SEMINAR ON THE LOCAL LANGLANDS CONJECTURE FOR GL_n OVER p-ADIC FIELDS

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In this seminar, we want to understand Scholze's proof of the local Langlands conjecture for GL_n over p-adic fields, cf. [11], which, among others, simplifies substantially some arguments in the proof given by Harris-Taylor, cf. [6]. Fix a p-adic field F, i.e. a finite extension of $\mathbb{Q}_p$. Denote by $\mathcal{A}_n(F)$ the set of equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(F)$. On the other hand, denote by $\mathcal{G}_n(F)$ the set of equivalence classes of Frobenius semisimple n-dimensional complex Weil-Deligne representations of the Weil group W_F. Recall that the statement of the local Langlands conjecture for GL_n over F, now a theorem of Harris-Taylor ([6]) and Henniart ([9]), is as follows.

Theorem 1 (Harris-Taylor, Henniart)**.** *There is a unique collection of bijections*

$$\sigma : \mathcal{A}_n(F) \longrightarrow \mathcal{G}_n(F), \quad \pi \mapsto \sigma(\pi)$$

satisfying the following properties:

(1) *For $n = 1$, $\sigma(\pi)$ is given by local class field theory.*
(2) *For $\pi \in \mathcal{A}_n(F), \chi \in \mathcal{A}_1(F)$, we have*

$$\sigma(\pi \otimes \chi \circ \det) = \sigma(\pi) \otimes \sigma(\chi).$$

(3) *For $\pi \in \mathcal{A}_n(F)$ with central character ω_π, we have*

$$\sigma(\omega_\pi) = \det \sigma(\pi).$$

(4) *For $\pi \in \mathcal{A}_n(F)$ with contragredient dual $\pi^\vee$, we have*

$$\sigma(\pi^\vee) = \sigma(\pi)^\vee.$$

(5) *For $\pi_1 \in \mathcal{A}_{n_1}(F), \pi_2 \in \mathcal{A}_{n_2}(F)$, we have*

$$L(\pi_1 \times \pi_2, s) = L(\sigma(\pi_1) \otimes \sigma(\pi_2), s), \quad \varepsilon(\pi_1 \times \pi_2, s, \psi) = \varepsilon(\sigma(\pi_1) \otimes \sigma(\pi_2), s, \psi),$$

where $\psi : F \to \mathbb{C}^\times$ is a fixed additive character.

This collection does not depend on the choice of the additive character ψ.

As in [6] and [9], the proof of the above theorem given by Scholze in [11] also uses some global arguments, which are based on the study of some unitary Shimura varieties. There are two key ingredients in Scholze's proof. The first one is a new definition of the test functions, which appear naturally in the counting points formula for bad reductions of the involved Shimrua varieties, generalizing the formula in [4]. These test functions are constructed by moduli spaces of p-divisible groups, and can be generalized to more general PEL setting. The second ingredient is the computation of the inertia-invariant nearby cycles for these special Shimrua varieties. This calculation leads to a direct proof of the bijectivity of the correspondence, without using the numerical local Langlands correspondence, [7], in contrast to both [6] and [9].

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As in [6], along the way to prove the local Langlands conjecture for GL_n, Scholze also proves the existence of certain automorphic Galois representations and a local-global compatibility statement, cf. Theorem 1.3 of [11].

0. Introduction. First give some overview of the local Langlands conjecture. Then outline the proof given in [11].

1. Weil-Deligne representations. Recall the definition and basic properties of complex Weil-Deligne representations of W_F. Then prove Theorem 4.2.1 and Corollary 4.2.2 of [12], see [3] section 8, which gives the bijection between complex Weil-Deligne representations and continuous l-adic representations of W_F for a prime $l \neq p$. Recall also that by Jacobson-Morozov theorem, we can interpret a Weil-Deligne representation as a continuous complex semisimple representation of the group $W_F \times SL_2(\mathbb{C})$, cf. [13] 3.1.10.

2. L- and ε-factors on the Galois side. Define the L- and ε-factors for a Weil-Deligne representation of W_F, cf. [13] 3.2. In particular, state the theorem of Deligne on the existence of ε-factors and give a proof, cf. [3] section 4.

3. Smooth representations of p-adic reductive groups and the Zelevinsky classification for $\mathbf{GL}_n$. Review some generalities on the smooth representations of p-adic reductive groups, cf. [13] 2.1. Define the Bernstein center, cf. [1]. Then recall the Zelevinsky classification for GL_n following [13] 2.2 and [14].

4. L- and ε-factors on the $\mathbf{GL}_n$ side. Define the L- and ε-factors on the GL_n side, cf. [13] 2.5. Then give a sketch of the proof of the main result of [8], in the formulation with generic representations; also state the version with supercuspidal representations as a corollary.

5. Cuspidal automorphic representations of $\mathbf{GL}_n$. Explain the general definition of a cuspidal automorphic representation for GL_n as in [2] section 3.3. State the theorem on the complete decomposition of L^2_0, the tensor product theorem, the existence of global Whittaker models and local uniqueness of Whittaker models and indicate proofs where possible (one can restrict to $n = 2$ and $\mathbb{Q}$ whenever this helps). Deduce some form of (strong) multiplicity 1. Explain briefly how to convert classical modular forms into automorphic forms (representations) on $\mathrm{GL}_2/\mathbb{Q}$, following [5], Proposition 1.4 and Theorem 4.1.

6. Automorphic forms on $D^\times$ and the simple unitary groups. Define the simple unitary group G associated to a division algebra D over a CM filed, as in [6] I.7. Then study automorphic forms on $D^\times$ and G by the Global Jacquet-Langlands correspondence and base change: state Theorems VI.1.1, VI.2.1 and VI. 2.9 of [6]. Indicate the ideas for the proofs whenever possible.

7. Harris-Taylor's Shimura varieties. Introduce Harris-Taylor's Shimura varieties, as [6] III.1, III.4, and [11] section 8. Applying Theorem 3.4 and Theorem 5.3 of [10] to prove Lemma 5.5 and Corollary 5.6 of [10].

8. Deformation spaces of p-divisible groups and the test functions. Define the test functions $\phi_{\tau,h}$ in section 2 of [11], then prove Theorem 2.6 there. Cover the contents in section 4.

9. Properties of the test functions. Prove Lemma 3.2 of [11]. Then cover the contents in section 5 and 6.

10. Counting points modulo p. Define the global test functions for Harris-Taylor's Shimura varieties by proving Corollary 7.4 of [11]. Then give the ideas in the counting points arguments, following [11] section 9 and [4]. Prove Theorem 9.3 and Corollary 9.4 of [11].

11. l-adic cohomology and Galois representations attached to automorphic forms. Prove Corollary 10.3 of [11]. Then assuming the statements (a) and (b) of Theorem 1.2 of loc. cit, prove Theorem 10.6.

12. The l-adic cohomology of Lubin-Tate tower. Recall the definition of Lubin-Tate tower. Cover section 11 of [11].

13. Bijectivity of the correspondence. Prove Theorems 12.1 and 12.3 of [11].

14. Non-Galois automorphic induction and compatibility of L- and ε-factors. Prove Theorems 13.6 and 14.1 of [11].

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