimes Z$ defined in [3] 2.0.1. By [3] 2.0.12, $\mathscr{A}(H)^\circ$ depends only on H^{der} and not on H.

Now we turn to the construction of integral models.

1.2.5. Let (G, X) be a Shimura datum of abelian type with good reduction at $p > 2$. By [18] Lemma 3.4.13, there is a Shimura datum of Hodge type (G_1, X_1) with good reduction at p, such that there is a central isogeny $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ inducing an isomorphism of Shimura data $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \xrightarrow{\sim} (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Let $G_{\mathbb{Z}_{(p)}}$ be a reductive group over $\mathbb{Z}_{(p)}$ with generic fiber G. By the proof of [18] Corollary 3.4.14, there exists a reductive model $G_{1,\mathbb{Z}_{(p)}}$ of G_1 over $\mathbb{Z}_{(p)}$, such that the central isogeny $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ extends to a central isogeny $G_{1,\mathbb{Z}_{(p)}}^{\mathrm{der}} \to G_{\mathbb{Z}_{(p)}}^{\mathrm{der}}$.

We can now follow discussions as in 1.1.3. Let $X_1^+ \subseteq X_1$ be a connected component. For $K_1 = K_{1,p}K_1^p$, let $\mathrm{Sh}_{K_1}(G_1, X_1)^+ \subseteq \mathrm{Sh}_{K_1}(G_1, X_1)$ be the geometrically connected component which is the image of $X_1^+ \times 1$. Then $\mathrm{Sh}_{K_1}(G_1, X_1)^+$ is defined over E_1^p, where E_1 is the reflex field of (G_1, X_1), and E_1^p is the maximal unramified extension of E_1. Let $O_{(p)}$ be the localization at (p) of the ring of integers of E_1^p, we write $\mathscr{S}_{K_1}(G_1, X_1)^+$ for the closure of $\mathrm{Sh}_{K_1}(G_1, X_1)^+$ in $\mathscr{S}_{K_1}(G_1, X_1) \otimes O_{(p)}$, and $\mathscr{S}_{K_{1,p}}(G_1, X_1)^+ := \varprojlim_{K_1^p} \mathscr{S}_{K_1}(G_1, X_1)^+$. The $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$-action on $\mathrm{Sh}_{K_{1,p}}(G_1, X_1)^+$ extends to $\mathscr{S}_{K_{1,p}}(G_1, X_1)^+$, which (after converting to a right action) induces an action of $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$. Here $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$ is as we introduced in 1.2.4.

The action of $G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ is described in [20] as follows. Let $(\mathcal{A}, \lambda, \varepsilon)$ be the pull back to $\mathscr{S}_{K_{1,p}}(G_1, X_1)$ of the universal abelian scheme (up to $\mathbb{Z}_{(p)}$-isogeny) with weak $\mathbb{Z}_{(p)}$-polarization and level structure, and Z be the center of $G_{1,\mathbb{Z}_{(p)}}$. By [20] Lemma 4.5.2, there is a natural embedding

$$Z \to \underline{\mathrm{Aut}}_{\mathbb{Z}_{(p)}}(\mathcal{A}),$$

where $\underline{\mathrm{Aut}}_{\mathbb{Z}_{(p)}}(\mathcal{A})$ is as in 1.2.3. For $\gamma \in G^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$, and $\mathcal{P}$ the fiber of $G_{1,\mathbb{Z}_{(p)}} \to G_{\mathbb{Z}_{(p)}}^{\mathrm{ad}}$ over γ, by 1.2.3 again, we have $\mathcal{A}^{\mathcal{P}}$, an abelian scheme up to $\mathbb{Z}_{(p)}$-isogeny. Moreover, by [20] Lemma 4.4.8 (resp. Lemma 4.5.4), λ (resp. ε) induces a weak $\mathbb{Z}_{(p)}$-polarization $\lambda^{\mathcal{P}}$(resp. level structure $\varepsilon^{\mathcal{P}}$) on $\mathcal{A}^{\mathcal{P}}$. By [20] Lemma 4.5.7, this gives a morphism

$$\mathscr{S}_{K_{1,p}}(G_1, X_1) \to \mathscr{S}_{K_{1,p}}(G_1, X_1),$$

such that on generic fiber it agrees with the morphism induced by conjugation by γ. This action stabilizes $\mathscr{S}_{K_{1,p}}(G_1, X_1)^+$.

Theorem 1.2.6. *The quotient*

$$\mathscr{S}_{K_p}(G, X) := [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p}}(G_1, X_1)^+]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$$

is represented by a scheme over $O_{(p)}$ which descends to $O_{E,(v)}$. Moreover, it is the integral canonical model of $\mathrm{Sh}_{K_p}(G, X)$.

Proof. This is [18] Theorem 3.4.10. See also [19] Errata for [Ki 2] for a fully corrected proof. □

We have also

$$\mathscr{S}_{K_p}(G,X)=[\mathscr{A}(G_{\mathbb{Z}_{(p)}})\times\mathscr{S}_{K_p}(G,X)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ,$$

where $\mathscr{S}_{K_p}(G,X)^+\subset\mathscr{S}_{K_p}(G,X)$ is a geometric connected component over $O_{(p)}$ given by

$$\mathscr{S}_{K_p}(G,X)^+:=\mathscr{S}_{K_{1,p}}(G_1,X_1)^+/\Delta$$

with

$$\Delta:=\mathrm{Ker}(\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ\to\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ).$$

For each open compact subgroup $K^p\subset G(\mathbb{A}_f^p)$ which is small enough, we get the integral canonical model $\mathscr{S}_{K_pK^p}(G,X):=\mathscr{S}_{K_p}(G,X)/K^p$ of $\mathrm{Sh}_{K_pK^p}(G,X)$. In this paper, we are mainly interested in the geometry of the special fiber $\mathscr{S}_{K_p,0}(G,X)$ of $\mathscr{S}_{K_p}(G,X)$, i.e. the special fibers $\mathscr{S}_{K_pK^p,0}(G,X)$ of $\mathscr{S}_{K_pK^p}(G,X)$ when K^p varies, so we will sometime work with $\mathscr{S}_{K_p}(G,X)\otimes O_{E,v}$. Here $O_{E,v}$ is the p-adic completion of $O_{E,(v)}$.

We consider the following example of Shimura varieties of abelian type, which will be investigated continuously in the rest of this paper. Another interesting example will be given in section 7.

Example 1.2.7. Let D be a quaternion algebra over a totally real extension F of $\mathbb{Q}$ of degree n. Let $\infty_1,\infty_2,\cdots,\infty_d$ be the infinite places of F at which D is split. We will always assume that $d>0$ in the discussion. Let $G=\mathrm{Res}_{F/\mathbb{Q}}(D^\times)$ and

$$h:\mathbb{S}\to\mathrm{GL}_{2,\mathbb{R}}^d\subseteq D_{\mathbb{R}}^\times=G_{\mathbb{R}}$$

be the homomorphism given by $z\mapsto(z,z,\cdots,z)\in\mathrm{GL}_{2,\mathbb{R}}^d$. One checks easily that h induces a Shimura datum denoted by (G,X). The associated Shimura variety is of dimension d, and it is defined over the totally real number field

$$E=\mathbb{Q}(\sum_{i=1}^d\infty_i(f)\mid f\in F)\subseteq\mathbb{C},$$

here we view ∞_i as an embedding $F\to\mathbb{R}$.

If $d=n$, then (G,X) is of PEL type; and if $d<n$, it is of abelian type but not of Hodge type, as the weight cocharacter is not defined over $\mathbb{Q}$. We are mainly interested in the second case here. By [40] Part I §1, fixing an imaginary quadratic extension K/F together with a subset P_K of archimedean places such that the restriction to F induces a bijection of from P_K to $\{\infty_{d+1},\infty_{d+2},\cdots\infty_n\}$, then one can construct a PEL moduli (pro-)variety M/E' with an open and closed embedding $\mathrm{Sh}(G,X)\otimes E'\to M$. Here $E'\supseteq E$ is the reflex field of the zero-dimensional Shimura datum determined by K and P_K.

If moreover D is split at p, the integral canonical model can be constructed as follows. Let v be a place of E over p, and $O_{E,v}$ be the p-adic completion of the ring of integers at v. By [40] Part I §2, one can choose K and P_K, such that $E'\subseteq E_v$, and M has a smooth model $\mathcal{M}/O_{E,v}$. The integral model of $\mathrm{Sh}(G,X)_{E_v}$ is then its closure in $\mathcal{M}$.

1.3. **Twisting p-divisible groups.** In order to study stratifications induced by p-divisible groups, it will be helpful to have a theory of twisting p-divisible group. For our applications, it suffices to think about those coming from abelian schemes. But we insist to give a general theory here, as it might be useful to study general Rapoport-Zink spaces.

1.3.1. Consider the setting of 1.2.3 with $R=\mathbb{Z}_p$. We will fix a group scheme Z over $\mathrm{Spec}\,R$ which is flat, affine and of finite type as well as a Z-torsor $\mathcal{P}$. Their ring of regular functions will be denoted by O_Z and $O_{\mathcal{P}}$ respectively.

Let $\mathcal{D}$ be a p-divisible group over a scheme S. Then $\mathrm{End}_S\mathcal{D}$ is a R-module. We will write $\underline{\mathrm{Aut}}_R(\mathcal{D})$ for the R-group whose points in an R-algebra A are given by

$$\underline{\mathrm{Aut}}_R(\mathcal{D})(A) = ((\mathrm{End}_S\mathcal{D}) \otimes_R A)^\times.$$

Suppose now that we are given a homomorphism of R-groups $Z \to \underline{\mathrm{Aut}}_R(\mathcal{D})$. For each positive integer n, we define a pre-sheaf $\mathcal{D}^{\mathcal{P}}[p^n]$ by setting

$$\mathcal{D}^{\mathcal{P}}[p^n](T) = (\mathcal{D}[p^n](T) \otimes_R O_{\mathcal{P}})^Z.$$

They form a direct system denoted by $\mathcal{D}^{\mathcal{P}}$.

Proposition 1.3.2. *$\mathcal{D}^{\mathcal{P}}[p^n]$ is represented by a truncated p-divisible group of level n over S, and $\mathcal{D}^{\mathcal{P}}$ is a p-divisible group.*

Proof. We proceed as in [20] Lemma 4.4.6, and take a finite, integral, torsion free R-algebra R' such that $\mathcal{P}(R')$ is non-empty. Specializing 1.2.2 by the map $O_{\mathcal{P}} \to R'$, we obtain an isomorphism $\mathcal{D}^{\mathcal{P}}[p^n] \otimes_R R' \cong \mathcal{D}[p^n] \otimes_R R'$. $\mathcal{D}^{\mathcal{P}}[p^n] \otimes_R R'$ is a truncated p-divisible group of level n as $\mathcal{D}[p^n] \otimes_R R'$ is isomorphic to the sum of $[R' : R]$ copies of $\mathcal{D}[p^n]$.

We may assume that $\mathrm{Fr}(R')$ is Galois over $\mathbb{Q}$, when $\mathcal{D}^{\mathcal{P}}[p^n]$ is the $\mathrm{Gal}(\mathrm{Fr}(R')/\mathbb{Q})$-invariants of $\mathcal{D}^{\mathcal{P}}[p^n] \otimes_R R'$. So $\mathcal{D}^{\mathcal{P}}[p^n]$ is the kernel of a homomorphism of truncated p-divisible groups of level n, and hence is a group scheme over S. It is necessarily flat as it is a direct summand of $\mathcal{D}^{\mathcal{P}}[p^n] \otimes_R R'$. By the same argument, after applying $(\ )^{\mathcal{P}}$ to

$$0 \longrightarrow \mathcal{D}[p^{n-i}] \longrightarrow \mathcal{D}[p^n] \xrightarrow{p^{n-i}} \mathcal{D}[p^i] \longrightarrow 0,$$

we have an exact sequence

$$0 \longrightarrow \mathcal{D}^{\mathcal{P}}[p^{n-i}] \longrightarrow \mathcal{D}^{\mathcal{P}}[p^n] \xrightarrow{p^{n-i}} \mathcal{D}^{\mathcal{P}}[p^i] \longrightarrow 0\ .$$

This implies that $\mathcal{D}^{\mathcal{P}}$ is a p-divisible group. □

Remark 1.3.3. The ways that we twist abelian schemes and p-divisible groups are compatible. More precisely, notations and hypothesis as in 1.2.3, but with $R \subseteq \mathbb{Z}_{(p)}$. Let $R' = \mathbb{Z}_p$ and $\mathcal{D} = \mathcal{A}[p^\infty]$. The map $Z \to \underline{\mathrm{Aut}}_R(\mathcal{A})$ induces a map $Z_{R'} \to \underline{\mathrm{Aut}}_{R'}(\mathcal{D})$, and we have $\mathcal{A}^{\mathcal{P}}[p^\infty] = \mathcal{D}^{\mathcal{P}_{R'}}$.

1.3.4. We will need to work with p-divisible groups with additional structure. Notations as in 1.3.1, we assume that S is an integral scheme which is flat over $\mathbb{Z}_{(p)}$, and that Z is *smooth with connected fibers*. Let $T_p(\mathcal{D})$ be the p-adic Tate module of $\mathcal{D}$ over the generic point of S, and $t \in T_p(\mathcal{D})^\otimes$ be a Z-invariant tensor. Using the proof of [19] Lemma 4.1.7, we have a canonical isomorphism $T_p(\mathcal{D}^{\mathcal{P}}) \cong T_p(\mathcal{D})^{\mathcal{P}}$, and tensor $t \in T_p(\mathcal{D})^\otimes$ is naturally an element of $T_p(\mathcal{D}^{\mathcal{P}})^\otimes$.

Corollary 1.3.5. *Assumptions as above, there exists an isomorphism $\mathcal{D}^{\mathcal{P}} \cong \mathcal{D}$ respecting t.*

Proof. Noting that Z is smooth with connected fibers, $\mathcal{P}$ is a trivial Z-torsor. Specializing 1.2.2 at $w \in \mathcal{P}(R)$, we get an isomorphism $\mathcal{D}^{\mathcal{P}} \cong \mathcal{D}$. It is by definition that its induced map on Tate modules respects t. □

2. Newton stratifications

We study the Newton stratifications on the special fibers of the Shimura varieties introduced in the last section.

2.1. **Group theoretic preparations.** Let G be a reductive group over $\mathbb{Z}_p$, and μ be a cocharacter of G defined over $W(\kappa)$ with κ a finite field. Let $W = W(\overline{\kappa})$, $L = W[1/p]$ and σ be the Frobenius on them. We need the following objects. Let $C(G)$ (resp. $B(G)$) be the set of $G(W)$-σ-conjugacy (resp. $G(L)$-σ-conjugacy) classes in $G(L)$, $C(G,\mu)$ be the set of $G(W)$-σ-conjugacy classes in $G(W)\mu(p)G(W)$, and $B(G,\mu)$ be the image of $C(G,\mu) \hookrightarrow C(G) \twoheadrightarrow B(G)$. The set $B(G)$ parametrizes isomorphism classes of F-isocrystals with G-structure over an algebraically closed field of characteristic p, cf. [39] Remark 3.4 (i).

2.1.1. Let T be a maximal torus of G, and $X_*(T)$ be its group of cocharacters. Let $\pi_1(G)$ be the quotient of $X_*(T)$ by the coroot lattice, and W_G be the Weyl group of G. Since G is unramified, we can fix a Borel subgroup $T \subset B \subset G$. To a $G(L)$-σ-conjugacy class $[b] \in B(G)$, Kottwitz defines two functorial invariants

$$\nu_G([b]) \in (X_*(T)_{\mathbb{Q}}/W_G)^{\Gamma} \cong X_*(T)^{\Gamma}_{\mathbb{Q},\mathrm{dom}}$$

and

$$\kappa_G([b]) \in \pi_1(G)_{\Gamma}$$

in [22]. Here $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and $X_*(T)_{\mathbb{Q},\mathrm{dom}}$ is the subspace spanned by dominant coweights corresponding to B. These two invariants determines $[b]$ uniquely. In the following, we will also write $\nu_G(b)$ and $\kappa_G(b)$ for an element $b \in G(L)$ for the two invariants of $[b] \in B(G)$, the $G(L) - \sigma$-conjugacy class of b.

We consider the partial order $\leq$ on $X_*(T)_{\mathbb{Q}}$ given by $\chi' \leq \chi$ if and only if $\chi - \chi'$ is a linear combination of positive coroots with positive rational coefficients. We write $\overline{v}$ for the average of its Γ-orbit. By [39] Theorem 4.2, we have $\nu_G(b) \leq \overline{\mu}$ and $\kappa_G(b) = \mu_*$ for $b \in G(W)\mu(p)G(W)$. Here μ_* is the image of μ in $\pi_1(G)_{\Gamma}$. By works of Gashi, Kottwitz, Lucarelli, Rapoport and Richartz, we have (See [48] 8.6))

$$B(G,\mu) = \{[b] \in B(G) \mid \nu_G(b) \leq \overline{\mu} \text{ and } \kappa_G(b) = \mu_*\}.$$

Remark 2.1.2. One can define for any algebraically closed field $k \supseteq \mathbb{F}_p$ a set $B'(G)$ exactly as how we define $B(G)$. But by [39] Lemma 1.3, the obvious map $B(G) \to B'(G)$ is bijective.

Remark 2.1.3. There is a unique maximal (resp. minimal) element in $B(G,\mu)$. For a variety X/κ with a map $X(\overline{\kappa}) \to B(G,\mu)$, the preimage of this element is called the μ-ordinary locus (resp. basic locus).

To each $G(L)$-σ-conjugacy class $[b]$, one defines M_b to be the centralizer in G of $\nu_G(b)$, and J_b be the group scheme representing

$$J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} L) \mid gb = b\sigma(g)\}.$$

The group J_b is a inner form of M_b which, up to isomorphism, does not depend on the choices of representatives in $[b]$ (see [22] 5.2).

Definition 2.1.4. For $[b] \in B(G)$, the defect of $[b]$ is defined by

$$\mathrm{def}_G(b) = \mathrm{rank}_{\mathbb{Q}_p} G - \mathrm{rank}_{\mathbb{Q}_p} J_b.$$

Hamacher gives a formula for $\mathrm{def}_G(b)$ using root data.

Proposition 2.1.5 ([12] Proposition 3.8)**.** *Let $w_1, \cdots, w_l$ be the sums over all elements in a Galois orbit of absolute fundamental weights of G. For $[b] \in B(G)$, we have*

$$\mathrm{def}_G(b) = 2 \cdot \sum_{i=1}^{l} \{\langle \nu_G(b), w_i \rangle\},$$

where $\{\cdot\}$ means the fractional part of a rational number.

2.2. **Newton stratifications on Shimura varieties of Hodge type.** No surprisingly, Newton strata on Shimura varieties of abelian type are, in some manner, induced by those on Shimura varieties of Hodge type. So we will first recall definition of Newton strata on Shimura varieties of Hodge type.

2.2.1. Notations as in 1.1. Let κ be the residue field of $O_{E,(v)}$. The Shimura datum (G,X) determines a G-orbit of cocharacters. It extends uniquely to a $G_{\mathbb{Z}_p}$-orbit of cocharacters, and hence has a representative $\mu:\mathbb{G}_m\to G_{W(\kappa)}$ which is unique up to conjugacy. We will assume that μ has weights 0 and 1 on $V^\vee_{\mathbb{Z}_p}\otimes W(\kappa)$. We will also write υ for $\sigma(\mu)$.

Let $W=W(\overline{\kappa})$ and $L=W[1/p]$. For $z\in\mathscr{S}_K(G,X)(\overline{\kappa})$, we will simply write D_z for $\mathbb{D}(\mathcal{A}_z[p^\infty])(W)$. Two points $x,y\in\mathscr{S}_K(G,X)(\overline{\kappa})$ are said to be in the same *Newton stratum* if there exists an isomorphism of isocrystals

$$D_x\otimes L\to D_y\otimes L$$

mapping $s_{\mathrm{cris},x}$ to $s_{\mathrm{cris},y}$. In fact, we have an F-crystal $\mathbb{D}(\mathcal{A}[p^\infty])$ with a crystalline Tate tensor s_{cris} over $\mathscr{S}_{K,0}(G,X)$, the special fiber of $\mathscr{S}_K(G,X)$.

