PERFECTOID SHIMURA VARIETIES OF ABELIAN TYPE

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Let p be a fixed prime. For a Shimura datum (G, X), we have the associated tower of Shimura varieties $(\mathrm{Sh}_K(G,X))_{K\subset G(\mathbb{A}_f)}$ over $\mathbb{C}_p$. Fix a sufficient small prime to p level $K^p \subset G(\mathbb{A}_f^p)$, and consider open compact subgroups K in the form $K = K_pK^p$ with $K_p \subset G(\mathbb{Q}_p)$. Let $\mathrm{Sh}_{K_pK^p}(G,X)^{ad}$ be the associated adic spaces over $\mathbb{C}_p$. As usual, associated to the Shimura datum (G, X), we have the flag variety $\mathcal{FL}_G$ over $\mathbb{C}_p$, which will be viewed as an adic space. In [4], we proved the following theorem.

Theorem 1. *Assume that the Shimura datum (G, X) is of abelian type.*

(1) *There exists a perfectoid space S_{K^p} over $\mathbb{C}_p$ such that $S_{K^p} \sim \varprojlim_{K_p} \mathrm{Sh}_{K_pK^p}(G,X)^{ad}$. For the meaning of $\sim$, see the Definition 2.4.1 of* [8].

(2) *There is a $G(\mathbb{Q}_p)$-equivariant map of adic spaces*

$$\pi_{\mathrm{HT}} : S_{K^p} \longrightarrow \mathcal{FL}_G,$$

which is invariant for the prime to p Hecke action on S_{K^p}, when K^p varies. Moreover, pullbacks of automorphic vector bundles over finite level Shimura varieties to S_{K^p} can be understood by using the map π_{HT} (for a precise statement, see [4] *subsection 3.4).*

Recall that the basic theory of perfectoid spaces was developed in [5]. Recall also that Shimura varieties of abelian type are exactly those studied by Deligne in [2], where he proved that the canonical models of these Shimura varieties exist. When the weight is rational, Shimura varieties of abelian type (over characteristic 0) are known as moduli spaces of abelian motives. The class of abelian type Shimura varieties is strictly larger than the class of Hodge type Shimura varieties. By Deligne's classification, the class of abelian type Shimura varieties is also the main class of Shimura varieties. Natural examples of abelian type Shimura varieties (which are usually not of Hodge type) include those associated to quaternion algebras over a totally real field, and those associated to special orthogonal groups over $\mathbb{Q}$ with signature $(2, n)$ for some integer $n \geq 1$.

Before stating the ideas in the proof of the theorem, let us first give some remarks. If (G, X) is of Hodge type, then the theorem was proved by Scholze in [6] (and the part (2) for Hodge-Tate period map was completed by Caraini-Scholze in [1]). In fact, Scholze proved a stronger version for some compactification of Shimura varieties, which is the key geometric ingredient for his construction of automorphic Galois representations.

By definition, a Shimura datum (G, X) is called of abelian type if there exists a Shimura datum of Hodge type (G_1, X_1), together with a central isogeny between the derived subgroups $G_1^{der} \to G^{der}$, such that it induces an isomorphism of the associated adjoint Shimura datum $(G_1^{ad}, X_1^{ad}) \simeq (G^{ad}, X^{ad})$. Therefore, the geometry of Shimura varieties of abelian type and of Hodge type are closely related. The ideas in the proof of the theorem are as follows.

Step 1. For any Shimura datum (G, X), fix a connected component $X^+ \subset X$. We show that the statement (1) in the theorem is equivalent to the statement that, there exists a perfectoid space $S^0_{K^p}$ over $\mathbb{C}_p$, such that $S^0_{K^p} \sim \varprojlim_{K_p} \mathrm{Sh}^0_{K_pK^p}(G,X)^{ad}$, where

$\mathrm{Sh}^0_{K_pK^p}(G,X)^{ad}$ are the connected (adic) Shimura varieties which over $\mathbb{C}$ come from $X^+\times\{e\}$ (e is the identity element in $G(\mathbb{A}_f)$).

