# A NOTE ON INTEGRAL STRUCTURES IN SOME LOCALLY ALGEBRAIC REPRESENTATIONS OF $\mathrm{GL}_2$

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Abstract. In the $p$-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, the following theorem of Berger and Breuil has played an important role: the locally algebraic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to crystabelline Galois representations admit a unique unitary completion. In this note, we give a new proof of the weaker statement that the locally algebraic representations admit *at most one* unitary completion and such a completion is automatically admissible. Our proof is purely representation theoretic, involving neither $(\varphi, \Gamma)$-module techniques nor global methods. When $F$ is a finite extension of $\mathbb{Q}_p$, we also get a simpler proof of a theorem of Vignéras for the existence of integral structures for (locally algebraic) special series and for (smooth) tamely ramified principal series.

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## 1. Introduction

Let $p$ be a prime number and $F$ be a finite extension of $\mathbb{Q}_p$ with $\mathcal{O}_F$ the ring of integers. We also fixe a finite extension $L$ of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$, which will serve as the coefficient field and be sufficiently large (in particular $L$ contains $F$).

Let $\Pi$ be a locally algebraic representation of $\mathrm{GL}_n(F)$ defined over $L$. It is a central and difficult question that whether there exist integral structures in $\Pi$. Here, by an integral structure we mean an $\mathcal{O}_L$-submodule $\mathcal{L}$ of $\Pi$ which is stable under $\mathrm{GL}_n(F)$, spans $\Pi$ over $L$ and contains no $L$-line (see for example [15, Def. 1.1]). This is equivalent to asking whether $\Pi$ admits non-zero $p$-adic unitary completion.

The first non-trivial examples were found by C. Breuil [5] in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$. One obvious necessary condition for the existence of integral structures is that the central character of $\Pi$ is unitary. In fact, Emerton's theory of Jacquet functor on locally analytic representations (in particular applicable to locally algebraic representations) provides other necessary conditions and, conjecturally, these conditions together with the unitarity of the central character are also *sufficient*. This is related to the so-called *Breuil-Schneider* conjecture, see [12], which turns out to be very difficult to prove in general. Here is a list of works surrounding this problem:[1]

(1) $G = \mathrm{GL}_2(\mathbb{Q}_p)$, see the work of Colmez [9] and Berger-Breuil [3] (both of the proofs use Fontaine's theory of $(\varphi,\Gamma)$-modules).
(2) $G = \mathrm{GL}_2(F)$, see the work of De Ieso [10], Vignéras [23], Kazhdan-De Shalit [15], and Assaf-Kazhdan-De Shalit [1]; the proofs are local and representation theoretic.
(3) $G = \mathrm{GL}_n(F)$, see the work of Sorensen [19] and Caraiani-Emerton-Geraghty-Gee-Paškūnas-Shin [7] (both of the proofs use global methods).

Note that, when $F \neq \mathbb{Q}_p$, the integral structures constructed in (2) do not give admissible unitary completions.

In this note, we (re)prove the following results (see below for the notation), firstly proved by Vignéras for (i) and (ii), and by Berger-Breuil for (iii).

**Theorem 1.1.** *(Theorems 2.8, 2.10, 3.8) Let $G = \mathrm{GL}_2(F)$ and $\Pi = \Pi_{\mathrm{sm}} \otimes \Pi_{\mathrm{alg}}$ be an irreducible locally algebraic $L$-representation of $G$. Assume that the central character of $\Pi$ is unitary.*

(i) *Assume $\Pi_{\mathrm{sm}}$ is a special series representation. Then $\Pi$ admits an integral structure.*
(ii) *Assume $\Pi = \Pi_{\mathrm{sm}} = \mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ is irreducible principal series with $\chi_1, \chi_2$ tamely ramified characters such that $\chi_1|_{\mathcal{O}_F^\times} \neq \chi_2|_{\mathcal{O}_F^\times}$. Then $\Pi$ admits an integral structure if and only if $1 \le |\chi_1(\varpi_F)| \le |q^{-1}|$.*
(iii) *Assume $F = \mathbb{Q}_p$ and $\Pi_{\mathrm{sm}} = \mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ (irreducible). If $\Pi$ admits an integral structure, say $\mathcal{L}$, then $\mathcal{L}$ is necessarily finitely generated as an $\mathcal{O}_L[G]$-module and is residually of finite length. Moreover, the universal unitary completion of $\Pi$ is irreducible.*

**Remark 1.2.** *Note that in* [8, §5]*, another proof of (iii) is given, but under mild restrictions. The proof, although local, involves certain projective envelopes of $\mathrm{GL}_2(\mathbb{Z}_p)$-representations. Our proof is elementary and simpler, using only basic results on diagrams and a key observation found in* [14, Prop. 4.1]*. More interestingly, our proof provides an interpretation of the $p$-adic Hodge coincidence that there exists only one weakly admissible filtration with given jumps on the underlying Weil-Deligne representation attached to two-dimensional crystabelline Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$; see §3.*

**Notations:** Let $G = \mathrm{GL}_2(F)$ and $K = \mathrm{GL}_2(\mathcal{O}_F)$ and $Z$ be the center of $G$. We fix a uniformizer $\varpi_F$ of $\mathcal{O}_F$ and let $q = |\mathcal{O}_F/\varpi_F|$. Define the following subgroups

[1] The list may not be complete and we refer to the cited papers for the precise conditions imposed. See also [20] for a nice exposition about this problem.

of $G$ (where $m \geq 1$):

$$I := \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ \varpi_F \mathcal{O}_F & \mathcal{O}_F^\times \end{pmatrix}, \quad I_m := \begin{pmatrix} 1+\varpi_F^m \mathcal{O}_F & \varpi_F^{m-1}\mathcal{O}_F \\ \varpi_F^m \mathcal{O}_F & 1+\varpi_F^m\mathcal{O}_F \end{pmatrix},$$

$$K_m := \begin{pmatrix} 1+\varpi_F^m \mathcal{O}_F & \varpi_F^{m}\mathcal{O}_F \\ \varpi_F^m \mathcal{O}_F & 1+\varpi_F^m\mathcal{O}_F \end{pmatrix}.$$

Let $\mathfrak{R}_0$ be the $G$-normalizer of $K$ so that $\mathfrak{R}_0 = KZ$, and $\mathfrak{R}_1$ be the $G$-normalizer of $I$ so that $\mathfrak{R}_1$ is generated by $I$ and $t := \left(\begin{smallmatrix} 0 & 1 \\ \varpi_F & 0 \end{smallmatrix}\right)$ as a group. One checks that $\mathfrak{R}_0 \cap \mathfrak{R}_1 = IZ$. Let $\mathrm{val}_F$ be the $p$-adic valuation on $F$ normalized as $\mathrm{val}_F(\varpi_F) := 1$.