For $x\in\mathscr{S}_K(G,X)(\overline{\kappa})$, choosing an isomorphism $t:V^\vee_{\mathbb{Z}_p}\otimes W\to D_x$ mapping s to $s_{\mathrm{cris},x}$, we get a Frobenius on $V^\vee_{\mathbb{Z}_p}\otimes W$ whose linearization $g_{x,t}$ lies in $G(W)\mu(p)G(W)$. Moreover, changing t to another isomorphism $V^\vee_{\mathbb{Z}_p}\otimes W\to D_x$ mapping s to $s_{\mathrm{cris},x}$ amounts to $G(W)$-σ-conjugacy of $g_{x,t}$. So we have a well defined map

$$\mathscr{S}_K(G,X)(\overline{\kappa})\to C(G,\mu).$$

Similarly, changing t to another isomorphism $V^\vee_{\mathbb{Z}_p}\otimes L\to D_x\otimes L$ mapping s to $s_{\mathrm{cris},x}$ amounts to $G(L)$-σ-conjugacy of $g_{x,t}$ (in $B(G)$), and we have a well defined map

$$\mathscr{S}_K(G,X)(\overline{\kappa})\to B(G,\mu).$$

It is clear that $x,y\in\mathscr{S}_K(G,X)(\overline{\kappa})$ are in the same Newton stratum if and only if they have the same image in $B(G,\mu)$.

Before stating the results about Newton strata on Shimura varieties of Hodge type, we need to fix some notations. When there is no confusion about the level K and the Shimura datum (G,X), we simply denote by $\mathscr{S}_0=\mathscr{S}_{K,0}(G,X)$ the special fiber of $\mathscr{S}_K(G,X)$. For $[b]\in B(G,\mu)$, we will write $\mathscr{S}_0^b$ for the Newton stratum corresponding to it. It is, a priori, just a subset of $\mathscr{S}_0(\overline{\kappa})$.

Theorem 2.2.2. *The Newton stratum $\mathscr{S}_0^b$ is a non-empty equi-dimensional locally closed subscheme of $\mathscr{S}_0$ of dimension*

$$\langle\rho,\mu+\nu_G(b)\rangle-\frac{1}{2}\mathrm{def}_G(b).$$

Here ρ is the half-sum of positive roots of G. Moreover, $\overline{\mathscr{S}_0^b}$, the closure of $\mathscr{S}_0^b$, is the union of strata $\mathscr{S}_0^{b'}$ with $[b']\leq[b]$, and $\overline{\mathscr{S}_0^b}-\mathscr{S}_0^b$ is either empty or pure of codimension 1 in $\overline{\mathscr{S}_0^b}$.

Proof. The non-emptiness of $\mathscr{S}_0^b$ is proved by Dong-Uk Lee, Kisin-Madapusi Pera and Chia-Fu Yu respectively, one can see for example [25]. That $\mathscr{S}_0^b$ is locally closed follows from [39] Theorem 3.6. In particular, it is defined over κ. The dimension formula is given in [13] Theorem 1.2, see also [54]. The last sentence follows from [13] Corollary 5.3. □

When the prime to p level K^p varies, by construction the Newton strata $\mathscr{S}^b_{K_pK^p,0}$ are invariant under the prime to p Hecke action. In this way we get also the Newton stratification on $\mathscr{S}_{K_p,0}=\varprojlim_{K^p}\mathscr{S}_{K_pK^p,0}$.

2.3. **Newton stratifications on Shimura varieties of abelian type.** The guiding idea of our construction is as follows. Let (G, X) be a Shimura datum of abelian type with good reduction at $p > 2$, $K^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small open compact subgroup, and $\mathscr{S}_{K,0}(G, X)$ be the special fiber of the associated integral canonical model (with $K = K_pK^p$). In order to define a stratification on $\mathscr{S}_{K,0}(G, X)$, the easiest way (and also the most direct way) one could think about is to do this for $\mathscr{S}_{K^{\mathrm{ad}},0}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ first, where $K^{\mathrm{ad}} = G^{\mathrm{ad}}(\mathbb{Z}_p)K^{p,\mathrm{ad}} \subset G^{\mathrm{ad}}(\mathbb{A}_f)$ containes the image of K under the induced map $G(\mathbb{A}_f) \to G^{\mathrm{ad}}(\mathbb{A}_f)$, and then pull it back via

$$\mathscr{S}_{K,0}(G, X) \to \mathscr{S}_{K^{\mathrm{ad}},0}(G^{\mathrm{ad}}, X^{\mathrm{ad}}).$$

The goal of this subsection is to explain how to define and study Newton stratifications for Shimura varieties of abelian type via this "passing to adjoints" approach.

We would like to begin with the following lemma, which says that if one wants to use $B(G, \mu)$ to parameterize all the Newton strata, then he could pass to the adjoint group freely.

Lemma 2.3.1. *Let $f : G \to H$ be a central isogeny of reductive groups over $\mathbb{Z}_p$, and μ be a cocharacter of G defined over $W(\kappa)$ with κ finite. Then the map $B(G, \mu) \to B(H, \mu)$ is a bijection respecting partial orders.*

Proof. This follows from [23] 6.5. □

The technical starting point is the following result of Kisin. It implies that for an adjoint Shimura datum of abelian type with good reduction at p, one can always realize it as the adjoint Shimura datum of a Hodge type one with very good properties.

Lemma 2.3.2 ([19] Lemma 4.6.6)**.** *Let (G, X) be a Shimura datum of abelian type with G an adjoint group. Then there exists a Shimura datum of Hodge type (G_1, X_1) such that*

1. *$(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \xrightarrow{\sim} (G, X)$ and Z_{G_1} is a torus;*
2. *if (G, X) has good reduction at p, then (G_1, X_1) in (1) can be chosen to have good reduction at p, and such that $E(G, X)_p = E(G_1, X_1)_p$.*

2.3.3. Let (G, X) be an *adjoint* Shimura datum of abelian type with good reduction at p, and (G_1, X_1) be a Shimura datum of Hodge type satisfying the two conditions in the above lemma. Then the center of $G_{1,\mathbb{Z}_{(p)}}$ is a torus.

Consider $\mathscr{S}_{K_p}(G, X)$. By Theorem 1.2.6, it is given by

$$\begin{aligned}\mathscr{S}_{K_p}(G, X) &= [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p}}(G_1, X_1)^+]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ \\ &= [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_p}(G, X)^+]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ,\end{aligned}$$

where on connected components we have

$$\mathscr{S}_{K_p}(G, X)^+ = \mathscr{S}_{K_{1,p}}(G_1, X_1)^+/\Delta$$

with

$$\Delta = \mathrm{Ker}(\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ \to \mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ).$$

By the last subsection,there is a Newton stratification on $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)$, we can restrict it to $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$ and then extend it trivially to $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$. We will sometimes call this the *induced Newton stratification* on $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$. Similarly for $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$.

Proposition 2.3.4. *The induced Newton stratification on $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$ (resp. $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$) is $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$-stable. Moreover, the induced Newton* *stratification on* $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$ *descends to the Newton stratification on* $\mathscr{S}_{K_{1,p},0}(G_1, X_1)$.

Proof. To see the first statement, for $([g,h],x) \in \mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p}}(G_1,X_1)^+$, with $g \in G(\mathbb{A}_f^p)$, $h \in G(\mathbb{Z}_{(p)})^+$ and $x \in \mathscr{S}_{K_{1,p}}(G_1,X_1)^+$, its p-divisible group is given by $\mathcal{A}_x[p^\infty]$. So, to prove the claim, it suffices to show that for any $([g',h']) \in \mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$ with $g' \in G_1(\mathbb{Z}_{(p)})_+^-$, $h' \in G(\mathbb{Z}_{(p)})^+$, the p-divisible group attached to $([g,h],x)\cdot(g',h')$ is isomorphic to $\mathcal{A}_x[p^\infty]$ respecting additional structure. But this follows from Corollary 1.3.5. By the same argument, we see that the induced Newton stratification on $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+$ descends to the Newton stratification on $\mathscr{S}_{K_{1,p},0}(G_1,X_1)$. $\square$

The induced Newton stratification on $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+$ descends to a stratification on $\mathscr{S}_{K_p,0}(G,X)$, and we will call it the Newton stratification. More formally, we have the following formulas for (G_1,X_1):

$$\mathscr{S}_{K_{1,p},0}(G_1,X_1) = \coprod_{[b]\in B(G_1,\mu_1)} \mathscr{S}_{K_{1,p},0}(G_1,X_1)^b,$$
$$\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+ = \coprod_{[b]\in B(G_1,\mu_1)} \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^{+,b},$$
$$\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^b = [\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^{+,b}]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ,$$

and for (G,X):

$$\mathscr{S}_{K_{,p},0}(G,X) = \coprod_{[b]\in B(G,\mu)} \mathscr{S}_{K_p,0}(G,X)^b,$$
$$\mathscr{S}_{K_p,\overline{\kappa}}(G,X)^+ = \coprod_{[b]\in B(G,\mu)} \mathscr{S}_{K_p,\overline{\kappa}}(G,X)^{+,b},$$
$$\mathscr{S}_{K_p,\overline{\kappa}}(G,X)^b = [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_p,\overline{\kappa}}(G,X)^{+,b}]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ.$$

Moreover, we have

$$\mathscr{S}_{K_p,\overline{\kappa}}(G,X)^{+,b} = \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^{+,b}/\Delta,$$
$$\mathscr{S}_{K_p,\overline{\kappa}}(G,X)^b = [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^{+,b}]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ.$$

The proposition also indicates how to relate Newton strata to the group theoretic object $B(G,\upsilon)$. For $x \in \mathscr{S}_{K_p,0}(G,X)(\overline{\kappa})$, we can find $x_0 \in \mathscr{S}_{K_p,0}(G,X)^+(\overline{\kappa})$ which is in the same Newton stratum as x. Noting that x_0 lifts to $\widetilde{x_0} \in \mathscr{S}_{K_{1,p},0}(G_1,X_1)^+(\overline{\kappa})$ whose image in $B(G_1,\mu_1) \simeq B(G,\mu)$ depends only on x, we get a well defined map

$$\mathscr{S}_{K_p,0}(G,X)(\overline{\kappa}) \to B(G,\mu)$$

whose fibers are Newton strata of $\mathscr{S}_{K_p,0}(G,X)$.

2.3.5. Now we are ready to think about general Shimura varieties of abelian type. Let (G,X) be a Shimura datum of abelian type (*not adjoint in general*) with good reduction at p. Let $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ be its adjoint Shimura datum, and (G_1,X_1) be a Shimura datum of Hodge type satisfying the two conditions in Lemma 2.3.2 with respect to $(G^{\mathrm{ad}},X^{\mathrm{ad}})$.

By the previous discussions, we have a commutative diagram

$$\begin{tikzcd}
 & \mathscr{S}_{K_{1,p},0}(G_1,X_1)(\overline{\kappa}) \arrow[r] \arrow[d] & B(G_1,\mu_1) \arrow[d, "\simeq"'] & \\
\mathscr{S}_{K_p,0}(G,X)(\overline{\kappa}) \arrow[r] & \mathscr{S}_{K_p^{\mathrm{ad}},0}(G^{\mathrm{ad}},X^{\mathrm{ad}})(\overline{\kappa}) \arrow[r] & B(G^{\mathrm{ad}},\mu) & B(G,\mu). \arrow[l, "\simeq"']
\end{tikzcd}$$

Here for μ (resp. μ_1), we use the same notation when viewing it as a cocharacter of G^{ad}, and we identified $B(G^{\mathrm{ad}},\mu)$ and $B(G^{\mathrm{ad}},\mu_1)$ silently. Now we can imitate the main results in Hodge type cases. Before stating the results, we fix notations as follows. Choose a sufficiently small open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$. We simply denote by $\mathscr{S}_0$ the special fiber of $\mathscr{S}_K(G,X)$, and by δ_{K^p} the induced Newton map $\mathscr{S}_0(\overline{\kappa}) \to B(G,\mu)$. For $[b] \in B(G,\mu)$, we will write $\mathscr{S}_0^b$ for the Newton stratum corresponding to it.

Theorem 2.3.6. *The Newton stratum $\mathscr{S}_0^b$ is non-empty, and it is an equi-dimensional locally closed subscheme of $\mathscr{S}_0$ of dimension*

$$\langle \rho, \mu + \nu_G(b) \rangle - \frac{1}{2}\mathrm{def}_G(b).$$

Here ρ is the half-sum of positive roots of G. Moreover, $\overline{\mathscr{S}_0^b}$, the closure of $\mathscr{S}_0^b$, is the union of strata $\mathscr{S}_0^{b'}$ with $[b'] \leq [b]$, and $\overline{\mathscr{S}_0^b} - \mathscr{S}_0^b$ is either empty or pure of codimension 1 in $\overline{\mathscr{S}_0^b}$.

Proof. For $\mathscr{S}_0(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, the statements for $\mathscr{S}_0(G^{\mathrm{ad}}, X^{\mathrm{ad}})^b$ follow by combining Theorem 2.2.2 with Proposition 2.3.4. On geometrically connected components, the morphism

$$\mathscr{S}_{\overline{\kappa}}(G,X)^+ \to \mathscr{S}_{\overline{\kappa}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^+$$

is a finite étale cover, and hence the statements for $\mathscr{S}_0^b$ hold. □

Thus for a Shimura datum (G,X) of abelian type with good reduction at $p>2$, we have the Newton stratification on the special fiber $\mathscr{S}_0$ of $\mathscr{S}_K(G,X)$

$$\mathscr{S}_0 = \coprod_{[b]\in B(G,\mu)} \mathscr{S}_0^b, \quad \overline{\mathscr{S}_0^b} = \coprod_{[b']\leq [b]} \mathscr{S}_0^{b'}.$$

As in Remark 2.1.3, there is a unique minimal (closed) strata $\mathscr{S}_0^{b_0}$, the basic locus, associated to the minimal element $[b_0] \in B(G,\mu)$; there is also a unique maximal (open) strata $\mathscr{S}_0^{b_\mu}$, the μ-ordinary locus, associated to the maximal element $[b_\mu] \in B(G,\mu)$.

Remark 2.3.7. Historically to study the geometry of Newton strata, one usually first proves that there exists some kind of almost product structure by introducing certain Igusa varieties over central leaves (cf. section 4) and the related Rapoport-Zink spaces, and then study the geometry of the associated Igusa varieties and Rapoport-Zink spaces respectively. This was done in the PEL type case in [31, 12] and in the Hodge type case in [13, 54]. In the abelian type case, we could also do this, using the Rapoport-Zink spaces constructed in [41]. However, we will not purse this aspect here. Novertheless, see [42] for the almost product structure of the Newton strata in the setting of perfectoid Shimura varieties of abelian type.

Example 2.3.8. Notations as in Example 1.2.7, we assume that D is split at p and that F is unramified at p. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ be places of F over p, and $F_{\mathfrak{p}_i}$ be the p-adic completion of F. We will fix an identification

$$\iota : \mathrm{Hom}(F,\mathbb{R}) \simeq \mathrm{Hom}(F, \overline{\mathbb{Q}}_p) = \sqcup_i \mathrm{Hom}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p).$$

After reordering the $\mathfrak{p}_i$, we can find $1 \leq s \leq t$, such that for $i \leq s$, $\mathrm{Hom}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p)$ contains some ∞_j with $j \leq d$; and for $i > s$, $\mathrm{Hom}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p)$ contains only ∞_j with $j > d$.

Then $G_{\mathbb{Q}_p} \cong \prod_i \mathrm{Res}_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{2,F_{\mathfrak{p}_i}} := \prod_i G_{\mathfrak{p}_i}$. The Shimura datum gives a cocharacter $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ as in 2.2.1. Under the isomorphism

$$G_{\overline{\mathbb{Q}}_p} \cong \prod_{i=1}^t \Big(\prod_{\sigma: F_{\mathfrak{p}_i} \hookrightarrow \overline{\mathbb{Q}}_p} \mathrm{GL}_{2,\overline{\mathbb{Q}}_p} \Big),$$

the cocharacter υ decomposes into

$$\mu_i : \mathbb{G}_m \to G_{\mathfrak{p}_i\overline{\mathbb{Q}}_p} = \prod_{\sigma: F_{\mathfrak{p}_i} \hookrightarrow \overline{\mathbb{Q}}_p} \mathrm{GL}_{2,\overline{\mathbb{Q}}_p},$$

where μ_i is trivial for $i > s$, and it is of the form

$$z \mapsto \left(\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

for $1 \leq i \leq s$. For $1 \leq i \leq s$, we will write a_i for the number of non-trivial factors of μ_i. Then

$$B(G, \mu) \cong \prod_{i=1}^{s} B(G_{\mathfrak{p}_i}, \mu_i) = \prod_{i=1}^{s} B(\mathrm{Res}_{F_{\mathfrak{p}_i}/\mathbb{Q}_p} \mathrm{GL}_{2,F_{\mathfrak{p}_i}}, \mu_i),$$

and we can use [8] 2.1 to compute $B(G_{\mathfrak{p}_i}, \mu_i)$.

Let $n_i = [F_{v_i} : \mathbb{Q}_p]$, for $[b_i] \in B(G_{\mathfrak{p}_i})$, $\nu_{G_{\mathfrak{p}_i}}(b_i)$ is of form $(\frac{\lambda_1}{n_i}, \frac{\lambda_2}{n_i})$ with $\lambda_1 \geq \lambda_2$. Here $\lambda_i = \frac{d_i}{h_i}$ is such that d_i, h_i are non-negative integers and $h_1 + h_2 = 2$. Let $B(G_{\mathfrak{p}_i}, \mu_i)_2$ (resp. $B(G_{\mathfrak{p}_i}, \mu_i)_1$) be the subset of $B(G_{\mathfrak{p}_i}, \mu_i)$ with 2 slopes (resp. 1 slope). Then $B(G_{\mathfrak{p}_i}, \mu_i)_2$ is the set of pairs $(\frac{d_1}{n_i}, \frac{d_2}{n_i})$ such that $d_1 > d_2$ and $d_1 + d_2 = a_i$, and $B(G_{\mathfrak{p}_i}, \mu_i)_1$ contains only one element which is the pair $(\frac{a_i}{2n_i}, \frac{a_i}{2n_i})$. It is then easy to see that the cardinality of $B(G_{\mathfrak{p}_i}, \mu_i)$ is $(\frac{a_i}{2}) + 1$, where

$$(\frac{a_i}{2}) = \begin{cases} m, & \text{if } a_i = 2m; \\ m+2, & \text{if } a_i = 2m+1. \end{cases}$$

The cardinality of $B(G, \mu)$ is the product of those of $B(G_{\mathfrak{p}_i}, \mu_i)$.

One sees easily that for each i, $B(G_{\mathfrak{p}_i}, \mu_i)$ is totally ordered. For $[b] \in B(G, \mu)$, its projection to $B(G_{\mathfrak{p}_i}, \mu_i)$ is of form $(\frac{\lambda_1}{n_i}, \frac{\lambda_2}{n_i})$ with $\lambda_1 \geq \lambda_2$ and $\lambda_1 + \lambda_2 = a_i$. The λ_is are integers unless $\lambda_1 = \lambda_2$. Let $l_i(b)$ be $[\lambda_1]$, where $[x]$ is the integer part of $[x]$. By Theorem 2.3.6 $\mathscr{S}_0^b$ is non-empty and equi-dimensional. One deduces easily from purity that it is of dimension $\sum_{i=1}^{s} l_i(b)$.

3. Ekedahl-Oort stratifications

We study the Ekedahl-Oort stratifications on the special fibers of the Shimura varieties introduced in the first section.

3.1. F-zips and G-zips. In this subsection, we will follow [33] and [37] to introduce F-zips and G-zips. They should be viewed as a kind of de Rham realizations of certain abelian motives. They are introduced by Moonen-Wedhorn and Pink-Wedhorn-Ziegler with the aim to study Ekedahl-Oort strata for Shimura varieties.

Let S be a scheme, and M be a locally free O_S-module of finite rank. By a descending (resp. ascending) filtration $C^\bullet$ (resp. $D_\bullet$) on M, we always mean a separating and exhaustive filtration such that $C^{i+1}(M)$ is a locally direct summand of $C^i(M)$ (resp. $D_i(M)$ is a locally direct summand of $D_{i+1}(M)$).

