Cohomological rigidity problem in toric topology

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What is toric topology?

Toric Topology
(Eds. Harada, Karshon, Masuda, Panov)
Contemporary of Math. AMS **460**, 2008

Content of this talk

Part I. Brief review of toric geometry

Part II. Cohomological rigidity problem

Part I. Brief review of toric geometry

Toric varieties and fans

Def. A toric variety is a normal algebraic variety of $\dim_{\mathbb{C}} n$ with $(\mathbb{C}^*)^n$-action having a dense orbit.

Exam. (1) $(\mathbb{C}^*)^n \curvearrowright V = \mathbb{C}^n$ standard representation

$$(z_1, \dots, z_n) \to (g_1 z_1, \dots, g_n z_n)$$

(2) $(\mathbb{C}^*)^n \curvearrowright P(V \oplus \mathbb{C}) = \mathbb{C}P^n$

(3) Projective bundles over a toric variety X^m

$$P(\gamma_1 \oplus \cdots \oplus \gamma_k \oplus \mathbb{C}) \xrightarrow{\mathbb{C}P^k} X^m$$

where $\gamma_1, \dots, \gamma_k$ are line bundles.

(4) $B_n \xrightarrow{\mathbb{C}P^1} B_{n-1} \xrightarrow{\mathbb{C}P^1} \cdots \xrightarrow{\mathbb{C}P^1} B_0 = \{*\}$ Bott tower

(Hirzebruch surface when $n = 2$)

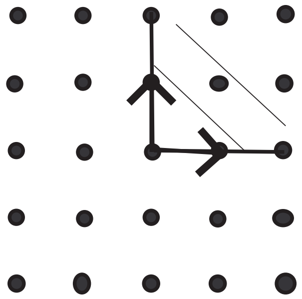
Toric <u>manifold</u> = smooth compact toric variety

Local charts of a toric manifold are $\mathbb{C}^n$ and transition functions are Laurent monomials.

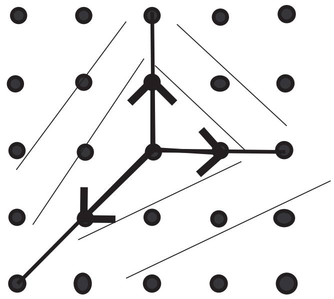
E.g. $n = 2$

$$\begin{array}{ccc}
(z_1, z_2) & \longrightarrow & (z_1^2 z_2^{-3}, z_1^{-1} z_2^2) = (w_1, w_2) \\
\big\downarrow & & \big\downarrow \\
(g_1 z_1, g_2 z_2) & \longrightarrow & (g_1^2 g_2^{-3} w_1, g_1^{-1} g_2^2 w_2)
\end{array}$$

where $(g_1, g_2) \in (\mathbb{C}^*)^2$.



Fan of $\mathbb{C}^2$



Fan of $\mathbb{C}P^2$

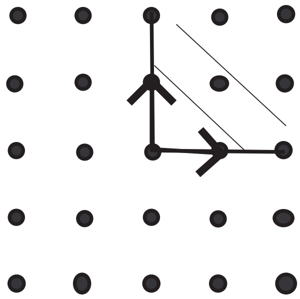
Def. A fan Δ of $\dim_{\mathbb{R}} n$ is a collection of rational polyhedral cones in $\mathbb{Z}^n \otimes \mathbb{R} = \mathbb{R}^n$ satisfying

(1) each face of a cone in Δ is a also a cone in Δ;

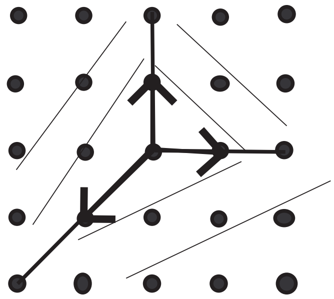
(2) the intersection of two cones in Δ is a face of each.

We may think of Δ as a pair of

a simplicial complex K and edge vectors $\{v_i\}$.



Fan of $\mathbb{C}^2$



Fan of $\mathbb{C}P^2$

Exam. For the fan of $\mathbb{C}P^2$

$K = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{3,1\}\}$ and

$v_1 = (1,0), \quad v_2 = (0,1), \quad v_3 = (-1,-1)$

To be more precise , we should regard

$$\mathbb{Z}^n = \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$$

$$(a_1, \ldots, a_n) \to \big(g \to (g^{a_1}, \ldots, g^{a_n})\big)$$

Fundamental theorem in toric geometry

$$\text{category of toric varieties} \overset{\text{equivalent}}{\Longleftrightarrow} \text{category of fans}$$

$$X \qquad\qquad\qquad \Delta(X)$$

The correspondence $\Longrightarrow$ is as follows when X is a toric manifold.

