Math. Z.
DOI 10.1007/s00209-009-0593-9

Mathematische Zeitschrift

Lipschitz equivalence of dust-like self-similar sets

Li-Feng Xi

Received: 21 October 2008 / Accepted: 3 July 2009

Abstract This paper proves a necessary and sufficient condition for two dust-like self-similar sets to be Lipschitz equivalent.

Keywords Fractal · Lipschitz equivalence · Self-similar set

Mathematics Subject Classification (2000) 28A80

1 Introduction

In the view of [3], topology may be considered as the study of equivalence classes of sets under homeomorphism, and then fractal geometry is sometimes regarded as the study of equivalence classes of sets under bi-Lipschitz mappings.

Many works have been devoted to the Lipschitz equivalence of fractals, for example, Cooper and Pignataro [1] studied the shape of Cantor sets, David and Semmes [2] discussed the BPI equivalence between Ahlfors regular fractals including self-similar sets and Moran sets generated by cubic patterns, Falconer and Marsh [3] obtained a necessary condition of Lipschitz equivalence between dust-like self-similar sets.

Definition 1 We say that $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are *Lipschitz equivalent*, denoted by $A \simeq B$, if there is a bijection $f : A \to B$ such that for all $a_1, a_2 \in A$, $C^{-1}|a_1 - a_2| \le |f(a_1) - f(a_2)| \le C|a_1 - a_2|$, where $C > 0$ is a constant.

First recall some notions of fractals. A mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ is a contracting similitude, if there is a ratio $\rho \in (0, 1)$ such that $|S(x) - S(y)| = \rho|x - y|$ for all $x, y \in \mathbb{R}^n$. Given

Research partially supported by NSFC (Grant No. 10671180), Natural Science Foundation of Ningbo (Grant No. 2009A610077) and Morningside Center of Mathematics.

L.-F. Xi (✉)
Institute of Mathematics, Zhejiang Wanli University,
315100 Ningbo, Zhejiang, People's Republic of China
e-mail: xilifengningbo@yahoo.com

Published online: 14 August 2009

contracting similitudes $\{S_i\}_{i=1}^n$, suppose $E = \cup_{i=1}^n S_i(E)$ is the corresponding self-similar set. We say E is *dust-like* [3], if $\cup_{i=1}^n S_i(E)$ is a disjoint union.

Example 1 Let $s = \log 2/\log 3$. Take $\beta \in (0, 1)$ with $3\beta^s = 1$ and let $h = (1-3\beta)/2$. Let $S_1(x) = \beta x$, $S_3(x) = (1-\beta)+\beta x$, and $S_2(x) = (h+\beta)+\beta x$. Suppose $E^* = \cup_{i=1}^3 S_i(E^*)$, then $\dim_H E^* = \log 2/\log 3 = \dim_H C$, where C is the Cantor ternary set. It is proved in [3] that E^* and C are not Lipschitz equivalent (also see [1] and [2]).

For any two dust-like self-similar set with the same Hausdorff dimension, the above example illustrates that they need not be Lipschitz equivalent. However it is proved in [3] that they are *nearly* Lipschitz equivalent. Furthermore, the *nearly* Lipschitz equivalence and *quasi*-Lipschitz equivalence between C^1 self-conformal sets were discussed in [7] and [8].

For self-similar sets not dust-like, we consider self-similar sets F_1 and F_2 satisfying

$$F_1 = F_1/5 \cup (2/5 + F_1/5) \cup (4/5 + F_1/5),$$
$$F_2 = F_2/5 \cup (3/5 + F_2/5) \cup (4/5 + F_2/5).$$

David and Semmes [2] asked the Lipschitz equivalence between F_1 and F_2, where F_2 is *not dust-like*. It is proved in [6] that they are Lipschitz equivalent. A generalized situation for three different ratios r_1, r_2, r_3 is discussed in [9]. Suppose $S_1(x) = T_1(x) = r_1 x$, $S_3(x) = T_3(x) = (1-r_3) + r_3 x$, and

$$S_2(x) = \frac{1+r_1-r_2-r_3}{2} + r_2 x, \quad T_2(x) = 1 - r_2 - r_3 + r_2 x.$$

Let $F_{\{r_1,r_2,r_3\}}$ and $G_{\{r_1,r_2,r_3\}}$ be self-similar sets generated by $\{S_i\}_{i=1}^3$ and $\{T_i\}_{i=1}^3$ respectively. It is proved in [9] that $F_{\{r_1,r_2,r_3\}}$ and $G_{\{r_1,r_2,r_3\}}$ are Lipschitz equivalent if and only $\log r_1/\log r_3$ is rational.

