SUMMER SCHOOL ON BEILINSON–BLOCK–KATO CONJECTURE ON RANKIN–SELBERG MOTIVES AND GAN–GROSS–PRASAD CYCLES

Week I (July 9–13): Selmer groups and vanishing orders of L-functions: case of Heegner points

Monday-Friday 10am-noon and 2-4pm, except that we have a break on Thursday (July 12) afternoon.

The main references are

- [BD] Bertolini and Darmon, Iwasawa's main conjecture for elliptic curves over anti-cyclotomic $\mathbb{Z}_p$-extensions.
- [NOTE I] and [NOTE II] two hand written notes by Liang Xiao (mind mistakes)
- [TexNote] A LaTeXed note (again, mind mistakes)

Talk 1: Geometry of modular curves (He, Wei)

In this workshop, we genuinely consider GL_2 as opposed to PGL_2 or SL_2.

(1) Introduce the moduli problem of the moduli curve $\mathbf{X}(K)$ over $\mathbb{Q}$ for a sufficiently small open subgroup $K \subset \mathrm{GL}_2(\mathbb{A}_f)$, as the adelic quotient. Namely, first define full level N over $\mathbb{Z}[\frac{1}{N}]$ and then take K-orbits. Explain why the complex points are given by the adelic quotient. Moreover, spend some time on the geometric connected component $\pi_0(\mathbf{X}(K))$ of the moduli curve and how Galois group act on $\pi_0(\mathbf{X}(K))$.

(2) Introduce the moduli problem of modular curves $\mathcal{X}(K^p)$ over $\mathbb{Z}_{(p)}$ and $\mathcal{X}(K^p \cap \Gamma_0(p))$ (the latter being the moduli problem of isogenies of degree p).

Explain briefly that the first one has good reduction and the second one has a semistable reduction.

(3) Discuss the supersingular locus of the special fibers of the modular curves $X_0(N)$ and $X_0(Np)$: on $X_0(N)$, the supersingular locus is the vanishing locus of the Hasse invariants (explain why); also say that the set of supersingular locus corresponds to the discrete Shimura variety associated to the definite quaternion algebra $B_{p\infty}^\times$. See e.g. Serre's paper: Two letters on quaternions and modular forms (mod p).

(4) Discuss the two natural maps between them $X_0(Np) \to X_0(N)$ (on one component it is an isomorphism, and the other one the Frobenius map). The ramification points are the supersingular points.

(5) Introduce basics of nearby cycles of étale sheaves, making explicit in the case of modular curves $X_0(Np)$. Basically, we start with a little bit more abstract case. A scheme $\mathcal{X}$ over $\mathcal{O}_K$ is called semistable if it is Zariski locally etale over $\mathcal{O}_K[x_1, \dots, x_n]/(x_1 \cdots x_k - \varpi)$ for some $1 \leq k \leq n$. Let X denote the special fiber of $\mathcal{X}$, and then $X = X_1 \cup \cdots \cup X_r$ is the union of r smooth divisors. For the purpose of the application, let's consider only the case when $r = 2$ (but each X_i need not be connected). Write $X_{12} = X_1 \cap X_2$. The nearby cycle of the etale cohomology sheaf $\Lambda = \mathbb{Z}_\ell, \mathbb{Q}_\ell, \mathbb{Z}_\ell/\ell^m$ in this case is

$$R^0\Psi\Lambda = \Lambda_X, \quad R^1\Psi\Lambda = \Lambda_{X_{12}}(-1).$$

Note that $R^0\Psi\Lambda$ can be alternatively computed using

$$0 \to \Lambda_X \to \Lambda_{X_1} \oplus \Lambda_{X_2} \to \Lambda_{X_{12}} \to 0$$

Then we have

$$H^i(\mathcal{X}_{\overline{\mathbb{Q}}_p}, \Lambda) = \mathbb{H}^i(X, \mathbf{R}\Psi\Lambda)$$

where the right $\mathbb{H}$ is the "hypercohomology" of the nearby cycle $\mathbf{R}\Psi\Lambda$. When $\mathcal{X} = \mathcal{X}_0(Np)$, use the nearby cycle to compute (exercise!) that the cuspidal part of the cohomology in the Grothendieck group of modules under the Hecke algebra:

$$\big[H^1(\mathcal{X}_0(Np)_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)\big] = 2\big[H^1(\mathcal{X}_0(N)_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)\big] + \big[H^1(\mathcal{X}_{B^\times_{pN},\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)\big] + \big[H^1(\mathcal{X}_{B^\times_{pN},\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell(-1)).\big]$$

Explain why this is consistent with the Langlands program.

