

L^r -Helmholtz decomposition and its application to the Navier-Stokes equations

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We show that every L^r -vector field on Ω can be uniquely decomposed into two spaces with scalar and vector potentials and the harmonic vector space via rot and div , where Ω is a bounded domain in \mathbb{R}^3 . This may be regarded as generalization of de Rham-Hodge decomposition for smooth k -forms on compact Riemannian manifolds. Our result holds not only smooth but also general L^r -vector fields. Basically, construction of harmonic vector fields is established by means of the theory of elliptic PDE system of boundary value problems due to Agmon-Douglis-Nirenberg. Since we deal with L^r -vector fields, such a general theory is not directly available. To get around this difficulty, we make use of certain variational inequalities associated with the quadratic forms defined by rot and div . Various kinds of boundary conditions which are compatible to rot and div and which determine the harmonic parts are fully discussed.

As applications, we first consider the stationary problem of the Navier-Stokes equations in multi-connected domains under the inhomogeneous boundary condition. Up to the present, it is an open question whether there exists a solution if the given boundary data satisfies the general flux condition. It will be clarified that if the harmonic extension of the boundary data into Ω is small in $L^3(\Omega)$ compared with the viscosity constant, then there is at least one weak solution.

The second application is on the global Div-Curl lemma. The classical Div-Curl lemma is stated in such a way that the convergence holds in the sense of distributions. Under the boundary condition determining the harmonic vector fields in the L^r -Helmholtz-Weyl decomposition in Ω , we show that the convergence holds in the whole domain Ω . Furthermore, we give a more general compensated compactness theorem in the Hilbert space associated with the global Div-Curl lemma.

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