How to classify the dust-like self-similar sets under the Lipschitz equivalence? Fix a ratios set $\mathcal{R} = \{r_i\}_{i=1}^n$, let $\mathcal{M}_{\mathcal{R}} = \mathcal{M}_{\{r_1,\ldots,r_n\}}$ be the collection of dust-like self-similar sets defined by

$$\mathcal{M}_{\mathcal{R}} = \left\{E = \cup_{i=1}^n S_i(E) : E \text{ is dust-like and } S_i \text{ has ratio } r_i \text{ for every } i\right\}.$$

The following fact is a version of "sliding" in [1]: If $E, F \in \mathcal{M}_{\mathcal{R}}$, then $E \simeq F$. In [1], some invariants of the Lipschitz equivalence for self-similar sets are discussed.

Suppose $\mathcal{R} = \{r_i\}_{i=1}^n$ and $\mathcal{T} = \{t_j\}_{j=1}^m$ are ratios sets with $\sum r_i^s = \sum t_j^s = 1$. If $E \in \mathcal{M}_{\mathcal{R}}$ and $F \in \mathcal{M}_{\mathcal{T}}$, the question is how to judge

$$\textit{whether E and F are Lipschitz equivalent or not.}$$

The paper [3] gives a *necessary condition* for E and F to be Lipschitz equivalent: If $E \simeq F$, then

(i) $\mathbb{Q}(r_1^s, \ldots, r_n^s) = \mathbb{Q}(t_1^s, \ldots, t_m^s)$;
(ii) There are positive integers p and q such that

$$sgp(r_1^p, \ldots, r_n^p) \subset sgp(t_1, \ldots, t_m),$$
$$sgp(t_1^q, \ldots, t_m^q) \subset sgp(r_1, \ldots, r_m),$$

where $sgp(a_1, \ldots, a_k)$ is the multiplicative semigroup generated by $\{a_1, \ldots, a_k\}$.

In this paper, we will give a *necessary and sufficient condition* to have $E \simeq F$. To state this condition, we need some notations. Write $\rho_j = t_j^s$ for $j = 1, \ldots, m$. Then $\sum_{i=1}^m \rho_j = 1$.

Let $\Sigma_m = \{1, \ldots, m\}^\infty$ be a symbolic system equipped with the Bernoulli measure $\mu = (\rho_1, \ldots, \rho_m)$. The *cylinder* $[i_1, \ldots, i_l]$ generated by the word $i_1, \ldots, i_l$ is defined by $[i_1, \ldots, i_l] = \{j_1, \ldots, j_l, \ldots, \in \Sigma_m : j_1, \ldots, j_l = i_1, \ldots, i_l\}$. Then for each cylinder $[i_1, \ldots, i_l]$,

$$\mu([i_1 \cdots i_l]) = \Pi_{t=1}^{l} \rho_{i_t}.$$

Let Σ_m^* denote the set of all finite words $\cup_{i=1}^{\infty}\{1, \ldots, m\}^i$. Let $\Lambda_{\mathcal{T}} = \{\Omega = \cup_{u=1}^{k}[\mathbf{i}_u^*] : k \in \mathbb{N}, \mathbf{i}_u^* \in \Sigma_m^*$ for all u and $[\mathbf{i}_u^*] \cap [\mathbf{i}_v^*] = \varnothing$ for any $u \neq v\}$. Suppose $\Omega \in \Lambda_{\mathcal{T}}$. Then the measure $\mu(\Omega) = \sum_{\mathbf{i}_u^*} \mu([\mathbf{i}_u^*])$, which is a *polynomial* in $\rho_1, \ldots, \rho_m$. Given a word $i_1, \ldots, i_k$ and a subset A of Σ_m, let $i_1, \ldots, i_k A = \{i_1, \ldots, i_k j_1, \ldots, j_l, \ldots, : j_1, \ldots, j_l, \ldots, \in A\}$. For $\Omega, \Omega' \in \Lambda_{\mathcal{T}}$, we denote $\Omega \prec \Omega'$, if either $\Omega = \Omega'$ or there is a word $i_1, \ldots, i_k$ such that $\Omega = i_1, \ldots, i_k \Omega'$.

Fix n, let $\Gamma_k = \{(i, j) : 1 \leq i \leq k, 1 \leq j \leq n\}$ for $k \geq 1$.