Talk 2: Geometric level raising in the unramified case (Yu, Fengqi)

Explain the meaning of level raising and state the main theorem of unramified level raising for modular forms. Using Abel-Jacobi map to construct the level-raising map, and show that its surjectivity is equivalent to the Ihara's lemma. Basically, follow Theorem 0.2, 0.3 and Corollary 0.4 of the [TexNote].

Talk 3: Geometry of Shimura curves (Yiwen Ding/Shengxin Zhang)

Introduce the moduli problem for Shimura curves over $\mathbb{Q}$ and discuss its semistable reduction and the geometry of its special fiber. This is essentially the same as what you see in Talk 1 with slight change.

One can also consider the moduli problem for Shimura curves over totally real field (Carayol's construction), but then the problem is: quaternionic Shimura curves do not have a PEL type moduli problem. Give a presentation of Carayol's construction relating the usual Shimura curve with the unitary Shimura curve. Mostly, one follows

`http://archive.numdam.org/article/CM_1986__59_2_151_0.pdf`

Moreover, one needs to include the case when p is ramified in the quaternion algebra, corresponding to the case when the polarization on the unitary Shimura variety is not prime-to-p (see my paper with Yichao

`https://arxiv.org/pdf/1308.0790.pdf`

Theorem 3.14 condition (b) (b2) and (b3) and all cases related to type $\beta^\sharp$.

Talk 4: Geometric Jacquet–Langlands and ramified level-raising (Han Zhou)

Continued with the previous talk, to show that the special fiber of $\mathcal{X}_{B^\times_{pN}}$ is the union of two collections of $\mathbb{P}^1$'s over $X_{B^\times_{N\infty}}$. Use this to construct the level-raising map in the ramified case (following the last section of [NOTE I]).

Talk 5: Anticyclotomic p-adic L-functions for modular forms (Bin Zhao)

Introduce the p-adic L-functions for modular forms twisted by characters in an anticyclotomic tower. Section 1 of [BD]. Personally, I am not a big fan of functions on graphs; I think of it as more of a distraction. I would prefer to redo everything just by looking at the adelic embedding

$$K^\times\backslash\mathbb{A}^\times_{K,f} \to B^\times\backslash B(\mathbb{A}_f)^\times.$$

and integrate a function on $B^\times\backslash B(\mathbb{A})^\times$ against finite Hecke characters of K. I don't know if a student can do this job well. Alternative plan would be to have Bin Zhao to give this talk.

Talk 6: Selmer groups in anticyclotomic Iwasawa theory (Xiaojun Yan)

Introduce the Iwasawa theory Selmer group for the Galois representation attached to a modular form of weight 2 (over totally real field F). Mostly follow Section 2 of [BD], but I would prefer not to use "Tate modules of elliptic curves", instead, we just say representations attached to modular forms. Even better, it would be great if the speaker can formulate everything without taking Galois cohomology of K_∞, but instead using Shapiro's lemma to talk about Galois cohomology with *coefficients* in the Iwasawa algebra (twisting the original representation by anti-cyclotomic characters): say we have a $\mathbb{Z}_p$-tower $\cdots \to K_n \to \cdots \to K$.

$$\varprojlim H^*(K_n, T) = \varprojlim H^*(K, T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[Gal(K_n/K)]) = H^*(K, T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[Gal(K_\infty/K)]]).$$

Also, the speaker should also introduce a notion of Selmer groups in a non-Iwasawa theory setup. (Basically, we just don't go up to the tower, or equivalently, we don't twist the representation.)

Talk 7: Non-vanishing of L-values and vanishing of Selmer groups(Xiuwu Zhu)

Prove that if the central L-value of the modular form of weight 2 (twisted with a quadratic character) is non-vanishing, then the corresponding Selmer group vanishes. Only prove this under the technical assumptions in Bertolini–Darmon, namely everything can be done modulo p. This talk should give a proof of the existence of admissible primes.

This is essentially the page 1-3 of [NOTE II].

