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VECTOR BUNDLES ON COMPACT RIEMANN SURFACES

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Abstract

VECTOR BUNDLES ON COMPACT RIEMANN SURFACES.

1. Cohomology of vector bundles and the duality theorem. 2. Divisors, line bundles and the Riemann-Roch theorem. 3. Projective embedding of a compact Riemann surface. 4. Genus and first Betti number. 5. Chern class and degree. 6. The Jacobian. 7. Line bundles and characters. 8. Poincaré bundle. 9. The Picard manifold of a compact Kähler manifold. 10. Vector bundles on a compact Riemann surface. 11. The Riemann-Roch theorem for vector bundles. 12. Indecomposable bundles and the Krull-Remak-Schmidt theorem. 13. Weil's theorem; unitary bundles. Appendix: Factors of automorphy.

1. COHOMOLOGY OF VECTOR BUNDLES AND THE DUALITY THEOREM

Let X be a compact Riemann surface, i.e. a compact connected complex manifold of complex dimension one. Let V be a (holomorphic) vector bundle of rank n on X. We shall denote by **V** the sheaf of holomorphic sections of V. If $\mathcal{O}$ denotes the sheaf of holomorphic functions on X, then **V** is a sheaf of $\mathcal{O}$-modules locally isomorphic to $\mathcal{O}^n = \mathcal{O} \oplus \ldots \oplus \mathcal{O}$ (n factors). Conversely, given such a locally free sheaf of $\mathcal{O}$-modules of rank n, it defines in a canonical way a vector bundle of rank n (Ref. [11], §4).

Let $H^i(X,V)$ denote the i^{th} cohomology space of X with coefficients in the sheaf **V**. We then have the following facts (see Refs [4, 11] and the paper by M.J. Field in these Proceedings).

(a) For all i, $H^i(X,V)$ are finite-dimensional vector spaces over $\mathbb{C}$ and $H^i(X,V) = 0$ for $i \geqslant 2$.

(b) **Duality theorem.** Let K denote the holomorphic cotangent bundle of X and V* the dual bundle of V. We then have

$$\dim_{\mathbb{C}} H^1(X,V) = \dim_{\mathbb{C}} H^0(X, K \otimes V^*)$$

(in fact, the spaces are canonically dual to each other).

Remarks

(i) The space $H^0(X,V)$ is the space of holomorphic sections of V over X. It is easy to prove its finite dimensionality, for instance by using Montel's theorem.

(ii) A vector bundle of rank 1 will be called a line bundle. The bundle K is called the **canonical bundle** of X.

2. DIVISORS, LINE BUNDLES AND THE RIEMANN-ROCH THEOREM

2.1. Divisors

An element of the free abelian group over the set X will be called a **divisor** on X. A divisor D is of the form

$$D = \sum_{P \in X} m_P P$$

where $m_P \in \mathbb{Z}$ and $m_P = 0$ for all but a finite number of points P of X. The integer

$$\sum_{P \in X} m_P$$

is defined to be the **degree** of the divisor D and is denoted by d(D).

Let U be an open subset of X and f a meromorphic function on U which is not identically zero on any connected component of U. For $P \in U$, define:

$$\nu_P(f) = \begin{cases} \text{order of the zero of f at P, if f is holomorphic at P} \\ -(\text{order of the pole of f at P}), \text{ if f has a pole at P} \end{cases}$$

Now if $f \not\equiv 0$ is a meromorphic function on X, then f defines a divisor

$$(f) = \sum_{P \in X} \nu_P(f) P$$

which is of degree 0 (see Ref. [7], Ch. III, §13, p. 43).

2.2. The line bundle associated to a divisor

Let $D = \sum m_P P$ be a divisor on X. For every point P of X, choose a meromorphic function $g_P \not\equiv 0$ in a connected neighbourhood U_P of P such that for $Q \in U_P$ we have

$$\nu_Q(g_P) = \begin{cases} m_P \text{ if } Q = P \\ 0 \text{ if } Q \neq P \end{cases}$$

Let L_D denote the sheaf which associates to an open set U of X the space of meromorphic functions f on U such that fg_P is holomorphic at P for all $P \in U$. It is easy to check that L_D is a locally free sheaf of rank 1 and that L_D depends only on D and not on the choice of $\{g_P\}$. We denote by L_D the line bundle determined by L_D and call it the line bundle associated to the divisor D.

Remarks.

(i) A set of transition functions for the line bundle L_D is given by the non-vanishing holomorphic functions

$$g_{ij} = \frac{g_{P_i}}{g_{P_j}} \text{ on } U_{P_i} \cap U_{P_j},\ P_i, P_j \in X$$

(ii) If D_1 and D_2 are divisors on X, $L_{D_1 + D_2} \simeq L_{D_1} \otimes L_{D_2}$; moreover, L_{D_1} and L_{D_2} are isomorphic as line bundles if and only if $D_1 - D_2$ is the divisor associated to a meromorphic function.

2.3. Euler characteristics of sheaves

Let $\mathscr{F}$ be a sheaf of $\mathbb{C}$-vector spaces over X such that $H^i(X,\mathscr{F})$ are finite dimensional for $i \geqslant 0$ and vanish for all large i. We define the Euler characteristic, $\chi(\mathscr{F})$, of $\mathscr{F}$ by:

$$\chi(\mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} H^i(X,\mathscr{F})$$

Proposition 2.3.1. *Let*

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

be an exact sequence of sheaves on X *where* $\mathscr{F}_i$ *satisfy the above condition. We then have*

$$\chi(\mathscr{F}_2) = \chi(\mathscr{F}_1) + \chi(\mathscr{F}_3)$$

We first prove

Lemma 2.3.2. *Let*

$$0 \to W_1 \to \ldots \to W_i \to \ldots \to W_k \to 0$$

be an exact sequence where W_i *are finite-dimensional vector spaces over* $\mathbb{C}$*, the maps being* $\mathbb{C}$*-linear. Then we have*

$$\sum_{i=1}^{k} (-1)^i \dim W_i = 0$$

Proof. The proof is by induction on k, the lemma being evident for $k \leqslant 3$. If W'_{k-1} is the kernel of $W_{k-1} \to W_k$, we have two exact sequences:

$$0 \to W_1 \to \ldots \to W'_{k-1} \to 0$$

$$0 \to W'_{k-1} \to W_{k-1} \to W_k \to 0$$

By induction hypothesis we have

$$\sum_{i=1}^{k-2} (-1)^i \dim W_i + (-1)^{k-1} \dim W'_{k-1} = 0$$

and $\dim W'_{k-1} - \dim W_{k-1} + \dim W_k = 0$ which yield the lemma.

Proof of Proposition 2.3.1. The proposition follows from Lemma 2.3.2 applied to the exact cohomology sequence:

$$0 \to H^0(X,\mathscr{F}_1) \to \ldots \to H^i(X,\mathscr{F}_1) \to H^i(X,\mathscr{F}_2) \to H^i(X,\mathscr{F}_3) \to H^{i+1}(X,\mathscr{F}_1) \to \ldots$$

arising from the exact sequence of sheaves:

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

2.4. Line bundles and divisors

Theorem 2.4.1. *Let* L *be a line bundle on* X. *Then there exists a divisor* D *on* X *such that* L *is isomorphic to the line bundle* L_D *associated to* D.

We first prove

Lemma 2.4.2. *Let* $P \in X$. *Then the line bundle* $L \otimes L_{kP}$ *admits a non-zero (holomorphic) section, for some* $k \in \mathbb{Z}$.

Proof. Let $\mathscr{I}_P = L_{-P}$ be the sheaf of holomorphic functions vanishing at P. Since $L \otimes L_{kP}$ is a locally free sheaf, tensoring the exact sequence

$$0 \to \mathscr{I}_P \to \mathscr{O} \to \mathscr{O}/\mathscr{I}_P \to 0$$

by $L \otimes L_{kP}$ gives rise to the exact sequence:

$$0 \to L \otimes L_{(k-1)P} \to L \otimes L_{kP} \to Q \to 0$$

where $Q = L \otimes L_{kP} \otimes_{\mathscr{O}} \mathscr{O}/\mathscr{I}_P$. Now the support of the sheaf Q is P and the stalk at P is a one-dimensional vector space over $\mathbb{C}$ (in fact the stalk is canonically isomorphic to the fibre at P of the line bundle $L \otimes L_{kP}$). Hence

$$\dim H^i(X,Q) = \begin{cases} 0 \text{ for } i \geqslant 1 \\ 1 \text{ for } i = 0 \end{cases}$$

so that $\chi(Q) = 1$. By Proposition 2.3.1,

$$\chi(L \otimes L_{kP}) = \chi(L \otimes L_{(k-1)P}) + 1$$

which implies that

$$\chi(L \otimes L_{kP}) = \chi(L) + k$$

Now

$$\dim H^0(X,L \otimes L_{kP}) = \dim H^1(X,L \otimes L_{kP}) + \chi(L) + k \geqslant \chi(L) + k$$

If we choose k such that $\chi(L) + k \geqslant 1$, we obtain $\dim H^0(X,L \otimes L_{kP}) \geqslant 1$, which proves the lemma.

Proof of Theorem 2.4.1. By Lemma 2.4.2 there exists a non-zero section s of $L \otimes L_{kP}$ for some k. Let D_1 be the divisor defined by the zeros of s, counted with multiplicity. Then $L \otimes L_{kP}$ is isomorphic to L_{D_1} and hence

$$L \simeq L_{D_1} \otimes L_{-kP} \simeq L_{(D_1-kP)}$$

2.4.3. Degree of a line bundle

Let L be a line bundle on X. Choose a divisor D with $L \simeq L_D$. We define the degree, d(L), of L to be the degree of the divisor D. This is well defined, for if $L \simeq L_{D'}$, then $D - D'$ is the divisor of a meromorphic function so that $d(D - D') = 0$. (See Remark (ii) in §2.2)

2.5. Riemann-Roch Theorem

The genus *of* X *is defined to be* $\dim_{\mathbb{C}} H^1(X,\mathcal{O})$ *and will be denoted by* g.