Theorem 1 *Suppose $\mathcal{R} = \{r_i\}_{i=1}^{n}$ and $\mathcal{T} = \{t_j\}_{j=1}^{m}$ are ratios sets with $\sum r_i^s = \sum t_j^s = 1$. Let $E \in \mathcal{M}_{\mathcal{R}}$ and $F \in \mathcal{M}_{\mathcal{T}}$. Then $E \simeq F$ if and only if there exist $\Omega_1, \Omega_2, \ldots, \Omega_k \in \Lambda_{\mathcal{T}}$ for some integer k, $\{\Omega_{i,j}\}_{(i,j)\in\Gamma_k} \subset \Lambda_{\mathcal{T}}$ and $\gamma : \Gamma_k \to \{1, \ldots, k\}$ such that*

(1) $\Omega_{i,j} \prec \Omega_{\gamma(i,j)}$ *for every* $(i, j) \in \Gamma_k$;
(2) *For every* $1 \leq i \leq k$, $\Omega_i = \cup_{j=1}^{n}\Omega_{i,j}$, *which is a disjoint union*;
(3) *For every* $(i, j) \in \Gamma_k$,

$$\mu(\Omega_{i,j})/\mu(\Omega_i) = r_j^s.$$

The paper is organized as follows. Section 2 gives some examples to illustrate the conditions of Theorem 1. Section 3 is devoted to the proof of Theorem 1. In this section, some lemmas from [1,3] and [6] are introduced, including graph-directed construction and structure of bi-Lipschitz mapping between self-similar sets. At the end of Sect. 3, we will obtain the necessary condition in [3] as a consequence of Theorem 1.

2 Examples

Let $\mathcal{R} = \{r_i\}_{i=1}^{n}$ and $\mathcal{T} = \{t_j\}_{j=1}^{m}$ with $\sum_i r_i^s = \sum_j t_j^s = 1$. We say the ratios set $\mathcal{R}$ is equivalent to $\mathcal{T}$, if any $E \in \mathcal{M}_{\mathcal{R}}$ and any $F \in \mathcal{M}_{\mathcal{T}}$ are Lipschitz equivalent.

Definition 2 Suppose $\mathcal{R}$ is equivalent to $\mathcal{T}$. We say that $\chi \in \mathbb{N}$ is the depth of the Lipschitz equivalence for $\mathcal{R}$ and $\mathcal{T}$ (or "depth for $\mathcal{R}$ and $\mathcal{T}$" in short), if there are $\{\Omega_i\}_i \cup \{\Omega_{i,j}\}_{(i,j)}$ as in Theorem 1 such that Ω_i is the union of finitely many cylinders generated by words of lengths not exceeding χ for every i, and $\Omega_{i,j} = \beta_{(i,j)}^{*}\Omega_{\gamma(i,j)}$ for every (i, j) where $\beta_{(i,j)}^{*}$ is an empty word or a finite word of length not exceeding χ.

Fixing $\mathcal{T}$ and χ, we can write a computer program to find all $\mathcal{R} = \{r_i\}_{i=1}^{n}$ which are equivalent to $\mathcal{T}$ with the depth for $\mathcal{T}$ and $\mathcal{R}$ not exceeding χ. We only need to search $\{\Omega_i\}_i$, $\{\beta_{(i,j)}^{*}\}_{(i,j)}$ and $\{\gamma(i, j)\}_{(i,j)}$ satisfying the disjoint union $\Omega_i = \cup_{j=1}^{m}\Omega_{i,j}$ with $\Omega_{i,j} = \beta_{(i,j)}^{*}\Omega_{\gamma(i,j)}$ and

$$\mu(\Omega_{1,j})/\mu(\Omega_1) = \mu(\Omega_{2,j})/\mu(\Omega_2) = \cdots = r_j^s.$$

The following example illustrates the case of $k = 1$

Example 2 Suppose Σ_m is a symbolic system with the Bernoulli measure

$$\mu_0 = (\rho_1, \dots, \rho_m).$$

Let $\Sigma_m = \cup_{u=1}^{n}[i_u^*]$ be a disjoint union. Write $\Omega_1 = [1] \cup \cdots \cup [m] = \Sigma_m$ and $\Omega_{1,u} = i_u^*\Omega_1 \prec \Omega_1$ for $u = 1, \dots, n$. Then

$$\Omega_1 = \cup_{u=1}^{n}\Omega_{1,u}$$

with $\mu_0(\Omega_{1,u})/\mu_0(\Omega_1) = \mu_0([i_u^*])$. Then the ratios set $\mathcal{R}_0 = \{\mu_0^{1/s}([i_u^*])\}_{u=1}^{n}$ is equivalent to $\mathcal{T}_0 = \{\rho_1^{1/s}, \dots, \rho_m^{1/s}\}$. We say that $\mathcal{R}_0$ is generated by splitting operator [1] from $\mathcal{T}_0$.

Remark 1 Fix $s > 0$. Suppose $\mathcal{R}$ is a ratios set equivalent to $\{(1/2)^{1/s}, (1/2)^{1/s}\}$. It is proved in [3] that $\mathcal{R}$ is generated by splitting operator from $\{(1/2)^{1/s}, (1/2)^{1/s}\}$. Then Example 1 follows from this result. The question is how to characterize the ratios class $\mathcal{T}$ satisfying the following property: any ratios set equivalent to $\mathcal{T}$ is generated by splitting operator from $\mathcal{T}$. The reader is referred to [1] and [3].