Let $\mathtt{LF}(S)$ be the category of locally free O_S-modules of finite rank, $\mathtt{FilLF}^\bullet(S)$ be the category of locally free O_S-modules of finite rank with descending filtration. For two objects $(M, C^\bullet(M))$ and $(N, C^\bullet(N))$ in $\mathtt{FilLF}^\bullet(S)$, a morphism

$$f : (M, C^\bullet(M)) \to (N, C^\bullet(N))$$

is a homomorphism of O_S-modules such that $f(C^i(M)) \subseteq C^i(N)$. We also denote by $\mathtt{FilLF}_\bullet(S)$ the category of locally free O_S-modules of finite rank with ascending filtration. For two objects $(M, C^\bullet)$ and $(M', C'^\bullet)$ in $\mathtt{FilLF}^\bullet(S)$, their tensor product is defined to be

$(M \otimes M', T^\bullet)$ with $T^i = \sum_j C^j \otimes C'^{i-j}$. Similarly for $\texttt{FilLF}_\bullet(S)$. For an object $(M, C^\bullet)$ in $\texttt{FilLF}^\bullet(S)$, one defines its dual to be

$$(M, C^\bullet)^\vee = ({}^\vee M := M^\vee, {}^\vee C^i := (M/C^{1-i})^\vee);$$

and for an object $(M, D_\bullet)$ in $\texttt{FilLF}_\bullet(S)$, one defines its dual to be

$$(M, D_\bullet)^\vee = ({}^\vee M := M^\vee, {}^\vee D_i := (M/D_{-1-i})^\vee).$$

It is clear from the convention that $(M, C^\bullet)^\vee = ({}^\vee M, {}^\vee C^\bullet) = (M^\vee, {}^\vee C^\bullet)$, and similarly for $D_\bullet$.

If S is over $\mathbb{F}_p$, we will denote by $\sigma : S \to S$ the morphism which is the identity on the topological space and p-th power on the sheaf of functions. For an S-scheme T, we will write $T^{(p)}$ for the pull back of T via σ. For a quasi-coherent O_S-module M, $M^{(p)}$ means the pull back of M via σ. For a σ-linear map $\varphi : M \to M$, we will denote by $\varphi^{\mathrm{lin}} : M^{(p)} \to M$ its linearization.

Definition 3.1.1. Let S be an $\mathbb{F}_p$-scheme.

(1) By an F-zip over S, we mean a tuple $\underline{M} = (M,\ C^\bullet,\ D_\bullet,\ \varphi_\bullet)$ where
- M is an object in $\texttt{LF}(S)$, i.e. M is a locally free sheaf of finite rank on S;
- $(M, C^\bullet)$ is an object in $\texttt{FilLF}^\bullet(S)$, i.e. $C^\bullet$ is a descending filtration on M;
- $(M, D_\bullet)$ is an object in $\texttt{FilLF}_\bullet(S)$, i.e. $D_\bullet$ is an ascending filtration on M;
- $\varphi_i : C^i/C^{i+1} \to D_i/D_{i-1}$ is a σ-linear map whose linearization

$$\varphi_i^{\mathrm{lin}} : (C^i/C^{i+1})^{(p)} \to D_i/D_{i-1}$$

is an isomorphism.

(2) By a morphism of F-zips

$$\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet) \to \underline{M'} = (M', C'^\bullet, D'_\bullet, \varphi'_\bullet),$$

we mean a morphism of O_S-modules $f : M \to N$, such that for all $i \in \mathbb{Z}$, $f(C^i) \subseteq C'^i$, $f(D_i) \subseteq D'_i$, and f induces a commutative diagram

$$\begin{array}{ccc} C^i/C^{i+1} & \xrightarrow{\varphi_i} & D_i/D_{i-1} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ C'^i/C'^{i+1} & \xrightarrow{\varphi'_i} & D'_i/D'_{i-1}. \end{array}$$

Example 3.1.2. ([37] Example 6.6) The Tate F-zips of weight d is

$$\mathbf{1}(d) := (O_S, C^\bullet, D_\bullet, \varphi_\bullet),$$

where

$$C^i = \begin{cases} O_S & \text{for } i \le d; \\ 0 & \text{for } i > d; \end{cases} \qquad D_i = \begin{cases} 0 & \text{for } i < d; \\ O_S & \text{for } i \ge d; \end{cases}$$

and φ_d is the Frobenius.

One can talk about tensor products and duals in the category of F-zips.

Definition 3.1.3. ([37] Definition 6.4) Let $\underline{M}$, $\underline{N}$ be two F-zips over S, then their tensor product is the F-zip $\underline{M} \otimes \underline{N}$, consisting of the tensor product $M \otimes N$ with induced filtrations $C^\bullet$ and $D_\bullet$ on $M \otimes N$, and induced σ-linear maps

$$\begin{array}{ccc} \mathrm{gr}^i_C(M \otimes N) & & \mathrm{gr}_i^D(M \otimes N) \\ \downarrow \cong & & \cong \uparrow \\ \bigoplus_j \mathrm{gr}^j_C(M) \otimes \mathrm{gr}^{i-j}_C(N) & \xrightarrow{\bigoplus_j \varphi_j \otimes \varphi_{i-j}} & \bigoplus_j \mathrm{gr}^D_j(M) \otimes \mathrm{gr}^D_{i-j}(N) \end{array}$$

whose linearization are isomorphisms.

Definition 3.1.4. ([37] Definition 6.5) The dual of an F-zip $\underline{M}$ is the F-zip $\underline{M}^\vee$ consisting of the dual sheaf of O_S-modules $M^\vee$ with the dual descending filtration of $C^\bullet$ and dual ascending filtration of $D_\bullet$, and σ-linear maps whose linearization are isomorphisms

$$(\mathrm{gr}^i_C(M^\vee))^{(p)} = ((\mathrm{gr}^{-i}_C M)^\vee)^{(p)} \xrightarrow{\left((\varphi^{\mathrm{lin}}_{-i})\right)^{-1\vee}} (\mathrm{gr}^D_{-i} M)^\vee \cong \mathrm{gr}^D_i(M^\vee).$$

For the Tate F-zips introduced in Example 3.1.2, we have natural isomorphisms $\mathbf{1}(d) \otimes \mathbf{1}(d') \cong \mathbf{1}(d+d')$ and $\mathbf{1}(d)^\vee \cong \mathbf{1}(-d)$. The d-th Tate twist of an F-zip $\underline{M}$ is defined as $\underline{M}(d) := \underline{M} \otimes \mathbf{1}(d)$, and there is a natural isomorphism $\underline{M}(0) \cong \underline{M}$.

Definition 3.1.5. A morphism between two objects in $\mathtt{LF}(S)$ is said to be admissible if the image of the morphism is a locally direct summand. A morphism $f : (M, C^\bullet) \to (M', C'^\bullet)$ in $\mathtt{FilLF}^\bullet(S)$ (resp. $f : (M, D_\bullet) \to (M', D'_\bullet)$ in $\mathtt{FilLF}_\bullet(S)$) is called admissible if for all i, $f(C^i)$ (resp. $f(D_i)$) is equal to $f(M) \cap C'^i$ (resp. $f(M) \cap D'_i$) and is a locally direct summand of M'. A morphism between two F-zips $\underline{M} \to \underline{M'}$ in F-$\mathtt{Zip}(S)$ is called admissible if it is admissible with respect to the two filtrations.

With admissible morphisms, tensor products, duals and the Tate object $\mathbf{1}(0)$ as above, F-$\mathtt{Zip}(S)$ becomes an $\mathbb{F}_p$-linear exact rigid tensor category (see [37] 6). By [37] Lemma 4.2 and Lemma 6.8, for a morphism in F-$\mathtt{Zip}(S)$, the property of being admissible is local for the fpqc topology.

We will introduce G-zips following [37]. These may be viewed as F-zips with G-structure. Note that the authors of [37] work with reductive groups over a general finite field $\mathbb{F}_q$ containing $\mathbb{F}_p$, and q-Frobenius. But we don't need the most general version of G-zips, as our reductive groups are connected and defined over $\mathbb{F}_p$.

3.1.6. Let G be a connected reductive group over $\mathbb{F}_p$, and χ be a cocharacter of G defined over κ, a finite extension of $\mathbb{F}_p$. Let $P_+ \subseteq G_\kappa$ (resp. $L \subseteq G_\kappa,\ P_- \subseteq G_\kappa$) be the subgroup whose Lie algebra is the submodule of $\mathrm{Lie}(G_\kappa)$ of non-negative weights (resp. of weight 0, of non-positive weights) with respect to μ composed with the adjoint action of G_κ on $\mathrm{Lie}(G_\kappa)$. The unipotent subgroup of P_+ (resp. P_-) will be denoted by U_+ (resp. U_-).

Definition 3.1.7. Let S be a scheme over κ.

(1) A G-zip of type χ over S is a tuple $\underline{I} = (I, I_+, I_-, \iota)$ consisting of
- a right G_k-torsor I over S,
- a right P_+-torsor $I_+ \subseteq I$ (i.e. the inclusion $I_+ \subseteq I$ is such that it is compatible for the P_+-action on I_+ and the G_κ-action on I),
- a right $P_-^{(p)}$-torsor $I_- \subseteq I$ (similarly as for $I_+ \subseteq I$), and
- an isomorphism of $L^{(p)}$-torsors $\iota : I_+^{(p)}/U_+^{(p)} \to I_-/U_-^{(p)}$.

(2) A morphism $(I, I_+, I_-, \iota) \to (I', I'_+, I'_-, \iota')$ of G-zips of type χ over S consists of equivariant morphisms $I \to I'$ and $I_\pm \to I'_\pm$ that are compatible with inclusions and the isomorphisms ι and ι'.

Here by a torsor over S of an fpqc group scheme G/S, we mean an fpqc scheme X/S with a G-action $\rho : X \times_S G \to X$ such that the morphism $X \times G \to X \times_S X$, $(x, g) \to (x, x \cdot g)$ is an isomorphism.

The category of G-zips of type χ over S will be denoted by G-$\mathtt{Zip}^\chi_\kappa(S)$. When $G = \mathrm{GL}_n$ we recover the category of F-zips, cf. [37] subsection 8.1. With the evident notation of pull back, the G-$\mathtt{Zip}^\chi_\kappa(S)$ form a fibered category over the category of schemes over κ, denoted by G-$\mathtt{Zip}^\chi_\kappa$. Noting that morphisms in G-$\mathtt{Zip}^\chi_\kappa(S)$ are isomorphisms, G-$\mathtt{Zip}^\chi_\kappa$ is a category fibered in groupoids.

Theorem 3.1.8. *([37] Corollary 3.12) The fibered category $G\text{-}\mathtt{Zip}^{\chi}_{\kappa}$ is a smooth algebraic stack of dimension 0 over κ.*

3.1.9. *Some technical constructions about G-zips.* We need more information about the structure of $G\text{-}\mathtt{Zip}^{\chi}_{\kappa}$. First, we need to introduce some standard G-zips as in [37].

Construction 3.1.10. ([37] Construction 3.4) Let S/κ be a scheme. For a section $g \in G(S)$, one associates a G-zip of type χ over S as follows. Let $I_g = S \times_{\kappa} G_{\kappa}$ and $I_{g,+} = S \times_{\kappa} P_+ \subseteq I_g$ be the trivial torsors. Then $I_g^{(p)} \cong S \times_{\kappa} G_{\kappa} = I_g$ canonically, and we define $I_{g,-} \subseteq I_g$ as the image of $S \times_{\kappa} P_-^{(p)} \subseteq S \times_{\kappa} G_{\kappa}$ under left multiplication by g. Then left multiplication by g induces an isomorphism of $L^{(p)}$-torsors

$$\iota_g : I_{g,+}^{(p)}/U_+^{(p)} = S \times_{\kappa} P_+^{(p)}/U_+^{(p)} \cong S \times_{\kappa} P_-^{(p)}/U_-^{(p)} \xrightarrow{\sim} g(S \times_{\kappa} P_-^{(p)})/U_-^{(p)} = I_{g,-}/U_-^{(p)}.$$

We thus obtain a G-zip of type χ over S, denoted by $\underline{I}_g$.

Lemma 3.1.11. *([37] Lemma 3.5) Any G-zip of type χ over S is étale locally of the form $\underline{I}_g$.*

Now we will explain how to write $G\text{-}\mathtt{Zip}^{\chi}_{\kappa}$ in terms of quotient of an algebraic variety by the action of a linear algebraic group following [37] Section 3.

Denote by $\mathrm{Frob}_p : L \to L^{(p)}$ the relative Frobenius of L, and by $E_{G,\chi}$ the fiber product

$$\begin{tikzcd} E_{G,\chi} \arrow[rr] \arrow[d] & & P_-^{(p)} \arrow[d] \\ P_+ \arrow[r] & L \arrow[r, "\mathrm{Frob}_p"] & L^{(p)}. \end{tikzcd}$$

Then we have

$$E_{G,\chi}(S) = \{(p_+ := lu_+,\ p_- := l^{(p)}u_-) : l \in L(S), u_+ \in U_+(S), u_- \in U_-^{(p)}(S)\}. \tag{3.1.12}$$

It acts on G_k from the left hand side as follows. For $(p_+, p_-) \in E_{G,\chi}(S)$ and $g \in G_k(S)$, set $(p_+, p_-) \cdot g := p_+ g p_-^{-1}$.

To relate $G\text{-}\mathtt{Zip}^{\chi}_{\kappa}$ to the quotient stack $[E_{G,\chi} \backslash G_{\kappa}]$, we need the following constructions in [37]. First, for any two sections $g, g' \in G_{\kappa}(S)$, there is a natural bijection between the set

$$\mathrm{Transp}_{E_{G,\chi}(S)}(g, g') := \{(p_+, p_-) \in E_{G,\chi}(S) \mid p_+ g p_-^{-1} = g'\}$$

and the set of morphisms of G-zips $\underline{I}_g \to \underline{I}_{g'}$ (see [37] Lemma 3.10). So we define a category $\mathcal{X}$ fibered in groupoids over the category of κ-schemes as follows. For any scheme S/κ, let $\mathcal{X}(S)$ be the small category whose underly set is $G(S)$, and for any two elements $g, g' \in G(S)$, the set of morphisms is the set $\mathrm{Transp}_{E_{G,\chi}(S)}(g, g')$.

Theorem 3.1.13. *([37] Proposition 3.11) Sending $g \in \mathcal{X}(S) = G(S)$ to $\underline{I}_g$ induces a fully faithful morphism $\mathcal{X} \to G\text{-}\mathtt{Zip}^{\chi}_{\kappa}$. Moreover, it induces an isomorphism $[E_{G,\chi} \backslash G_{\kappa}] \to G\text{-}\mathtt{Zip}^{\chi}_{\kappa}$.*

3.2. **Some group theoretic descriptions for the geometry of $[E_{G,\chi} \backslash G_{\kappa}]$.** Let $B \subseteq G$ be a Borel subgroup, and $T \subseteq B$ be a maximal torus. Note that such a B exists by [24] Theorem 2, and such a T exists by [4] XIV Theorem 1.1. Let $W(B,T) := \mathrm{Norm}_G(T)(\overline{\kappa})/T(\overline{\kappa})$ be the Weyl group, and $I(B,T)$ be the set of simple reflections defined by $B_{\overline{\kappa}}$. Let φ be the Frobenius on G given by the p-th power. It induces an isomorphism

$$(W(B,T), I(B,T)) \xrightarrow{\sim} (W(B,T), I(B,T))$$

of Coxeter systems still denoted by φ.

A priori the pair $(W(B,T), I(B,T))$ depends on the pair (B,T). However, any other pair (B',T') with $B' \subseteq G_{\overline{\kappa}}$ a Borel subgroup and $T' \subseteq B'$ a maximal torus is obtained on conjugating $(B_{\overline{\kappa}}, T_{\overline{\kappa}})$ by some $g \in G(\overline{\kappa})$ which is unique up to right multiplication by $T_{\overline{\kappa}}$. So conjugation by g induces isomorphisms $W(B,T) \to W(B',T')$ and $I(B,T) \to I(B',T')$ that are independent of g. Moreover, the morphisms attached to any three of such pairs are compatible, so we will simply write (W,I) for $(W(T), I(B,T))$, and view it as 'the' Weyl group with 'the' set of simple reflections.

The cocharacter $\mu : \mathbb{G}_m \to G_\kappa$ as in 3.1 gives a parabolic subgroup P_+, and hence a subset $J \subseteq I$ by taking simple roots whose inverse are roots of P_+. Let W_J the subgroup of W generated by J, and ${}^JW$ be the set of elements w such that w is the element of minimal length in some coset W_Jw'. Note that there is a unique element in W_Jw' of minimal length, and each $w \in W$ can be uniquely written as $w = w_J{}^Jw$ with $w_J \in W_J$ and ${}^Jw \in {}^JW$. In particular, ${}^JW$ is a system of representatives of $W_J\backslash W$.

Furthermore, if K is a second subset of I, then for each w, there is a unique element in W_JwW_K which is of minimal length. We will denote by ${}^JW^K$ the set of elements of minimal length, and it is a set of representatives of $W_J\backslash W/W_K$.

Let ω_0 be the element of maximal length in W, set $K := {}^{\omega_0}\varphi(J)$. Here we write ${}^gJ$ for gJg^{-1}. Let $x \in {}^KW^{\varphi(J)}$ be the element of minimal length in $W_K\omega_0W_{\varphi(J)}$. Then x is the unique element of maximal length in ${}^KW^{\varphi(J)}$ (see [48] 5.2). There is a partial order $\preceq$ on ${}^JW$, defined by $w' \preceq w$ if and only if there exists $y \in W_J$, such that

$$yw'x\varphi(y^{-1})x^{-1} \leq w$$

(see [48] Definition 5.8). Here $\leq$ is the Bruhat order (see A.2 of [48] for the definition). The partial order $\preceq$ makes ${}^JW$ into a topological space.

Now we can state the the main result in [36] of Pink-Wedhorn-Ziegler that gives a combinatory description of the topological space of $[E_{G,\mu}\backslash G_\kappa]$ (and hence $G\text{-}\mathtt{Zip}^\mu_\kappa$).

Theorem 3.2.1. *For $w \in {}^JW$, and $T' \subseteq B' \subseteq G_{\overline{\kappa}}$ with T' (resp. B') a maximal torus (resp. Borel) of $G_{\overline{\kappa}}$ such that $T' \subseteq L_{\overline{\kappa}}$ and $B' \subseteq P^{(p)}_{-,\overline{\kappa}}$, let $g, \dot{w} \in \mathrm{Norm}_{G_{\overline{\kappa}}}(T')$ be a representative of $\varphi^{-1}(x)$ and w respectively, and $G^w \subseteq G_{\overline{\kappa}}$ be the $E_{G,\mu}$-orbit of $gB'\dot{w}B'$. Then*

1. *The orbit G^w does not depends on the choices of $\dot{w}$, T', B' or g.*
2. *The orbit G^w is a locally closed smooth subvariety of $G_{\overline{\kappa}}$. Its dimension is $\dim(P) + l(w)$. Moreover, G^w consists of only one $E_{G,\mu}$-orbit. So G^w is actually the orbit of $g\dot{w}$.*
3. *Denote by $\big|[E_{G,\mu}\backslash G_\kappa] \otimes \overline{\kappa}\big|$ the topological space of $[E_{G,\mu}\backslash G_\kappa] \otimes \overline{\kappa}$, and still write ${}^JW$ for the topological space induced by the partial order $\preceq$. Then the association $w \mapsto G^w$ induces a homeomorphism ${}^JW \to \big|[E_{G,\mu}\backslash G_\kappa] \otimes \overline{\kappa}\big|$.*

Remark 3.2.2. There is a unique maximal (resp. minimal) element in ${}^JW$ (with respect to $\preceq$). For a variety $X/\overline{\kappa}$ with a map $X(\overline{\kappa}) \to {}^JW$, the preimage of this element is called the ordinary locus (resp. superspecial locus).

3.3. **Ekedahl-Oort stratifications on Shimura varieties of Hodge type.** Now we will explain how to construct Ekedahl-Oort stratification following [53]. Notations as in 1.1, we will write $\mathcal{V}$, s and s_{dR} respectively for its reduction mod p, and L (resp. G, $\mathscr{S}_0$) for the special fiber of $V_{\mathbb{Z}_{(p)}}$ (resp. $G_{\mathbb{Z}_{(p)}}$, $\mathscr{S}_K(G,X)$). By [53] Lemma 2.3.2 1), the scheme $I = \mathrm{Isom}_{\mathscr{S}_0}\big((L^\vee, s) \otimes O_{\mathscr{S}_0}, (\mathcal{V}, s_{\mathrm{dR}})\big)$ is a right G-torsor.