Step 2. Let (G,X) be of abelian type and (G_1,X_1) be of Hodge type as above. Consider the scheme over $\mathbb{C}_p$ defined by

$$\mathrm{Sh}^0_{K^p}(G,X)=\varprojlim_{K_p}\mathrm{Sh}^0_{K_pK^p}(G,X).$$

Then we can show that there exists some $K_1^p\subset G_1(\mathbb{A}_f^p)$ such that

$$\mathrm{Sh}^0_{K^p}(G,X)=\mathrm{Sh}^0_{K_1^p}(G_1,X_1)/\Delta,$$

where Δ is some finite group, acting freely on the scheme

$$\mathrm{Sh}^0_{K_1^p}(G_1,X_1)=\varprojlim_{K_{1p}}\mathrm{Sh}^0_{K_{1p}K_1^p}(G_1,X_1).$$

We would like a perfectoid version of this construction. By Step 1 and Scholze's result for (G_1,X_1), there is a perfectoid Shimura variety $S^0_{K_1^p}(G_1,X_1)$, such that $S^0_{K_1^p}(G_1,X_1)\sim\varprojlim_{K_{1p}}\mathrm{Sh}^0_{K_{1p}K_1^p}(G_1,X_1)^{ad}$. The key points are now

(1) Δ acts freely on $S^0_{K_1^p}(G_1,X_1)$, which implies that $S^0_{K_1^p}(G_1,X_1)/\Delta$ exists as a diamond (cf. [7]).
(2) In fact, $S^0_{K_1^p}(G_1,X_1)/\Delta$ exists as an adic space. Moreover, there is a finite étale Galois cover $S^0_{K_1^p}(G_1,X_1)\to S^0_{K^p}:=S^0_{K_1^p}(G_1,X_1)/\Delta$ with Galois group Δ.

Then by a theorem of Kedlaya-Liu (cf. [3] Proposition 3.6.22), $S^0_{K^p}$ is perfectoid. By construction, we have $S^0_{K^p}\sim\varprojlim_{K_p}\mathrm{Sh}^0_{K_pK^p}(G,X)^{ad}$. By Step 1 again, the statement (1) of the theorem holds.

Step 3. Let (G,X) and (G_1,X_1) be as in Step 2. Then we have $\mathcal{FL}_G=\mathcal{FL}_{G_1}$. By the results of Caraiani-Scholze, statement (2) of theorem holds for (G_1,X_1). Let $\pi'_{HT}:S_{K_1^p}(G_1,X_1)\longrightarrow\mathcal{FL}_G$ be the Hodge-Tate period map for the Hodge type perfectoid Shimura variety $S_{K_1^p}(G_1,X_1)$. The key point is then $\pi'_{HT}|_{S^0_{K_1^p}(G_1,X_1)}$ is Δ-invariant. So we get a map

$$\pi_{HT}:S^0_{K^p}\to\mathcal{FL}_G.$$

Then applying the $G(\mathbb{Q}_p)$-action, and the theory of connected components of Shimura varieties, we get that the statement (2) of the theorem holds.

Here we give briefly some applications of the theorem. First, if (G,X) is a Shimura datum of abelian type such that the associated Shimura varieties are compact, then we can use the theorem to deduce the vanishing of degree i-th completed cohomology of these varieties, where $i>dim\mathrm{Sh}_K$. Next, we can prove that the moduli spaces of polarized K3 surfaces with infinity level at p are perfectoid, by applying our theorem and the global Torelli theorem for K3 surfaces. We hope that this result will lead more interesting applications to the arithmetic of K3 surfaces.

References

[1] A. Caraiani, P. Scholze, *On vanishing of torsion in the cohomology of Shimura varieties*, in preparation.
[2] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, in "Automorphic forms, representations and L-functions" (Corvallis 1977), Proc. Sympos. Pure Math. XXXIII, Amer. Math. Soc., 247-289, 1979.
[3] K. S. Kedlaya, R. Liu, *Relative p-adic Hodge theory: Foundations*, arXiv:1301.0792, to appear in Astérisque.
[4] X. Shen, *Perfectoid Shimura varieties of abelian type*, arXiv: 1507.01824
[5] P. Scholze, *Perfectoid spaces*, Publ. Math. I.H.E.S.116 (2012), no.1, 245-313.

[6] P. Scholze, *On torsion in the cohomology of locally symmetric varieties*, Preprint, to appear in Ann. Math.
[7] P. Scholze, *Lectures on p-adic geometry*, notes taken by J. Weinstein, Fall 2014.
[8] P. Scholze, J. Weinstein, *Moduli of p-divisible groups*, Cambridge Journal of Mathematics 1 (2013), 145-237.