Theorem 2.5.1 **(Riemann-Roch).** *Let* L *be a line bundle on* X. *Let* L^* *denote the dual bundle of* L, *and* K *the canonical line bundle on* X. *We then have:*

(i) $\dim H^0(X,L) - \dim H^1(X,L) = d(L) - g + 1$

(ii) $\dim H^0(X,L) - \dim H^0(X,K \otimes L^*) = d(L) - g + 1$

when g *is the genus of* X *and* d(L) *is the degree of* L.

Proof. We shall prove (i). The duality theorem and (i) would then give (ii). By Theorem 2.4.1 we can assume that $L \simeq L_{D_0}$ for some divisor D_0. Note that for the divisor $D = 0$, $\chi(L_D) = \chi(\mathcal{O}) = 1-g$, since $H^0(X, \mathcal{O}) \simeq \mathbb{C}$. The theorem follows from

Lemma 2.5.2. $\chi(L_D) - d(D)$ *is independent of the divisor* D.

Proof. It is enough to show that, for $P \in X$,

$$\chi(L_D) - d(D) = \chi(L_{D+P}) - d(D+P)$$

As in the proof of Lemma 2.4.2, we have an exact sequence

$$0 \to L_D \to L_{D+P} \to L_{D+P} \underset{\mathcal{O}}{\otimes} \mathcal{O}/\mathcal{I}_P \to 0$$

yielding

$$\chi(L_{D+P}) = \chi(L_D) + 1$$

Since $d(D+P) = d(D) + 1$, we have

$$\chi(L_{D+P}) - d(D+P) = \chi(L_D) - d(D)$$

2.6. Some consequences of the Riemann-Roch theorem

(1) There exist non-constant meromorphic functions on X. In fact, let D be any divisor with $d(D) \geqslant (g+1)$. Applying the Riemann-Roch theorem, we have $\dim H^0(X,L_D) \geqslant 2$. Since $H^0(X,L_D)$ can be identified with a space of meromorphic functions (see §2.2), it follows that there exist non-constant meromorphic functions on X.

(2) The degree of the canonical line bundle K is (2g-2). Applying the Riemann-Roch theorem for K, we get

$$\dim H^0(X,K) - \dim H^0(X,\mathcal{O}) = d(K) - g + 1$$

while $\dim H^0(X,K) = \dim H^1(X,\mathcal{O}) = g$, by duality. Hence $d(K) = (2g-2)$.

2.7. Vanishing theorem

Proposition. *Let* L *be a line bundle with* $d(L) \geqslant (2g-1)$. *Then* $H^1(X,L) = 0$.

Proof. By duality, $\dim H^1(X,L) = \dim H^0(X,K \otimes L^*)$; but $d(K \otimes L^*) = d(K) + d(L^*) = (2g-2) - d(L) < 0$. This implies that $H^0(X,K \otimes L^*) = 0$. For, if M is a line bundle with a non-zero section s, then $d(M) \geqslant 0$ as $M \simeq L_D$, where D is the divisor defined by the zeros of s.

3. PROJECTIVE EMBEDDING OF A COMPACT RIEMANN SURFACE

If W is a finite-dimensional vector space over $\mathbb{C}$, we denote by $\mathbb{P}(W)$ the projective space of one-dimensional subspaces of W. The space $\mathbb{P}(W)$ has a natural structure of a compact complex manifold (see Ref. [10], Lecture 16).

Definition 3.1. *Let* L *be a line bundle on* X. *We say that* L *is generated by its sections if the evaluation map* $H^0(X,L) \to L(P)$ *is surjective for all* $P \in X$, *where* L(P) *is the fibre of* L *at* P. (This is equivalent to requiring that, given $P \in X$, there exists a holomorphic section s of L with $s(P) \neq 0$.)

Proposition 3.2. *Let* L *be a line bundle on* X *generated by its sections. Then there exists a canonical holomorphic map*

$$\varphi_L : X \to \mathbb{P}(H^0(X,L)^*) = \mathbb{P}_L$$

where $H^0(X,L)^*$ *denotes the dual space of* $H^0(X,L)$.

Proof. Let $P \in X$. By hypothesis, the evaluation map at P, $H^0(X,L) \to L(P)$, is surjective; hence the dual map $L(P)^* \to H^0(X,L)^*$ is injective and we define $\varphi(P) \in \mathbb{P}_L$ to be the image. (The space of sections vanishing at P is of codimension one in $H^0(X,L)$ and $\varphi(P)$ is the subspace of $H^0(X,L)^*$ orthogonal to this space.)

Let $s_0, \ldots, s_N$ be a basis of $H^0(X,L)$ and let s_i be given, with respect to a trivialization of L in a neighbourhood of P, by the function f_i. Then it is easily checked that φ is given locally by

$$z \mapsto (f_0(z), \ldots, f_N(z))$$

in terms of homogeneous co-ordinates with respect to the dual basis $s_0^*, \ldots, s_N^*$. This proves that φ_L is holomorphic.

Definition 3.3. *A line bundle* L *on* X *is said to be* **very ample** *if it is generated by its sections and the map* $\varphi_L : X \to \mathbb{P}_L$ *is an embedding.* (The second condition means that φ_L is injective and that the differential of φ_L is injective at every point of X.)

Theorem 3.4. *Let* L *be a line bundle on* X *with* $d(L) \geqslant 2g + 1$. *Then* L *is very ample.*

Corollary. *Any compact Riemann surface can be embedded in a complex projective space.*

We first prove

Lemma 3.5. *A line bundle* L *with* $d(L) \geqslant 2g$ *is generated by its sections.*

Proof. Let $P \in X$. Tensoring the exact sequence

$$0 \to \mathcal{I}_P \to \mathcal{O} \to \mathcal{O}/\mathcal{I}_P \to 0$$

with L gives

$$0 \to L \underset{\mathcal{O}}{\otimes} \mathcal{I}_P \to L \to L \underset{\mathcal{O}}{\otimes} \mathcal{O}/\mathcal{I}_P \to 0$$

The associated exact cohomology sequence yields the exact sequence

$$H^0(X,L) \xrightarrow{\eta} H^0(X,L \underset{\mathcal{O}}{\otimes} \mathcal{O}/\mathcal{I}_P) \to H^1(X,L \otimes L_{-P})$$

By the vanishing theorem 2.7, we have $H^1(X,L \otimes L_{-P}) = 0$, since $d(L \otimes L_{-P}) = d(L) - 1 \geqslant 2g-1$, so that $H^0(X,L) \xrightarrow{\eta} H^0(X,L \otimes \mathcal{O}/\mathcal{I}_P)$ is surjective. But $H^0(X,L \otimes \mathcal{O}/\mathcal{I}_P)$ is canonically isomorphic to the fibre L(P) of L at P and the map η is then the evaluation map.

Lemma 3.6. *Let* L *be a line bundle with* $d(L) \geqslant 2g + 1$. *Given* $P, Q \in X$ *with* $P \neq Q$, *there exists a section* s *of* L *with* $s(P) = 0$ *and* $s(Q) \neq 0$.

Proof. Let $\mathcal{I}_{P,Q} = L_{-P-Q}$ denote the sheaf of holomorphic functions vanishing at P and Q. We then have the exact sequence of sheaves

$$0 \to L \otimes L_{-P-Q} \to L \to L \underset{\mathcal{O}}{\otimes} \mathcal{O}/\mathcal{I}_{P,Q} \to 0$$

Since $d(L \otimes L_{-P-Q}) = d(L) - 2 \geqslant 2g - 1$, we have $H^1(X,L \otimes L_{-P-Q}) = 0$ by 2.7. This shows that

$$H^0(X,L) \to H^0(X,L \underset{\mathcal{O}}{\otimes} \mathcal{O}/\mathcal{I}_{P,Q}) \simeq L(P) \oplus L(Q)$$

is surjective. The lemma follows by interpreting the map $H^0(X,L) \to L(P) \oplus L(Q)$ as the sum of evaluation maps at P and Q.

Proof of Theorem 3.4. Let L be a line bundle with $d(L) \geqslant 2g + 1$. By Lemma 3.5, the map φ_L is defined. By Lemma 3.6, given P,Q with $P \neq Q$, the space of sections vanishing at P is different from the space of sections vanishing at Q. It follows that φ_L is injective.

We shall now show that φ_L is of rank 1 at any $P \in X$. Let $\mathcal{I}_P^2 = L_{-2P}$ be the sheaf of functions vanishing to the second order at P. From the exact sequence

$$0 \to L \otimes \mathcal{I}_P^2 \to L \otimes \mathcal{I}_P \to L \otimes \mathcal{I}_P/\mathcal{I}_P^2 \to 0$$

we obtain the exact sequence

$$0 \to H^0(X, L \otimes \mathcal{I}_P^2) \to H^0(X, L \otimes \mathcal{I}_P) \to H^0(X, L \otimes \mathcal{I}_P/\mathcal{I}_P^2) \to 0$$

since $H^1(X, L \otimes \mathcal{I}_P^2) = H^1(X, L \otimes L_{-2P}) = 0$. But $\dim H^0(X, L \otimes \mathcal{I}_P/\mathcal{I}_P^2) = 1$ while $H^0(X, L \otimes \mathcal{I}_P)$ and $H^0(X, L \otimes \mathcal{I}_P^2)$ are, respectively, the space of sections vanishing at P and of sections vanishing to the second order at P. Hence there exists a basis $s_0, \ldots, s_N$ for $H^0(X,L)$ such that $s_0(P) \neq 0$, $s_i(P) = 0$ for $1 \leqslant i \leqslant N$, and s_1 does not vanish to the second order at P. If f_i is the local expression of s_i, the mapping φ_L is given in a neighbourhood of P by

$$z \mapsto \left(\frac{f_1(z)}{f_0(z)}, \ldots, \frac{f_N(z)}{f_0(z)}\right)$$

Further,

$$\frac{d(f_1/f_0)}{dz}(P) = \frac{d f_1}{dz}(P) / f_0(P) \neq 0$$

Thus the differential of φ_L at P is of rank 1.