Setting 3.3.1. Let $F : \mathcal{V}^{(p)} \to \mathcal{V}$ and $V : \mathcal{V} \to \mathcal{V}^{(p)}$ be the Frobenius and Verschiebung on $\mathcal{V}$ respectively. Let $\delta : \mathcal{V} \to \mathcal{V}^{(p)}$ be the semi-linear map sending v to $v \otimes 1$. Then we have a semi-linear map $F\circ\delta:\mathcal{V}\to\mathcal{V}$. There is a descending filtration

$$\mathcal{V}\supseteq\ker(F\circ\delta)\supseteq 0$$

and an ascending filtration

$$0\subseteq\mathrm{im}(F)\subseteq\mathcal{V}.$$

The morphism V induces an isomorphism

$$\mathcal{V}/\mathrm{im}(F)\to\ker(F)$$

whose inverse will be denoted by V^{-1}. Then F and V^{-1} induce isomorphisms

$$\varphi_0:(\mathcal{V}/\ker(F\circ\delta))^{(p)}\to\mathrm{im}(F)$$

and

$$\varphi_1:(\ker(F\circ\delta))^{(p)}\to\mathcal{V}/(\mathrm{im}(F)).$$

Setting 3.3.2. Let μ be as in 2.2.1, we use the same symbol for its reduction mod p. The cocharacter $\mu:\mathbb{G}_{m,\kappa}\to G_\kappa\subseteq\mathrm{GL}(L_\kappa)\cong\mathrm{GL}(L_\kappa^\vee)$ induces an F-zip structure on $L_\kappa^\vee$ as follows. Let $(L_\kappa^\vee)^0$ (resp. $(L_\kappa^\vee)^1$) be the subspace of $L_\kappa^\vee$ of weight 0 (resp. 1) with respect to μ, and $(L_\kappa^\vee)_0$ (resp. $(L_\kappa^\vee)_1$) be the subspace of $L_\kappa^\vee$ of weight 0 (resp. 1) with respect to $\mu^{(p)}$. Then we have a descending filtration

$$L_\kappa^\vee\supseteq(L_\kappa^\vee)^1\supseteq 0$$

and an ascending filtration

$$0\subseteq(L_\kappa^\vee)_0\subseteq L_\kappa^\vee.$$

Let $\xi:L_\kappa^\vee\to(L_\kappa^\vee)^{(p)}$ be the isomorphism given by $l\otimes k\mapsto l\otimes 1\otimes k$, $\forall\ l\in L^\vee$, $\forall\ k\in\kappa$. Then ξ induces isomorphisms

$$\phi_0:(L_\kappa^\vee)^{(p)}/((L_\kappa^\vee)^1)^{(p)}\xrightarrow{\mathrm{pr}_2}((L_\kappa^\vee)^0)^{(p)}\xrightarrow{\xi^{-1}}(L_\kappa^\vee)_0$$

and

$$\phi_1:((L_\kappa^\vee)^1)^{(p)}\xrightarrow{\xi^{-1}}((L_\kappa^\vee)_1\simeq L_\kappa^\vee/(L_\kappa^\vee)_0.$$

The first main result of [53] is as follows.

Theorem 3.3.3. *([53] Theorem 2.4.1)*

(1) *Let $I_+\subseteq I$ be the closed subscheme*

$$I_+:=\mathrm{Isom}_{\mathscr{S}_0}\big((L_\kappa^\vee\supseteq(L_\kappa^\vee)^1,s)\otimes O_{\mathscr{S}_0},\ (\mathcal{V}\supseteq\ker(F\circ\delta),s_{\mathrm{dR}})\big).$$

Then I_+ is a P_+-torsor over $\mathscr{S}_0$.

(2) *Let $I_-\subseteq I$ be the closed subscheme*

$$I_-:=\mathrm{Isom}_{\mathscr{S}_0}\big(((L_\kappa^\vee)_0\subseteq L_\kappa^\vee,s)\otimes O_{\mathscr{S}_0},\ (\mathrm{im}(F)\subseteq\mathcal{V},s_{\mathrm{dR}})\big).$$

Then I_- is a $P_-^{(p)}$-torsor over $\mathscr{S}_0$.

(3) *Let $\iota:I_+^{(p)}/U_+^{(p)}\to I_-/U_-^{(p)}$ be the morphism induced by*

$$\begin{aligned}&I_+^{(p)}\to I_-/U_-^{(p)}\\&f\mapsto(\varphi_0\oplus\varphi_1)\circ\mathrm{gr}(f)\circ(\phi_0^{-1}\oplus\phi_1^{-1}),\forall\ S/\mathscr{S}_0\text{ and }\forall\ f\in I_+^{(p)}(S).\end{aligned}$$

Then ι is an isomorphism of $L^{(p)}$-torsors.

Hence the tuple (I,I_+,I_-,ι) is a G-zip of type μ over $\mathscr{S}_0$.

The G-zip (I, I_+, I_-, ι) induces a morphism $\mathscr{S}_0 \to G\text{-}\mathtt{Zip}^\mu_\kappa \simeq [E_{G,\chi}\backslash G_\kappa]$ over κ. In the following we will simply write $\mathscr{S}_{\overline{\kappa}} = \mathscr{S}_{0,\overline{\kappa}} = \mathscr{S}_{K,\overline{\kappa}}(G,X)$. We will write the induced morphism over $\overline{\kappa}$ as

$$\zeta : \mathscr{S}_{\overline{\kappa}} \to G\text{-}\mathtt{Zip}^\mu_\kappa \otimes \overline{\kappa} \simeq [E_{G,\chi}\backslash G_\kappa] \otimes \overline{\kappa},$$

whose fibers are called Ekedahl-Oort strata of $\mathscr{S}_{\overline{\kappa}}$. In the following we will sometimes abbreviate "Ekedahl-Oort" to "E-O" for short. The main results about the Ekedahl-Oort stratifications are as follows.

Theorem 3.3.4. (1) *The morphism ζ is smooth and surjective. In particular,*
  (a) *each E-O stratum is a non-empty, smooth and locally closed subscheme of $\mathscr{S}_{\overline{\kappa}}$, the closure of an E-O stratum is a union of strata;*
  (b) *all the strata are in bijection with ${}^JW$, and for $w \in {}^JW$, the corresponding stratum is of dimension $l(w)$, the length of w.*

(2) *Each E-O stratum is quasi-affine.*

Proof. For (1), all the statements but non-emptiness follows from [53]. To see the non-emptiness of E-O strata, by Theorem 4.1.1, each central leaf in the basic locus is non-empty, and by the proof of [48] Proposition 9.17, the minimal E-O stratum is a central leaf and hence non-empty. By flatness of ζ, all the E-O strata are non-empty.

For (2), by [11] Theorem 3.3.1 (2), each E-O is a locally closed subscheme of an affine scheme, and hence quasi-affine. □

When the prime to p level K^p varies, by construction the Ekedahl-Oort strata $\mathscr{S}^w_{K_pK^p,\overline{\kappa}}$ are invariant under the prime to p Hecke action. In this way we get also the Ekedahl-Oort stratification on $\mathscr{S}_{K_p,\overline{\kappa}} = \varprojlim_{K^p} \mathscr{S}_{K_pK^p,\overline{\kappa}}$.

3.4. **Ekedahl-Oort stratifications on Shimura varieties of abelian type.** We now explain how to define Ekedahl-Oort stratifications on Shimura varieties of abelian type. As what we did for Newton strata, we would like to begin with the following lemma, which says that if one wants to use the topological space of the quotient stack $[E_{G,\mu}\backslash G_\kappa]$ to parameterize all the Ekedahl-Oort strata, then he could pass to the adjoint group freely.

Lemma 3.4.1. *Let $f : G \to H$ be a homomorphism of reductive groups over $\mathbb{F}_p$ and μ be a cocharacter of G defined over a finite field κ. Denote also by μ the induced cocharacter of H by f. Let $U_{G,-}$, $U_{H,-}$ and $E_{G,\mu}$, $E_{H,\mu}$ be as in 3.1.6 and 3.1.12 respectively.*

(1) *If $U_{G,-} \to U_{H,-}$ induced by f is smooth, then $f_* : [E_{G,\mu}\backslash G_\kappa] \to [E_{H,\mu}\backslash H_\kappa]$ is smooth.*
(2) *If f is a central isogeny, then f_* is a smooth homeomorphism.*

Proof. To see (1), for $g \in G(\overline{\kappa})$, by the last paragraph of the proof of [53] Theorem 3.1.2, the $E_{G,\mu}$-equivariant morphism $U_{G,-} \times E_{G,\mu} \to G_\kappa$ given by $(u, g') \mapsto g' \cdot (ug)$ is smooth at $(1,1) \in U_{G,-} \times E_{G,\mu}$. So the induced morphism $U_{G,-} \to [E_{G,\mu}\backslash G_\kappa]$ is smooth at the identity. Similarly $f(g) \in H(\overline{\kappa})$ induces a morphism $U_{H,-} \to [E_{H,\mu}\backslash H_\kappa]$ which is smooth at the identity. Consider the commutative diagram

$$\begin{tikzcd} U_{G,-} \arrow[r, "f|_{U_{G,-}}"] \arrow[d] & U_{H,-} \arrow[d] \\ {[E_{G,\mu}\backslash G]} \arrow[r, "f_*"] & {[E_{H,\mu}\backslash H],} \end{tikzcd}$$

the composition $U_{G,-} \to U_{H,-} \to [E_{H,\mu}\backslash H]$ is smooth at the identity, and hence f_* is smooth in a neighborhood of g. But g can be any point, so f_* is smooth.

To see (2), the smoothness follows from (1), as $U_{G,-} \to U_{H,-}$ is an isomorphism. The homomorphism f is faithfully flat, so is $f_* : [E_{G,\mu}\backslash G] \to [E_{H,\mu}\backslash H]$. The induced map on topological spaces is then an open surjection. Noting that they both have cardinality $|^J W|$, it will then be a homeomorphism. $\square$

3.4.2. As what we did for Newton stratifications, we consider adjoint groups first. More precisely, let (G, X) be an *adjoint* Shimura datum of abelian type with good reduction at p, and (G_1, X_1) be a Shimura datum of Hodge type satisfying the two conditions in 2.3.2.

There is an E-O stratification on $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)$, we can, as in 2.3.3, restrict it to $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$ and then extend it trivially to $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$. We will sometimes call this the *induced E-O stratification* on $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$. Similarly for $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$.

Proposition 3.4.3. *The induced E-O stratification on $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$ (resp. $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$) is $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$-stable. Moreover, the induced E-O stratification on $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$ descends to the E-O stratification on $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)$.*

Proof. The proof is identical to that of Proposition 2.3.4. $\square$

The induced E-O stratification on $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$ descends to a stratification on $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G, X)$, this will be called the E-O stratification. More formally, we have the following formulas for (G_1, X_1):

$$\begin{aligned}
\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1) &= \coprod_{w \in {}^J W_{G_1}} \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^w, \\
\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+ &= \coprod_{w \in {}^J W_{G_1}} \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^{+,w}, \\
\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^w &= [\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^{+,w}]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ,
\end{aligned}$$

and for (G, X):

$$\begin{aligned}
\mathscr{S}_{K_{,p},\overline{\kappa}}(G, X) &= \coprod_{w \in {}^J W_G} \mathscr{S}_{K_p,\overline{\kappa}}(G, X)^w, \\
\mathscr{S}_{K_p,\overline{\kappa}}(G, X)^+ &= \coprod_{w \in {}^J W_G} \mathscr{S}_{K_p,\overline{\kappa}}(G, X)^{+,w}, \\
\mathscr{S}_{K_p,\overline{\kappa}}(G, X)^w &= [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_p,\overline{\kappa}}(G, X)^{+,w}]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mathscr{S}_{K_p,\overline{\kappa}}(G, X)^{+,w} &= \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^{+,w}/\Delta, \\
\mathscr{S}_{K_p,\overline{\kappa}}(G, X)^w &= [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^{+,w}]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ.
\end{aligned}$$

One can also define E-O stratifications as follows.

Proposition 3.4.4. *We have a commutative diagram of smooth morphisms*

$$\begin{tikzcd}
\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1) \arrow[r, "\zeta_1"] \arrow[d, "f"] & {[E_{G_1,\mu} \backslash G_{1,\kappa}] \otimes \overline{\kappa}} \arrow[d, "f_*"] \\
\mathscr{S}_{K_p,\overline{\kappa}}(G, X) \arrow[r, "\zeta_2"] & {[E_{G,\mu} \backslash G_\kappa] \otimes \overline{\kappa}}
\end{tikzcd}$$

Proof. The morphism $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1) \to \mathscr{S}_{K_p,\overline{\kappa}}(G, X)$ is étale. Smoothness of ζ_1 (resp. f_*) follows from Theorem 3.3.4 (1) (resp. Lemma 3.4.1). We only need to show how to construct $\zeta_2 : \mathscr{S}_{K_p,\overline{\kappa}}(G, X) \to [E_{G,\mu} \backslash G_\kappa] \otimes \overline{\kappa}$ and why it is smooth.

We use notations as in 1.3.4. Let $\mathcal{D}$ be the p-divisible group over $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$, $\mathcal{D}[p]$ gives a $G_{1,\kappa}$-zip, and hence a G_κ-zip over $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)$. For $\gamma \in G(\mathbb{Z}_{(p)})^+$, we write $\mathcal{P}$ for the fiber in $G_{1,\mathbb{Z}_p}$ of γ viewed as an element in $G(\mathbb{Z}_p)$. It is a trivial torsor under the center of $G_{1,\mathbb{Z}_p}$. For $\widetilde{\gamma} \in \mathcal{P}(\mathbb{Z}_p)$, the isomorphism $\widetilde{\gamma} : \mathcal{D}^{\mathcal{P}}[p] \to \mathcal{D}[p]$ induces an isomorphism of $G_{1,\kappa}$-zips, which depends only on γ (i.e. is independent of choices of $\widetilde{\gamma}$) when passing to G_κ-zips. But this means that the G_κ-zip attached to $\mathcal{D}[p]$ on $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)^+$ descends to $\mathscr{S}_{K_p,\overline{\kappa}}(G, X)^+$, and hence induces a morphism $\mathscr{S}_{K_p,\overline{\kappa}}(G, X)^+ \to [E_{G,\mu}\backslash G_\kappa] \otimes \overline{\kappa}$ which is necessarily smooth. Putting together these morphisms on geometrically connected components, we get ζ_2 which is necessarily smooth. $\square$

Remark 3.4.5. By 5.3.3, ζ_2 is actually defined over κ, the field of definition of $\mathscr{S}_{K_p,0}(G, X)$.

3.4.6. Now consider general Shimura varieties of abelian type. Let (G, X) be a Shimura datum of abelian type (*not adjoint in general*) with good reduction at p. Let $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ be its adjoint Shimura datum, and (G_1, X_1) be a Shimura datum of Hodge type satisfying the two conditions in Lemma 2.3.2 with respect to $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$.

By the previous discussions, we have a commutative diagram

$$\begin{tikzcd}
 & \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1) \arrow[r] \arrow[d] & {[E_{G_1,\mu}\backslash G_{1,\kappa}] \otimes \overline{\kappa}} \arrow[d, "\simeq"] & \\
\mathscr{S}_{K_p,\overline{\kappa}}(G, X) \arrow[r] & \mathscr{S}_{K_p^{\mathrm{ad}},\overline{\kappa}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \arrow[r] & {[E_{G^{\mathrm{ad}},\mu}\backslash G^{\mathrm{ad}}_\kappa] \otimes \overline{\kappa}} & {[E_{G,\mu}\backslash G_\kappa] \otimes \overline{\kappa}.} \arrow[l, "\simeq"']
\end{tikzcd}$$

Now we can imitate the main results in Hodge type cases. Fix a sufficiently small open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$. Let us simply write $\mathscr{S}_{\overline{\kappa}} = \mathscr{S}_{K,\overline{\kappa}}(G, X)$, and

$$\zeta : \mathscr{S}_{\overline{\kappa}} \to [E_{G,\mu}\backslash G_\kappa] \otimes \overline{\kappa}.$$

Theorem 3.4.7. (1) *The morphism ζ is smooth and surjective. In particular,*
  (a) *each stratum is a non-empty, smooth and locally closed subscheme of $\mathscr{S}_{\overline{\kappa}}$, the closure of a stratum is a union of strata;*
  (b) *all the strata are in bijection with ${}^JW$, and for $w \in {}^JW$, the corresponding stratum is of dimension $l(w)$, the length of w.*

(2) *Each E-O stratum is quasi-affine.*

Proof. Noting that $\mathscr{S}_{K,0}(G, X) \to \mathscr{S}_{K^{\mathrm{ad}},0}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is étale, the smoothness of ζ is a direct consequence of the previous proposition. All the other statements follow by combining Theorem 3.3.4 with Proposition 3.4.3. $\square$

Recall by Remark 3.2.2, there is a unique maximal length element $w_\mu \in {}^JW$. We call the associated open E-O stratum the ordinary E-O stratum. By the above closure relation, it is dense in $\mathscr{S}_{\overline{\kappa}}$.

Corollary 3.4.8. *The μ-ordinary locus in $\mathscr{S}_{\overline{\kappa}}$ coincides with the ordinary E-O stratum. In particular, the μ-ordinary locus is open dense.*

Proof. In the Hodge type case, this follows from [51] Theorem 6.10. The abelian type case follows from the Hodge type case by our construction. $\square$

Thus for a Shimura datum (G, X) of abelian type with good reduction at $p > 2$, we have the Ekedahl-Oort stratification on the geometric special fiber $\mathscr{S}_{\overline{\kappa}}$ of $\mathscr{S}_K(G, X)$

$$\mathscr{S}_{\overline{\kappa}} = \coprod_{w \in {}^JW} \mathscr{S}_{\overline{\kappa}}^w, \quad \overline{\mathscr{S}_{\overline{\kappa}}^w} = \coprod_{w' \preceq w} \mathscr{S}_{\overline{\kappa}}^{w'}.$$

As in Remark 3.2.2, there is a unique closed (minimal) stratum $\mathscr{S}_{\overline{\kappa}}^{w_0}$ (the superspecial locus), associated to the element $w_0 = 1 \in {}^JW$; there is also a unique open (maximal) stratum $\mathscr{S}_{\overline{\kappa}}^{w_\mu}$ (the ordinary locus), associated to the maximal element $w_\mu \in {}^JW$.

Example 3.4.9. Notations as in 1.2.7, but we will write G for the special fiber and W for its Weyl group. Then $W \cong (\mathbb{Z}/2\mathbb{Z})^n$, ${}^JW \cong (\mathbb{Z}/2\mathbb{Z})^d$ and $W_J \cong (\mathbb{Z}/2\mathbb{Z})^{n-d}$, and the partial order $\preceq$ on ${}^JW$ is the Bruhat order. More explicitly, for $w, w' \in {}^JW$, $w \preceq w'$ if and only if w is obtained from w' by changing some of the 1 to 0. The dimension of $\mathscr{S}_{\overline{\kappa}}^{w}$ is the number of 1s in w. In particular, for $0 \leq i \leq d$, there are $\binom{d}{i}$ strata of dimension i. We refer the reader to [43] for some related (more refined) construction for these quaternionic Shimura varieties.

4. Central leaves

In this section, we consider a refinement for both the Newton and the Ekedahl-Oort stratifications studied in the previous two sections.

4.1. **Central leaves on Shimura varieties of Hodge type.** Central leaves were first introduced and studied by Oort in the Siegel case, cf. [35]. They were generalized by Mantovan in the PEL type case in [31] and independently by P. Hamacher ([13]) and C. Zhang ([54]) in the Hodge type case.

Notations as in 1.1, for $z \in \mathscr{S}_K(G,X)(\overline{\kappa})$, we simply write D_z for $\mathbb{D}(\mathcal{A}_z[p^\infty])(W)$, here $W = W(\overline{\kappa})$. We will also write $L = W[1/p]$ as in 2.1. Two points $x, y \in \mathscr{S}_K(G,X)(\overline{\kappa})$ are said to be in the same *central leaf* if there exists an isomorphism of Dieudonné modules $D_x \to D_y$ mapping $s_{\mathrm{cris},x}$ to $s_{\mathrm{cris},y}$. It is clear from the definition that the $\overline{\kappa}$-points of a Ekedahl-Oort stratum (resp. Newton stratum) is a union of central leaves. We can also define *classical central leaves* by putting together $\overline{\kappa}$-points with isomorphic Dieudonné modules. Each classical central leaf is locally closed in $\mathscr{S}_{K,\overline{\kappa}}(G,X)$.

Let $C(G,\mu)$ and $B(G,\mu)$ be as at the beginning of 2.1. For $x \in \mathscr{S}_K(G,X)(\overline{\kappa})$, choosing an isomorphism $t : V_{\mathbb{Z}_p}^\vee \otimes W \to D_x$ mapping s to $s_{\mathrm{cris},x}$, we get a Frobenius on $V_{\mathbb{Z}_p}^\vee \otimes W$ whose linearization $g_{x,t}$ is of form $G(W)\mu(p)G(W)$. Moreover, changing t to another isomorphism $V_{\mathbb{Z}_p}^\vee \otimes W \to D_x$ mapping s to $s_{\mathrm{cris},x}$ amounts to $G(W)$-σ-conjugacy of $g_{x,t}$. So we have a well defined map

$$\mathscr{S}_K(G,X)(\overline{\kappa}) \to C(G,\mu)$$

whose fibers are central leaves. The composition

$$\mathscr{S}_K(G,X)(\overline{\kappa}) \to C(G,\mu) \to B(G,\mu)$$

has Newton strata as fibers.