We write $\mathcal{O} = \mathcal{O}_L$ and let $k = k_L$ be the residue field of $\mathcal{O}$. Fix a uniformizer $\varpi = \varpi_L$ of $\mathcal{O}$ and let $\mathrm{val}_L$ be the normalized $p$-adic valuation on $L$.

Let $B$ be the upper Borel subgroup of $G$. Given two characters $\chi_1, \chi_2 : F^\times \to L^\times$, we consider $\chi_1 \otimes \chi_2$ as a character of $B$ sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $\chi_1(a)\chi_2(d)$ and let $\mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ denote the principal series representation of $G$.

Finally, if $H$ is a group, $A$ is a commutative ring, $W$ is an $A[H]$-module and $W_1 \subset W$ is any subset, we let $\langle H.W_1 \rangle$ denote the sub-$A[H]$-module of $W$ generated by $W_1$.

## 2. Diagrams

Let $A$ be a topological commutative ring, typically $A = L, \mathcal{O}, k$. By a diagram $D$ (for $\mathrm{GL}_2$) of (continuous) $A$-modules, we mean the data $(D_0, D_1, r)$, where $D_0$ (resp. $D_1$) is an $A$-module with a continuous action of $\mathfrak{R}_0$ (resp. $\mathfrak{R}_1$), and $r : D_1 \to D_0$ is an $IZ$-equivariant continuous homomorphism of $A$-modules. Diagrams of $A$-modules with obvious morphisms form an abelian category. Attached to a diagram, we can define a $G$-equivariant homomorphism $\partial : \text{c-Ind}_{\mathfrak{R}_1}^G D_1 \otimes \delta_{-1} \to \text{c-Ind}_{\mathfrak{R}_0}^G D_0$ (see [6, §9] or [17, §3]), where $\delta_{-1}$ denotes the (continuous) character of $\mathfrak{R}_1$ (to $A$) of order 2 sending $g$ to $(-1)^{\mathrm{val}_F(\det g)}$, and $\partial$ is the $G$-equivariant morphism determined by

$$\partial([\mathrm{Id}, x]) = [\mathrm{Id}, r(x)] - [t, r(t^{-1} \cdot x)] \in \text{c-Ind}_{\mathfrak{R}_0}^G D_0, \quad \forall x \in D_1 \otimes \delta_{-1}. \tag{1}$$

The kernel and cokernel of $\partial$ are denoted by $H_1(D)$ and $H_0(D)$ respectively, so that we have an exact sequence

$$0 \to H_1(D) \to \text{c-Ind}_{\mathfrak{R}_1}^G D_1 \otimes \delta_{-1} \xrightarrow{\partial} \text{c-Ind}_{\mathfrak{R}_0}^G D_0 \to H_0(D) \to 0.$$

By definition, a short exact sequence of diagrams of $A$-modules $0 \to D' \to D \to D'' \to 0$ gives a long exact sequence

$$0 \to H_1(D') \to H_1(D) \to H_1(D'') \to H_0(D') \to H_0(D) \to H_0(D'') \to 0. \tag{2}$$

Note that, if $\pi$ is a continuous $A$-representation of $G$, we get trivially a diagram $\mathcal{K}(\pi) := (\pi|_{\mathfrak{R}_0}, \pi|_{\mathfrak{R}_1}, \mathrm{Id})$. One has that $H_0(\mathcal{K}(\pi)) \cong \pi$ by [16, Lem. 5.4.2] and $H_1(\mathcal{K}(\pi)) = 0$ by Lemma 2.1 below.

### 2.1. Mod $p$ diagrams with trivial $H_0$.

In this subsection, we only consider diagrams of $k$-modules. Since $k$ is equipped with the discrete topology, the action of $\mathfrak{R}_i$ on $D_i$ is smooth. We first recall the following result.

**Lemma 2.1.** *Let $D$ be a diagram of $k$-modules such that $D_0$ is an admissible $\mathfrak{R}_0$-representation and $r$ is injective. Then $H_0(D) \neq 0$ and $H_1(D) = 0$.*

*Proof.* The first assertion is [16, Lem. 5.3.2] and the second is [23, Lem. 1.3]. $\square$

**Proposition 2.2.** *Let $D=(D_0,D_1,r)$ be a diagram of $k$-modules, not necessarily finite dimensional, such that $H_0(D)=0$. Then $D$ has a filtration by sub-diagrams such that each graded piece has one of the following three forms $(Q_0,Q_1,r)$:*

(I) *$Q_1=k\cdot v$, $Q_0=0$, $r=0$;*

(II) *$Q_1=k\cdot v\oplus k\cdot t(v)$ where $IZ$ acts on $v$ via some character $\psi$, $Q_0\cong \mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(k\cdot t(v))$, $r|_{k\cdot v}=0$ and $r|_{k\cdot t(v)}$ is the natural map;*

(III) *$Q_1=k\cdot v\oplus k\cdot t(v)$ where $IZ$ acts on $v$ via some character $\psi$, $Q_0$ is a quotient of $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(k\cdot t(v))$ such that $\dim_k Q_0\leq q$ (possibly $0$), $r|_{k\cdot v}=0$ and $r|_{k\cdot t(v)}$ is the natural map.*

*In particular, if $D_0$ is of finite dimension, then $\dim_k D_0\leq \dim_k D_1\cdot\frac{q+1}{2}$ and the equality holds if and only if only diagrams of type (II) appear as graded pieces of the filtration.*

**Remark 2.3.** *Consider a diagram $Q$ of type* (III). *Since $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(k\cdot t(v))$ has dimension $q+1$, the condition $\dim_k Q_0\leq q$ is equivalent to demanding that $Q_0$ is a proper quotient of $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(k\cdot t(v))$. When $F=\mathbb{Q}_p$, this is again equivalent to demanding that $Q_0$ is irreducible or zero, since $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0}\psi$ has length $2$ for any smooth character $\psi: IZ\to k^\times$.*

*Proof.* Since $H_0(D)=0$, Lemma 2.1 implies that $r$ is not injective. Choose a non-zero vector $v\in\big(\ker(r)\big)^{I_1}$ and write $M=k\cdot v$. Since the order of $I/I_1$ is prime to $p$, we may choose $v$ to be an eigenvector for $I$, i.e. $M$ is stable under $I$. Consider the sub-diagram $Q:=(Q_0,Q_1,r_Q)$ of $D$ defined by

$$Q_1=M+t(M),\quad Q_0=\langle\mathfrak{R}_0.r(t(M))\rangle,\quad r_Q=r|_{Q_1}.$$

In particular, $r_Q=0$ on $M$. Remark that we *do not* guarantee that $v$ and $t(v)$ are linearly independent over $k$; indeed $Q_0$ could be zero and $r_Q$ be identically zero; in this case $Q$ is of type (I) in the statement. If $v$ and $t(v)$ are linearly independent then $\dim_k Q_1=2$. By Frobenius reciprocity, $Q_0$ is a quotient of $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0}t(M)$ and $Q$ is of type (II) if $Q_0\cong\mathrm{Ind}_{IZ}^{\mathfrak{R}_0}t(M)$, or equivalently $\dim_k Q_0=q+1$, and of type (III) otherwise. Note that $r$ could be identically zero, hence $Q_0=0$, in case of type (III).