Remark. It can be shown that X is a projective variety, i.e. X is the set of zeros in the projective space $\mathbb{P}_L$ of a family of homogeneous polynomials. This is a special case of Chow's theorem (see Ref. [1]).

4. GENUS AND FIRST BETTI NUMBER

Proposition 4.1. *Let* $\mathbb{C}$ *denote the constant sheaf on* X *with stalk* $\mathbb{C}$. *We then have*

$$\dim_{\mathbb{C}} H^1(X, \mathbb{C}) = 2g$$

where g *is the genus of* X.

Proof. Consider the exact sequence of sheaves (of $\mathbb{C}$-modules)

$$0 \to \mathbb{C} \to \mathcal{O} \xrightarrow{d} K \to 0 \tag{4.2}$$

where d is given by the exterior derivation, $f \mapsto df$. The sequence is exact at K, as any holomorphic differential is locally the derivative of a holomorphic function (Cauchy's theorem). Since $H^0(X, \mathbb{C}) \xrightarrow{\sim} H^0(X, \mathcal{O})$ and $H^2(X, \mathcal{O}) = 0$, we then have an exact sequence:

$$0 \to H^0(X,K) \to H^1(X,\mathbb{C}) \to H^1(X,\mathcal{O}) \to H^1(X,K) \to H^2(X,\mathbb{C}) \to 0$$

But $H^2(X,\mathbb{C}) \simeq \mathbb{C}$ by Poincaré duality (see also §5.3) and $\dim H^1(X,K) = \dim H^0(X,\mathcal{O}) = 1$; we obtain the exact sequence;

$$0 \to H^0(X,K) \to H^1(X,\mathbb{C}) \to H^1(X,\mathcal{O}) \to 0$$

However, $\dim H^0(X,K) = \dim H^1(X,\mathcal{O}) = g$. This proves the proposition.

Remark. The proposition shows that the genus depends only on the topology of X.

5. CHERN CLASS AND DEGREE

5.1. Chern classes

Let $\mathcal{O}^*$ denote the sheaf of non-vanishing holomorphic functions on X. The group $H^1(X,\mathcal{O}^*)$ is canonically isomorphic to the group (under tensor product) of isomorphism classes of holomorphic line bundles on X. If $\ell \in H^1(X,\mathcal{O}^*)$ is represented, with respect to an open covering $\{U_i\}$, by the non-vanishing holomorphic functions g_{ij} in $U_i \cap U_j$, then under this isomorphism ℓ is mapped into the isomorphism class of the line bundle with transition functions g_{ij}.

Consider the exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \tag{5.1.1}$$

where the map $\mathcal{O} \to \mathcal{O}^*$ is given by $f \mapsto e^{2\pi i f}$ $(i = \sqrt{-1})$. We then have the connecting homomorphism

$$b : H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z}) \tag{5.1.2}$$

Definition 5.1.3. *Let* L *be a line bundle on* X. *We define the (first) Chern class* $C_1(L)$ *of* L *by*

$$C_1(L) = -b(\tilde{L}) \in H^2(X,\mathbb{Z})$$

where $\tilde{L}$ *denotes the isomorphism class of* L.

5.2. Topological line bundles

The Chern class is defined for any topological line bundle on X. In fact, let $\mathscr{C}$ and $\mathscr{C}^*$ denote, respectively, the sheaf of continuous complex-valued functions and non-vanishing continuous functions. Then the group of isomorphism classes of topological line bundles on X is canonically identified with the group $H^1(X,\mathscr{C}^*)$ and the Chern class is defined by means of the connecting homomorphism

$$H^1(X,\mathscr{C}^*) \to H^2(X,\mathbb{Z})$$

arising from the exact sequence

$$0 \to \mathbb{Z} \to \mathscr{C} \xrightarrow{e^{2\pi i(\;)}} \mathscr{C}^* \to 0 \tag{5.2.1}$$

Comparing the exact cohomology sequences arising from (5.1.1) and (5.2.1), it is clear that the Chern class of a holomorphic line bundle depends only on the underlying structure of topological line bundle.

Proposition 5.2.2

(1) *The homomorphism* $H^1(X,\mathscr{C}^*) \to H^2(X,\mathbb{Z})$ *is an isomorphism. In particular, a line bundle is characterized topologically by its Chern class.*

(2) *Every topological line bundle on a compact Riemann surface can be endowed with the structure of a holomorphic line bundle.*

Proof

(1) Since $\mathscr{C}$ is a fine sheaf, we have $H^1(X,\mathscr{C}) = H^2(X,\mathscr{C}) = 0$. From the cohomology sequence associated to (5.2.1) we see that $H^1(X,\mathscr{C}^*) \to H^2(X,\mathbb{Z})$ is an isomorphism.

(2) From the exact sequence of sheaves (5.1.1) we get an exact sequence:

$$H^1(X, \mathcal{O}^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O})$$

But $H^2(X,\mathcal{O}) = 0$; hence $H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z})$ is surjective. Comparing the exact cohomology sequences arising from (5.1.1) and (5.2.1), we get a commutative diagram:

$$\begin{tikzcd}
H^1(X,\mathcal{O}^*) \arrow[r] \arrow[d] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, "id"] & 0 \\
H^1(X,\mathscr{C}^*) \arrow[r] & H^2(X,\mathbb{Z}) &
\end{tikzcd}$$

It now follows from (1) that $H^1(X,\mathcal{O}^*) \to H^1(X,\mathscr{C}^*)$ is surjective.

From now on we shall mean, as before, by a line bundle a holomorphic line bundle.

5.3. Chern class and degree

We now define a canonical homomorphism:

$$I' : H^2(X,\mathbb{C}) \to \mathbb{C}$$

Let $v \in H^2(X,\mathbb{C})$ and let v be represented by a (closed) 2-form ω under the de Rham isomorphism. Define

$$I'(v) = \int_X \omega$$

the integration being with respect to the canonical orientation on X given by the complex structure on X. ($I'(v)$ does not depend on the choice of ω, by Stokes' theorem.) The linear map $I' : H^2(X,\mathbb{C}) \to \mathbb{C}$ is an isomorphism.

The canonical map $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ is injective. We denote by I:

$$I : H^2(X,\mathbb{Z}) \to \mathbb{C}$$

the composite of this map and I'. (It follows from the next proposition that I maps $H^2(X,\mathbb{Z})$ onto $\mathbb{Z}$.)

Proposition 5.3.1. *Let* D *be a divisor on* X *and* L_D *the associated line bundle on* X. *We then have*

$$I(C_1(L_D)) = d(D)$$

where d(D) *is the degree of* D *and* $C_1(L_D)$ *is the Chern class of* L_D.

Corollary 5.3.2. *A line bundle* L *on* X *is of degree zero if and only if* $C_1(L) = 0$.

We first prove a lemma which gives an alternative description of the Chern class.

Lemma 5.3.3. *Consider the homomorphism* $\mathcal{O}^* \to K$ *given by* $f \mapsto \frac{1}{2\pi i} \cdot \frac{df}{f}$ *and let* $b' : H^1(X,\mathcal{O}^*) \to H^1(X,K)$ *be the induced homomorphism. If* $\ell \in H^1(X,\mathcal{O}^*)$, *we have*

$$b(\ell) = (\eta \circ b')(\ell)$$

where $\eta : H^1(X,K) \to H^2(X,\mathbb{C})$ *is the isomorphism given by (4.2)* (for the definition of b see (5.1.2)).

Proof. Consider the commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r, "e^{2\pi i(\ )}"] \arrow[d] & \mathcal{O}^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathcal{O} \arrow[r, "d"] & K \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{O}^* \to K$ is given by $f \mapsto \frac{1}{2\pi i} \cdot \frac{df}{f}$. This gives rise to the commutative diagram:

$$\begin{tikzcd}
H^1(X,\mathcal{O}^*) \arrow[r, "b"] \arrow[d, "b'"] & H^2(X,\mathbb{Z}) \arrow[d] \\
H^1(X,K) \arrow[r, "\eta"] & H^2(X,\mathbb{C})
\end{tikzcd}$$

which proves the lemma.

Remark 5.3.4. If $\xi \in H^1(X,K)$ is represented (under the Dolbeault isomorphism) by a form ψ of type $(1,1)$ then $\eta(\xi)$ is represented in $H^2(X,\mathbb{C})$ by the same form ψ (under the de Rham isomorphism).