The next example gives the Lipschitz equivalence for two different ratios sets $\{r_1, r_2\}$ and $\{t_1, t_2\}$ with $r_1^s + r_2^s = t_1^s + t_2^s = 1$. Here $m = n = k = \chi = 2$.

Example 3 For the symbolic system Σ_2 equipped with the Bernoulli measure $\mu = (\rho_1, \rho_2)$, let

$$\Omega_1 = [1] \cup [2], \Omega_2 = [1] \cup [21].$$

Then $\mu(\Omega_1) = 1$, $\mu(\Omega_2) = \rho_1 + \rho_1\rho_2$. Here

$$\begin{aligned}\Omega_1 &= [22] \cup ([1] \cup [21]) = 22\Omega_1 \cup \Omega_2 = \Omega_{1,1} \cup \Omega_{1,2},\\ \Omega_2 &= [1] \cup [21] = 1\Omega_1 \cup 21\Omega_1 = \Omega_{2,1} \cup \Omega_{2,2}.\end{aligned}$$

The condition (3) of Theorem 1 requires

$$\begin{cases}\alpha_1 = \mu(22\Omega_1)/\mu(\Omega_1) = \mu(1\Omega_1)/\mu(\Omega_2),\\ \alpha_2 = \mu(\Omega_2)/\mu(\Omega_1) = \mu(21\Omega_1)/\mu(\Omega_2).\end{cases}$$

Therefore, we have

$$\begin{cases}\rho_2^2 = \rho_1/(\rho_1 + \rho_1\rho_2),\\ \rho_1 + \rho_1\rho_2 = \rho_1\rho_2/(\rho_1 + \rho_1\rho_2),\\ \rho_1 + \rho_2 = 1, \rho_1, \rho_2 > 0.\end{cases}$$

That means

$$\rho_2^2(1 + \rho_2) = 1, \rho_1 = 1 - \rho_2.$$

We have

$$\rho_2 = 0.7548\cdots \text{ and } \rho_1 = 0.24\cdots.$$

We also have

$$\alpha_1 = \rho_1 + \rho_1\rho_2 = 0.43\cdots, \quad \alpha_2 = \rho_2^2 = 0.56\cdots.$$

For any $s > 0$, let $r_i = (\alpha_i)^{1/s}$ $(i = 1, 2)$ and $t_j = (\rho_j)^{1/s}$ $(j = 1, 2)$. Then $\{r_1, r_2\}$ is equivalent to $\{t_1, t_2\}$. Here $\{r_1, r_2\}$ can not be generated by splitting operator from $\{t_1, t_2\}$.

The last example gives the Lipschitz equivalence for ratios sets $\{r'_i\}_{i=1}^3$ and $\{t'_j\}_{j=1}^4$ satisfying $\sum_i (r'_i)^s = \sum_j (t'_i)^s = 1$. Here $m = 3$, $n = 4$, $k = 2$ and depth $\chi = 3$.

Example 4 For symbolic system Σ_3 with the Bernoulli measure $\mu' = (\rho'_1, \rho'_2, \rho'_3)$, let

$$\Omega'_1 = [1] \cup [2],\ \Omega'_2 = [3].$$

Then $\mu'(\Omega_1) = \rho'_1 + \rho'_2$, $\mu'(\Omega_2) = \rho'_3$. Here

$$\begin{aligned}\Omega'_1 &= [1] \cup [2] = 1([1] \cup [2]) \cup 1[3] \cup 2([1] \cup [2]) \cup 2[3]\\ &= 1\Omega'_1 \cup 1\Omega'_2 \cup 2\Omega'_1 \cup 2\Omega'_2 = \Omega'_{1,1} \cup \Omega'_{1,2} \cup \Omega'_{1,3} \cup \Omega'_{1,4},\\ \Omega'_2 &= 333([1] \cup [2]) \cup 333[3] \cup 33([1] \cup [2]) \cup 3([1] \cup [2])\\ &= 333\Omega'_1 \cup 333\Omega'_2 \cup 33\Omega'_1 \cup 3\Omega'_1 = \Omega'_{2,1} \cup \Omega'_{2,2} \cup \Omega'_{2,3} \cup \Omega'_{2,4}.\end{aligned}$$

The condition (3) of Theorem 1 requires

$$\begin{cases}\alpha'_1 = \mu'(1\Omega'_1)/\mu'(\Omega'_1) = \mu'(333\Omega'_1)/\mu'(\Omega'_2),\\ \alpha'_2 = \mu'(1\Omega'_2)/\mu'(\Omega'_1) = \mu'(333\Omega'_2)/\mu'(\Omega'_2),\\ \alpha'_3 = \mu'(2\Omega'_1)/\mu'(\Omega'_1) = \mu'(33\Omega'_1)/\mu'(\Omega'_2),\\ \alpha'_4 = \mu'(2\Omega'_2)/\mu'(\Omega'_1) = \mu'(3\Omega'_1)/\mu'(\Omega'_2).\end{cases}$$