Since $H_0(D/Q)=0$ by (2), we can continue the above construction for $D/Q$ and in this way get a filtration of $D$ by sub-diagrams whose graded pieces are one of the three types (I)-(III). If $D_0$ is of finite dimension, the filtration is also finite. The last assertion follows from the corresponding dimension inequality for the graded pieces $Q$. □

2.2. **Naive diagrams.** In this subsection, we classify diagrams of $k$-modules with trivial $H_0$ and $H_1$.

**Definition 2.4.** *Let $D=(D_0,D_1,r)$ be a diagram of $k$-modules such that $D_0$ and $D_1$ are both finite dimensional. We say that $D$ satisfies the* dimension relation *if there exists $d\in\mathbb{Z}_{\geq 0}$ such that*

$$\dim_k D_1=2d,\quad \dim_k D_0=d(q+1).$$

We give some examples of diagrams which satisfy the dimension relation. For an absolutely irreducible $k$-representation $\sigma$ of $\mathfrak{R}_0$, $\lambda\in k$ and $\chi: F^\times\to k^\times$ a smooth character, we recall the usual notation [4]:

$$\pi(\sigma,\lambda,\chi):=\big(\text{c-Ind}_{\mathfrak{R}_0}^{G}\sigma/(T-\lambda)\big)\otimes\chi\circ\det,$$

where $T \in \mathrm{End}_G(\text{c-Ind}_{\mathfrak{R}_0}^G \sigma)$ is the Hecke operator defined in [2].

**Exemple 2.5.** *Let $\pi = \pi(\sigma, \lambda, \chi)$ for some $\sigma, \lambda, \chi$ as above and assume $F = \mathbb{Q}_p$ if $\lambda = 0$. Then the canonical diagram (see* [13]*) $D(\pi) := (D_0(\pi), D_1(\pi), \mathrm{can})$ defined by*

$$D_1(\pi) := \pi^{I_1}, \quad D_0(\pi) := \langle \mathfrak{R}_0.D_1(\pi) \rangle \subset \pi, \quad \mathrm{can} : D_1(\pi) \hookrightarrow D_0(\pi)$$

*satisfies the dimension relation. In fact, using results of* [2] *and* [4] *(when $\lambda = 0$ and $F = \mathbb{Q}_p$), one checks easily that $\dim_k D_1(\pi) = 2$ and $\dim_k D_0(\pi) = q+1$ (resp. $p+1$ when $\lambda = 0$ in which case $F = \mathbb{Q}_p$).*

Note that the canonical diagram $D(\mathrm{Sp})$ (resp. $D(\mathbf{1})$) of the Steinberg representation Sp (resp. the trivial representation $\mathbf{1}$) does not satisfy the dimension relation (but $D(\mathrm{Sp}) \oplus D(\mathbf{1})$ does). Another example of diagrams satisfying the dimension relation is a diagram of type (II) in Proposition 2.2. We give it a name for convenience.

**Definition 2.6.** *A diagram $D = (D_0, D_1, r)$ of $k$-modules is said to be* naive *if it is of type* (II) *as in Proposition 2.2.*

By definition, if $D = (D_0, D_1, r)$ is a naive diagram, then $\dim_k D_1 = 2$ and $\dim_k D_0 = q+1$, hence $D$ satisfies the dimension relation with $d = 1$ in Definition 2.4.

**Lemma 2.7.** *(i) If $D$ is a naive diagram, then $H_0(D) = H_1(D) = 0$.*

*(ii) Conversely, if $D = (D_0, D_1, r)$ is a diagram of $k$-modules such that $H_0(D) = H_1(D) = 0$, then $D$ can be written as a successive extension of naive diagrams. In particular, if $D_0$ and $D_1$ are finite dimensional, then $D$ satisfies the dimension relation.*

*Proof.* (i) By definition of $D$, there exists some $D_1^+ \subset D_1$, a sub-$IZ$-representation, such that

$$\text{c-Ind}_{\mathfrak{R}_1}^G D_1 \otimes \delta_{-1} \cong \text{c-Ind}_{IZ}^G t(D_1^+) \cong \text{c-Ind}_{\mathfrak{R}_0}^G D_0.$$

Moreover, one checks that if we identify both the source and the target with $\text{c-Ind}_{IZ}^G t(D_1^+)$, then $\partial$ is exactly the identity morphism. The result follows.

(ii) We may assume $D$ is non-zero. First, by Proposition 2.2, $D$ admits a sub-diagram $Q$ which is one of the three types (I)-(III). It suffices to show $Q$ is naive. Since $H_1(Q) \hookrightarrow H_1(D)$ and $H_1(D) = 0$ by assumption, we have $H_1(Q) = 0$. Therefore, it suffices to show that diagrams of type (I) or (III) always have non-zero $H_1$. This is an easy exercise. □

2.3. **Diagrams in characteristic** $0$. Let $\Pi_{\mathrm{sm}}$ be a finite length smooth representation of $G$ on an $L$-vector space. Let $c \geq 1$ be an integer such that $\Pi_{\mathrm{sm}}$ is generated by its $K_c$-invariants. To $\Pi_{\mathrm{sm}}$ one may associate a diagram $\Pi_{\mathrm{sm}}^{I_c} \hookrightarrow \Pi_{\mathrm{sm}}^{K_c}$. As a special case of a theorem of Schneider-Stuhler [21, Thm. V.1], we know that

$$H_0(\Pi_{\mathrm{sm}}^{I_c} \hookrightarrow \Pi_{\mathrm{sm}}^{K_c}) \cong \Pi_{\mathrm{sm}}.$$

If moreover, $\Pi_{\mathrm{alg}}$ is an irreducible algebraic $L$-representation of $G$, we set $\Pi = \Pi_{\mathrm{sm}} \otimes_L \Pi_{\mathrm{alg}}$ and

$$X = (X_1 \xrightarrow{r} X_0) := (\Pi_{\mathrm{sm}}^{I_c} \hookrightarrow \Pi_{\mathrm{sm}}^{K_c}) \otimes \Pi_{\mathrm{alg}}.$$

Then we have (see [23, Prop. 0.4])

$$H_0(X) \cong \Pi. \tag{3}$$

By a diagram of sub-$\mathcal{O}$-lattices $\mathcal{X}$ in $X$, we mean $\mathcal{X}_0$ (resp. $\mathcal{X}_1$) is an $\mathcal{O}$-lattice inside $X_0$ (resp. $X_1$) and the morphism $\mathcal{X}_1 \to \mathcal{X}_0$ is the restriction of $r : X_1 \to X_0$. We have a natural morphism $H_0(\mathcal{X}) \to H_0(X) \cong \Pi$ which, however, need not be injective.