Proof of Proposition 5.3.1. It is sufficient to prove the proposition in the case $D = (P)$ where P is a point of X. Choose a local co-ordinate system z at P in a disc U. Let $U_1 = U$ and $U_2 = X\text{-}P$. Then $\{U_1,U_2\}$ form an open covering of X, and L_P is given by the transition function $f_{12} = z$, $f_{21} = z^{-1}$ in $U_{12} = U_1 \cap U_2$. We wish to find a $(1,1)$ form representing $b'(L_P)$ and then integrate it over X. Let

$$\omega_{12} = \frac{1}{2\pi i} \cdot \frac{df_{12}}{f_{12}} \text{ and } \omega_{21} = -\omega_{12}$$

in U_{12}. We shall find explicitly $C^\infty(1,0)$ forms ω_1 in U_1 and ω_2 in U_2 such that $\omega_2 - \omega_1 = \omega_{12}$ in $U_1 \cap U_2$; then a form ψ (defined over X) representing $\{\omega_{12}\}$ would be given by

$$\psi = \bar{\partial}\omega_i \text{ in } U_i$$

Note that $\psi = \bar{\partial}\omega_i = d\omega_i$, as every $(2,0)$ form on X is zero.

To find ω_i, let W and V be discs round P with $\overline{W} \subset V$ and $\overline{V} \subset U$. Let φ be a C^∞ function in U whose support is contained in V and such that $\varphi(Q) = 1$ for all $Q \in W$. Consider φ as a C^∞ function on X by extending it by zero outside U. Set:

$\omega_1 = -(1-\varphi)\,\omega_{12}$ in U_1 (defined to be 0 at P)

$\omega_2 = \varphi\omega_{12}$ in U_2 (defined to be 0 outside V)

Then $\omega_2 - \omega_1 = \omega_{12}$ in $U_1 \cap U_2$. If $\psi = d\omega_1 = d\omega_2$, then

$$\int_X \psi = \int_V \psi \text{ (as } \psi \text{ is 0 outside V)} = \int_V d\omega_1 = \int_{\partial V} \omega_1 = -\int_{\partial V} \omega_{12} = -\frac{1}{2\pi i}\int_{\partial V} \frac{dz}{z} = -1$$

(∂V denotes the boundary of V). Now the proposition follows from Lemma 5.3.3 and Remark 5.3.4, recalling that $C_1(\ell) = -b(\ell)$ for $\ell \in H^1(X,\mathcal{O}^*)$.

6. THE JACOBIAN

Theorem 6.1. *The group of isomorphism classes of (holomorphic) line bundles on* X *of degree zero has a natural structure of a complex torus of complex dimension* g.

This complex torus is called the **Jacobian** of X. We first prove

Lemma 6.2. *The map* $H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O})$, *induced by the natural inclusion of the constant sheaf* $\mathbb{R}$ *in* $\mathcal{O}$, *is an isomorphism of real vector spaces.*

Proof. By Proposition 4.1,

$$\dim_{\mathbb{R}} H^1(X,\mathbb{R}) = \dim_{\mathbb{R}} H^1(X,\mathcal{O})$$

So Lemma 6.2 follows from

Lemma 6.3. *The map* $H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O})$ *is injective.*

Proof. Let $\omega \in H^1(X,\mathbb{R})$ be in the kernel. We can then find an open covering $\{U_i\}$ of X in which ω is represented by a cocycle $r_{ij} \in H^0(U_i \cap U_j, \mathbb{R})$ and holomorphic functions f_i in U_i such that $f_j - f_i = r_{ij}$ in $U_i \cap U_j$. Let $h_j = \mathrm{Im}\, f_j$, where Im denotes the imaginary part. Since r_{ij} are real-valued, $h_i = h_j$ in $U_i \cap U_j$ and hence define a global harmonic function h on X. By the maximum principle h is constant. It follows that $\mathrm{Rl}\, f_i$ is constant on each connected component of U_i, so that it defines a section of the sheaf $\mathbb{R}$ over U_i. We have

$$\mathrm{Rl}\, f_j - \mathrm{Rl}\, f_i = r_{ij} \text{ in } U_i \cap U_j$$

which shows that r_{ij} is a coboundary, i.e. $\omega = 0$ in $H^1(X, \mathbb{R})$.

Remark. Lemma 6.3. and its proof are valid in any compact complex manifold.

Proof of Theorem 6.1. Starting from the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

we obtain the exact sequence

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O}^*) \xrightarrow{b} H^2(X,\mathbb{Z})$$

If J denotes the group of isomorphism classes of line bundles of degree 0, we see by Corollary 5.3.2 that J = ker b so that we have an exact sequence

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}) \to J \to 0$$

Now $H^1(X,\mathbb{Z})$ is a lattice in $H^1(X,\mathbb{R})$ and, by Lemma 6.2, $H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O})$ is an isomorphism of real vector spaces. Thus $H^1(X,\mathbb{Z})$ is a lattice in $H^1(X,\mathcal{O})$ and since $H^1(X,\mathcal{O})$ is a complex vector space, J acquires a natural structure of a complex torus.

Remark. If D_1 and D_2 are divisors on X, we say that D_1 and D_2 are **linearly equivalent** if the divisor $D_1 - D_2$ is the divisor of a meromorphic function $f \not\equiv 0$ on X. In view of Remark (ii) in §2.2, we can rephrase Theorem 6.1 as follows: the group of linear equivalence classes of divisors of degree 0 on X has a natural structure of a complex torus of dimension g.

7. LINE BUNDLES AND CHARACTERS

By $H_1(X,\mathbb{Z})$ and π we denote, respectively, the first homology group of X with integer coefficients and the fundamental group of X. Let $\rho : \pi \to \mathbb{C}^*$ be a homomorphism; since the quotient of π by the commutator subgroup of π is $H_1(X,\mathbb{Z})$, ρ is the same as a homomorphism of $H_1(X,\mathbb{Z})$ into $\mathbb{C}^*$. We denote by L_ρ the line bundle on X associated to ρ. We recall that L_ρ is defined as follows. Let $\tilde{X} \to X$ be the universal covering of X, considered as a holomorphic principal fibre space with structure group π. The group π acts on $\tilde{X} \times \mathbb{C}$, the action $\pi \times X \times \mathbb{C} \to \tilde{X} \times \mathbb{C}$ being given by $(\gamma,x,v) \mapsto (x\gamma,\rho(\gamma)^{-1}\, v)$ for $\gamma \in \pi$, $x \in X$ and $v \in \mathbb{C}$. The orbit space for this action is then a (holomorphic) line bundle L_ρ over X. If $\{\gamma_{ij}\}$, $\gamma_{ij} \in \pi$, are the transition functions for the principal bundle $\tilde{X} \to X$ with respect to a suitable open covering U_i of X, then $\rho\,(\gamma_{ij})$ give a set of transition functions for L_ρ.

Remark 7.1. More generally if $\rho : \pi \to GL(n,\mathbb{C})$ is a homomorphism, we can construct similarly a holomorphic vector bundle E_ρ of rank n on X. We say that E_ρ is associated to the representation ρ of the fundamental group of X.

Remark 7.2. In particular, if $\chi : H_1(X,\mathbb{Z}) \to \mathbb{C}_1^*$ denotes a character of $H_1(X,\mathbb{Z})$, where $\mathbb{C}_1^*$ denotes the group of complex numbers of modulus 1, we have a line bundle L_χ associated to the character χ. If $\mathbb{C}_1^*$ also denotes the constant sheaf with stalk $\mathbb{C}_1^*$, the group $H^1(X,\mathbb{C}_1^*)$ is canonically identified with Hom $(H_1(X,\mathbb{Z}), \mathbb{C}_1^*)$. (This follows, e.g., from the universal coefficient theorem or from A.10 of the Appendix.) The map $H^1(X,\mathbb{C}_1^*) \to H^1(X,\mathcal{O}^*)$ induced by the natural inclusion $\mathbb{C}_1^* \to \mathcal{O}^*$ of sheaves associates then to a character χ, the isomorphism class of the line bundle L_χ.

Theorem 7.3. *A line bundle associated to a character of* $H_1(X,\mathbb{Z})$ *is of degree* 0 *and any line bundle of degree* 0 *is associated to a character. Moreover, two characters of* $H_1(X,\mathbb{Z})$ *are the same if and only if the associated line bundles are isomorphic.*

Remark. The content of the theorem is that the underlying topological torus of the Jacobian J is the character group of $H_1(X,\mathbb{Z})$.

Proof. Consider the commutative diagram of exact sequences of sheaves:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{R} \arrow[r, "e^{2\pi i(\ )}"] \arrow[d] & \mathbb{C}_1^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}^* \arrow[r] & 0
\end{tikzcd}$$

Since $H^2(X,\mathbb{Z})$ has no torsion, the map $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$ is injective and we get a commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(X,\mathbb{R}) \arrow[r] \arrow[d] & H^1(X,\mathbb{C}_1^*) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}) \arrow[r] & J \arrow[r] & 0
\end{tikzcd}$$

But $H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O})$ is an isomorphism of real vector spaces (Lemma 6.2). It follows that $H^1(X,\mathbb{C}_1^*) \to J$ is an isomorphism. This proves the theorem in view of Remark 7.2.

8. POINCARE BUNDLE

Theorem 8.1. *Let* J *be the Jacobian of* X. *There exists a (holomorphic) line bundle* $\mathscr{P}$ *on* $J \times X$, *such that for each* $j \in J$ *the restriction of* $\mathscr{P}$ *to* $j \times X$, *considered as a line bundle on* X, *is in the isomorphism class* j.

