RIGID CONNECTIONS
(MCM–YMSC p-ADIC GEOMETRY LEARNING SEMINAR, SPRING 2024)

KOJI SHIMIZU (YMSC) AND DAXIN XU (MCM)

Purpose: The main goal of this seminar is to study various results around rigid connections: existence of a Frobenius structure on a rigid connection due to Esnault–Groechenig [EG23], application to Simpson's integral conjecture, and the Fourier transform algorithm for rigid connections on $\mathbb{P}^1$ due to Katz [Kat96] and Arinkin [Ari10].

Time: 2:30-4:00 pm on Mondays

Location: MCM 110

Website: `https://ymsc.tsinghua.edu.cn/info/1053/3152.htm`

Mailing List: We make the seminar announcements via the mailing list. To join the mailing list, please contact Koji.

Schedule: The following is an outline and suggestion for each talk. Sometimes, too many topics are assigned to one talk. Please reorganize the materials to give a 90-minute-long *comprehensible* talk, rather than copying the lecture notes. Junior speakers are encouraged to talk with us during the lecture preparation.

Lecture 1. *Overview.* (3/4, Daxin)
Give an overview of the seminar.

Lecture 2. *Higgs-de Rham flow.* (3/11, Mao)
Give an overview of the theory of Higgs-de Rham flow following Lan–Sheng–Zuo [LSZ19].

Lecture 3. *p-adic topology on W-points of a scheme or a stack.* (3/18,)
Discuss [EG23, §2] for a p-adic topology on W-points of a scheme or a linear quotient stack over W. See [Čes15, §2] for more details.

Lecture 4. *Frobenius pullback functor for flat connections.* (3/25, Koji)
Discuss [EG23, §3.1-3.4]. Recall the Frobenius pullback functor for flat log-connections (Proposition 3.3) and explain its relationship with Cartier transform (Definition 3.8). Prove Proposition 3.9 following Olsson's stacky point of view on log-connections [EG23, §3.4].

Lecture 5. *Rational F-isocrystal property.* (4/1,)
Discuss [EG23, §4-5.2]. Show that F^* is an open embedding (Theorem 4.1) and explain that F^* preserves isolated points (Corollaries 4.7 and 4.9). Use above corollaries to prove first part of Theorem 1.2 (§5.2).

Lecture 6. *Integral Frobenius structure.* (4/8,)
Discuss [EG23, §3.5-3.7, 5.3]. Explain the flow functor Φ and finish the proof of the second part of Theorem 1.2 (§5.3).

Date: February 26, 2024.

Lecture 7. *Applications to integrality & p-curvature conjectures.* (4/15,)
Explain the application of previous local results to global problems: Simpson's integral conjecture for rigid connections; an irreducible rigid connection with vanishing p-curvature has unitary monodromy. Reference: [EG17, §7] (this section is not included in [EG20]) and [EG20, §6].

Lecture 8. *Review of formal connections over a punctured disc* (4/22,)
Review a fundamental result of Levelt for (formal) connections over a punctured disc [Mal91, III 1.2], discuss its application to the slope decomposition for formal connections and state the result about canonical extension (also known as Katz's extension) [Kat87, II 2.1-2.4].

Lecture 9. *Review of Fourier transform and local Fourier transforms.* (4/29,)
Discuss [BE04, §3]. Review the Fourier transform for $\mathcal{D}$-modules over $\mathbf{A}^1$ and its description in terms of Gauss–Manin connections [BE04, Lemma 3.2] and discuss local Fourier transforms and their properties.

Lecture 10. *Fourier transform preserves cohomological rigidity.* (5/6,)
Review some properties of the intermediate extension and prove [BE04, Theorem 4.3] using local Fourier transform.

Lecture 11. *Rigidity index and rank of the Fourier transform.* (5/13,)
Discuss [Ari10, §3]. Show that cohomological rigidity is equivalent to rigidity [BE04, Theorem 4.7, 4.10], and prove a formula about the rank of the Fourier transform [Mal91, V.1.5].

Lecture 12. *Proof of main results.* (5/20,)
Discuss [Ari10, §4] and finish the proof of main results [Ari10, Theorem A, Corollary 2.5].

References

[Ari10] D. Arinkin, *Rigid irregular connections on $\mathbb{P}^1$*, Compos. Math. **146** (2010), no. 5, 1323–1338. MR 2684302

[BE04] Spencer Bloch and Hélène Esnault, *Local Fourier transforms and rigidity for D-modules*, Asian J. Math. **8** (2004), no. 4, 587–605. MR 2127940

[Čes15] Kestutis Česnavičius, *Topology on cohomology of local fields*, Forum Math. Sigma **3** (2015), Paper No. e16, 55. MR 3482265

[EG17] Hélène Esnault and Michael Groechenig, *Rigid connections, F-isocrystals and integrality*, Preprint, arXiv:1707.00752v1 (2017).

[EG20] ———, *Rigid connections and F-isocrystals*, Acta Math. **225** (2020), no. 1, 103–158. MR 4176065

[EG23] ———, *Cristallinity of rigid flat connections revisited*, Preprint, arXiv:2309.15949v2 (2023).

[Kat87] Nicholas M. Katz, *On the calculation of some differential Galois groups*, Invent. Math. **87** (1987), no. 1, 13–61. MR 862711

[Kat96] ———, *Rigid local systems*, Annals of Mathematics Studies, vol. 139, Princeton University Press, Princeton, NJ, 1996. MR 1366651

[LSZ19] Guitang Lan, Mao Sheng, and Kang Zuo, *Semistable higgs bundles, periodic higgs bundles and representations of algebraic fundamental groups.*, Journal of the European Mathematical Society (EMS Publishing) **21** (2019), no. 10.

[Mal91] B. Malgrange, *Équations différentielles à coefficients polynomiaux*, Progress in Mathematics, vol. 96, Birkhäuser Boston, Inc., Boston, MA, 1991. MR 1117227