Starting from a diagram of $\mathcal{O}$-lattices $\mathcal{X}$ in $X$, Vignéras constructs in [23] a sequence of diagrams of $\mathcal{O}$-lattices $(\mathcal{X}^{(n)})_{n\geq 0}$ with $\mathcal{X}_0 = \mathcal{X}$ (denoted by $(z^n(\mathcal{X}))_{n\geq 1}$ in *loc. cit.*). The construction is as follows: knowing $\mathcal{X}^{(n)}$, we let inductively

- $\mathcal{X}_1^{(n+1)} = \mathcal{X}_0^{(n)} + t(\mathcal{X}_0^{(n)})$;
- $\mathcal{X}_0^{(n+1)} = \langle \mathfrak{R}_0.\mathcal{X}_1^{(n+1)}\rangle$.

By construction the natural map $H_0(\mathcal{X}^{(n)}) \to H_0(\mathcal{X}^{(n+1)})$ is surjective for any $n$. Moreover, by [23, Cor. 0.3], $\Pi$ admits an integral structure if and only if the sequence $(\mathcal{X}^{(n)})_{n\geq 0}$ stabilizes.

2.4. **Application I.** Our first application of the techniques developed above is a simple proof of the following result of Vignéras [23, Prop. 0.9]. Let St denote the smooth Steinberg $L$-representation of $G$.

**Theorem 2.8.** *Let* $\Pi = \mathrm{St} \otimes \mathrm{Sym}^k L^2 \otimes |\det|^{k/2}$ *for some integer* $k \geq 0$. *Then* $\Pi$ *admits an integral structure.*

*Proof.* In the notation of §2.3, we may take $c = 1$ so that

$$X_0 = \mathrm{St}^{K_1} \otimes \mathrm{Sym}^k L^2 \otimes |\det|^{k/2}, \quad X_1 = \mathrm{St}^{I_1} \otimes \mathrm{Sym}^k L^2 \otimes |\det|^{k/2}. \tag{4}$$

It is clear that the central character of $\Pi$ is unitary. Since $\mathfrak{R}_1/Z$ is compact, there exist open bounded $\mathfrak{R}_1$-stable $\mathcal{O}$-lattices inside $X_1$. We fix such a lattice $\mathcal{X}_1$ and let $\mathcal{X}_0 := \langle \mathfrak{R}_0.\mathcal{X}_1\rangle$, which is an open bounded $\mathcal{O}$-lattice in $X_0$. Let $\mathcal{X}^{(0)} := \mathcal{X}$ and $(\mathcal{X}^{(n)})_{n\geq 0}$ be the sequence of diagrams of $\mathcal{O}$-modules obtained by applying Vignéras' algorithm. If the sequence is finite, we are done; so we assume it is infinite in the rest of the proof. Since $X_1$ is irreducible as an $\mathfrak{R}_1$-representation[2] and since the coefficient field $L$ is discretely valued, there are only finitely many homothety classes of $\mathfrak{R}_1$-invariant $\mathcal{O}$-lattices in $X_1$. Therefore there exist integers $n < n'$ such that $\mathcal{X}_1^{(n)}$ and $\mathcal{X}_1^{(n')}$ lie in the same homothety class, that is, there exists $\lambda \in L^\times$ such that

$$\mathcal{X}_1^{(n')} = \lambda \mathcal{X}_1^{(n)}.$$

Since $\mathcal{X}_0^{(n)}$ (resp. $\mathcal{X}_0^{(n')}$) is generated by $\mathcal{X}_1^{(n)}$ (resp. $\mathcal{X}_1^{(n')}$), we get

$$\mathcal{X}^{(n')} = \lambda \mathcal{X}^{(n)}.$$

Moreover, since $\mathcal{X}^{(n)} \subsetneq \mathcal{X}^{(n')}$, we have $\mathrm{val}_L(\lambda) < 0$.

To simplify the notation we assume $\mathcal{X} = \mathcal{X}^{(n)}$, i.e. $n = 0$. Since the natural morphism $H_0(\mathcal{X}) \to H_0(\lambda\mathcal{X})$ is surjective, we have $H_0(\lambda\mathcal{X}/\mathcal{X}) = 0$. Noting that $\lambda\mathcal{X} = \varpi^{\mathrm{val}_L(\lambda)}\mathcal{X}$, we deduce by dévissage that $H_0(\varpi^{-1}\mathcal{X}/\mathcal{X}) = 0$, equivalently $H_0(\mathcal{X} \otimes_{\mathcal{O}} k) = 0$. By Proposition 2.2, this implies that

$$\dim_k(\mathcal{X}_0 \otimes_{\mathcal{O}} k) \leq \dim_k(\mathcal{X}_1 \otimes_{\mathcal{O}} k) \cdot \frac{q+1}{2},$$

but this is not the case (4) since $\dim_k(\mathcal{X}_i \otimes_{\mathcal{O}} k) = \mathrm{rank}_{\mathcal{O}}\mathcal{X}_i = \dim_L X_i$ for $i \in \{0, 1\}$. ∎

[2]Indeed, $\mathrm{St}^{I_1}$ is 1-dimensional and $\mathrm{Sym}^k L^2$ is irreducible as an $\mathfrak{R}_1$-representation

**Remark 2.9.** *It is known, at least in the case $F = \mathbb{Q}_p$ and $k \geq 1$, that the universal unitary completion of* $\mathrm{St} \otimes \mathrm{Sym}^k L^2 \otimes |\det|^{k/2}$ *is not admissible (in the sense of Schneider-Teitelbaum* [22]*).*

2.5. **Application II.** In this subsection, we reprove (under a mild extra condition) a result of Vignéras [23, Thm. 0.10] about the existence of integral structures in (smooth) tamely ramified principal series. Kazhdan and De Shalit have given another proof using Kirillov models, see [15, Thm. 1.2]. Our proof is motivated by Vignéras', but has the advantage that the computation needed is very small.