Such a bundle $\mathscr{P}$ will be called a **Poincaré bundle.**

Remarks: factors of automorphy. Let $B \to M$ be a holomorphic principal bundle with structure group a complex Lie group G, acting on the right. A holomorphic function $f : B \times G \to \mathbb{C}^*$ is called a **factor of automorphy** on $B \times G$ with values in $\mathbb{C}^*$, if for $x \in B$, $g_1, g_2 \in G$, we have

$$f(x,g_1 g_2) = f(x,g_1)\, f(xg_1,g_2) \tag{8.2}$$

Writing $f_g(x) = f(x,g)$, the above condition can be written as

$$f_{g_1 g_2}(x) = f_{g_1}(x)\, f_{g_2}(xg_1)$$

Given a factor of automorphy f, we can construct a line bundle L_f on M as follows. The map

$$B \times \mathbb{C} \times G \to B \times \mathbb{C}$$

given by

$$(x,v,g) \mapsto (xg, f(x,g)^{-1} v),\ x \in B,\ g \in G,\ v \in \mathbb{C}$$

is an action of G on $B \times \mathbb{C}$ in view of the condition (8.2) and the quotient space L_f is a line bundle on M.

For more details on the factors of automorphy, see the Appendix.

Proof of Theorem 8.1. We first show that $H^1(X,\mathbb{C}^*)$ can be considered as a holomorphic principal bundle over J with structure group $H^0(X,K)$. The commutative diagram of exact sequences of sheaves

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[u] & \mathcal{O}^* \arrow[r, "df/2\pi if"] \arrow[u] & K \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[u, "e^{2\pi i(\ )}"'] & \mathcal{O} \arrow[r, "d"] \arrow[u, "e^{2\pi i(\ )}"'] & K \arrow[r] \arrow[u, equal] & 0 \\
 & \mathbb{Z} \arrow[u] \arrow[r, "\sim"] & \mathbb{Z} \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

gives rise to the following commutative diagram of exact sequences:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & H^0(X,K) \arrow[r] & H^1(X,\mathbb{C}^*) \arrow[r] \arrow[u] & J \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & H^0(X,K) \arrow[r] \arrow[u, equal] & H^1(X,\mathbb{C}) \arrow[r] \arrow[u] & H^1(X,\mathcal{O}) \arrow[r] \arrow[u] & 0 \\
 & & H^1(X,\mathbb{Z}) \arrow[r, "\sim"] \arrow[u] & H^1(X,\mathbb{Z}) \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Introduce on $H^1(X,\mathbb{C}^*)$ the complex structure induced by the natural complex structure on the quotient $H^1(X,\mathbb{C})/H^1(X,\mathbb{Z})$. Since $H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ and $H^1(X,\mathcal{O}) \to J$ are local isomorphisms, we see that $H^1(X,\mathbb{C}^*) \to J$ is a principal bundle with structure group $H^0(X,K)$.

Let $\tilde{X}$ be the universal covering of X. Put $R = H^1(X,\mathbb{C}^*)$. Then $R \times \tilde{X} \to J \times X$ is a principal fibre bundle with structure group $G = G_0 \times \pi$, where $G_0 = H^0(X,K)$ and π is the fundamental group of X. We shall construct a Poincaré bundle on $J \times X$ by means of a factor of automorphy on $(G_0 \times \pi) \times (R \times X)$ with values in $\mathbb{C}^*$.

We identify $H^1(X,\mathbb{C}^*)$ with $\mathrm{Hom}(\pi,\mathbb{C}^*)$. If $\omega \in H^0(X,K)$, denote by χ_ω the corresponding element in $\mathrm{Hom}(\pi,\mathbb{C}^*)$ given by the map $H^0(X,K) \to H^1(X,\mathbb{C}^*)$. Let $\tilde{\omega}$ denote the pullback of ω to $\tilde{X}$. Since $\tilde{X}$ is simply connected, we can find a holomorphic function $\varphi_\omega : \tilde{X} \to \mathbb{C}^*$ such that $\varphi_\omega^{-1}\, d\varphi_\omega = -2\pi i\,\tilde{\omega}$ and $\varphi_\omega(x_0) = 1$, where x_0 is a (fixed) point of $\tilde{X}$. Note that φ_ω is uniquely determined by these conditions and, in fact,

$$\varphi_\omega(x) = \exp \int_{x_0}^{x} (-2\pi i\tilde{\omega}),\ x \in \tilde{X} \tag{8.3}$$

This shows that $\varphi_\omega(x) = \varphi(\omega,x)$ is holomorphic in (ω,x) and that

$$\varphi_\omega(x\gamma) = \chi_\omega(\gamma)\, \varphi_\omega(x) \text{ for } \gamma \in \pi \tag{8.4}$$

We now define the factor of automorphy. Set

$$f_{\omega,\gamma}(\rho,x) = \chi_\omega(\gamma)\, \rho(\gamma)\, \varphi_\omega(x)$$

for $\rho \in R$, $x \in X$, $\omega \in G_0$ and $\gamma \in \pi$. We have

$$\begin{aligned} f_{\omega_1+\omega_2,\gamma_1,\gamma_2}(\rho,x) &= \chi_{\omega_1+\omega_2}(\gamma_1\gamma_2)\, \rho(\gamma_1\gamma_2)\, \varphi_{\omega_1+\omega_2}(x) \\ &= \chi_{\omega_1}(\gamma_1\gamma_2)\, \chi_{\omega_2}(\gamma_1\gamma_2)\, \rho(\gamma_1\gamma_2)\, \varphi_{\omega_1}(x)\, \varphi_{\omega_2}(x) \\ &= \chi_{\omega_1}(\gamma_1)\, \chi_{\omega_1}(\gamma_2)\, \chi_{\omega_2}(\gamma_1)\, \chi_{\omega_2}(\gamma_2)\, \rho(\gamma_1)\, \rho(\gamma_2)\, \varphi_{\omega_1}(x)\, \varphi_{\omega_2}(x) \end{aligned}$$

On the other hand, we have

$$f_{\omega_1,\gamma_1}(\rho,x) = \chi_{\omega_1}(\gamma_1)\, \rho(\gamma_1)\, \varphi_{\omega_1}(x)$$

and

$$\begin{aligned} f_{\omega_2,\gamma_2}((\rho,x)(\omega_1,\gamma_1)) &= f_{\omega_2,\gamma_2}(\chi_{\omega_1}\rho, x\gamma_1) \\ &= \chi_{\omega_2}(\gamma_2)(\chi_{\omega_1}\rho)(\gamma_2)\,\varphi_{\omega_2}(x\gamma_1) \\ &= \chi_{\omega_2}(\gamma_2)\,\chi_{\omega_1}(\gamma_2)\,\rho(\gamma_2)\,\chi_{\omega_2}(\gamma_1)\,\varphi_{\omega_2}(x) \end{aligned}$$

using (8.4). Hence

$$f_{\omega_1+\omega_2,\gamma_1\gamma_2}(\rho,x) = f_{\omega_1,\gamma_1}(\rho,x)\, f_{\omega_2,\gamma_2}((\rho,x)(\omega_1,\gamma_1))$$

which shows that f is a factor of automorphy.

The factor of automorphy f defines a line bundle $\mathscr{P}$ on $R \times \widetilde{X}/G_0 \times \pi = J \times X$. We claim that $\mathscr{P}$ satisfies the requirement in Theorem 8.1. If $j \in J$ and ρ in $H^1(X,\mathbb{C}^*)$ is mapped into j we shall show that the restriction of $\mathscr{P}$ to $j \times X$ is isomorphic to the line bundle associated to the homomorphism $\rho : \pi \to \mathbb{C}^*$. We have the commutative diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "e_\rho"] \arrow[d] & R \times \widetilde{X} \arrow[d] \\ X \arrow[r, "e_j"] & J \times X \end{tikzcd}$$

where $e_\rho(x) = (\rho,x)$ for $x \in X$ and $e_j(y) = (j,y)$, $y \in X$. The map e_ρ clearly satisfies the condition

$$e_\rho(x\gamma) = e_\rho(x)(0,\gamma),\ (0,\gamma) \in G_0 \times \pi$$

It follows that the inverse image $e_j^*(\mathscr{P})$ of $\mathscr{P}$ by the map e_j is given by the factor of automorphy $f_{0,\gamma}(\rho,x) = \rho(\gamma)$ on $\pi \times \widetilde{X}$, i.e. $e_j^*(\mathscr{P}) \approx L_\rho$ (see also A.9 of the Appendix). This proves the theorem.

Remark. The idea of the proof of Theorem 8.1 is the following. Let $R = H^1(X,\mathbb{C}^*) = \mathrm{Hom}(\pi,\mathbb{C}^*)$. It is easy to construct a line bundle $\mathscr{L}$ on $R \times X$ such that $\mathscr{L}|\rho \times X \approx L_\rho$ for $\rho \in R$, where L_ρ is the line bundle associated to ρ. In fact $\mathscr{L}$ is the quotient space for the action of π on $R \times \widetilde{X} \times \mathbb{C}$ given by

$$(\rho,x,v,\gamma) \mapsto (\rho,x\gamma,\rho(\gamma)^{-1}v)$$

$\rho \in R$, $x \in \widetilde{X}$, $v \in \mathbb{C}$ and $\gamma \in \pi$. Note that if $\rho_1, \rho_2 \in \mathrm{Hom}(\pi,\mathbb{C}^*)$, the bundles L_{ρ_1} and L_{ρ_2} are isomorphic if there exists a holomorphic function $\varphi : \widetilde{X} \to \mathbb{C}^*$ satisfying

$$\varphi(x\gamma) = \rho_1(\gamma)\,\varphi(x)\,\rho_2(\gamma)^{-1},\ x \in \widetilde{X},\ \gamma \in \pi$$

If ρ_1 and ρ_2 differ by an element $\omega \in H^0(X,K)$, the exponential of the abelian integral, φ_ω in (8.3), associated to ω gives an isomorphism between L_{ρ_1} and L_{ρ_2} (see (8.4)). Moreover, the abelian integrals depend holomorphically on ω if suitably normalized. These isomorphisms can then be used to "descend" the line bundle $\mathscr{L}$ into a line bundle $\mathscr{P}$ on $J \times X$.