We denote by $\mathscr{S}_0$ the special fiber of $\mathscr{S}_K(G,X)$, and by $\nu_G(-)$ the Newton map. For $[b] \in B(G,\mu)$ (resp. $[c] \in C(G,\mu)$), we write $\mathscr{S}_{\overline{\kappa}}^b$ (resp. $\mathscr{S}_{\overline{\kappa}}^c$) for the corresponding Newton stratum (resp. central leaf). The main results for central leaves on Shimura varieties are as follows.

Theorem 4.1.1. *For $[c] \in C(G,\mu)$, $\mathscr{S}_{\overline{\kappa}}^c$ is a smooth, equi-dimensional locally closed subscheme of $\mathscr{S}_{\overline{\kappa}}$. It is open and closed in the classical central leaf containing it, and closed in the Newton stratum containing it. Any central leaf in a Newton stratum $\mathscr{S}_{\overline{\kappa}}^b$ is of dimension $\langle 2\rho, \nu_G(b)\rangle$. Here ρ is the half sum of positive roots.*

Proof. The non-emptiness of $\mathscr{S}_{\overline{\kappa}}^c$ follows from non-emptiness of Newton strata and [19] Proposition 1.4.4. All other statements are proved in [13] and [54] respectively, using different methods. □

When the prime to p level K^p varies, by construction the central leaves $\mathscr{S}^c_{K_pK^p,\overline{\kappa}}$ are invariant under the prime to p Hecke action. In this way we get also the central leaves on $\mathscr{S}_{K_p,\overline{\kappa}} = \varprojlim_{K^p} \mathscr{S}_{K_pK^p,\overline{\kappa}}$.

4.2. **Central leaves on Shimura varieties of abelian type.** We now explain how to define central leaves on a Shimura varieties of abelian type. As what did before, we should start with a group theoretic result which says that if one wants to use $C(G,\mu)$ to parameterize all central leaves, then he could pass to the adjoint group freely. But due to technical difficulties, we can only prove the following special case.

Lemma 4.2.1. *Let $f : G \to H$ be a central isogeny of reductive groups over $\mathbb{Z}_p$ with connected kernel, and μ be a cocharacter of G defined over $W(\kappa)$ with κ finite. Then the map $f_* : C(G,\mu) \to C(H,\mu)$ is a bijection.*

Proof. Let W be $W(\overline{\kappa})$ and L be $W[1/p]$ as before. To see that f_* is surjective, noting that any element in $C(H,\mu)$ has a representative in $G(L)$ of form $h\mu(p)$ with $h \in H(W)$, it suffices to show that $G(W) \to H(W)$ is surjective. But f is smooth as it has connected center, so $G(W) \to H(W)$ is surjective as it is so for $\overline{\kappa}$-points.

Now we prove that f_* is injective. Assume that $g_1\mu(p), g_2\mu(p) \in G(L)$ has the same image in $C(H,\mu)$, then there is $h \in H(W)$ such that $h^{-1}\overline{g_1}\mu(p)\sigma(h) = \overline{g_2}\mu(p)$ in $H(L)$. Here for $i = 1,2$, $\overline{g_i}$ is the image in $H(W)$ of g_i. Take $g \in G(W)$ mapping to h, then $g^{-1}g_1\mu(p)\sigma(g)\mu(p)^{-1}g_2^{-1} = z \in Z(W)$, here $Z = \ker(f)$ is a torus by assumption. We rewrite the equation as $g^{-1}g_1\mu(p)\sigma(g) = zg_2\upsilon(p)$, to prove that f_* is injective, it suffices to show that $z = t^{-1}\sigma(t)$ for some $t \in Z(W)$.

Noting that Z splits over an unramified extension and we are working with W-points, we can assume that $Z = \mathbb{G}_{m,W}$. Consider the equation $\sigma(x) = xy$. Writing $x = (x_0, x_1, \cdots)$ and $y = (y_0, y_1 \cdots)$ as Witt vectors, the equation becomes $(x_0^p, x_1^p, \cdots) = (x_0, x_1, \cdots)(y_0, y_1 \cdots)$. The multiplication on the right is given by a polynomial P_n of degree p^n with the assignment $\deg(x_i) = \deg(y_i) = p^i$, so for given $(x_0, x_1, \cdots, x_{n-1})$ and $(y_0, y_1 \cdots, y_n)$, $x_n^p - P_n(x,y) = 0$ is of form $x_n^p + a_1x_n + a_0 = 0$, and hence always has solution in k. But $x_0^p = x_0y_0$ has a non-zero solution for any $y_0 \neq 0$, so by induction, $\sigma(x) = xy$ has a solution in $W^\times$ for any $y \in W^\times$. $\square$

4.2.2. Let (G,X) be a *adjoint* Shimura datum of abelian type with good reduction at p, and (G_1,X_1) be a Shimura datum of Hodge type satisfying the two conditions in Lemma 2.3.2. Then the center of $G_{1,\mathbb{Z}_{(p)}}$ is a torus.

By the last subsection, we have central leaves on $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)$. We can restrict them to $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+$ and then extend them trivially to $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+$. We will sometimes call these the *induced central leaves* on $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+$. Similarly for $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+$.

Proposition 4.2.3. *Each induced central leaf on*

$$\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+$$

(resp. $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+$) is $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ$-stable. Moreover, induced central leaves on $\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+$ descend to central leaves on $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)$.

Proof. The proof is identical to that of Proposition 2.3.4. $\square$

The central leaves of $\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+$ descend to locally closed subschemes of $\mathscr{S}_{K_p,\overline{\kappa}}(G,X)$, and we will call them central leaves of $\mathscr{S}_{K_p,\overline{\kappa}}(G,X)$. More formally, we have the following formulas:

$$\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^c = [\mathscr{A}(G_{1,\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^{+,c}]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ,$$
$$\mathscr{S}_{K_p,\overline{\kappa}}(G,X)^c = [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_p,\overline{\kappa}}(G,X)^{+,c}]/\mathscr{A}(G_{\mathbb{Z}_{(p)}})^\circ.$$

Moreover, we have

$$\mathscr{S}_{K_p,\overline{\kappa}}(G,X)^{+,c} = \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^{+,c}/\Delta,$$
$$\mathscr{S}_{K_p,\overline{\kappa}}(G,X)^c = [\mathscr{A}(G_{\mathbb{Z}_{(p)}}) \times \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^{+,c}]/\mathscr{A}(G_{1,\mathbb{Z}_{(p)}})^\circ.$$

The proposition also indicates how to relate central leaves to the group theoretic object $C(G,\mu)$. For $x \in \mathscr{S}_{K_p,\overline{\kappa}}(G,X)(\overline{\kappa})$, we can find $x_0 \in \mathscr{S}_{K_p,\overline{\kappa}}(G,X)^+(\overline{\kappa})$ which is in the same central leaf as x. Noting that x_0 lifts to $\widetilde{x_0} \in \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)^+(\overline{\kappa})$ whose image in $C(G,\mu)$ depends only on x, we get a well defined map $\mathscr{S}_{K_p,\overline{\kappa}}(G,X)(\overline{\kappa}) \to C(G,\mu)$ whose fibers are central leaves of $\mathscr{S}_{K_p,\overline{\kappa}}(G,X)$.

4.2.4. Now we can pass to general Shimura varieties of abelian type. Let (G,X) be a Shimura datum of abelian type (*not adjoint in general*) with good reduction at p. Let $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ be its adjoint Shimura datum, and (G_1,X_1) be a Shimura datum of Hodge type satisfying the two conditions in Lemma 2.3.2 with respect to $(G^{\mathrm{ad}},X^{\mathrm{ad}})$.

By Lemma 4.2.1, we have $C(G_1,\mu_1) \cong C(G^{\mathrm{ad}},\mu)$, and by the above discussions, we have a commutative diagram

$$\begin{tikzcd}
 & \mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)(\overline{\kappa}) \arrow[r] \arrow[d] & C(G_1,\mu_1) \arrow[d, "\simeq"'] \\
\mathscr{S}_{K_p,\overline{\kappa}}(G,X)(\overline{\kappa}) \arrow[r] & \mathscr{S}_{K_p^{\mathrm{ad}},\overline{\kappa}}(G^{\mathrm{ad}},X^{\mathrm{ad}})(\overline{\kappa}) \arrow[r] & C(G^{\mathrm{ad}},\mu).
\end{tikzcd}$$

Here as in 2.3.5, we use the same notation when viewing μ (resp. μ_1) as a cocharacter of G^{ad}, and identify $B(G^{\mathrm{ad}},\mu)$ with $B(G^{\mathrm{ad}},\mu_1)$ silently.

Now we can imitate the main results in Hodge type cases. We fix a prime to p level K^p. Let $\mathscr{S}_0$ be the special fiber of $\mathscr{S}_K(G,X)$, and by $\nu_G(-)$ the Newton map. For $[b] \in B(G,\mu) \simeq B(G^{\mathrm{ad}},\mu)$ (resp. $[c] \in C(G^{\mathrm{ad}},\mu)$), we write $\mathscr{S}^b_{\overline{\kappa}}$ (resp. $\mathscr{S}^c_{\overline{\kappa}}$) for the corresponding Newton stratum (resp. central leaf).

Theorem 4.2.5. *Each central leaf is a smooth, equi-dimensional locally closed subscheme of $\mathscr{S}_{\overline{\kappa}}$. It is closed in the Newton stratum containing it. Any central leaf in a Newton stratum $\mathscr{S}^b_{\overline{\kappa}}$ is of dimension $\langle 2\rho, \nu_G(b)\rangle$. Here ρ is the half sum of positive roots.*

Proof. For $\mathscr{S}_0(G^{\mathrm{ad}},X)$ and $[b] \in B(G^{\mathrm{ad}},\mu)$, the statement follows by combining Theorem 4.1.1 with Proposition 4.2.3. But then the general case follows by noting that $\mathscr{S}_0(G,X) \to \mathscr{S}_0(G^{\mathrm{ad}},X^{\mathrm{ad}})$ is finite étale. □

Example 4.2.6. Notations as in 2.3.8. For $[b] \in B(G,\mu)$, its projection to $B(G_{\mathfrak{p}_i},\mu_i)$ is of form $(\frac{\lambda_1}{n_i},\frac{\lambda_2}{n_i})$ with $\lambda_1 \geq \lambda_2$, $\lambda_1+\lambda_2 = a_i$ and these λ_i are integers unless $\lambda_1 = \lambda_2$. Let $c_i(b) := \lambda_1 - \lambda_2$, then central leaves in $\mathscr{S}^b_{\overline{\kappa}}$ are smooth varieties of dimension $\sum_{i=1}^s c_i(b)$.

5. Filtered F-crystals with G-structure and stratifications

In this section, we will revist the Newton stratification, the Ekedahl-Oort stratification, and the central leaves on Shimura varieties of abelian type studied previously from the point of view of p-adic Hodge theory.

5.1. **Filtered F-crystals with G-structure.** For a scheme X, we will write Bun_X for the groupoid of vector bundles (of finite rank) over X. By a *filtration* $\mathrm{Fil}^\bullet$ on a vector bundle N/X, we mean a separating exhaustive descending filtration such that Fil^{i+1} is a locally direct summand of Fil^i. The groupoid of vector bundles over X with a filtration is denoted by Fil_X. Both Bun_X and Fil_X are rigid exact tensor categories.

5.1.1. *G-bundles and filtered G-bundles.* Let G be an fppf affine group scheme over $S = \operatorname{Spec} R$. We write $\mathrm{Rep}_R(G)$ for the category of algebraic representations of G taking values in finite projective R-modules. Let X be a scheme which is faithfully flat over S. By a *G-bundle* on X, we mean a faithful exact R-linear tensor functor $\mathrm{Rep}_R(G) \to \mathrm{Bun}_X$. By a *filtered G-bundle* on X, we mean a faithful exact R-linear tensor functor $\mathrm{Rep}_R(G) \to \mathrm{Fil}_X$.

For simplicity, we assume that $R = \mathbb{Z}_p$ and G is reductive from now on.

By [1] Theorem 1.2, to give a G-bundle on X is the same as to give a G-torsor on X. As explained in [28] 2.2.8, by putting together Propositions 2.1.5 and 2.2.7 of loc. cit., we find that to give a filtered G-bundle on X is the same as to give a G-torsor I/X together with a G-equivariant morphism $I \to \mathcal{P}$. Here $\mathcal{P}$ is the scheme of parabolic subgroups of G.

One can also talk about the *type* of a filtered G-bundle. More precisely, we fix the type τ of a conjugacy class of parabolic subgroups in G, it is defined over a finite étale extension A of R. Assume that the structure map $X \to S = \operatorname{Spec} R$ factors through $\operatorname{Spec} A$. Then a filtered G-bundle is said to be of type τ if the associated morphism $I \to \mathcal{P}$ factors through $\mathcal{P}^\tau$. Here $\mathcal{P}^\tau \subseteq \mathcal{P}$ is the subscheme of parabolic subgroups of G. It is smooth over A with geometrically connected fibers.

5.1.2. *The functor* R. For $(N, \mathrm{Fil}^\bullet) \in \mathrm{Fil}_X$, we define

$$\mathrm{R}(N) := \sum_i p^{-i}\mathrm{Fil}^i \subseteq N[p^{-1}].$$

By the proof of [28] Proposition 2.1.5, $\mathrm{R}(-)$ is an exact tensor functor from Fil_X to Bun_X.

5.1.3. *Filtered F-crystals.* Let k be a finite field and $Y/W(k)$ be a smooth scheme. We denote by Bun_Y^∇ (resp. Fil_Y^∇) the category of vector bundles on Y with integrable connection (resp. filtered vector bundles on Y with integrable connection satisfying the Griffiths transversality). Let $K = W(k)[1/p]$, X be the formal scheme obtained by p-adic completion of Y, and X_K be the rigid generic fibre over $\mathrm{Spa}(K, W)$. We write Bun_X^∇ (resp. $\mathrm{Bun}_{X_K}^\nabla$, Fil_X^∇, $\mathrm{Fil}_{X_K}^\nabla$) for the similar category but with the condition that ∇ is topologically quasi-nilpotent. An object in Bun_X^∇ (resp. $\mathrm{Bun}_{X_K}^\nabla$, Fil_X^∇, $\mathrm{Fil}_{X_K}^\nabla$) is called a crystal (resp. an isocrystal, a filtered crystal, a filtered isocrystal).

Let $U \subseteq X$ be open affine, and σ_U be a lift of the Frobenius on the special fiber of U. An *F-isocrystal* is an isocrystal M/X_K together with for each pair (U, σ_U) an isomorphism $\varphi_{\sigma_U} : \sigma_U^* M_U \to M_U$, such that the φ_{σ_U} are horizontal with respect to the natural connections on both sides, and that the composition

$$\sigma_U^* M_{U\cap U'} \xrightarrow{\varphi_{\sigma_U}} M_{U\cap U'} \xleftarrow{\varphi_{\sigma_{U'}}} \sigma_{U'}^* M_{U\cap U'}$$

is the natural isomorphism induced by the connection ∇. One can define an *F-crystal* to be a "lattice" of an F-isocrystal. More precisely, it is an F-isocrystal M/X_K together with a crystal N/X and an identification $N[1/p] \cong M$. The category of F-isocrystals (resp. F-crystals) over X is denoted by $\mathrm{FIsoCrys}_{X_K}$ (resp. FCrys_X). We have a natural functor $\mathrm{FCrys}_X \to \mathrm{FIsoCrys}_{X_K}$.

A *filtered F-crystal* on X is then a filtered crystal $(M, \mathrm{Fil}^\bullet, \nabla) \in \mathrm{Fil}_X^\nabla$ together with for each pair (U, σ_U) as above a horizontal isomorphism

$$\varphi_U : \mathrm{R}(\sigma_U^* M_U) \to M_U$$

which forms an isocrystal after inverting p. Here $\mathrm{R}(\sigma_U^* M_U)$ as in 5.1.2 is canonically a submodule of $\sigma_U^*(M_{U[p^{-1}]})$, and is equipped with a canonical flat connection by [6] Page 34. In particular, the words "horizontal" and "isocrystal" make sense. The category of filtered F-crystals on X is denoted by FFCrys_X. Similarly we have the category of filtered F-isocrystals $\mathrm{FFIsoCrys}_{X_K}$. There is an obvious commutative diagram

$$\begin{tikzcd} \mathrm{FFCrys}_X \arrow[r] \arrow[d] & \mathrm{FFIsoCrys}_{X_K} \arrow[d] \\ \mathrm{FCrys}_X \arrow[r] & \mathrm{FIsoCrys}_{X_K}. \end{tikzcd}$$

A *filtered F-crystal with G-structure* is then a $\mathbb{Z}_p$-linear exact tensor functor

$$\omega : \mathrm{Rep}_{\mathbb{Z}_p}(G) \to \mathrm{FFCrys}_X.$$

Similarly, a *filtered F-crystal with G-structure* is then a $\mathbb{Q}_p$-linear exact tensor functor

$$\omega : \mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathrm{FFIsoCrys}_{X_K}.$$

These objects can be equivalently defined as filtered G-bundles with flat topologically quasi-nilpotent connection and certain further structures, for more details, see [28] 2.4.7 and 2.4.9.

5.2. **Filtered F-crystals on Shimura varieties.** Notations an assumptions as in 1.2.5. We will write $\widehat{\mathscr{S}_{K_p}}$ for the p-adic completion of the integral canonical model $\mathscr{S}_{K_p} := \varprojlim_{K^p} \mathscr{S}_K(G,X)$. This is a formal scheme over $O_{E_v} = W(\kappa)$ which is formally smooth. Its generic fiber, as an adic space over $\mathrm{Spa}(E_v, O_{E_v})$, is still denoted by $\mathrm{Sh}_{K_p}(G,X)$. We will sometimes simply write Sh_{K_p} for it.

5.2.1. Let $Z_{nc} \subseteq Z_G$ be the largest subtorus of Z_G that is split over $\mathbb{R}$ but anisotropic over $\mathbb{Q}$, and set $G^c = G/Z_{nc}$. If (G,X) is a Shimura datum of Hodge type, then we have $G = G^c$. Let $G_{\mathbb{Z}_p}$ (resp. $G^c_{\mathbb{Z}_p}$) be the reductive model of $G_{\mathbb{Q}_p}$ (resp. $G^c_{\mathbb{Q}_p}$). We will write $\mathrm{Rep}_{\mathbb{Q}_p}(G)$ (resp. $\mathrm{Rep}_{\mathbb{Z}_p}(G)$) for the category of algebraic representations of $G_{\mathbb{Q}_p}$ (resp. $G_{\mathbb{Z}_p}$) taking values in finite dimensional $\mathbb{Q}_p$-vector spaces (finite free $\mathbb{Z}_p$-modules). Similarly for G^c.

By [26] page 340-341, the pro-Galois $G^c(\mathbb{Z}_p)$-cover $\mathrm{Sh}(G,X) \to \mathrm{Sh}_{K_p}(G,X)$ gives a $\mathbb{Z}_p$-linear faithful exact tensor functor

$$\omega_{\text{ét}} : \mathrm{Rep}_{\mathbb{Z}_p}(G^c) \to \mathrm{Lisse}_{\mathbb{Z}_p}(\mathrm{Sh}_{K_p}),$$

which induces a $\mathbb{Q}_p$-linear tensor functor

$$\omega_{\text{ét},\mathbb{Q}_p} : \mathrm{Rep}_{\mathbb{Q}_p}(G^c) \to \mathrm{Lisse}_{\mathbb{Q}_p}(\mathrm{Sh}_{K_p}).$$

By the main theorem in [26] of Liu and Zhu, it is de Rham and thus by comparision theoerem it extends to a functor

$$\omega_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}(G^c) \to \mathrm{Fil}^{\nabla}_{\mathrm{Sh}_{K_p}}.$$

This ω_{dR} factors via $\mathrm{Rep}_{E_v}(G^c_{E_v}) \to \mathrm{Fil}^{\nabla}_{\mathrm{Sh}_{K_p}}$ which defines a filtered G^c-bundle I_{E_v} with flat connection on Sh_{K_p}. Liu and Zhu conjecture (see [26] Remark 4.1 (ii)) that this should agree with the analytification of the canonical model of the automorphic vector constructed by Milne in the case when $Z(G)^\circ$ is split by a CM field. By using the theory of abelian motives, this is true in the abelian type case (compare [28] 3.1.3.).