**Theorem 2.10.** *Let $\Pi = \mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ be a smooth principal series $L$-representation (i.e. the algebraic part is trivial). Assume that $\chi_1, \chi_2 : F \to L^\times$ are tamely ramified and $\chi_1|_{\mathcal{O}_F^\times} \neq \chi_2|_{\mathcal{O}_F^\times}$. Then $\Pi$ admits an integral structure if and only if $\chi_1\chi_2$ is unitary and $1 \leq |\chi_1(\varpi_F))| \leq |q^{-1}|$.*

*Proof.* The necessity is well-known, see [23] or [15, §3.1].

For the sufficiency, note that we may take

$$X_0 = \Pi^{K_1}, \quad X_1 = \Pi^{I_1}$$

in the notation of §2.3. In particular, we have $\dim_L X_1 = 2$ and $\dim_L X_0 = q+1$. Assume that $\Pi$ does not admit an integral structure. Since $X_1$ is irreducible as an $\mathfrak{R}_1$-representation by the assumption on $\chi_i$, the proof of Theorem 2.8 produces a diagram of $\mathcal{O}$-modules $\mathcal{X} \subset X$ such that $H_0(\mathcal{X} \otimes_{\mathcal{O}} k) = 0$. Write $D = \mathcal{X} \otimes_{\mathcal{O}} k$. The assumption on $\chi_1, \chi_2$ also implies that $D_1$ is irreducible as an $\mathfrak{R}_1$-representation. Since the dimension relation holds for $D$, Proposition 2.2 implies that $D$ is a naive diagram.

Again using the assumption $\chi_1 \neq \chi_2$ on $\mathcal{O}_F^\times$, we have $\mathcal{X}_1 = \mathcal{O} \cdot v \oplus \mathcal{O} \cdot t(v)$, where $v$ is a non-zero vector on which $I$ acts via $(\chi_1 \otimes \chi_2)|_I$, hence $D_1 = k \cdot v \oplus k \cdot \overline{t(v)}$. Since $D$ is naive, *exactly* one of $\overline{v}$ and $\overline{t(v)}$ is sent to zero via the natural morphism $\overline{r} : D_1 \to D_0$. Without loss of generality, we assume that $\overline{r}(\overline{t(v)}) = 0$, i.e. $t(v) \in \varpi\mathcal{X}_0$. Then we obtain

$$D_0 = \langle \mathfrak{R}_0 . \overline{v} \rangle \cong \mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(k \cdot \overline{v}).$$

Using Nakayama's lemma, this implies that $\mathcal{X}_0 = \mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(\mathcal{O} \cdot v)$ (here we use the assumption that $\chi_1, \chi_2$ are tamely ramified). In particular, $\mathcal{X}_0$ has an $\mathcal{O}$-basis given by

$$v, \quad \sum_{\lambda \in \mathbb{F}_q} [\lambda]^i \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} v, \quad 0 \leq i \leq q-1.$$

However, an easy computation shows that

$$t(v) = \chi_1(\varpi_F) \sum_{\lambda \in \mathbb{F}_q} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} v,$$

so $t(v) \in \varpi\mathcal{X}_0$ if and only if $|\chi_1(\varpi_F)| < 1$. The assumption on $\chi_1(\varpi_F)$ then forces that $D$ is not a naive diagram, giving the desired contradiction. □

## 3. The case of $\mathrm{GL}_2(\mathbb{Q}_p)$

In this section, we assume $F = \mathbb{Q}_p$ so that $G = \mathrm{GL}_2(\mathbb{Q}_p)$. In [3], Berger and Breuil proved that the locally algebraic representations associated to crystabelline Galois representations admit a unique non-zero unitary completion. This fact, very important in the $p$-adic local Langlands programme for $\mathrm{GL}_2(\mathbb{Q}_p)$, corresponds to the $p$-adic Hodge theoretic coincidence that there exists only one weakly admissible filtration on the underlying Weil-Deligne representation with given jumps (determined by the Hodge-Tate weights of the Galois representation). This phenomenon only happens for the group $\mathrm{GL}_2(\mathbb{Q}_p)$ and crystabelline representations. In [8, §5], the uniqueness part is reproved under mild conditions. We give a new proof here, based on the techniques developed in last section. We will see that the coincidence can be interpreted as a certain dimension relation.

3.1. **Standard diagrams.** Let $\pi$ be a smooth $k$-representation of $G$ of finite length and $D$ be a sub-diagram of $\mathcal{K}(\pi)$. Note that $r : D_1 \to D_0$ is injective and $H_1(D) = 0$.

**Definition 3.1.** *We say that $D$ is a* standard diagram[3] *of $\pi$ if $D_0$ is finite dimensional and the natural morphism $H_0(D) \to \pi$ is an isomorphism.*

The assumption $G = \mathrm{GL}_2(\mathbb{Q}_p)$ guarantees the existence of standard diagrams $D$ of $\pi$; see [9, Chap. III]. When $\pi$ is irreducible, we know that

$$D(\pi) = (D_0(\pi), D_1(\pi), \mathrm{can}) := (\langle \mathfrak{R}_0.\pi^{I_1}\rangle, \pi^{I_1}, \mathrm{can})$$

is a standard diagram of $\pi$ ([6, §10] or [9, Chap. III]). It is also called the canonical diagram associated to $\pi$ in [13], in the sense that $D(\pi)$ is the *smallest* standard diagram of $\pi$. We give a proof of this fact for completeness.

**Lemma 3.2.** *Let $\pi$ be an absolutely irreducible smooth $k$-representation of $G$ with a central character and $D$ be a standard diagram of $\pi$. Then $D$ contains the diagram $D(\pi)$.*

*Proof.* We need to show the inclusions (i) $D_1(\pi) \subseteq D_1$ and (ii) $D_0(\pi) \subseteq D_0$. First remark that it suffices to check either of them. In fact, since $D_0(\pi) = \langle \mathfrak{R}_0.D_1(\pi)\rangle$ and $D_0 = \langle \mathfrak{R}_0.D_1\rangle$, (ii) follows from (i); on the other hand, we have $D_1(\pi) = D_0(\pi) \cap t(D_0(\pi))$ and $D_1 = D_0 \cap t(D_0)$, so (i) follows from (ii). Note that, we always have $D_1 \cap D_1(\pi) \neq 0$ (as $D_1^{I_1} \neq 0$) and $D_0 \cap D_0(\pi) \neq 0$ as we can check that $D_0(\pi) \supseteq \mathrm{soc}_K\pi$, where $\mathrm{soc}_K\pi$ denotes the $K$-socle of $\pi$.