9. THE PICARD MANIFOLD OF A COMPACT KÄHLER MANIFOLD

The results of §7 and §8 carry over with little change to the case of a compact connected Kähler manifold M [13].

(1) The set of isomorphism classes of holomorphic line bundles on M with zero Chern class has a natural structure of a complex torus, $\mathrm{Pic}_0(M)$, called the Picard manifold of M. We have $\dim_{\mathbb{C}} \mathrm{Pic}_0(M) = \dim_{\mathbb{C}} H^1(M,\mathcal{O})$. The proof of Theorem 6.1 carries over once one knows that $H^1(M,\mathbb{R}) \to H^1(M,\mathcal{O})$ is an isomorphism over $\mathbb{R}$; this isomorphism results from Hodge theory of Kähler manifolds [11,13].

(2) The underlying topological torus of $\mathrm{Pic}_0(M)$ is the character group of the free abelian group $\{H_1(M,\mathbb{Z})/\text{Torsion subgroup of } H_1(M,\mathbb{Z})\}$. By slightly modifying the proof of Theorem 8.1 we can construct a Poincaré bundle on $\mathrm{Pic}_0(M) \times M$ (e.g. one has to work with the sheaf Ω^1_c of closed holomorphic 1-forms on M instead of K).

10. VECTOR BUNDLES ON A COMPACT RIEMANN SURFACE

Let V be a vector bundle of rank n on X. The line bundle $\overset{n}{\wedge} V$, the nth exterior power of V, will be denoted by det V. If g_{ij} is a set of transition functions for V, det g_{ij} form a set of transition functions for det V, where det $g_{ij}(x)$ denotes the determinant of the matrix $g_{ij}(x)$.

We define the degree of V, d(V), by

$$d(V) = d(\det V)$$

Remark 10.1

(a) Let V_1 and V_2 be vector bundles on X and $f : V_1 \to V_2$ be a (holomorphic) homomorphism. Let $f(P) : V_1(P) \to V_2(P)$ denote the induced map on the fibres of V_1 and V_2 at $P \in X$. Then if $P_0 \in X$, there exists a neighbourhood U of P_0 such that

$$\dim \ker f(P) \leqslant \dim \ker f(P_0), \text{ for } P \in U$$

(b) If dim ker f(P) is independent of P in X, we say that f is of constant rank. If f is of constant rank, then ker f (kernel of f) and Im f (Image of f) form subbundles of V_1 and V_2 respectively (for proofs see Ref. [5], Ch.III).

Lemma 10.2. *Let* V_1 *be a subbundle of* V *and let* V_2 *be the quotient bundle. We have*

$$d(V) = d(V_1) + d(V_2)$$

Proof. We can assume that the transition functions $\{g_{ij}\}$ of V are of the form

$$g_{ij} = \begin{pmatrix} h_{ij} & * \\ 0 & k_{ij} \end{pmatrix}$$

where h_{ij} and k_{ij} are transition functions for V_1 and V_2. Hence $\det V \cong \det V_1 \otimes \det V_2$, which gives $d(V) = d(V_1) + d(V_2)$.

Lemma 10.3. *Let* $P \in X$. *Then the vector bundle* $V \otimes L_{kP}$ *admits a non-zero section for some* $k \in \mathbb{Z}$.

Proof. The proof is similar to that of Lemma 2.4.2. We have an exact sequence

$$0 \to V \otimes L_{(k-1)P} \to V \otimes L_{kP} \to Q \to 0$$

where Q is a sheaf with support at P and whose stalk at P is an n-dimensional vector space (canonically isomorphic to the fibre of $V \otimes L_{kP}$ at P). Hence

$$\chi(V \otimes L_{kP}) = \chi(V \otimes L_{(k-1)P}) + n$$

which implies that

$$\chi(V \otimes L_{kP}) = \chi(L) + kn$$

Choosing k large, we obtain

$$\dim H^0(X, V \otimes L_{kP}) \geqslant 1$$

10.4. The projective bundle associated to a vector bundle

Let V be a vector bundle on X. The group $\mathbb{C}^*$ acts on $V - \sigma_0(X)$ by multiplication, where σ_0 is the zero section of V, and the quotient is a locally trivial holomorphic fibre bundle, $\mathbb{P}(V)$, over X of fibre type $\mathbb{P}(\mathbb{C}^n)$. The fibre of $\mathbb{P}(V)$ at $P \in X$ is identified canonically with the projective space of one-dimensional subspaces of the fibre of V at P. A line subbundle of V defines clearly a (holomorphic) section of $\mathbb{P}(V)$ over X, and conversely a section of $\mathbb{P}(V)$ over X defines a line subbundle of V.

Theorem 10.5. *A vector bundle (of rank $\geqslant 1$) on a compact Riemann surface admits a filtration by subbundles such that the successive quotients are line bundles. That is, if* V *is a vector bundle of rank* $n \geqslant 1$, *there exist subbundles* $V_1, \ldots, V_{n-1}$ *of* V *with rank* $V_i = i$ *and* $V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V$.

Proof. It is sufficient to show that V has a line subbundle L, for then we can prove the theorem by induction on the dimension of V and using the induction hypothesis for V/L. In turn, it suffices to show that for some line bundle L' on X, $V \otimes L'$ has a line subbundle; for if L'' is a subbundle of $V \otimes L'$, the bundle $(L')^* \otimes L''$ would be a subbundle of V. Thus we may assume, in view of Lemma 10.3, that V has a non-zero section s and prove that V has a line subbundle.

If $s(P) \neq 0$ for all $P \in X$, s clearly defines a section of the projective bundle $\mathbb{P}(V)$ and hence V has a line subbundle (see §10.4). Otherwise let S denote the finite set of points P with $s(P) = 0$. Then s defines a section, $\tilde{s}$, of $\mathbb{P}(V)$ in the complement of S. We shall show that $\tilde{s}$ extends to a section of $\mathbb{P}(V)$ over X, which will complete the proof of the theorem.

Let $P \in S$. We may suppose that in a connected co-ordinate neighbourhood U of P, the section s is given by

$$s = (f_1, \ldots, f_m, 0, \ldots, 0)$$

where f_i are holomorphic functions in U, $f_i \not\equiv 0$ and $f_i(P) = 0$ for $1 \leqslant i \leqslant m$. If z is a local co-ordinate system in U with $z(P) = 0$, write $f_i = z^{k_i} g_i$, $1 \leqslant i \leqslant m$, where g_i are holomorphic and non-vanishing in a neighbourhood U' of P with $U' \subset U$. Let

$$k = \min_{1 \leqslant i \leqslant m} k_i$$

be the minimum of k_i. We may assume that $k = k_1$. Consider

$$s' = (f_1/z^k, \ldots, f_m/z^k, 0, \ldots, 0)$$

Noting that $(f_1/z^k)(Q) = (f_1/z^{k_1})(Q) = g_1(Q) \neq 0$, for $Q \in U_1$, we see that s' defines a section $\widetilde{s}'$ of $\mathbb{P}(V)$ in U'. Since the function $1/z^k$ does not vanish in $U' - P$, we have $\widetilde{s}' = \widetilde{s}$ in $U' - P$. It follows that $\widetilde{s}$ extends to a section of $\mathbb{P}(V)$ over X.

Proposition 10.6. *Let* V *be a vector bundle on* X. *There exists a real number* C_0, *depending on* V, *such that for all subbundles* W *of* V *we have* $d(W) \leqslant C_0$.

Proof. Note that if W is a subbundle of rank $k \geqslant 2$ of V, $\overset{k}{\wedge} W$ is a line subbundle of $\overset{k}{\wedge} V$ and $d(\overset{k}{\wedge} W) = d(W)$. Hence it is sufficient to prove that for a vector bundle V the degree of line subbundles of V is bounded above.

Let L be a line subbundle of V. Let $0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = V$ be a filtration on V, where V_i are subbundles of V and $V_i/V_{i-1} = L_i$ are line bundles for $1 \leqslant i \leqslant n$ (Theorem 10.5). Now it is clear that the natural map $L \to L_i$ is non-zero for some i. This shows that $d(L) \leqslant d(L_i)$; for $L^* \otimes L_i$ has a non-zero section so that $d(L^* \otimes L_i) \geqslant 0$. Thus

$$d(L) \leqslant \sup_i d(L_i)$$

11. THE RIEMANN-ROCH THEOREM FOR VECTOR BUNDLES

Theorem 11.1. *Let* V *be a vector bundle of rank* n *on* X. *Then*

(1) $\dim H^0(X,V) - \dim H^1(X,V) = d(V) + n(1-g)$

(2) $\dim H^0(X,V) - \dim H^0(X,K \otimes V^*) = d(V) + n(1-g)$

where V^* *is the dual bundle of* V, $d(V)$ *is the degree of* V *and* g *is the genus of* X.

Proof. It is sufficient to prove (1), in view of the duality theorem. The theorem has been proved already for $n = 1$ (Theorem 2.5.1). Assume that the theorem has already been proved for vector bundles of rank $n-1$. Let L be a line subbundle of V, which exists by Theorem 10.5. From the exact sequence

$$0 \to L \to V \to V/L \to 0$$

we have, by Proposition 2.3.1,

$$\begin{aligned}\chi(V) &= \chi(L) + \chi(V/L)\\ &= d(L) + (1-g) + d(V/L) + (n-1)(1-g)\\ &\qquad \text{(by induction hypothesis)}\\ &= d(V) + n(1-g)\\ &\qquad \text{(by Lemma 10.2).}\end{aligned}$$

12. INDECOMPOSABLE BUNDLES AND THE KRULL-REMAK-SCHMIDT THEOREM

Definition 12.1. *A vector bundle* V *on* X *is said to be indecomposable if it cannot be written as a direct sum of proper subbundles of* V.