5.2.2. Lovering constructs in [28] a certain filtered F-crystal with $G^c_{\mathbb{Z}_p}$-structure over $\widehat{\mathscr{S}_{K_p}}$ whose underlying filtered isocrystal on the generic fiber is ω_{dR}. Lovering calls it the "crystalline canonical model" of ω_{dR} (or I_{E_v}). It is characterized by a CPLF condition (means "crystalline points lattice + Frobenius", see [28] 3.1.5 for the precisely definition). Roughly speaking, this condition is imposed to ensure that one can have certain integral crystalline comparison theorem between $\omega_{\text{ét}}$ and ω_{cris} (see below). By [28] Proposition 3.1.6, crystalline canonical model, if exists, is unique up to isomorphism. We will write

$$\omega_{\mathrm{cris}}:\mathrm{Rep}_{\mathbb{Z}_p}(G^c)\to \mathrm{FFCrys}_{\widehat{\mathscr{S}_{K_p}}},$$

and sometimes I, for the crystalline canonical model of ω_{dR}.

By [27] Lemma 3.1.3, a morphism $(G,X)\to(G',X')$ of Shimura data induces a homomorphism $G^c\to G'^c$. If moreover, it comes from a morphism of reductive group schemes $G_{\mathbb{Z}_{(p)}}\to G'_{\mathbb{Z}_{(p)}}$, we have a natural homomorphism $G^c_{\mathbb{Z}_{(p)}}\to G'^c_{\mathbb{Z}_{(p)}}$.

Theorem 5.2.3. *(*[28] *3.4.8, Proposition 3.1.6)*

1. *If (G,X) is of abelian type, then the crystalline canonical model of ω_{dR} exists.*
2. *Let $f:(G,X)\to(G',X')$ be a morphism of Shimura data of abelian type induced by a homomorphism $G_{\mathbb{Z}_{(p)}}\to G'_{\mathbb{Z}_{(p)}}$ of reductive groups over $\mathbb{Z}_{(p)}$, and I (resp. I') be the crystalline canonical model over $\widehat{\mathscr{S}_{K_p}}$ (resp. $\widehat{\mathscr{S}'_{K'_p}}$). Then we have a canonical isomorphism $I\times^{G^c_{\mathbb{Z}_p}}G'^c_{\mathbb{Z}_p}\cong f^*I'$ of filtered F-crystals over $\widehat{\mathscr{S}_{K_p}}$ with $G'^c_{\mathbb{Z}_p}$-structure.*

Remark 5.2.4. Notations as in the above theorem. The morphism $I\times^{G^c_{\mathbb{Z}_p}}G'^c_{\mathbb{Z}_p}\cong f^*I'$ in (2) is stated in [28] Proposition 3.1.6 (2) as an isomorphism of *weak* filtered F-crystals with $G'^c_{\mathbb{Z}_p}$-structure. But $I\times^{G^c_{\mathbb{Z}_p}}G'^c_{\mathbb{Z}_p}$ given by

$$\mathrm{Rep}_{\mathbb{Z}_p}(G'^c)\to\mathrm{Rep}_{\mathbb{Z}_p}(G^c)\to\mathrm{FFCrys}_{\widehat{\mathscr{S}_{K_p}}}$$

is by definition a filtered F-crystal with $G'^c_{\mathbb{Z}_p}$-structure, and hence f^*I' is a filtered F-crystal with $G'^c_{\mathbb{Z}_p}$-structure. It is in general difficult to determine whether the base-change of a filtered F-crystal is again a filtered F-crystal.

Remark 5.2.5. Notations as above. Let τ be a type of parabolic subgroups of $G_{\mathbb{Z}_p}$ defined over $W(\kappa)$. Then a filtered F-crystal with $G^c_{\mathbb{Z}_p}$-structure (over $\widehat{\mathscr{S}_{K_p}}$) is said to be of type τ if its underly filtered $G^c_{\mathbb{Z}_p}$-bundle is of type τ. Here we view τ as a type of parabolic subgroups of $G^c_{\mathbb{Z}_p}$. The crystalline canonical model ω_{cris} of ω_{dR} is of type μ. Here we write μ for the type of $P_+\subseteq G^c_{W(\kappa)}$ where μ is viewed as a cocharacter of $G^c_{W(\kappa)}$.

5.3. **Stratifications via filtered F-crystals.** We will explain in this section, how to define and study stratifications on Shimura varieties of abelian type using the filtered F-crystal with $G^c_{\mathbb{Z}_p}$-structure ω_{cris}. The good point is that, this filtered F-crystal with $G^c_{\mathbb{Z}_p}$-structure is intrinsically determined by the Shimura datum, and once we can define stratifications using it, they will be automatically intrinsically determined by the Shimura datum.

5.3.1. Let A be the p-adically completion of a formally smooth $W(\kappa)$-algebra, and σ be a lifting of the Frobenius of $A_0:=A\otimes_{W(\kappa)}\kappa$. It is well known that an F-isocrystal (resp. F-crystal) over A depends only on A_0 up to isomorphism. We will simply call an F-isocrystal (resp. F-crystal) over A (or equivalently, over A_0) an F-isocrystal (resp. F-crystal), and the corresponding category is denoted by $\mathrm{FIsoCrys}_{A_0}$ (resp. FCrys_{A_0}).

Let $\omega_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Z}_p}(G^c) \to \mathrm{FFCrys}_{\widehat{\mathscr{S}_{K_p}}}$ be the filtered F-crystal with $G^c_{\mathbb{Z}_p}$-structure over $\widehat{\mathscr{S}_{K_p}}$, by forgetting the filtrations, we get a faithful exact tensor functor

$$\omega : \mathrm{Rep}_{\mathbb{Z}_p}(G^c) \to \mathrm{FCrys}_{\mathscr{S}_{K_p,0}}.$$

Now we can define stratifications on $\mathscr{S}_{K_p,0}$. We will define Newton strata and central leaves pointwise first using ω, and then define Ekedahl-Oort strata using F-zips. For $x \in \mathscr{S}_{K_p,0}(\overline{\kappa})$, pulling back the F-crystal with $G^c_{\mathbb{Z}_p}$-structure ω over $\mathscr{S}_{K_p,0}$ to x induces an F-crystal with $G^c_{\mathbb{Z}_p}$-structure over $\overline{\kappa}$, i.e. a faithful exact $\mathbb{Z}_p$-linear functor $\omega_x : \mathrm{Rep}_{\mathbb{Z}_p}(G^c) \to \mathrm{FCrys}_{\overline{\kappa}}$. Passing to isocrystals, we get an F-isocrystal with $G^c_{\mathbb{Q}_p}$-structure, i.e. an exact $\mathbb{Q}_p$-linear functor $\omega_{x,\mathbb{Q}_p} : \mathrm{Rep}_{\mathbb{Q}_p}(G^c_{\mathbb{Q}_p}) \to \mathrm{FIsoCrys}_{\overline{\kappa}}$.

Definition 5.3.2. Two points $x, y \in \mathscr{S}_{K_p,0}(\overline{\kappa})$ are said to be in the same *central leaf* if the F-crystals with $G^c_{\mathbb{Z}_p}$-structure ω_x and ω_x are isomorphic. They are said to be in the same *Newton stratum* if the F-isocrystals with $G^c_{\mathbb{Q}_p}$-structure $\omega_{x,\mathbb{Q}_p}$ and $\omega_{x,\mathbb{Q}_p}$ are isomorphic.

We will now indicate how to relate these stratifications to group theoretic constructions defined in 2.2.1. Let υ be the cocharacter of $G_{W(\kappa)}$ with the induced cocharacter of $G^c_{W(\kappa)}$ denoted by the same notation. For $x \in \mathscr{S}_{K_p,0}(\overline{\kappa})$ with a lift $\widetilde{x} \in \mathscr{S}_{K_p}(W(\overline{\kappa}))$, the torsor $I_{\widetilde{x}}$ is trivial, and we can take $t \in I_{\widetilde{x}}(W(\overline{\kappa}))$ such that the filtration in the filtered F-crystal is induced by υ. For a representation $G^c_{\mathbb{Z}_p} \to \mathrm{GL}(L)$, $I_{\widetilde{x}}$ gives a filtered F-crystal structure on $L_{W(\overline{\kappa})}$, and the Frobenius φ is of form $g\upsilon(p)$, where $g \in \mathrm{GL}(L)(W(\overline{\kappa}))$ is the composition

$$L_{W(\overline{\kappa})} \xrightarrow{\xi} L^{\sigma}_{W(\overline{\kappa})} \xrightarrow{\upsilon(p)^{-1}} \mathrm{R}(L^{\sigma}_{W(\overline{\kappa})}) \xrightarrow{\varphi} L_{W(\overline{\kappa})} .$$

Here we use the filtration induced by υ to construct $\mathrm{R}(L^{\sigma}_{W(\overline{\kappa})})$, and the isomorphism $\xi : L_{W(\overline{\kappa})} \to L^{\sigma}_{W(\overline{\kappa})}$ is given by $l \otimes k \mapsto l \otimes 1 \otimes k$. Let $s \in L^{\otimes}$ be a tensor fixed by $G^c_{\mathbb{Z}_p}$, then it is also in $\mathrm{R}(L^{\otimes}_{W(\overline{\kappa})})$, and such that $\varphi(s) = s$. In paticular, $g \in G^c_{\mathbb{Z}_p}(W(\overline{\kappa}))$, and the assignment $x \mapsto \sigma^{-1}(g)$ gives well defined maps $\mathscr{S}_{K_p,0}(\overline{\kappa}) \to C(G^c, \mu)$ and $\mathscr{S}_{K_p,0}(\overline{\kappa}) \to B(G^c, \mu)$. The fibers of there maps are central leaves and Newton strata respectively.

5.3.3. We now explain how to define Ekedahl-Oort stratification. Unlike for central leaves or Newton strata, we can work directly with families using [33] Example 7.3. Let $\omega_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Z}_p}(G^c) \to \mathrm{FFCrys}_{\widehat{\mathscr{S}_{K_p}}}$ be the crystalline canonical model of ω_{dR} over $\widehat{\mathscr{S}_{K_p}}$. To define the morphism

$$\zeta : \mathscr{S}_{K_p,0} \to [E_{G^c,\mu} \backslash G^c_{\kappa}],$$

we need to construct a G^c_0-zip $(I_0, I_{0,+}, I_{0,-}, \iota)$ of type μ on $\mathscr{S}_0$. Here G^c_0 is the special fiber of $G^c_{\mathbb{Z}_p}$.

One could get I_0 and $I_{0,+}$ (almost) directly from the underly filtered $G^c_{\mathbb{Z}_p}$-bundle of ω_{cris}, and $I_{0,-}, \varphi$ from the filtered F-crystal structure. To get started, we fix a faithful representation

$$G^c_{\mathbb{Z}_p} \to \mathrm{GL}(L)$$

and a tensor $s \in L^{\otimes}$ defining $G^c_{\mathbb{Z}_p}$. Then ω_{cris} gives a filtered F-crystal $(M, \mathrm{Fil}^{\bullet}, \nabla)$ and an embedding of filtered F-crystals $s_{\mathrm{cris}} : O_{\mathscr{S}_{K_p}} \to M^{\otimes}$. The reduction mod p of M (resp. $\mathrm{Fil}^{\bullet}$, s_{cris}) is denoted by M_0 (resp. $C^{\bullet}$, $s_{\mathrm{cris},0}$).

Now set

$$I_0 = \mathbf{Isom}((L_{\kappa}, s), (M_0, s_{\mathrm{cris},0})).$$

We can also see it without choosing any embedding, as it is the special fiber of the underly $G^c_{\mathbb{Z}_p}$-bundle I. Let $L^\bullet$ be the descending filtration on $L_{W(\kappa)}$ induced by μ, then set

$$I_{0,+} = \mathbf{Isom}((L_\kappa, L^\bullet_\kappa, s), (M_0, C^\bullet, s_{\mathrm{cris},0})).$$

We still need to show that $(M_0, C^\bullet)$ can be "extended" to an F-zip. Let A be an open affine of $\mathscr{S}_{K_p}$ with a Frobenius lifting σ of $A_0 := A/(p)$. Let $D_i|_{A_0}$ be elements $m \in M_0 \otimes A_0$ such that there exists $n \in M_A$ with $p^{-i}\varphi(n) \in M_A$ and the image in $M_0 \otimes A_0$ of $p^{-i}\varphi(n)$ is m. By the discussions in [33] Example 7.3, $D_\bullet|_{A_0}$ is a descending filtration on $M_0 \otimes A_0$ as in Definition 3.1.1, and $p^{-i}\varphi$ induces an F-zip

$$(M_0 \otimes A_0, C^\bullet \otimes A_0, D_\bullet|A_0, p^{-i}\varphi).$$

We remark that we assumed that φ is strongly divisible with respect to all (A, σ), so

$$(M_0 \otimes A_0, C^\bullet \otimes A_0, D_\bullet|A_0, p^{-i}\varphi)$$

is always an F-zip. The flat connection induces canonical isomorphism for different choices of σ. In particular, these $(M_0 \otimes A_0, C^\bullet \otimes A_0, D_\bullet|A_0, p^{-i}\varphi)$ can be glued into an F-zip $(M_0, C^\bullet, D_\bullet, \phi_\bullet)$ on $\mathscr{S}_{K_p,0}$.

Let $L_\bullet$ be the ascending filtration on $L_{W(\kappa)}$ induced by $\sigma(\mu)$, set

$$I_{0,-} = \mathbf{Isom}((L_\kappa, L_{\bullet,\kappa}, s), (M_0, D_\bullet, s_{\mathrm{cris},0})),$$

and let ι be simply the isomorphism induced by $\phi_\bullet$. We remark that the isomorphism $\varphi : \mathrm{R}(M^\sigma) \to M$ respects s_{cris}, this implies that the morphism $\iota : I_{0,+}/U_+ \to I_{0,-}/U_-^{(p)}$ is well defined.

Definition 5.3.4. Two points $x, y \in \mathscr{S}_{K_p,0}(\overline{\kappa})$ are in the same *Ekedahl-Oort stratum* if and only if their attached G^c_0-zip functors are isomorphic.

It is clear by construction that fibers of $\zeta \otimes \overline{\kappa}$ are the Ekedahl-Oort strata in $\mathscr{S}_{K_p,\overline{\kappa}}$.

5.4. **Properties of stratifications.** We will study properties of various stratifications here. We will mainly deduce these properties from what we know for those of Hodge type, and also compare the definitions here and those we gave before. It should be possible to study stratications directly using the filtered F-crystal with $G^c_{\mathbb{Z}_p}$-structure, but we would not do it here.

5.4.1. *Functoriality: some fundamental diagrams.* Notations as in Theorem 5.2.3. We assume moreover that both (G, X) and (G', X') are of abelian type. As we have remarked, f^*I' is a filtered F-crystal with G'^c-structure over $\widehat{\mathscr{S}_{K_p}}(G, X)$.

We have a canonical identification $I \times^{G^c} G'^c \cong f^*I'$ which induces, by our discussion in the previous parts, commutative diagrams

$$\begin{tikzcd}
\mathscr{S}_{K_p}(G,X)(\overline{\kappa}) \arrow[r] \arrow[d] & B(G^c,\mu) \arrow[d] \\
\mathscr{S}_{K'_p}(G',X')(\overline{\kappa}) \arrow[r] & B(G'^c,\mu)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{S}_{K_p,\kappa}(G,X) \arrow[r] \arrow[d] & {[E_{G^c,\mu}\backslash G^c_\kappa]} \arrow[d] \\
\mathscr{S}_{K'_p,\kappa}(G',X') \arrow[r] & {[E_{G'^c,\mu}\backslash G'^c_\kappa]}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{S}_{K_p}(G,X)(\overline{\kappa}) \arrow[r] \arrow[d] & C(G^c,\mu) \arrow[d] \\
\mathscr{S}_{K'_p}(G',X')(\overline{\kappa}) \arrow[r] & C(G'^c,\mu).
\end{tikzcd}$$

5.4.2. *Settings.* To study properties of stratifications defined using the filtered F-crystal with $G^c_{\mathbb{Z}_p}$-structure, as well as to compare them with those we defined via passing to adjoint groups (as we will see, they are sometimes the same thing), we introduce the following settings.

Let (G, X) be Shimura datum of abelian type as above, and (G_1, X_1) be a Shimura datum of Hodge type with Z_{G_1} a torus and $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}})$ (see Lemma 2.3.2). Let $(\mathcal{B}, X')$ be the Shimura datum constructed in [28] Proposition 3.4.2 (see also [27] 4.6) using G_1^{der} and the reflex field of (G_1, X_1), then there is a commutative diagram of Shimura data

$$\begin{tikzcd}
(\mathcal{B}, X') \arrow[r] \arrow[d] & (G_1, X_1) \arrow[d] \\
(G, X) \arrow[r] & (G^{\mathrm{ad}}, X^{\mathrm{ad}})
\end{tikzcd}$$

inducing a commutative diagram of (integral models of) Shimura varieties

$$\begin{tikzcd}
\mathscr{S}_{K_{\mathcal{B},p}}(\mathcal{B}, X') \arrow[r] \arrow[d] & \mathscr{S}_{K_{1,p}}(G_1, X_1) \arrow[d] \\
\mathscr{S}_{K_p}(G, X) \arrow[r] & \mathscr{S}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}).
\end{tikzcd}$$

The reflex field of $(\mathcal{B}, X')$ is the same as that of (G_1, X_1) by construction (cf. [27] 4.6). By Lemma 2.3.2 (2), the local reflex fields of the Shimura varieties in the above diagram are the same. As before, we denote by κ the common residue field of the local reflex field E_v.

5.4.3. *Newton stratifications.* Using the fundamental diagram for Newton strata, we find a commutative diagram

$$\begin{tikzcd}
\mathscr{S}_{K_{\mathcal{B},p}}(\mathcal{B}, X')(\overline{\kappa}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathscr{S}_{K_{1,p}}(G_1, X_1)(\overline{\kappa}) \arrow[dr] \arrow[dd] & \\
& B(\mathcal{B}^c, \mu) \arrow[rr] \arrow[dd] & & B(G_1^c, \mu) \arrow[dd] \\
\mathscr{S}_{K_p}(G, X)(\overline{\kappa}) \arrow[rr] \arrow[dr] & & \mathscr{S}_{K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})(\overline{\kappa}) \arrow[dr] & \\
& B(G^c, \mu) \arrow[rr] & & B(G^{\mathrm{ad}}, \mu).
\end{tikzcd}$$

This implies that the Newton stratification on $\mathscr{S}_{K_{1,p},\kappa}(G_1, X_1)$ (resp. $\mathscr{S}_{K_p,\kappa}(G, X)$) is a refinement of the pullback of that on $\mathscr{S}_{K_p^{\mathrm{ad}},\kappa}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, and the Newton stratification on $\mathscr{S}_{K_{\mathcal{B},p},\kappa}(\mathcal{B}, X')$ is a refinement of both the pullback of that on $\mathscr{S}_{K_{1,p},\kappa}(G_1, X_1)$ and that on $\mathscr{S}_{K_p,\kappa}(G, X)$. However, noting that the maps on $B(-, \mu)$ are bijective, the Newton stratification on $\mathscr{S}_{K_{\mathcal{B},p},\kappa}(\mathcal{B}, X')$ (resp $\mathscr{S}_{K_{1,p},\kappa}(G_1, X_1,$ $\mathscr{S}_{K_p,\kappa}(G, X))$ is just the pullback of that on $\mathscr{S}_{K_p^{\mathrm{ad}},\kappa}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$.

By the construction of ω_{cris} in the Hodge type case (see [28] Theorem 3.3.3), the Newton stratification on $\mathscr{S}_{K_{1,p},\kappa}(G_1, X_1)$ we defined here coincides with that we defined before. So the above discussions also show that the Newton stratification on $\mathscr{S}_{K_p^{\mathrm{ad}},\kappa}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ (and hence the Newton stratification on $\mathscr{S}_{K_p,\kappa}(G, X)$) we defined here coincides with the one we defined in 2.3.5.