Therefore, we have

$$\begin{cases}\rho'_1 = (\rho'_3)^2(\rho'_1 + \rho'_2), & (1)\\ \rho'_1\rho'_3/(\rho'_1 + \rho'_2) = (\rho'_3)^3, & (2)\\ \rho'_2 = \rho'_3(\rho'_1 + \rho'_2), & (3)\\ \rho'_2\rho'_3/(\rho'_1 + \rho'_2) = \rho'_1 + \rho'_2, & (4)\\ \rho'_1 + \rho'_2 + \rho'_3 = 1,\ \rho'_1, \rho'_2, \rho'_3 > 0. & (5)\end{cases}$$

Adding (1) and (3), we get

$$\rho'_1 + \rho'_2 = (\rho'_1 + \rho'_2)[\rho'_3 + (\rho'_3)^2],$$

which implies $\rho'_3 + (\rho'_3)^2 = 1$, i.e., $\rho'_3 = (\sqrt{5} - 1)/2$. Then, from (1) and (3), we get

$$\rho'_1 = \frac{7 - 3\sqrt{5}}{2} = 0.145\cdots, \quad \rho'_2 = \sqrt{5} - 2 = 0.236\cdots, \quad \rho'_3 = \frac{\sqrt{5} - 1}{2} = 0.618\cdots.$$

Here formulas (1)–(5) hold. We also have

$$\begin{aligned}&\alpha'_1 = \tfrac{3-\sqrt{5}}{2} = 0.381\cdots, \quad \alpha'_2 = \sqrt{5} - 2 = 0.236\cdots,\\ &\alpha'_3 = \sqrt{5} - 2 = 0.236\cdots, \ \alpha'_4 = \tfrac{7-3\sqrt{5}}{2} = 0.145\cdots.\end{aligned}$$

For any $s > 0$, let $r'_i = (\alpha'_i)^{1/s}$ $(i = 1, 2, 3, 4)$ and $t'_j = (\rho'_j)^{1/s}$ $(j = 1, 2, 3)$. Then $\{r'_1, r'_2, r'_3\}$ is equivalent to $\{t'_1, t'_2, t'_3, t'_4\}$.

3 Proof of Theorem 1

3.1 Graph-directed construction

Suppose that $\{1, \ldots, M\}$ is the vertices set of a directed graph G. For any $1 \le i, j \le M$, let $\Gamma_{i,j} = \{e' : e' \in G$ is a directed edge from i to $j\}$. For any edge $e \in \Gamma_{i,j}$, there is a corresponding similitude $T_e : \mathbb{R}^n \to \mathbb{R}^n$ with the similarity ratio $\rho_e \in (0, 1)$, that is

$|T_e(x) - T_e(y)| = \rho_e|x - y| \ \forall \ x, y \in \mathbb{R}^n$. The compact sets $\{E_i\}_{i=1}^M$ are called the graph-directed sets [5] on the graph G, if for each i,

$$E_i = \bigcup_j \bigcup_{e \in \Gamma_{i,j}} T_e(E_j).$$

We say $\{E_i\}_{i=1}^M$ are *dust-like*, if the above union is a disjoint union for each i.

Lemma 1 [6] *Suppose $\{E_i\}_{i=1}^M$ and $\{F_i\}_{i=1}^M$ are dust-like graph-directed sets on the same graph G satisfying*

$$E_i = \bigcup_j \bigcup_{e \in \Gamma_{i,j}} T_e(E_j) \quad \text{and} \quad F_i = \bigcup_j \bigcup_{e \in \Gamma_{i,j}} S_e(F_j),$$

where S_e and T_e have the same ratio ρ_e for each edge e. Then $E_i \simeq F_i$ for all i.

By using the above lemma, we can get the following results [6]: fix an integer $n \geq 2$. For any subset $A \subset \{1, 2, \ldots, n\}$, let $E_A = \cup_{i \in A}[E_A/n + (i-1)/n]$. If $A, B \subset \{1, 2, \ldots, n\}$ with $\#A = \#B$, then E_A and E_B are Lipschitz equivalent. A special case with three different ratios was discussed in [9].

3.2 Copy and finite-copy

For a subset $A \subset \mathbb{R}^l$, we say $B = g(A)$ is a *copy* of A, if $g : \mathbb{R}^l \to \mathbb{R}^l$ is a similitude with ratio $r \in \mathbb{R}^+$. The set $B = \cup_{i=1}^k g_i(A)$ is called a *finite-copy* of A, if $\{g_i\}_{i=1}^k$ are similitudes and $\{g_i(A)\}_{i=1}^k$ are pairwise disjoint.