Lemma 12.2. *Let* W *be a vector bundle on* X *and* p_1, p_2 *be endomorphisms of* W *such that* $p_i^2 = p_i (i = 1,2)$ *and* $p_1 + p_2 = \mathrm{Id}_W$ *(identity map of* W*). Then* $\mathrm{Im}\, p_1$ *(image of* p_1*) and* $\mathrm{Im}\, p_2$ *are subbundles of* W *and* $W = \mathrm{Im}\, p_1 + \mathrm{Im}\, p_2$.

Proof. If $P \in X$ and for $i = 1,2$, $p_i(P) : V(P) \to V(P)$ is the induced map on the fibre at P, we have $V(P) = \operatorname{Im} p_1(P) \oplus \operatorname{Im} p_2(P)$. By Remark 10.1(a) it follows that p_1 and p_2 are of constant rank. By Remark 10.1(b), Im p_i are subbundles and $W = \operatorname{Im} p_1 + \operatorname{Im} p_2$.

Theorem 12.3. *Let* V *be a vector bundle on* X. *Then*

(1) V *can be written as a finite direct sum of indecomposable subbundles of* V;

(2) *If*

$$V = \bigoplus_{i=1}^{k} W_i = \bigoplus_{j=1}^{m} W'_j$$

where W_i *and* W'_j *are indecomposable subbundles* ($\neq 0$) *of* V, *we have* $k = m$ *and there exists a permutation* σ *of* $[1, ..., k]$ *and an automorphism* T *of* V *such that* $TW_i = W'_{\sigma(i)}$, *for* $1 \leqslant i \leqslant k$.

For the proof we need

Theorem 12.4 (Krull-Remak-Schmidt). *Let* A *be a ring and* M *a module over* A *of finite length (i.e.* M *satisfies ascending and descending chain conditions for submodules). Then* M *is a finite direct sum of indecomposable submodules. Moreover, if*

$$M = \bigoplus_{i=1}^{k} M_i = \bigoplus_{j=1}^{m} M'_i$$

are two decompositions of M *into indecomposable submodules* $\neq 0$, *then* $k = m$ *and there exists a permutation* σ *of* $[1, ..., k]$ *and an automorphism* T_0 *of* M *such that* $T_0 M_i = M'_{\sigma(i)}$, *for* $1 \leqslant i \leqslant k$.

For a proof see Ref. [3], Ch.8, §2.2, p.23.

Remark 12.5. We shall prove Theorem 12.3 by applying Theorem 12.4 in the following situation. Let A denote the ring of endomorphisms of the vector bundle V. We take M = A, considered as a right module over A. Since $A = H^0(X, V \otimes V^*)$, we see that A is a finite-dimensional algebra over $\mathbb{C}$ and is hence of finite length over A.

Lemma 12.6. *Suppose that* $V = W \oplus W'$ *where* W ($\neq 0$) *is indecomposable. Let* $p : V \to W$ *be the corresponding projection. Then the right ideal generated by* p *in* A *is indecomposable, as a right submodule of* A (A = *the ring of endomorphisms of* V).

Proof. Let (p) be the right ideal generated by p. Suppose that (p) is decomposable and $(p) = C_1 \oplus C_2$, $C_i \neq 0$ being right ideals. Write $p = r_1 + r_2$ with $r_i \in C_i$. We claim that

$$r_1 r_2 = r_2 r_1 = 0 \text{ and } r_i^2 = r_i \tag{12.7}$$

To prove this, note that $r_1 r_2 \in C_1$. But $r_1 r_2$ also belongs to C_2; for if $r_2 = p\,a$, for some $a \in A$, we have

$$r_1 r_2 = pr_2 - r_2^2 = p^2 a - r_2^2 = pa - r_2^2 = r_2 - r_2^2 \in C_2$$

Thus $r_1 r_2 \in C_1 \cap C_2$ and hence $r_1 r_2 = 0$. Similarly $r_2 r_1 = 0$. Since $(r_1 + r_2)^2 = r_1 + r_2$, it follows that $r_i^2 = r_i$.

We have, from Eq. (12.7), that $pr_i = r_i$. Hence r_i (i = 1,2) maps W into W and $\mathrm{Id}_W = r_1|_W + r_2|_W$. By Lemma 12.2, W must then be decomposable.

Proof of Theorem 12.3. Note that the first part of the theorem is clear by induction on the rank of V. For, a line bundle is indecomposable and if a bundle is decomposable it is the direct sum of subbundles of strictly lower rank.

We use the notation of Remark 12.5. Let p_i and p_j' denote, respectively, the projection onto W_i and W_j'; (p_i) and (p_j') denote, respectively, the right ideal generated by p_i and p_j' in A. By Lemma 12.6

$$A = \bigoplus_{i=1}^{k} (p_i) = \bigoplus_{j=1}^{m} (p_j')$$

give two decompositions of A into indecomposable submodules. By Theorem 12.4, we have k = m and there exists a permutation σ of $[1, \ldots, n]$ and an automorphism T_0 of A (considered as a right module over itself) with $T_0((p_i)) = (p'_{\sigma(i)})$. But T_0 is given by multiplication on the left by an element T (= $T_0(\mathrm{Id}_V)$) which is invertible in A. Thus T is an automorphism of V mapping W_i onto $W'_{\sigma(i)}$ as $T p_i = p'_{\sigma(i)} a$ for some $a \in A$. This proves (2).

Proposition 12.8. *Let* V *be a vector bundle on* X. *Then* V *is indecomposable if and only if every endomorphism of* V *is of the form* $\lambda \,\mathrm{Id}_V + N$, *where* $\lambda \in \mathbb{C}$ *and* N *is nilpotent.* (Id_V *is the identity endomorphism* of V.)

Proof. Suppose $V = V_1 \oplus V_2$, $V_i \neq 0$. Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and T be the endomorphism of V such that $T|V_1 = \lambda \,\mathrm{Id}_V$ and $TV_2 = 0$. Suppose that T is of the form $\nu \,\mathrm{Id}_V + N$ with $\nu \in \mathbb{C}$ and N nilpotent. Then we would have $N = -\nu \,\mathrm{Id}_{V_2}$ on V_2, and as N is nilpotent, $\nu = 0$. But then $T|V_1 = N|V_1 = \lambda \,\mathrm{Id}_{V_1}$, which implies that $\lambda = 0$, a contradiction.

Let V be a vector bundle of rank n and T an endomorphism of V. For $x \in X$ let T_x be the endomorphism of the fibre V_x induced by T and

$$P_x(t) = \sum_{0 \leqslant i \leqslant n} a_i(x)\, t^i$$

be the characteristic polynomial of T_x. Since $a_i(x)$ are holomorphic functions on X, $a_i(x)$ are constant equal to, say, a_i. Let

$$P(t) = \sum_{0 \leqslant i \leqslant n} a_i\, t^i = \prod_{j=1}^{k} (t - \lambda_j)^{n_j}$$

with $\lambda_j \in \mathbb{C}$ distinct. Let $V_{j,x}$ be the kernel of $(T_x - \lambda_j)^{n_j}$, i.e. the generalized eigenspace of T_x corresponding to λ_j. Then, as is well known, dim $V_{j,x} = n_j$, a constant. Hence by Remark 10.1(b), $V_{j,x}$ build a subbundle V_j of V and $V = \oplus V_j$. So if V is indecomposable $P(t) = (t - \lambda)^n$, which means that $(T - \lambda \,\mathrm{Id}_V)$ is nilpotent.

Remark. Theorem 12.3 and Proposition 12.8 (and their proofs) are valid for holomorphic vector bundles on any compact connected complex manifold.

13. WEIL'S THEOREM; UNITARY BUNDLES

In this section we state without proofs two theorems on vector bundles on compact Riemann surfaces.

Theorem (A. Weil). *A vector bundle on a compact Riemann surface* X *is associated to a representation of the fundamental group of* X (see §7.1) *if and only if each of its indecomposable components is of degree zero.*

For a proof see Ref. [2], §7, and Ref. [13].

We next state a generalization of Theorem 7.3 for vector bundles of higher rank.

Definition. *A vector bundle* V *of degree zero is said to be* stable *if for every proper subbundle* W *of* V *we have* $d(W) < 0$.

Let ρ be an n-dimensional unitary representation of $\pi_1(X)$, i.e. ρ is a homomorphism of $\pi_1(X)$ into U(n), the group of n × n unitary matrices. We then have a holomorphic vector bundle E_ρ of rank n associated to ρ (§7.1). The bundle E_ρ is said to be associated to the unitary representation ρ and is called a unitary bundle.

Let us recall that two n-dimensional unitary representations ρ_1 and ρ_2 are said to be equivalent if there exists $T \in U(n)$ such that $\rho_1(\gamma) = T\,\rho_2(\gamma)\,T^{-1}$, $\forall\, \gamma \in \pi_1(X)$.

We then have

Theorem. *Let* $g \geqslant 2$. *A vector bundle on* X *is associated to a* unitary *representation of the fundamental group of* X *if and only if each of its indecomposable components is* stable *and of degree* 0. *Moreover, two vector bundles associated to unitary representations are isomorphic (as holomorphic bundles) if and only if the representations are equivalent.*

For a proof see Refs [8, 9].