We refer to [9, §III.3] or [6, §10] for the explicit structure of $D(\pi)$. If $\pi$ is a special series representation or a character, then $D_1(\pi) = \pi^{I_1}$ is 1-dimensional, hence is contained in $D_1$ (since $D_1^{I_1} \neq 0$) and the proof is finished. If $\pi$ is a ramified principal series representation (see [6, §10,(iv)]), then $D_1(\pi) = \pi^{I_1}$ is of the form $\psi \oplus \psi^s$ with $\psi \neq \psi^s$. Since $D_1^{I_1}$ is non-zero and stable under $t$, we must have $D_1(\pi) \subseteq D_1$ and the result follows again. Finally, in all other cases, that is $\pi$ is either supersingular or an unramified principal series representation, we have $D_0(\pi)|_K = \mathrm{soc}_K\pi \cong \sigma_1 \oplus \sigma_2$ is the direct sum of two *non-isomorphic* irreducible $K$-representations and $D_1(\pi) = \sigma_1^{I_1} \oplus \sigma_2^{I_1}$ is two-dimensional (however, the eigencharacters of $I$ acting on $\sigma_i^{I_1}$ are possibly equal). Since $D_1 \cap \pi^{I_1} \neq 0$, $D_1$ contains a non-zero vector $v \in \pi^{I_1}$. If $v \in \sigma_1$ (resp. $v \in \sigma_2$), then by the explicit description of $D(\pi)$ (see [6, §10,(iii),(iv)]), we have $t(v) \in \sigma_2$ (resp. $t(v) \in \sigma_1$) so that the inclusion (i) holds. If $v \notin \sigma_1$ and $v \notin \sigma_2$, then $\langle \mathfrak{R}_0.v\rangle$ is equal to $\sigma_1 \oplus \sigma_2$ because $\sigma_1$ and $\sigma_2$ are non-isomorphic, which implies the inclusion (ii). This finishes the proof. $\square$

[3]The notion comes from Colmez's "*présentation standard*", see [9].

**Lemma 3.3.** *Let $\pi$ be a smooth $k$-representation of $G$ of finite length and $D$ be a standard diagram of $\pi$. Let $\pi' \subset \pi$ be a sub-$G$-representation and $\pi''$ be the corresponding quotient. Then $D\cap\mathcal{K}(\pi')$ is a standard diagram of $\pi'$ and $D/(D\cap\pi')$ is a standard diagram of $\pi''$.*

*Proof.* Write $D' := D\cap\pi'$ and $D'' := D/D'$. By definition, we know that $D'$ (resp. $D''$) is a sub-diagram of $\mathcal{K}(\pi')$ (resp. $\mathcal{K}(\pi'')$). In particular, we have $H_1(D') = H_1(D'') = 0$. The exact sequence $0 \to D' \to D \to D'' \to 0$ then gives a short exact sequence

$$0 \to H_0(D') \to H_0(D) \to H_0(D'') \to 0.$$

In particular, $H_0(D')$ and $H_0(D)$ are both of finite length since $H_0(D)$ is. It then follows from [14, Prop. 4.1] that the natural morphisms $H_0(D') \to \pi'$ and $H_0(D'') \to \pi''$ are both injective, hence are also surjective for the reason of lengths. □

We introduce one more notion. If $\pi$ is a $k$-representation of $G$ of finite length and if $\tau$ is an irreducible $k$-representation of $G$, we set

$$[\pi : \tau] := \dim \mathrm{Hom}_G(\tau, \pi^{\mathrm{ss}}),$$

i.e. the multiplicity with which $\tau$ appears in the semisimplification $\pi^{\mathrm{ss}}$. Similarly we have the notion for $\mathfrak{R}_0$-representations.

**Proposition 3.4.** *Let $\pi$ be a smooth $k$-representation of $G$ of finite length and with central character $\chi$. Let $D \hookrightarrow \mathcal{K}(\pi)$ be a standard diagram of $\pi$. The following statements hold.*

*(i) There exists $r \in \mathbb{N}$ such that*

$$\begin{cases} \dim_k D_1 = 2r + ([\pi : \chi\circ\det] - [\pi : \mathrm{Sp}\otimes\chi\circ\det]) \\ \dim_k D_0 = (p+1)r + ([\pi : \chi\circ\det] - [\pi : \mathrm{Sp}\otimes\chi\circ\det]). \end{cases} \tag{5}$$

*(ii) Let $\sigma$ be an absolutely irreducible smooth $k$-representation of $\mathfrak{R}_0$. If $\sigma \notin \{\chi\circ\det, \mathrm{st}\otimes\chi\circ\det\}$, then $[D_0 : \sigma] = [D_0 : \sigma^{[s]}]$; otherwise we have*

$$[D_0 : \chi\circ\det] - [D_0 : \mathrm{st}\otimes\chi\circ\det] = [\pi : \chi\circ\det] - [\pi : \mathrm{Sp}\otimes\chi\circ\det].$$

*Proof.* Using Lemma 3.3, we may assume that $\pi$ is semi-simple, say $\pi \cong \oplus_{i=1}^s \pi_i$. Moreover, by twisting we may assume the central character $\chi$ is trivial.

(i) For each $\pi_i$ let $D(\pi_i)$ be the associated canonical diagram. Then $D(\pi) := \oplus_{i=0}^s D(\pi_i)$ is a standard diagram of $\pi$. We claim that the equalities (5) hold for $D(\pi)$. In fact, an induction shows that we may assume $\pi$ irreducible, in which case the assertion is obvious by Example 2.5 and the explicit description of $D(\mathbf{1})$ and $D(\mathrm{Sp})$ (see [6, §10]).

Now, by Lemmas 3.2 and 3.3, $D$ contains $D(\pi)$ as a sub-diagram. If we denote by $Q$ the quotient $D/D(\pi)$, then the long exact sequence (2) associated to $0 \to D(\pi) \to D \to Q \to 0$ shows that $H_1(Q) = H_0(Q) = 0$, hence $Q$ satisfies the dimension relation by Lemma 2.7(ii). This implies the equalities (5) for $D$.

(ii) The proof is similar as in (i) using two facts: a) the statement holds for $D(\pi)$, b) for a naive diagram $Q$, one has $[Q_0 : \mathbf{1}] = [Q_0 : \mathrm{st}]$. □

We record an obvious corollary of Proposition 3.4.

**Corollary 3.5.** *With notation in Proposition 3.4, we have*

$$\dim_k D_0 \le \dim_k D_1 \cdot (p+1)/2, \quad (\text{resp. } \dim_k D_0 \ge \dim_k D_1 \cdot (p+1)/2)$$

*if and only if*

$$[\pi:\chi\circ\det]\geq[\pi:\mathrm{Sp}\otimes\chi\circ\det],\quad (\text{resp. } [\pi:\chi\circ\det]\leq[\pi:\mathrm{Sp}\otimes\chi\circ\det]).$$

3.2. **Criteria.** In this subsection we give two criteria for a diagram to be standard.