Suppose $H = \cup_{i=1}^{\eta} S_i(H)$ is a dust-like self-similar set. For $j^* = j_1, \ldots, j_k \in \Sigma_\eta^*$, write $S_{j^*} = S_{j_1} \circ \cdots \circ S_{j_k}$ and $H_{j^*} = S_{j^*}(H)$. H_{j^*} is called a *fine copy* of H. Given a finite subset Ξ of Σ_η^*, we say that $\cup_{j^* \in \Xi} H_{j^*}$ is a *fine finite-copy* of H.

Lemma 2 [1] *Suppose A is a dust-like self-similar set. Then A is Lipschitz equivalent to any of its finite-copy.*

3.3 Structure of bi-Lipschitz mapping

In this subsection, we *always* assume that E, F are dust-like self-similar sets and $E \simeq F$. Let $f : E \to F$ be the corresponding bi-Lipschitz bijection. Then by [1] and [3], we have the following two lemmas.

Lemma 3 [3] *There exists an integer N such that for any fine copy $\bar{E}$ of E, there exists a subset $\Lambda \subset \{1, \ldots, m\}^N$ so that*

$$f(\bar{E}) = \bigcup_{j^* \in \Lambda} F_{j_1 \cdots j_k j^*},$$

where $F_{j_1, \ldots, j_k}$ is the smallest fine copy containing $f(\bar{E})$.

Let $s = dim_H(E) = dim_H(F)$ and $\mathcal{H}^s$ the s-dimensional Hausdorff measure.

Lemma 4 [1] *There are a fine copy $\bar{E}$ of E and a fine finite-copy $\bar{F}$ of F such that the restriction $f|_{\bar{E}} : \bar{E} \to \bar{F}$ is bijective, and $f|_{\bar{E}}$ is measure linear, i.e., for any Borel subset $A \subset \bar{E}$ with $\mathcal{H}^s(A) > 0$,*

$$\frac{\mathcal{H}^s(f(A))}{\mathcal{H}^s(A)} \equiv \frac{\mathcal{H}^s(\bar{F})}{\mathcal{H}^s(\bar{E})}.$$

3.4 Proof of Theorem 1: necessity

Assume that $E \in \mathcal{M}_{\mathcal{R}}$, $F \in \mathcal{M}_{\mathcal{T}}$ are dust-like self-similar sets and $E \simeq F$.

Let $f : E \to F$ be the corresponding bi-Lipschitz bijection. By Lemma 4, there exist a fine copy $\bar{E}$ of E and a fine finite-copy $\bar{F}$ of F such that the bi-Lipschitz mapping $f|_{\bar{E}} : \bar{E} \to \bar{F}$ is measure linear.

By Lemma 3, for any fine copy $E' \subset \bar{E}$, there exists $\Lambda \subset \{1, \dots, m\}^N$ such that

$$f(E') = \cup_{j^* \in \Lambda} F_{j_1 \cdots j_t j^*}, \tag{1}$$

where $F_{j_1,\dots,j_t}$ is the smallest fine copy containing $f(E')$. We say $\Lambda \subset \{1, \dots, m\}^N$ is an *admissible* set if there exists $E' \subset \bar{E}$ such that (1) holds. Since $\{1, \dots, m\}^N$ is a finite set, there are only finitely many admissible sets $\Lambda_1, \dots, \Lambda_k \subset \{1, \dots, m\}^N$.

For every Λ_i, there exists a fine copy $E^i \subset \bar{E}$ such that $f(E^i) = \cup_{j^* \in \Lambda_i} F_{u_1,\dots,u_p j^*}$ where $F_{u_1 \cdots u_p}$ the smallest fine copy containing $f(E^i)$. Let $E^i = E_{v_1,\dots,v_q}$. Then $E^i = \cup_{j=1}^m E_{v_1,\dots,v_q j}$. Let φ_i denote the natural similitude from E to $E_{v_1,\dots,v_q} (= E^i)$ and χ_i the natural similitude from $F_{u_1,\dots,u_p}$ to F. Therefore, we have

$$h_i \hat{=} \chi_i \circ f|_{E^i} \circ \varphi_i : E \xrightarrow{\varphi_i} E_{v_1 \dots v_q} \xrightarrow{f|_{E^i}} \cup_{j^* \in \Lambda_i} F_{u_1 \dots u_p j^*} \xrightarrow{\chi_i} \cup_{j^* \in \Lambda_i} F_{j^*}. \tag{2}$$

Here $h_i(E_j) \subset \cup_{j^* \in \Lambda_i} F_{j^*} \subset F$, which implies that there is an admissible set $\Lambda_{\gamma(i,j)}$ and a finite word or an empty word $\beta^*_{(i,j)}$ such that

$$h_i(E_j) = \cup_{j^* \in \Lambda_{\gamma(i,j)}} F_{\beta^*_{(i,j)} j^*}. \tag{3}$$