APPENDIX

FACTORS OF AUTOMORPHY

A.1. Let G and G′ be (complex Lie) groups and M a (complex) manifold. Let $p : P \to M$ and $p' : P' \to M$, respectively, be principal bundles with structure group G and G′, the groups acting as usual on the right. Suppose that the pullback $p^*(P')$ is trivial on P. Let then $\sigma : P \to p^*(P')$ be a section and set $i_\sigma = \sigma \circ \widetilde{p}$, where $\widetilde{p} : p^*(P') \to P'$ is the natural projection:

$$\begin{tikzcd}
p^*(P') \arrow[r, "\widetilde{p}"] \arrow[d] & P' \arrow[d, "p'"] \\
P \arrow[u, bend left, "\sigma"] \arrow[ur, "i_\sigma"'] \arrow[r, "p"'] & M
\end{tikzcd}$$

For $g \in G$ and $x \in P$ the points $i_\sigma(xg)$ and $i_\sigma(x)$ lie on the same fibre of the map $p' : P' \to M$ and hence there exists a unique element $f(x,g) \in G'$ such that

$$i_\sigma(xg) = i_\sigma(x)\, f(x,g)$$

We have for $x \in P$ and $g_1, g_2 \in G$,

$$i_\sigma(x)\, f(x,g_1 g_2) = i_\sigma(x g_1 g_2) = i_\sigma(x g_1)\, f(x g_1, g_2)$$

$$= i_\sigma(x)\, f(x,g_1)\, f(x g_1, g_2)$$

so that

$$f(x, g_1 g_2) = f(x, g_1)\, f(x g_1, g_2)$$

If σ' is another section of $p^*(P')$ over P we have $\sigma'(x) = \sigma(x)\, h(x)$, $x \in P$ and $h(x) \in G'$. Let $i_{\sigma'}(xg) = i_{\sigma'}(x)\, f'(x,g)$. We then have $i_{\sigma'}(xg) = i_{\sigma'}(x)\, f'(x,g) = i_\sigma(x)\, h(x)\, f'(x,g)$ and

$$i_{\sigma'}(xg) = i_\sigma(xg)\, h(xg) = i_\sigma(x)\, f(x,g)\, h(xg)$$

Hence

$$h(x)\, f'(x,g) = f(x,g)\, h(xg)$$

which may be written as

$$f'(x,g) = h(x)^{-1}\, f(x,g)\, h(xg)$$

This motivates

Definition A.2. *Let* G *and* G′ *be (complex Lie) groups and* $P \to M$ *be a principal* G*-bundle. A* **factor of automorphy** *on* $P \times G$ *with values in* G′ *is a (holomorphic) function* $f : P \times G \to G'$ *satisfying*

$$f(x,g_1 g_2) = f(x,g_1)\, f(x g_1, g_2) \qquad (\mathcal{A})$$

for $x \in P$ *and* $g_1, g_2 \in G$. *Two factors of automorphy* $f, f' : P \times G \to G'$ *are said to be* **equivalent** *if there exists a (holomorphic) function* $h : P \to G'$ *satisfying*

$$f'(x,g) = h(x)^{-1}\, f(x,g)\, h(xg)$$

for $x \in P$, $g \in G$.

Remark A.3. If we write $f_g(x) = f(g,x)$ the condition $(\mathcal{A})$ becomes

$$f_{g_1 g_2}(x) = f_{g_1}(x)\, f_{g_2}(x g_1)$$

Thus f may be viewed as a 1-cocycle of G with values in the group of (holomorphic) functions on P with values in G′. Equivalence of two factors of automorphy means that the corresponding cocycles are cohomologous.

Remark A.4. If $\rho : G \to G'$ is a homomorphism, the function $f(x,g) = \rho(g)$ defines a factor of automorphy.

Remark A.5. Suppose that G′ is **discrete** and P is connected. Then all the factors of automorphy are of the form $f(x,g) = \rho(g)$ where $\rho : G \to G'$ is a (holomorphic) homomorphism; moreover, two factors of automorphy ρ and ρ' are equivalent if and only if there exists $T \in G'$ with $\rho'(g) = T^{-1}\rho(g)T$ for all $g \in G$. These results follow if we remark that, for $g \in G$, the functions $x \mapsto f(x,g)$ and $x \mapsto h(x)$ are constant.

A.6. Bundle associated to a factor of automorphy

Let $p : P \to M$ be a principal G-bundle and G' a group. Suppose we are given a factor of automorphy $f : P \times G \to G'$. We can then construct a (holomorphic) principal G'-bundle E_f on M whose pullback on P has a canonical trivialization such that the factor of automorphy associated to this trivialization as in (A.1) is the given f.

In fact consider the map

$$(P \times G') \times G \to P \times G'$$

$$(x, g', g) \mapsto (xg, f(x,g)^{-1} g')$$

for $x \in P$, $g \in G$, $g' \in G'$. The condition ($\mathscr{A}$) precisely means that this gives a right action of G on $P \times G'$. The orbit space E_f exists as a complex manifold. Moreover the action

$$(P \times G') \times G' \to P \times G'$$

$$(x, g', g'') \mapsto (x, g'g'')$$

of G' on $P \times G'$ induces an action of G' on E_f and makes of it a principal G'-bundle with base M [6]. If $\eta : P \times G' \to E_f$ is the canonical projection then the map $i_\sigma : x \mapsto \eta(x,e)$, $x \in P$, e the identity element of G', gives rise to a trivialization of $p^*(E_f)$. We have

$$i_\sigma(xg) = \eta(xg,e) = \eta((x,f(x,g))g)$$

$$= \eta(x,f(x,g)) = i_\sigma(x)\, f(x,g) \text{ for } x \in P, g \in G$$

Let $f, f' : P \times G \to G'$ be two factors of automorphy. Then the G'-bundles E_f and $E_{f'}$ are isomorphic if and only if f and f′ are equivalent. In fact if

$$f'(x,g) = h(x)^{-1}\, f(xg)\, h(xg)$$

the map $P \times G' \to P \times G'$, $(x,g') \mapsto (x,h(x)^{-1} g')$, induces an isomorphism between E_f and $E_{f'}$. Conversely let $\varphi : E_f \to E_{f'}$ be an isomorphism of G'-bundles. Let $i_\sigma : P \to E_f$ and $i_{\sigma'} : P \to E_{f'}$ be the canonical maps defined above. We then have

$$i_{\sigma'}(x) = (\varphi \circ i_\sigma)(x)\, h(x) \text{ for } h(x) \in G'$$

It is easy to check, using φ as a G'-morphism, that

$$f'(x,g) = h(x)^{-1}\, f(xg)\, h(xg)$$

Moreover, if f is a factor of automorphy defined by a section σ of $p^*(P')$, with the notation of §A.1, we have an isomorphism between E_f and P' induced by the map

$$P \times G' \to P', \quad (x,g') \mapsto i_\sigma(x)g'$$

Remark A.7. The bundle E_ρ associated to the factor of automorphy corresponding to a homomorphism $\rho : G \to G'$ (A.4) is known as the bundle obtained from P by extending the structure group by ρ to G'.

The above discussion may be summarized in

Proposition A.8. *Let $P \to M$ be a principal G-bundle and G' a group. Then the set of isomorphism classes of principal G'-bundles on M whose pullbacks on P are trivial is in canonical bijective correspondence with the set of equivalence classes of factors of automorphy on $P \times G$ with values in G'.*

Remark A.9. Let $P_1 \to M_1$ and $P_2 \to M_2$ be, respectively, principal bundles with G_1 and G_2 as structure group and let $\rho : G_1 \to G_2$ be a homomorphism. A map $\Phi : P_1 \to P_2$ is said to be a ρ-morphism if $\Phi(xg) = \Phi(x)\,\rho(g)$ for $x \in P_1$, $g \in G_1$. We see that Φ induces a map $\widetilde{\Phi} : M_1 \to M_2$. Let $f : P_2 \times G_2 \to G_3$ be a factor of automorphy with values in a group G_3. Let P' be the G_3-bundle on M_2 associated to f (A.6). Then $\widetilde{\Phi}^*(P')$ is isomorphic (on M_1) to the bundle associated to the factor of automorphy $f' : P_1 \times G_1 \to G_3$ given by

$$f'(x,g) = f(\Phi(x), \rho(g)),\ x \in P_1,\ g \in G_1$$

A.10. Representations of the fundamental group and bundles with discrete structure group

Let M be a connected manifold and $P : \widetilde{M} \to M$ its universal covering considered as a principal $\pi_1(M)$-bundle. If $\rho : \pi_1(M) \to G'$ is a homomorphism we denote by E_ρ the bundle obtained from $\widetilde{X}$ by extending the structure group by ρ to G' (A.7). Recall that two homomorphisms ρ_1, ρ_2 from $\pi_1(M)$ to a group G' are said to be equivalent if there exists $T \in G'$ such that $\rho_1(\gamma) = T^{-1}\rho_2(\gamma)\,T$, for $\gamma \in \pi_1(M)$.

Proposition. *Let G' be a discrete group and M a connected manifold. There is a canonical bijective correspondence between the set of equivalence classes of homomorphisms of $\pi_1(M)$ into G' and the set of isomorphism classes of principal G'-bundles over M. (The map is induced by $\rho \mapsto$ isomorphism class of E_ρ.)*

Proof. In view of Remark A.5 and Proposition A.8, it is enough to show that if P' is a principal G'-bundle on M, the pullback of P' to $\widetilde{M}$ is trivial. But if M' is a connected component of $p^*(P')$ it is easily checked that the restriction of the map $p^*(P') \to \widetilde{M}$ to M' is a covering space of $\widetilde{M}$ and hence is an isomorphism, as $\widetilde{M}$ is simply connected. This means that $p^*(P')$ admits a section over $\widetilde{M}$.

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