5.4.4. *Ekedahl-Oort stratifications.* By the fundamental diagram for E-O stratification, we have a commutative diagram of morphisms of stacks

$$\begin{tikzcd}
\mathscr{S}_{K_{\mathcal{B},p},\kappa}(\mathcal{B},X') \arrow[rr] \arrow[dr] \arrow[dd] & & \mathscr{S}_{K_{1,p},\kappa}(G_1,X_1) \arrow[dr] \arrow[dd] & \\
& {[E_{\mathcal{B}^c,\mu}\backslash \mathcal{B}^c_\kappa]} \arrow[rr] \arrow[dd] & & {[E_{G_1^c,\mu}\backslash G^c_{1,\kappa}]} \arrow[dd] \\
\mathscr{S}_{K_p,\kappa}(G,X) \arrow[rr] \arrow[dr] & & \mathscr{S}_{K_p^{\mathrm{ad}},\kappa}(G^{\mathrm{ad}},X^{\mathrm{ad}}) \arrow[dr] & \\
& {[E_{G^c,\mu}\backslash G^c_\kappa]} \arrow[rr] & & {[E_{G^{\mathrm{ad}},\mu}\backslash G^{\mathrm{ad}}_\kappa]}.
\end{tikzcd}$$

Similar to Newton stratifications, the E-O stratification on $\mathscr{S}_{K_{\mathcal{B},p},\overline{\kappa}}(\mathcal{B},X')$ (resp. $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1, X_1)$, $\mathscr{S}_{K_p,\overline{\kappa}}(G,X)$) is just the pullback of that on $\mathscr{S}_{K_p^{\mathrm{ad}},\overline{\kappa}}(G^{\mathrm{ad}},X^{\mathrm{ad}})$, and the E-O stratification on $\mathscr{S}_{K_p^{\mathrm{ad}},\overline{\kappa}}(G^{\mathrm{ad}},X^{\mathrm{ad}})$ (and hence the E-O stratification on $\mathscr{S}_{K_p,\overline{\kappa}}(G,X)$) we defined in 3.4.6 coincides with the E-O stratification we defined here. In particular, the morphism $\mathscr{S}_{K_p,\overline{\kappa}}(G,X)\to [E_{G^{\mathrm{ad}},\mu}\backslash G^{\mathrm{ad}}_\kappa]\otimes\overline{\kappa}$ is smooth surjective.

5.4.5. *Central leaves.* We have a similar commutative diagram as in 5.4.3 (one only needs to replace $B(-,\mu)$ by $C(-,\mu)$). It implies that central leaves on $\mathscr{S}_{K_{1,p},\overline{\kappa}}(G_1,X_1)$ (resp. $\mathscr{S}_{K_p,\overline{\kappa}}(G,X)$) are refinements of the pullback of those on $\mathscr{S}_{K_p^{\mathrm{ad}},\overline{\kappa}}(G^{\mathrm{ad}},X^{\mathrm{ad}})$, and central leaves on $\mathscr{S}_{K_{\mathcal{B},p},\overline{\kappa}}(\mathcal{B},X')$ are refinements of both the pullback of those on $\mathscr{S}_{K_{1,p},\kappa}(G_1,X_1)$ and those on $\mathscr{S}_{K_p,\overline{\kappa}}(G,X)$. Noting that the map $C(G_1,\mu)\to C(G^{\mathrm{ad}},\mu)$ is bijective, central leaves on $\mathscr{S}_{K_{1,p},\kappa}(G_1,X_1)$ are just the pullback of those on $\mathscr{S}_{K_p^{\mathrm{ad}},\overline{\kappa}}(G^{\mathrm{ad}},X^{\mathrm{ad}})$, and the central leaves on $\mathscr{S}_{K_p^{\mathrm{ad}},\overline{\kappa}}(G^{\mathrm{ad}},X^{\mathrm{ad}})$ we defined in 4.2.2 coincides with the central leaves we defined here.

We remark that we do NOT know in general whether central leaves on $\mathscr{S}_{K_p,\overline{\kappa}}(G,X)$ we defined here coincide with what we defined before. But in the case Z_G is connected, this is indeed true by Lemma 4.2.1.

6. Comparing Ekedahl-Oort and Newton stratifications

In this section, we will study the relations between Ekedahl-Oort strata and Newton strata by group theoretic methods.

6.1. **Group theoretic results.** We will recall some group theoretic results first. The settings are as follows. We start with a pair (G,μ) where G is a reductive group over $\mathbb{Z}_p$, and $\mu:\mathbb{G}_m\to G_{W(\kappa)}$ is a minuscule cocharacter defined over $W(\kappa)$ with κ a finite field. We will write G_0 for the special fiber of G, $W=W(\overline{\kappa})$, $L=W[1/p]$, $K=G(W)$, and

$$K_1=\mathrm{Ker}(K\to G(\overline{\kappa})).$$

Let $B\subseteq G$ be a Borel subgroup, $T\subseteq B$ be a maximal torus, and $\mathcal{I}$ be the Iwahori subgroup attached to B_0, the special fiber of B. Let W_G be the Weyl group with respect to T. Let $\widetilde{W}_G:=\mathrm{Norm}_G(T)(L)/T(W)\cong W_G\ltimes X_*(T)$ be the extended affine Weyl group and W_a be the affine Weyl group. There is a canonical exact sequence

$$0\longrightarrow X_*(T)\longrightarrow \widetilde{W}_G\longrightarrow W_G\longrightarrow 0\ .$$

Let $\Omega \subseteq \widetilde{W}_G$ be the stabilizer of the alcove corresponding to the Iwahoric subgroup of $G(L)$ given by the preimage of $B(\overline{\kappa})$. We define the length function on $\widetilde{W}_G$ by

$$l(wr) = l(w), \text{ for } w \in W_a, r \in \pi_1(G). \tag{6.1.1}$$

The choice of B (resp. $\mathcal{I}$) determines simple reflections (resp. simple reflections and simple affine roots) in W_G (resp. $\widetilde{W}_G$) denoted by S (resp. $\widetilde{S}$). It also gives the Bruhat order on W_G (resp. $\widetilde{W}_G$), denoted by $\leq$. Clearly, we have $S \subseteq \widetilde{S}$.

6.1.2. *Minimal elements and fundamental elements.* An element $x \in G(F)$ is called *minimal* if for any $y \in K_1 x K_1$, there is a $g \in K$ such that $y = gx\sigma(g)^{-1}$. By [48] Remark 9.1, if x is minimal, then any element in the K-σ-orbit of x is again minimal.

An element $x \in \widetilde{W}_G$ is *fundamental* if $\mathcal{I}x\mathcal{I}$ lies in a single $\mathcal{I}$-σ-orbit. For an element $w \in \widetilde{W}_G$, we consider the element $w\sigma \in \widetilde{W}_G \rtimes \langle\sigma\rangle$. There exists $n \in \mathbb{N}$ such that $(w\sigma)^n = t^\lambda$ for some $\lambda \in X_*(T)$. Let ν_w be the unique dominant element in the W_G-orbit of λ/n. It is known that ν_w is independent of the choice of n, and it is the Newton point of w when regarding w as an element in $G(L)$. We say that an element w is *σ-straight* if

$$l((w\sigma)^n) = nl(w).$$

Here $l(-)$ is the length. This is equivalent to saying that

$$l(w) = \langle \nu_w, 2\rho \rangle,$$

where ρ is the half sum of all positive roots in the root system of the affine Weyl group. A σ-conjugacy class of $\widetilde{W}_G$ is called straight if it contains a σ-straight element.

The main results in the above setting are as follows.

Theorem 6.1.3. (1) *For $w \in \widetilde{W}_G$, it is fundamental if and only if it is σ-straight.*
(2) *An element $g \in G(L)$ is minimal if and only if it lies in a K-σ-conjugacy class of some fundamental element of $\widetilde{W}_G$. Moreover, when G is split, each σ- conjugacy class of $G(L)$ contains one and only one K-σ-conjugacy class of minimal elements.*
(3) *If μ is a minuscule cocharacter of T, then each σ-conjugacy class intersecting $K\mu(p)K$ contains a fundamental element in $W_G\mu(p)W_G$.*

Proof. They are [34] Theorem 1.3, Theorem 1.4 and Proposition 1.5 respectively. $\square$

It is sometimes helpful to keep in mind the following commutative diagram, which is a direct consequence of the above theorem. Let $\mathrm{EO}(\mu)$ be as in 6.1.4, and $\mathrm{EO}(\mu)_{\sigma-\mathrm{str}} \subseteq \mathrm{EO}(\mu) \subseteq \widetilde{W}_G$ the subset of σ-straight elements. Then we have

$$\begin{tikzcd}
 & & B(G,\mu) \\
\mathrm{EO}(\mu)_{\sigma-\mathrm{str}} \arrow[urr, two heads] \arrow[r, hook] \arrow[drr, hook, two heads] & C(G,\mu) \arrow[ur, two heads] \arrow[dr, two heads] & \\
 & & {[E_{G,\mu}\backslash G_\kappa](\overline{\kappa}).}
\end{tikzcd}$$

6.1.4. $\mathrm{Adm}(\mu)$, $B(G,\mu)$ *and* $\mathrm{EO}(\mu)$. We will introduce some distinguished sets follow [9].

For any subset J of $\widetilde{S}$, we denote by W_J the subgroup of $\widetilde{W}_G$ generated by the simple reflections in J and by ${}^J\widetilde{W}_G$ (resp. $\widetilde{W}_G^J$) the set of minimal length elements for the cosets $W_J\backslash\widetilde{W}_G$ (resp. $\widetilde{W}_G\backslash W_J$). We simply write ${}^J\widetilde{W}_G^K$ for ${}^J\widetilde{W}_G \cap \widetilde{W}_G^K$.

The *μ-admissible set* $\mathrm{Adm}(\mu)$ is defined to be

$$\mathrm{Adm}(\mu) = \{w \in \widetilde{W}_G \mid w \leq t^{x\mu} \text{ for some } x \in W_G\}.$$

Here we write t^λ for elements in the affine part of $\widetilde{W}_G$.

By [10] Theorem 1.3 (1), the map

$$\Psi : B(\widetilde{W}_G)_{\sigma-\mathrm{str}} \to B(G)$$

induced by the inclusion $N(T)(L) \subset G(L)$ is bijective. Let $\mathrm{Adm}(\mu)_{\sigma-\mathrm{str}}$ be the set of σ-straight elements in the admissible set $\mathrm{Adm}(\mu)$ and $B(\widetilde{W}_G, \mu)_{\sigma-\mathrm{str}}$ be its image in $B(\widetilde{W}_G)_{\sigma-\mathrm{str}}$. Then by [10] Theorem 1.3 (2), we have

$$\Psi(B(\widetilde{W}_G, \mu)_{\sigma-\mathrm{str}}) = B(G, \mu).$$

The *set of EO elements* $\mathrm{EO}(\mu)$ is defined to be

$$\mathrm{EO}(\mu) = \mathrm{Adm}^S(\mu) \cap {}^S\widetilde{W}_G = \mathrm{Adm}(\mu) \cap {}^S\widetilde{W}_G,$$

where $\mathrm{Adm}^S(\mu) = W_S\mathrm{Adm}(\mu)W_S$. Here for the second equality, see [15] Theorem 6.10 for example.

There is a partial order $\preceq$ on ${}^S\widetilde{W}_G$ as follows. For $w, w' \in {}^S\widetilde{W}_G$, $w \preceq w'$ if and only if there exists $x \in W_G$, such that $xw\sigma(x)^{-1} \leq w'$. This partial order restrict to $\mathrm{EO}(\mu)$ and will still be denoted by $\preceq$.

6.1.5. $\mathrm{EO}(\mu)$ *and* G_0*-zips.* Before moving on, let's explain how to identify $\mathrm{EO}(\mu)$ (with the partial order $\preceq$) with the topological space of $[E_{G,\mu}\backslash G_{0,\kappa}]$.

Let $\mathcal{T} \subseteq \widetilde{W}_G$ be given by

$$\mathcal{T} = \{(w, \mu) \in W_G \times X_*(T) \mid w \in {}^\mu W\}.$$

It is naturally identified with $\mathrm{EO}(\mu)$. Let $x_\mu = w_0 w_{0,\mu}$ where w_0 denotes the longest element of W_G and where $w_{0,\mu}$ is the longest element of W_μ, the Weyl group of the centralizer of μ. Then $\tau_\mu = x_\mu \mu(p)$ is the shortest element of $W_G\mu(p)W_G$.

By [46] Theorem 1.1 (1), the map assigning to $(w, \mu) \in \mathcal{T}$ the K-σ-conjugacy class of $K_1 w\tau_\mu K_1$ is a bijection between $\mathcal{T}$ and the set of K-σ-conjugacy classes in $K_1\backslash K\mu(p)K/K_1$. By [51] Proposition 6.7, the assignment

$$g_1\mu(p)g_2 \mapsto E_{G,\mu} \cdot (\overline{\sigma^{-1}(g_2)g_1})$$

induces a bijection from the set of K-σ-conjugacy classes in $K_1\backslash K\upsilon(p)K/K_1$ to the set of $\overline{\kappa}$-points of $[E_{G,\mu}\backslash G_{0,\kappa}]$. By Theorem 3.2.1,

$$[E_{G,\mu}\backslash G_{0,\kappa}](\overline{\kappa}) \cong {}^\mu W = \mathcal{T},$$

and by [46] corollary 4.7, the induced partial order coincides with $\preceq$.

6.1.6. Notations as in 6.1.4, we have

$$Y = K\mu(p)K = \bigcup_{w \in \mathrm{Adm}(\mu)} KwK = \bigcup_{w \in \mathrm{Adm}^S(\mu)} \mathcal{I}w\mathcal{I}.$$

There is a K-action on $G(L) \times Y$ given by $g \cdot (h, y) = (hg^{-1}, gy\sigma(g)^{-1})$. Let Z be the quotient of this action. The map $(h, y) \mapsto (hy\sigma(h)^{-1}, hK)$ gives an isomorphism

$$Z \cong \{(b, gK) \in G(L) \times G(L)/K \mid g^{-1}b\sigma(g) \in Y\}.$$

The projection to the first factor induces a map $Z \to G(L)$, and its image is a union of σ-conjugacy classes indexed by $B(G, \mu)$.

For a σ-conjugacy class $\mathcal{O} \in B(G,\mu)$, we write $Z_{\mathcal{O}} \subseteq Z$ for the corresponding subset. The decomposition

$$Z = \coprod_{\mathcal{O} \in B(G,\mu)} Z_{\mathcal{O}}$$

is called the *Newton stratification* of Z. For the basic class $\mathcal{O}_0 \in B(G,\mu)$, the corresponding stratum $Z_{\mathcal{O}_0}$ is called the *basic locus* in Z.

Writing $x \cdot_\sigma y$ for $xy\sigma(x)^{-1}$, by [9] Theorem 3.2.1, we have

$$Y = \coprod_{w \in \mathrm{EO}(\mu)} K \cdot_\sigma \mathcal{I}w\mathcal{I}.$$

But then

$$Z = \coprod_{w \in \mathrm{EO}(\mu)} Z_w,$$

where $Z_w = G(L) \times^K (K \cdot_\sigma \mathcal{I}w\mathcal{I})$. This decomposition is called the *Ekedahl-Oort stratification* on Z.

Given $w \in \mathrm{EO}(\mu)$ and $\mathcal{O} \in B(G,\mu)$, the intersection $Z_w \cap Z_{\mathcal{O}}$ is a fiber bundle over $\mathcal{O}$, and the fiber over $b \in \mathcal{O}$ is given by

$$X_w(b) := \{gK \mid g^{-1}b\sigma(g) \in K \cdot_\sigma \mathcal{I}w\mathcal{I}\} \subseteq G(L)/K.$$

Recall that attached to the triple $(G, \{\mu\}, b)$ we have the affine Deligne-Lusztig variety

$$X(\mu, b) = \{gK \mid g^{-1}b\sigma(g) \in K\mu(p)K\}.$$

It admits a perfect scheme structure over $\overline{\kappa}$ by [55]. By our discussions in 6.1.4, 6.1.5 and [10] 1.4, we have the following decomposition

$$X(\mu, b) = \coprod_{w \in {}^J W} X_w(b).$$

We remark that not every subset $X_w(b)$ in the above decomposition is non-empty (see Proposition 6.2.5).

6.1.7. (G,μ) *of Coxeter type.* We also need a subset $\mathrm{EO}_{\sigma,\mathrm{cox}}(\mu)$ of $\mathrm{EO}(\mu)$. It is the subset of elements w such that $\mathrm{supp}_\sigma(w)$ is a proper subset of $\widetilde{S}$ and w that is a σ-Coxeter element of $W_{\mathrm{supp}_\sigma(w)}$. We will not explain this but just refer to [9] 2.2.

A pair (G,μ) with $G_{\mathbb{Q}_p}$ absolutely quasi-simple is said to be *of Coxeter type* if

$$Z_{\mathcal{O}_0} = \coprod_{w \in \mathrm{EO}_{\sigma,\mathrm{cox}}(\mu)} Z_w.$$

A complete list for pairs (G,μ) of Coxeter type is given in [9] Theorem 5.1.2. The Newton and Ekedahl-Oort stratifications on Z have very nice properties which we will recall.

Recall that a ranked poset is a partially ordered set (poset) equipped with a rank function ρ such that whenever y covers x, $\rho(y) = \rho(x) + 1$. We say that the partial order of a poset is almost linear if the poset has a rank function ρ such that for any x, y in the poset, $x < y$ if and only if $\rho(x) < \rho(y)$.

Theorem 6.1.8. *Let (G,μ) be of Coxeter type.*

1. *Every Newton stratum of Z is a union of Ekedahl-Oort strata.*
2. *For any $w \in \mathrm{EO}(\mu) - \mathrm{EO}_{\sigma,\mathrm{cox}}(\mu)$ and $b \in \mathcal{O}_w$, the σ-centralizer J_b acts transitively on $X_w(b)$.*
3. *The partial order of $B(G,\mu)$ (inherited from $B(G)$) is almost linear.*
4. *The partial order $\preceq$ of $\mathrm{EO}_{\sigma,\mathrm{cox}}(\mu)$ coincides with the usual Bruhat order and is almost linear. Here the rank is the length function.*

Proof. The first two statements are in [9] Theorem 5.2.1, the last two statements are in [9] Theorem 5.2.2. □

6.1.9. *(G, μ) of fully Hodge-Newton decomposable type.* Görtz, He and Nie define and study in [10] a much more general class of pairs (G, μ) with the name of being fully Hodge-Newton decomposable. They prove that this is equivalent to property (1) in the previous theorem, and they also give a classification of such pairs. It turns out all the groups in such pairs are classical groups (i.e. reductive groups of type A, B, C, and D), cf. [10] Theorem 2.5.

Let's recall the definition of being fully Hodge-Newton decomposable. As what we used to do, we restrict to good reduction cases only.

Definition 6.1.10. (1) Let $M \subsetneqq G_L$ be a σ-stable standard Levi subgroup. We say that $b \in B(G, \mu)$ is Hodge-Newton decomposable with respect to M if $M_{\nu(b)} \subseteq M$ and $v^\diamond - \nu(b) \in \mathbb{R}_{\geq 0}\Phi_M^\vee$. Here $M_{\nu(b)} \subseteq G_L$ is the centralizer of $\nu(b)$, and $\mu^\diamond = \frac{1}{n_0}\sum_{i=0}^{n_0-1}\sigma^i(\mu)$ with $n_0 \in \mathbb{N}$ the order of σ.

(2) We say that a pair (G, μ) is fully Hodge-Newton decomposable if every non-basic σ-conjugacy class b is Hodge-Newton decomposable with respect to some proper standard Levi.

The following is part of [10] Theorem 2.3 which suffices for our applications.

Theorem 6.1.11. *The following statements for (G, μ) are equivalent.*

(1) *It is fully Hodge-Newton decomposable.*
(2) *For any $w \in \mathrm{EO}(\mu)$, there is a unique $b \in B(G, \mu)$ such that $X_w(b) \neq \emptyset$; i.e. every Newton stratum of Z is a union of Ekedahl-Oort strata. Here Z, $\mathrm{EO}(\mu)$ and $X_w(b)$ are as in 6.1.4.*
(3) *For any non-basic $b \in B(G, \mu)$, $\dim X(\mu, b) = 0$.*

We remark that [10] Theorem 2.3 is stated only for quasi-simple groups, but by discussions just after the theorem there, it holds in general. We also remark that although it is not stated in the main theorem there, it is true that if (G, μ) is fully Hodge-Newton decomposable, non-basic elements in $\mathrm{EO}(\mu)$ are σ-straight (see [10] Proposition 4.5).

6.2. **Applications to stratifications.** We will explain how to use group theoretic results above to study relations between E-O stratifications and Newton stratifications. Unlike in [9] or [10], we will do this directly and without assuming any results on existence of Rapoport-Zink uniformizations.