Since h_i is a homeomorphism, we have

$$h_i(E_{j_1}) \cap h_i(E_{j_2}) = \varnothing \quad \text{for any } j_1 \neq j_2.$$

Let

$$\Omega_i = \cup_{j^* \in \Lambda_i} [j^*] \quad \text{and} \quad \Omega_{i,j} = \cup_{j^* \in \Lambda_{\gamma(i,j)}} [\beta^*_{(i,j)} j^*] = \beta^*_{(i,j)} \Omega_{\gamma(i,j)}. \tag{4}$$

Then

$$\Omega_{i,j} \prec \Omega_{\gamma(i,j)} \quad \text{and} \quad \Omega_i = \cup_{j=1}^n \Omega_{i,j}, \tag{5}$$

which is a disjoint union since $h_i(E_{j_1}) \cap h_i(E_{j_2}) = \varnothing$ for any $j_1 \neq j_2$.

In formula (2), χ_i, $f|_{E^i}$ and φ_i are measure linear, which implies h_i is measure linear. Therefore, for every pair (i, j),

$$r_j^s = \frac{\mathcal{H}^s(E_j)}{\mathcal{H}^s(E)} = \frac{\mathcal{H}^s(\cup_{j^* \in \Lambda_{\gamma(i,j)}} F_{\beta^*_{(i,j)} j^*})}{\mathcal{H}^s(\cup_{j^* \in \Lambda_i} F_{j^*})} = \frac{\mu(\Omega_{i,j})}{\mu(\Omega_i)}, \tag{6}$$

where for any Borel set $A \subset \Sigma_m$,

$$\mu(A) = \mathcal{H}^s(\pi A)/\mathcal{H}^s(F) \ [4],$$

with $\pi : \Sigma_m \to F$ defined by $\{\pi(j_1, \dots, j_k, \dots)\} = \cap_{k=1}^\infty F_{j_1,\dots,j_k}$.

Therefore, the necessary condition follows from (4)–(6).

3.5 Proof of Theorem 1: sufficiency

Suppose $\pi : \Sigma_m \to F$ is defined by

$$\{\pi(j_1, \ldots, j_k, \ldots)\} = \cap_{k=1}^{\infty} F_{j_1,\ldots,j_k}.$$

Denote $X_i = \pi(\Omega_i)/[\mu(\Omega_i)]^{1/s}$. Then we get a disjoint union

$$X_i = \cup_{j=1}^{n} \pi(\Omega_{i,j})/[\mu(\Omega_i)]^{1/s}. \tag{7}$$

Since $\Omega_{i,j} \prec \Omega_{\gamma(i,j)}$, we notice that $\pi(\Omega_{i,j})/[\mu(\Omega_i)]^{1/s}$ is a copy of

$$X_{\gamma(i,j)} = \pi(\Omega_{\gamma(i,j)})/[\mu(\Omega_{\gamma(i,j)})]^{1/s}$$

with ratio r satisfying

$$\begin{aligned} r^s &= \frac{\mathcal{H}^s[\pi(\Omega_{i,j})/[\mu(\Omega_i)]^{1/s}]}{\mathcal{H}^s\{\pi(\Omega_{\gamma(i,j)})/[\mu(\Omega_{\gamma(i,j)})]^{1/s}\}} \\ &= \frac{\mu(\Omega_{\gamma(i,j)})}{\mu(\Omega_i)} \frac{\mathcal{H}^s[\pi(\Omega_{i,j})]/\mathcal{H}^s(F)}{\mathcal{H}^s[\pi(\Omega_{\gamma(i,j)})]/\mathcal{H}^s(F)} \\ &= \frac{\mu(\Omega_{\gamma(i,j)})}{\mu(\Omega_i)} \frac{\mu(\Omega_{i,j})}{\mu(\Omega_{\gamma(i,j)})} \\ &= \frac{\mu(\Omega_{i,j})}{\mu(\Omega_i)} \\ &= r_j^s, \end{aligned}$$

which implies $r = r_j$. Therefore there exists an isometry $T_{(i,j)}$ such that

$$\pi(\Omega_{i,j})/[\mu(\Omega_i)]^{1/s} = T_{(i,j)}(r_j X_{\gamma(i,j)}). \tag{8}$$

It follows from (7) and (8) that for every $i = 1, \ldots, k$,

$$X_i = \cup_{j=1}^{n} T_{(i,j)}(r_j X_{\gamma(i,j)}), \tag{9}$$

which is a disjoint union.