Talk 8-9: Main Theorems of Bertolini–Darmon(Liu, Yu)

Explain Bertolini–Darmon's inductive proof of anti-cyclotomic Iwasawa main conjecture (as in their paper). Follow section 4 of [BD]. Recollect results from previous sections if needed.

Note that Theorem 4.1 and Theorem 4.2 are essentially proved in Talk 2 and 4! (Theorem 4.1 follows from the diagram of middle of page 3 of [NOTE II] or equivalently the last diagram of [NOTE I]; this already appears in Talk 7. Theorem 4.2 is just the top diagram on page 4 of [NOTE I].)

Week II (July 16–20): Selmer groups and vanishing orders of L-functions: case of Gan-Gross-Prasad cycles

Monday-Friday 10am-noon and 2-4pm, except that we have a break on Thursday (July 19) afternoon.

The goal of this week is to understand the five authors paper relating the special value of certain Rankin-Selberg L-function and the Selmer groups of the corresponding motive.

Talk 1-2: Introduction to unitary Shimura varieties (Xu Shen)

A brief introduction to untiary Shimura varieties, its relation to moduli space of abelian varieties with variant PEL structure. (See section 4.1) Several places to pay attention:

- the construction of the unitary level structure avoiding the isomorphism $\widehat{\mathbb{Z}} \cong \widehat{\mathbb{Z}}(1)$ (see §4.1.9)
- explain the subtlety for failure of Hasse principle (see Lemma 4.1.8).
- Give a practical introduction to Dieudonné modules of p-divisible groups, explain the meaning of Serre-Tate theorem, explain Grothendieck-Messing. Explain how to compute the tangent space of Shimura varieties using crystalline deformation theory. (If time permits, talk about general theory of local model. The discussion can follow [Deligne-Pappas].)

Talk 3: Statement of the main theorem (Zheng Liu)

Associate Galois representations to RACSDC automorphic representations, explain base change theorems from GL_n to unitary groups, explain the definition of Bloch–Kato Selmer groups, state the main theorem. (Include Vogan packets in this lecture.)

Talk 4: Kolyvagin argument (Ashay Burungale)

Introduce unramified and singular parts of the Galois cohomology and their duality (including the case at p), explain carefully "Kolyvagin system argument" for bounding Selmer groups (but only do this under stronger hypotheses on $\bar{\rho}$).

Explain the existence of certain "very good" prime ℓ. Depending on the time, either do it under general setup (with effort to control the error term) or do it under stronger hypothesis. The speaker should make the choice.

Talk 5: Supersingular locus of Shimura varieties for $\widetilde{\mathrm{GU}}(1, 2r)$ (Han Zhou)

Explain the proof that describes the supersingular locus of unitary Shimura varieties $\widetilde{\mathrm{GU}}(1, 2r)$, in the good reduction case. State the results related to the Tate conjecture. Follow Section 4.2 and 4.3.

Talk 6: Level raising map and rank 1 $\Rightarrow$ rank 0 argument (Cai, Li)

Explain the level raising map for $\widetilde{\mathrm{GU}}(1, 2r-1)$-Shimura varieties. Explain the reduction from rank 1 to rank 0. Recall the Ichino–Ikeda–Harris period formula. Follow Section 4.4.

Talk 7: Geometry of unitary Shimura varieties with semistable reduction (Haining Wang)

Describe the geometry of the unitary Shimura variety in case of semistable reduction, in particular, describe the supersingular locus on the special fiber. Follow Section 5.1–5.2 and maybe Section 5.3.

Half day break.

Talks 8: Computation of Rapoport–Zink weight spectral sequence (Chao Li)

Follow Section 5.4–5.6. Introduce the Rapoport-Zink weight spectral sequence. Explain the preliminary computation of its terms. The key lies in the computation of the middle part of the RZ spectral sequence. When n is odd, explain the "ramified Tate conjecture type" question. When n is even, explain the computation of the cohomology of Sh_2 and show its torsion-freeness (there are two parts: injectivity of the description of H^{n-2} (Theorem 5.6.6); and a reduction to $R = T$ theorem (Theorem 5.7.5). Make sure cover the first one and explain the Ihara's lemma as a corollary (Corollary 5.6.13). Defer the second one to the next talk.

Talks 9: Nearby cycle computation II (Liang Xiao)

Explain the reduction process to $R = T$ theorem as mentioned above.

Explain the relation between the Rapoport–Zink weight spectral sequence with the cycle class map and complete the proof of rank 0 argument.