**Theorem 3.6.** *Let $\pi$ be a smooth $k$-representation of $G$ of finite length and let $W=(W_0,W_1,r)\hookrightarrow\mathcal{K}(\pi)$ be a sub-diagram such that $W_0$ is of finite dimension and the natural morphism $\theta: H_0(W)\to\pi$ is surjective. Assume that*

(i)

$$(\star)\qquad \dim_k W_0\leq \dim_k W_1\cdot(p+1)/2;$$

(ii) *there exists one (hence any) standard diagram $D$ of $\pi$ such that* $\dim_k D_0\geq \dim_k D_1\cdot(p+1)/2$.

*Then $W$ is a standard diagram of $\pi$. In particular, $H_0(W)$ is of finite length and the inequalities in (i) and (ii) are both equalities.*

*Proof.* By [9, Cor. III.1.15], we can choose a standard diagram $D$ of $\pi$ containing $W$. Let $Q$ be the quotient $D/W$. Then we have an exact sequence

$$0\to H_1(Q)\to H_0(W)\xrightarrow{\theta}\pi\to H_0(Q)\to 0.$$

Since $\theta$ is assumed to be surjective, we get $H_0(Q)=0$. Write $Q=(Q_0,Q_1,r_Q)$, then $\dim_k Q_0\leq\dim_k Q_1\cdot\frac{p+1}{2}$ by Proposition 2.2. By $(\star)$, we deduce the same inequality for $\dim_k D_i$. Hence, by (ii) we have

$$\dim_k D_0=\dim_k D_1\cdot\frac{p+1}{2}$$

and that $(\star)$ is in fact an equality. Moreover, $Q$ also satisfies the dimension relation, hence $H_0(Q)=H_1(Q)=0$ by Proposition 2.2, and $\theta$ is an isomorphism. □

For application later, we need a variant of Theorem 3.6 as follows. The advantage is that we do not need to fix a prior (finite length) representation $\pi$ of $G$.

**Theorem 3.7.** *Let $W=(W_0,W_1,r)$ be a diagram of $k$-modules with central character such that $r$ is an injection and that $W_0$ is of finite dimension. Assume the following conditions:*

(a) $(\star)$ *holds;*

(b) *up to semi-simplification, $W_0$ is isomorphic to a direct sum of $\mathrm{Ind}_{IZ}^{\mathfrak{K}_0}\chi_i$, for a finite set of smooth characters $\chi_i: I\to k^\times$.*

*Then $H_0(W)$ is of finite length and $(\star)$ is an equality.*

*Proof.* The idea of the proof is as follows: starting with a finite dimensional diagram $W$, we produce $\pi$ via the construction of Breuil-Paškūnas [6], then verify condition (ii) of Theorem 3.6 under the assumption (b) on $W_0$.

Up to twist we assume that the central character of $W$ is trivial. By [6, §9], we can embed $W$ into $\mathcal{K}(\Omega)$, where $\Omega$ is a smooth $G$-representation (with central character) such that $\Omega|_K$ is isomorphic to an injective envelope of $\mathrm{soc}_K W_0$. Let $\pi\subset\Omega$ be the sub-$G$-representation generated by $W_0$. Since $\Omega$ is admissible and $W_0$ is of finite dimension, $\pi$ is of finite length[4]; see for example [11, Cor. 4.9]. Write

[4]this is a special property for smooth $k$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$; it is unknown whether it remains true if $F\neq\mathbb{Q}_p$.

$m_{\mathbf{1}} = [\pi : \mathbf{1}]$, the multiplicity of $\mathbf{1}$ in $\pi^{\mathrm{ss}}$, and $m_{\mathrm{Sp}} = [\pi : \mathrm{Sp}]$. We will show that $m_{\mathbf{1}} = m_{\mathrm{Sp}}$, so that the result follows from Theorem 3.6 using Proposition 3.4.

Let $D$ be a standard diagram of $\pi$ containing $W$. We claim that $m_{\mathbf{1}} \geq m_{\mathrm{Sp}}$. Indeed, if $m_{\mathbf{1}} < m_{\mathrm{Sp}}$, then we would have $\dim_k D_0 > \dim_k D_1 \cdot (p+1)/2$ by Corollary 3.5, which would contradict Theorem 3.6. So we assume $m_{\mathbf{1}} \geq m_{\mathrm{Sp}}$ in the rest of the proof. Note that Proposition 3.4(i) implies the following equality

$$\dim_k D_1 = \frac{2}{p-1}(\dim_k D_0 - \dim_k D_1) + (m_{\mathbf{1}} - m_{\mathrm{Sp}}). \tag{6}$$

Since $H_0(D/W) = 0$, we can find a finite filtration of $D/W$ whose graded pieces $Q$ has one of the shapes (I)-(III) in Proposition 2.2. In all cases, the condition $(\star)$ holds for $Q$. We let $s_1$ (resp. $s_2$) be the number of $Q$ of type (I) (resp. type (II)) and $s_{3,\sigma}$ (resp. $s_{3,0}$) be the number of $Q$ of type (III) with $Q_0 \cong \sigma$, for each irreducible $\sigma$ (resp. $Q_0 = 0$); see Remark 2.3. The assumption (b) and Proposition 3.4(ii) imply that

$$s_{3,\sigma} = s_{3,\sigma^{[s]}}, \quad \text{if } \sigma \notin \{\mathbf{1}, \mathrm{st}\} \tag{7}$$

$$m_{\mathbf{1}} - m_{\mathrm{Sp}} = s_{3,\mathbf{1}} - s_{3,\mathrm{st}}. \tag{8}$$

On the other hand, if we let $s = \frac{1}{2}\dim W_1 + s_2 \in \mathbb{Q}_{\geq 0}$ and $s' = s_1 + 2s_{3,0}$, then

$$\begin{aligned} \dim_k D_1 &= \dim_k W_1 + s_1 + 2s_2 + \sum_\sigma 2s_{3,\sigma} + 2s_{3,0} \\ &= 2s + s' + 2\sum_\sigma s_{3,\sigma} \end{aligned}$$

and using (7),

$$\dim_k D_0 = \dim_k W_0 + (p+1)s_2 + \frac{(p+1)}{2} \sum_{\sigma \notin \{\mathbf{1},\mathrm{st}\}} s_{3,\sigma} + s_{3,\mathbf{1}} + p \cdot s_{3,\mathrm{st}}.$$

Using $(\star)$, we deduce

$$\begin{aligned} &\dim_k D_0 - \dim_k D_1 \\ \leq\ & (p-1)s - s' + \tfrac{p-3}{2} \cdot \Big( \sum_{\sigma \notin \{\mathbf{1},\mathrm{st}\}} s_{3,\sigma} \Big) - s_{3,\mathbf{1}} + (p-2)s_{3,\mathrm{st}} \\ =\ & (p-1)\big(s + \tfrac{s'}{2} + \sum_\sigma s_{3,\sigma}\big) - \tfrac{p+1}{2}\Big(s' + \sum_{\sigma \notin \{\mathbf{1},\mathrm{st}\}} s_{3,\sigma}\Big) - ps_{3,\mathbf{1}} - s_{3,\mathrm{st}}. \end{aligned}$$