On the other hand, we have

$$E = \cup_{j=1}^{n} S_j(E) \text{ with the ratio of } S_j \text{ being } r_j.$$

For every $i = 1, \ldots, k$, let

$$Y_i = E. \tag{10}$$

Then

$$Y_i = \cup_{j=1}^{n} S_j(Y_{\gamma(i,j)}). \tag{11}$$

We construct a graph containing k vertices $\{1, \ldots, k\}$ such that for every pair $(i, j) \in \Gamma_k$, there is an edge $e_{(i,j)}$ starting at vertex i and ending at vertex $\gamma(i, j)$. For this edge $e_{(i,j)}$, the corresponding mappings are S_j with respect to $\{Y_i\}_{i=1}^k$ and $T_{(i,j)} \circ r_j$ with respect to $\{X_i\}_{i=1}^k$, these two mappings having the same ratio r_j. It follows from (9), (11) and Lemma 1, X_i and Y_i are Lipschitz equivalent for every i. Here Y_i is E and X_i is a finite-copy of F. By Lemma 2, we have $E \simeq F$.

3.6 Necessary Condition in [3]

In this subsection, by using Theorem 1, we check that: If $E \simeq F$, then

(i) $\mathbb{Q}(r_1^s, \ldots, r_n^s) \subset \mathbb{Q}(t_1^s, \ldots, t_m^s)$;
(ii) $sgp(r_1^p, \ldots, r_n^p) \subset sgp(t_1, \ldots, t_m)$ for some $p \in \mathbb{N}$.

In fact, (i) follows from $\mu(\Omega_{i,j})/\mu(\Omega_i) = r_j^s$, where $\mu(\Omega_{i,j})$ and $\mu(\Omega_i)$ are polynomial of $\{t_1^s, \ldots, t_m^s\}$ with integer coefficients.

To prove (ii), it suffices to show that $(r_j^s)^{p_j} \in sgp(\rho_1, \ldots, \rho_m)$ for some $p_j \in \mathbb{N}$. Fix j, there is an infinite sequence $\{c_u\}_{u=1}^{\infty} \subset \{1, \ldots, k\}$ such that $c_{u+1} = \gamma(c_u, j)$ for every u. Then there are $l < l'$ such that $c_l = c_{l'}$. We have $\mu(\Omega_{c_l}) = \mu(\Omega_{c_{l'}})$, and

$$\begin{aligned}(r_j^s)^{l'-l+1} &= \frac{\mu(\Omega_{c_l,j})}{\mu(\Omega_{c_l})}\frac{\mu(\Omega_{c_{l+1},j})}{\mu(\Omega_{c_{l+1}})}\cdots\frac{\mu(\Omega_{c_{l'-1},j})}{\mu(\Omega_{c_{l'-1}})}\\ &= \frac{\mu(\Omega_{c_l,j})}{\mu(\Omega_{c_{l+1}})}\frac{\mu(\Omega_{c_{l+1},j})}{\mu(\Omega_{c_{l+2}})}\cdots\frac{\mu(\Omega_{c_{l'-2},j})}{\mu(\Omega_{c_{l'-1}})}\frac{\mu(\Omega_{c_{l'-1},j})}{\mu(\Omega_{c_{l'}})},\end{aligned}$$

where $\Omega_{c_t,j} \prec \Omega_{c_{t+1}}$ and $\frac{\mu(\Omega_{c_t,j})}{\mu(\Omega_{c_{t+1}})} \in sgp(\rho_1, \ldots, \rho_m)$. That means

$$(r_j^s)^{l'-l+1} \in sgp(\rho_1, \ldots, \rho_m).$$

References

1. Cooper, D., Pignataro, T.: On the shape of Cantor sets. J. Differ. Geom. **28**, 203–221 (1988)
2. David, G., Semmes, S.: Fractured Fractals and Broken Dreams: Self-Similar Geometry Through Metric and Measure. Oxford University Press, Oxford (1997)
3. Falconer, K.J., Marsh, D.T.: On the Lipschitz equivalence of Cantor sets. Mathematika **39**, 223–233 (1992)
4. Hutchinson, J.E.: Fractals and self similarity. Indiana Univ. Math. J. **30**, 713–747 (1981)
5. Mauldin, R.D., Williams, S.C.: Hausdorff dimension in graph directed constructions. Trans. Am. Math. Soc. **309**(1-2), 811–839 (1998)
6. Rao, H., Ruan, H.J., Xi, L.F.: Lipschitz equivalence of self-similar set. C. R. Math. Acad. Sci. Paris **342**(3), 191–196 (2006)
7. Xi, L.F.: Quasi-Lipschitz equivalence of fractals. Israel J. Math. **160**, 1–21 (2007)
8. Xi, L.F.: Lipschitz equivalence of self-conformal sets. J. London Math. Soc. **70**(2), 369–382 (2004)
9. Xi, L.F., Ruan, H.J.: Lipschitz equivalence of generalized {1,3,5}-{1,4,5} self-similar sets. Sci. China Ser. A **50**(11), 1537–1551 (2007)