Notations as in 6.1.6, for $(b, gK) \in Z$ with $b \in G(L)$ and $gK \in G(L)/K$ such that $g^{-1}b\sigma(g) \in K\mu(p)K$, the assignment $(b, gK) \mapsto g^{-1}b\sigma(g)$ induces a well defined map

$$Z \to C(G, \mu).$$

Moreover, the maps $Z \to B(G, \mu)$ and $Z \to \mathrm{EO}(\mu)$ factor through $C(G, \mu)$. Let $\overline{Z}_w$ (resp. $\overline{Z}_{\mathcal{O}}$) be the image of Z_w (resp. $Z_{\mathcal{O}}$) in $C(G, \mu)$ for $w \in \mathrm{EO}(\mu)$ (resp. $\mathcal{O} \in B(G, \mu)$). Then $\overline{Z}_{\mathcal{O}} = \coprod_i \overline{Z}_{w_i}$ if and only if $Z_{\mathcal{O}} = \coprod_i Z_{w_i}$. Moreover, $\overline{Z}_w \cap \overline{Z}_{\mathcal{O}} \neq \emptyset$ if and only if $Z_w \cap Z_{\mathcal{O}} \neq \emptyset$ which is then equivalent to that $X_w(b) \neq \emptyset$ for some (and hence any) $b \in \mathcal{O}$.

We fix a prime to p level K^p and simply denote the integral canonical model over O_{E_v} by $\mathscr{S} = \mathscr{S}_{K_pK^p}(G, X)$ for a Shimura datum (G, X) of abelian type with good reduction at p. Its geometric special fiber is denoted by $\mathscr{S}_{\overline{\kappa}}$. In the rest of this section, we will study the Newton stratification, Ekedahl-Oort stratification, and central leaves on $\mathscr{S}_{\overline{\kappa}}$. We start with the following commutative diagrams.

6.2.1. *General relations.* If we consider stratifications defined by passing to the adjoint ones first, we have a commutative diagram induced by a similar diagram attached to certain Shimura datum of Hodge type satisfying Lemma 2.3.2:

$$\begin{tikzcd}
 & & B(G^{\mathrm{ad}},\mu) \\
\mathscr{S}(\overline{\kappa}) \arrow[r] \arrow[urr] \arrow[drr] & C(G^{\mathrm{ad}},\mu) \arrow[ur] \arrow[dr] & \\
 & & {[E_{G^{\mathrm{ad}},\mu}\backslash G^{\mathrm{ad}}_{\kappa}](\overline{\kappa}).}
\end{tikzcd}$$

Similarly, by 5.3.3 and the discussions just before it, for stratifications given by F-crystals with additional structure, we have a commutative diagram:

$$\begin{tikzcd}
 & & B(G^{c},\mu) \\
\mathscr{S}(\overline{\kappa}) \arrow[r] \arrow[urr] \arrow[drr] & C(G^{c},\mu) \arrow[ur] \arrow[dr] & \\
 & & {[E_{G^{c},\mu}\backslash G^{c}_{\kappa}](\overline{\kappa}).}
\end{tikzcd}$$

Note that by Lemma 4.2.1, we have $C(G,\mu) \simeq C(G^c,\mu)$ and the natural map $C(G^c,\mu) \to C(G^{\mathrm{ad}},\mu)$ is a bijection if Z_G is connected. We also remind the readers that the above two diagrams do NOT bring any differences if we just look at the E-O and Newton stratifications. *So in the following discussions we will mean either of these two cases when talking about E-O or Newton stratification.*

Definition 6.2.2. An E-O stratum is said to be minimal[5] if it is a central leaf.

Proposition 6.2.3. *Each Newton stratum contains a minimal E-O stratum. Moreover, if G splits, then each Newton stratum contains a unique minimal E-O stratum.*

Proof. The statements follow from Theorem 6.1.3. □

Example 6.2.4. By Corollary 3.4.8, the ordinary E-O stratum coincides with the μ-ordinary locus (i.e. the open Newton stratum), which is a central leaf by Proposition 6.2.3.

Proposition 6.2.5. *For any $b \in B(G,\mu)$ and $w \in \mathrm{EO}(\mu) \simeq {}^J W$, we have*

$$\mathscr{S}^b_{\overline{\kappa}} \cap \mathscr{S}^w_{\overline{\kappa}} \neq \emptyset \Longleftrightarrow X_w(b) \neq \emptyset.$$

Proof. This follows from the fact that each central leaf is non-empty (cf. Theorem 4.2.5) and [51] 6.2 consequences (3). □

6.2.6. *Special relations.* By Görtz, He and Nie's classification of fully Hodge-Newton decomposable pairs ([10] Theorem 2.5) and Delgine's classification of Shimura varieties of abelian type ([3] Table 2.3.8), it is natural to discuss fully Hodge-Newton decomposable Shimura data in the framework of Shimura data of abelian type, in view of Kisin's work [18]. If (G,X) is fully Hodge-Newton decomposable, we have the followings.

[5]We remind the readers that this notion is (in general) different from the superspecial locus, i.e. the unique closed E-O stratum attached to $1 \in {}^J W$.

Proposition 6.2.7. *Let (G, X) be a Shimura datum of abelian type with good reduction at p whose attached pair $(G_{\mathbb{Q}_p}, \mu)$ is fully Hodge-Newton decomposable, then*

1. *each Newton stratum of $\mathscr{S}_{\overline{\kappa}}$ is a union of Ekedahl-Oort strata;*
2. *each E-O stratum in a non-basic Newton stratum is a central leaf, and it is open and closed in the Newton stratum, in particular, non-basic Newton strata are smooth;*
3. *if (G, μ) is of Coxeter type, then for two E-O stratum $\mathscr{S}^1_{\overline{\kappa}}$ and $\mathscr{S}^2_{\overline{\kappa}}$, $\mathscr{S}^1_{\overline{\kappa}}$ is in the closure of $\mathscr{S}^2_{\overline{\kappa}}$ if and only if $\dim(\mathscr{S}^2_{\overline{\kappa}}) > \dim(\mathscr{S}^1_{\overline{\kappa}})$.*

Proof. Statement (1) follows directly from Theorem 6.1.11. For (2), the first half follows from our remarks after Theorem 6.1.11 (which is just [10] Proposition 4.5); and the second half follows from Theorem 4.2.5. Statement (3) follows from Theorem 6.1.8 (4). □

Example 6.2.8. Notations as in Example 2.3.8. The pair (G, μ) is fully Hodge-Newton decomposable if and only of all the a_is are either 1 or 2. The if part is clear. To see the only if part, it there is some $a_i \geq 3$, by the dimension formula in Example 2.3.8 and Example 4.2.6, the dimension of the maximal non-ordinary Newton stratum is strictly bigger than that of its central leaves, and hence it is not fully Hodge-Newton decomposable.

Examples 6.2.9.
1. The unitary Shimura varieties with signature $(1, n-1) \times (0, n) \times (0, n)$ at a split prime p studied by Harris-Taylor in [14] is fully Hodge-Newton decomposable.
2. Consider $G = \mathrm{GU}(V, \langle,\rangle)$, the unitary similitude group over $\mathbb{Q}_p$ associated to a Hermintain space $(V, \langle,\rangle)$. Take $\{\mu\}$ such that it corresponds to $\big((1, \cdots, 1, 0), 0\big)$. Then $(G, \{\mu\})$ is fully Hodge-Newton decomposable by the explicit description of the set $B(G, \mu)$ as in [2] 3.1. Globally, these are the unitary Shimura varieties studied by Bütel-Wedhorn in [2].
3. The pair $(\mathrm{GSp}_4, \{\mu\})$ is fully Hodge-Newton decomposable, where μ is the cocharacter corresponding to $(1, 1, 0, 0)$. Globally, these are the Siegel modular varieties with genus $g = 2$ (Siegel threefolds).
4. Consider $G = \mathrm{SO}(V, Q)$, the special orthogonal group over $\mathbb{Q}_p$ associated to a quadratic space (V, Q) of dimension $n + 2$. Take $\{\mu\}$ such that it corresponds to $(1, 0, \cdots, 0, -1)$. Then $(G, \{\mu\})$ is fully Hodge-Newton decomposable by the explicit description of the set $B(G, \mu)$. Globally, these are the SO-Shimura varieties of orthogonal type, cf. the next section.

7. Shimura varieties of orthogonal type

We discuss our main results in the setting of Shimura varieties of orthogonal type.

7.1. **Good reductions of Shimura varieties of orthogonal type.**

7.1.1. *The* SO-*Shimura varieties.* Let V be a $n+2$-dimensional $\mathbb{Q}$-vector space equipped with a non-degenerate bilinear form B (whose associated quadratic form is) of signature $(n, 2)$. Let $\mathrm{SO}(V)$ be the special orthogonal group attached to (V, B), and

$$h : \mathbb{S} \to \mathrm{SO}(V)_{\mathbb{R}}$$

be such that

1. its induced Hodge structure on V is of type $(-1, 1) + (0, 0) + (1, -1)$ with $\dim V^{-1,1} = 1$;
2. B is a polarization of this Hodge structure.

It is well known that h gives a Shimura datum $(\mathrm{SO}(V), X)$.

7.1.2. *The* GSpin-*Shimura varieties.* Let $C(V)$ and $C^+(V)$ be the Clifford algebra and even Clifford algebra respectively. Note that there is an embedding $V \hookrightarrow C(V)$ and an anti-involution $*$ on $C(V)$ (see [30], 1.1).

Let $\mathrm{GSpin}(V)$ be the stabilizer in $C^+(V)^\times$ of $V \hookrightarrow C(V)$ with respect to the conjugation action of $C^+(V)^\times$ on $C(V)$. Then $\mathrm{GSpin}(V)$ is a reductive group over $\mathbb{Q}$, and the conjugation action of $\mathrm{GSpin}(V)$ on V induces a homomorphism $\mathrm{GSpin}(V) \to \mathrm{SO}(V)$. We actually have an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1\ ,$$

where $\mathbb{G}_m$ are identified with invertible scalars in $C^+(V)$.

The homomorphism h in 7.1.1 lifts to $\mathrm{GSpin}(V)$ and induces a Shimura datum $(\mathrm{GSpin}(V), X')$ with $X' \simeq X$. Consider the left action of $\mathrm{GSpin}(V)$ on $C^+(V)$, there is a perfect alternating form ψ on $C^+(V)$, such that the embedding $\mathrm{GSpin}(V) \hookrightarrow \mathrm{GL}(C^+(V))$ factors through $\mathrm{GSp}(C^+(V), \psi)$ and induces an embedding of Shimura data

$$(\mathrm{GSpin}(V), X') \to (\mathrm{GSp}(C^+(V), \psi), \mathbb{H}^\pm).$$

We refer to [30] 1.8, 1.9, 3.4, 3.5 for details.

To sum up, $(\mathrm{GSpin}(V), X')$ is a Shimura datum of Hodge type and $(\mathrm{SO}(V), X)$ is a Shimura datum of abelian type. One can also see that the reflex field of $(\mathrm{SO}(V), X)$ (resp. $(\mathrm{GSpin}(V), X')$) is $\mathbb{Q}$ if $n > 0$. We will assume that $n > 0$ from now on.

Let (G, Y) be either $(\mathrm{SO}(V), X)$ or $(\mathrm{GSpin}(V), X')$. Let $K \subseteq G(\mathbb{A}_f)$ be a compact open subgroup which is small enough, then

$$\mathrm{Sh}_K := G(\mathbb{Q})\backslash Y \times (G(\mathbb{A}_f)/K)$$

has a canonical model over $\mathbb{Q}$ which will again be denoted by Sh_K. Let $K \subset \mathrm{GSpin}(V)(\mathbb{A}_f)$ be a sufficently small open compact subgroup, and $K_1 \subset \mathrm{SO}(V)(\mathbb{A}_f)$ be its image induced by the map $\mathrm{GSpin}(V) \to \mathrm{SO}(V)$. Then the induced map between the corresponding Shimura varieties

$$\mathrm{Sh}_K(\mathrm{GSpin}(V), X') \to \mathrm{Sh}_{K_1}(\mathrm{SO}(V), X)$$

is a finite étale Galois cover, cf. [30] 3.2.

7.1.3. *Good reductions.* Let $p > 2$ be a prime and $L \subseteq V$ be a $\mathbb{Z}_{(p)}$-lattice such that the bilinear form B is perfect on it. Then $\mathrm{SO}(L)$ is a reductive group over $\mathbb{Z}_{(p)}$ with generic fiber $\mathrm{SO}(V)$. Similarly, we have $C(L)$, $C^+(L)$ and $\mathrm{GSpin}(L)$, and $\mathrm{GSpin}(L)$ is a reductive group over $\mathbb{Z}_{(p)}$ with generic fiber $\mathrm{GSpin}(V)$.

Let (G, Y) be either $(\mathrm{SO}(V), X)$ or $(\mathrm{GSpin}(V), X')$ as above, and we still write G for its reductive model over $\mathbb{Z}_{(p)}$ by abuse of notation. Let $K_p = G(\mathbb{Z}_p)$ and $K^p \subseteq G(\mathbb{A}_f^p)$ be a compact open subgroup which is small enough. Let $K = K_pK^p$, then by Theorem 1.2.6, Sh_K has an integral canonical model over $\mathbb{Z}_{(p)}$ denoted by $\mathscr{S}_K$. Let $K^p \subset \mathrm{GSpin}(V)(\mathbb{A}_f^p)$ be a sufficently small open compact subgroup, and $K_1 \subset \mathrm{SO}(V)(\mathbb{A}_f^p)$ be its image induced by the map $\mathrm{GSpin}(V) \to \mathrm{SO}(V)$. Set $K = \mathrm{GSpin}(V)(\mathbb{Z}_p)K^p$, and $K_1 = \mathrm{SO}(V)(\mathbb{Z}_p)K_1^p$. Then the induced map between the corresponding integral canonical models

$$\mathscr{S}_K(\mathrm{GSpin}(V), X') \to \mathscr{S}_{K_1}(\mathrm{SO}(V), X)$$

is a finite étale Galois cover, cf. [30] Theorem 4.4.

When the level K is clear, the special fiber of $\mathscr{S}_K$ is denoted by $\mathscr{S}_0$, and the geometric special fiber is denoted by $\mathscr{S}_{\overline{\kappa}}$.

7.2. **Ekedahl-Oort stratifications.** Let (G,Y) and $\mathscr{S}_0$ be as above. The Shimura datum determines a cocharacter $\mu:\mathbb{G}_{m,\mathbb{Z}_p}\to G_{\mathbb{Z}_p}$ which is unique up to conjugation. The special fiber of μ will still be denoted by μ. The cocharacter μ determines a parabolic subgroup $P_+\subseteq G_{\mathbb{F}_p}$, whose type will be denoted by J. Let W be the Weyl group of $G_{\mathbb{F}_p}$, and ${}^JW$ together with the partial order $\preceq$ be as in 3.3 (before Theorem 3.2.1). Then Theorem 3.4.7 implies that the structure of Ekedahl-Oort stratification on $\mathscr{S}_{\overline{\kappa}}$ is described by ${}^JW$ together with the partial order $\preceq$.

7.2.1. *A description of* $({}^JW,\preceq)$. Let's recall the description of $({}^JW,\preceq)$ in [50] (see also [9] 6.4 and 6.6). Let m be the dimension of a maximal torus in $\mathrm{SO}(L_{\mathbb{F}_p})$. There are two cases:

Case 1. If n is odd, then the partial order $\preceq$ on ${}^JW$ is a total order, and the length function induces an isomorphism of totally ordered sets $({}^JW,\preceq)\xrightarrow{\sim}\{0,1,2,\cdots,n\}$.

Case 2. If n is even, noting that in this case $n+2=2m$, then W is generated by simple refections $\{s_i\}_{i=1,\cdots,m}$, where

$$s_i=\begin{cases}(i,i+1)(n-i+2,n-i+3), & \text{for } i=1,\cdots,m-1;\\ (m-1,m+1)(m,m+2), & \text{for } i=m.\end{cases}$$

Let

$$w_i=\begin{cases}s_1s_2\cdots s_i, & \text{for } i\leq m-1;\\ s_1s_2\cdots s_m, & \text{for } i=m;\\ s_1s_2\cdots s_ms_{m-2}\cdots s_{2m-i-1}, & \text{for } i\geq m+1.\end{cases}$$

and $w'_{m-1}=s_1s_2\cdots s_{m-2}s_m$. Then ${}^JW=\{w_i\}_{0\leq i\leq n}\cup\{w'_{m-1}\}$, and the partial order $\preceq$ is given by

$$\begin{aligned}w_0=\mathrm{id}\preceq w_1\preceq\cdots&\preceq w_{m-2}\\ &\preceq w_{m-1},w'_{m-1}\\ &\preceq w_m\preceq\cdots\preceq w_n.\end{aligned}$$

Applying Theorem 3.4.7 together with 7.2.1, we get the following description for the E-O stratification on $\mathscr{S}_{\overline{\kappa}}$.

Corollary 7.2.2. *Let m and n be as before.*

(1) *There are $2m$ Ekedahl-Oort strata on $\mathscr{S}_{\overline{\kappa}}$.*
(2) (a) *If n is odd, then for any integer $0\leq i\leq n$, there is precisely one stratum $\mathscr{S}^i_{\overline{\kappa}}$ such that $\dim(\mathscr{S}^i_{\overline{\kappa}})=i$. These are all the Ekedahl-Oort strata on $\mathscr{S}_{\overline{\kappa}}$. Moreover, the Zariski closure of $\mathscr{S}^i_{\overline{\kappa}}$ is the union of all the $\mathscr{S}^{i'}_{\overline{\kappa}}$ such that $i'\leq i$.*
  (b) *If n is even, then for any integer i such that $0\leq i\leq n$ and $i\neq n/2$, there is precisely one stratum $\mathscr{S}^i_{\overline{\kappa}}$ such that $\dim(\mathscr{S}^i_{\overline{\kappa}})=i$. There are 2 strata of dimension $n/2$. These are all the Ekedahl-Oort strata on $\mathscr{S}_{\overline{\kappa}}$. Moreover, the Zariski closure of the stratum $\mathscr{S}^w_{\overline{\kappa}}$ is the union of $\mathscr{S}^w_{\overline{\kappa}}$ with all the strata whose dimensions are smaller than $\dim(\mathscr{S}^w_{\overline{\kappa}})$.*

7.3. **Newton stratifications.** The pair $(G^{\mathrm{ad}}_{\mathbb{Q}_p},\mu)$ is of Coxeter type if $n\neq 2$, and it is always fully Hodge-Newton decomposable. More precisely,

- if $n\geq 5$ and odd, then it corresponds to type $(B_m,\omega_1^\vee,\mathbb{S})$ in [9] Theorem 5.1.2;
- if $n\geq 6$ and even, then it corresponds to type $(D_m,\omega_1^\vee,\mathbb{S})$ (resp. $({}^2D_m,\omega_1^\vee,\mathbb{S})$) there, if $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is residually split (not residually split).

For the exceptions,

- if $n=1$, it is $(A_1,\omega_1^\vee,\mathbb{S})$;
- if $n=3$, it is $(C_2,\omega_2^\vee,\mathbb{S})$;

- if $n = 4$, it is $(A_3, \omega_1^\vee, \mathbb{S})$ (resp. $({}^2A_3', \omega_1^\vee, \mathbb{S})$) when $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is residually split (not residually split).

When $n = 2$, it is no longer of Coxeter type as $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is no longer absolutely quasi-simple. But it is still fully Hodge-Newton decomposable. It is of type $(A_1, \omega_1^\vee, \mathbb{S}) \times (A_1, \omega_1^\vee, \mathbb{S})$ if $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is not ($\mathbb{Q}_p$-)simple, and $(A_1 \times A_1, (\omega_1^\vee, \omega_1^\vee), {}^1\varsigma_0)$ (see [10] 2.6) otherwise.

We are going to state relations between E-O strata, Newton strata and central leaves. In order to prevent listing them case by case, we introduce the following terminologies. For $n \geq 2$ an even number, we say that $\mathscr{S}_0$ is *of split type* if $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is either residually split when $n \geq 4$ or not ($\mathbb{Q}_p$-)simple when $n = 2$.

Corollary 7.3.1. *Let $\mathscr{S}_0$ be as in the end of 7.1.3, then each of its Newton stratum is a union of E-O strata, and each of its non-basic E-O is a central leaf in the Newton stratum containing it. Moreover, denote by m the rank of G^{ad},*

(1) *if n is odd, then the basic locus is of dimension $m - 1$;*
(2) *if n is even, then the basic locus is of dimension $m - 1$ if it is of split type, and of dimension m if it is not of split type.*

Proof. The first sentence follows from Proposition 6.2.7.

To see the dimension of basic locus, one can either use [9] 6.4 and 6.6, and compute the length of maximal elements in the basic locus, or reduce to GSpin-Shimura varieties and use [16] Theorem 6.4.1 directly. □

Finally, we refer the readers to [41] section 7 for some further discussions in the case $n = 19$ for applications to K3 surfaces and their moduli in mixed characteristic.

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