Using the relations (6), (7), (8), we get

$$0 \leq -\frac{p+1}{p-1}\Big(s' + \sum_{\sigma \notin \{\mathbf{1},\mathrm{st}\}} s_{3,\sigma}\Big) - \frac{p+1}{p-1}s_{3,\mathbf{1}} - \frac{p+1}{p-1}s_{3,\mathrm{st}}.$$

This implies $s' = 0$ and $s_{3,\sigma} = 0$ for all $\sigma$, that is, only Type (II) diagrams appear in the filtration of $D/W$. Hence $H_0(D/W) = H_1(D/W) = 0$ and $W$ is a standard diagram. □

3.3. **Application III.** Assume $\Pi_{\mathrm{sm}} = \mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ is irreducible principal series and $\Pi = \Pi_{\mathrm{sm}} \otimes \Pi_{\mathrm{alg}}$ is an irreducible locally algebraic representation of $G$. The following theorem is a part of a result of Berger and Breuil [3]. It is reproved under mild conditions in [8, Thm. 5.1].

**Theorem 3.8.** *Let $c \geq 1$ be such that $\Pi_{\mathrm{sm}}^{K_c} \neq 0$ and define $X$ as in §2.3.*

*(i) If $\mathcal{L}$ is an integral structure inside $\Pi$ and $\mathcal{X} \subset X$ is the induced diagram of $\mathcal{O}$-modules, then $H_0(\mathcal{X}) \cong \mathcal{L}$ and $\mathcal{L}$ is residually of finite length.*

*(ii) The universal unitary completion of $\Pi$, if non-zero, is automatically admissible (in the sense of* [22]*).*

*(iii) The universal unitary completion of $\Pi$, if non-zero, is topologically irreducible.*

*Proof.* (i) We first show that $H_0(\mathcal{X} \otimes k)$ is of finite length as a $G$-representation. It suffices to check that the conditions in Theorem 3.7 hold for $\mathcal{X} \otimes k$: indeed the inequality $(\star)$ follows from the fact that

$$\Pi_{\mathrm{sm}}^{K_c} \cong \mathrm{Ind}_{J_c}^{K}\theta, \quad \Pi_{\mathrm{sm}}^{I_c} \cong \mathrm{Ind}_{J_c}^{I}\theta$$

where $\theta := \chi_1 \otimes \chi_2$ and $J_c := (K \cap B)K_c = (I \cap B)K_c$; the condition (b) is verified using the isomorphism

$$X_0 = \Pi_{\mathrm{sm}}^{K_c} \otimes \Pi_{\mathrm{alg}} \cong \mathrm{Ind}_{J_c}^{K}(\theta \otimes \Pi_{\mathrm{alg}}) \cong \mathrm{Ind}_{I}^{K}\mathrm{Ind}_{J_c}^{I}(\theta \otimes \Pi_{\mathrm{alg}})$$

and the fact that irreducible $k$-representations of $I$ are characters.

By construction the morphisms $\mathcal{X}_1 \otimes k \hookrightarrow \mathcal{X}_0 \otimes k \hookrightarrow \mathcal{L} \otimes k$ are all injective. So [14, Prop. 4.1] is applicable and implies that the morphism $H_0(\mathcal{X} \otimes k) \to \mathcal{L} \otimes k$ is injective. Hence, we get $H_0(\mathcal{X}) = \mathcal{L}$ by [14, Lem. 4.5].

(ii) By (i), any $G$-invariant open bounded $\mathcal{O}$-lattice (if exists) inside $\Pi$ is finitely generated as an $\mathcal{O}[G]$-module. Therefore, any two such lattices are commensurable and the universal unitary completion $\hat{\Pi}$ of $\Pi$ is exactly the completion of $\Pi$ with respect to any such lattice. Hence, if $\hat{\Pi}$ is non-zero, it is admissible.

(iii) If $\hat{\Pi}$ is not topologically irreducible, it admits a non-trivial quotient, say $\hat{\Pi}'$. Since $\Pi$ is itself absolutely irreducible, the composition $\Pi \to \hat{\Pi}'$ is still injective[5]. Let $\mathcal{L}' := \Pi \cap \hat{\Pi}'^0$ be the induced lattice of $\Pi$, where $\hat{\Pi}'^0$ denotes the unit ball of $\hat{\Pi}'$, and let $\mathcal{X}' := (X_0 \cap \mathcal{L}', X_1 \cap \mathcal{L}', \mathrm{can})$. Then $\mathcal{L}' \cong H_0(\mathcal{X}')$ by (i). It is clear that $\mathcal{L}'/\varpi\mathcal{L}' \cong \hat{\Pi}'^0/\varpi\hat{\Pi}'^0$. Since $\mathcal{L}$ and $\mathcal{L}'$ are commensurable, $\mathcal{L}/\varpi\mathcal{L}$ and $\mathcal{L}'/\varpi\mathcal{L}'$ have the same length as $G$-representations. But this would contradict the assumption that $\hat{\Pi}'$ is a non-trivial quotient of $\hat{\Pi}$ (cf. [18, Lem. 5.5]). □

**Remark 3.9.** *(1) Keep the notation in Theorem 3.8. In* [14, Thm. 4.6]*, the authors proved the isomorphism $H_0(\mathcal{X}) \cong \mathcal{L}$ by* assuming *$\mathcal{L}$ is residually of finite length (i.e. theorem of Berger-Breuil). The proof also used crucially* [14, Prop. 4.1]*. The main innovation of Theorem 3.8(i) is to prove $H_0(\mathcal{X}) \cong \mathcal{L}$ without the assumption: in fact we deduce it as a byproduct.*

*(2) Note that our proof of topological irreducibility of $\hat{\Pi}$ is different of the original proof of Berger-Breuil which uses $(\varphi, \Gamma)$-modules (cf.* [3, Cor. 5.3.2]*).*

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[5]Otherwise, $\Pi$ would be contained in the kernel, say $\hat{\Pi}''$, which is a Banach sub-representation of $\hat{\Pi}$. The universal property of $\hat{\Pi}$ gives a morphism $\hat{\Pi} \to \hat{\Pi}''$ such that the composition with the natural inclusion $\hat{\Pi}'' \hookrightarrow \hat{\Pi}$ is identity, hence the equality $\hat{\Pi}'' = \hat{\Pi}$, a contradiction.

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