X_i $(i = 1, \dots, m)$ invariant divisors of X

(1) $K_X := \{I \subset \{1, \dots, m\} \mid \cap_{i \in I} X_i \neq \emptyset\}$
a simplicial complex

(2) $v_i \in \mathbb{Z}^n = \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ fixes X_i.

The fan $\Delta(X)$ is formed from K_X and $\{v_i\}$.

Fundamental theorem implies

$$X \cong_{w.eq} X' \Longleftrightarrow \Delta(X) \cong \Delta(X')$$

- $X \cong_{w.eq} X'$ means that
  $\exists f\colon X \to X'$ (iso.) and $\exists \rho \in \mathrm{Aut}((\mathbb{C}^*)^n)$ s.t.
  $$f(tx) = \rho(t) f(x) \quad (t \in (\mathbb{C}^*)^n,\ x \in X).$$

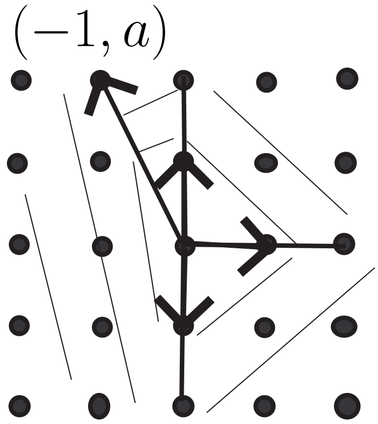
- $\Delta(X) \cong \Delta(X')$ means that
  $\exists g \in \mathrm{GL}(n;\mathbb{Z})$ s.t. $g(\Delta(X)) = \Delta(X')$.

Fact. $X \cong X'$ as varieties $\iff X \cong_{w.eq} X'$.

This implies

Classification of toric manifolds as varieties
$=$ Classification of fans up to isomorphism

Exam. $a \in \mathbb{Z}$
$H_a = P(\gamma^a \oplus \mathbb{C}) \to \mathbb{C}P^1$ (Hirzebruch surface)
$H_a \cong H_{a'}$ as varieties $\iff |a| = |a'|$.



$(a = 2)$

Equivariant cohomology

$T = (\mathbb{C}^*)^n$. We have a fibration

$$X \longrightarrow ET \times_T X \xrightarrow{\pi} BT = (\mathbb{C}P^\infty)^n$$

and

$$H_T^*(X) := H^*(ET \times_T X)$$

is not only a ring but also an <u>algebra</u> over $H^*(BT)$ via π^*.

Thm. *$X \cong X'$ as varieties $\Longleftrightarrow H_T^*(X) \cong H_T^*(X')$.*

<u>Sketch of Proof</u>

($\Longrightarrow$) follows from Fact mentioned before.

($\Longleftarrow$) We show we can reproduce $\Delta(X)$ from $H_T^*(X)$.

To be more precise,

X_i $(i = 1, \ldots, m)$: invariant divisors

$$X_i \quad \overset{P.D.}{\longleftrightarrow} \quad \tau_i \in H^2_T(X)$$

$$\bigcap_{i \in I} X_i \quad \overset{P.D.}{\longleftrightarrow} \quad \prod_{i \in I} \tau_i \in H^*_T(X)$$

Lem. *As a ring*

$$H^*_T(X) = \mathbb{Z}[\tau_1, \ldots, \tau_m] \Big/ \Big(\prod_{i \in I} \tau_i \mid \bigcap_{i \in I} X_i = \emptyset \Big)$$

Lemma shows that $H^*_T(X)$ is the face ring of the simplicial complex

$$K_X = \{ J \subset \{1, \ldots, m\} \mid \bigcap_{j \in J} X_j \neq \emptyset \}$$

Recall $H_T^*(X)$ is an algebra over $H^*(BT)$ via π^* where $X \longrightarrow ET \times_T X \xrightarrow{\pi} BT$.

Lem. $\exists_1\ v_i \in H_2(BT)\ (i = 1, \ldots, m)$ *s.t.*

$$\pi^*(u) = \sum_{i=1}^{m} \langle u, v_i \rangle \tau_i \quad \textit{for } \forall u \in H^2(BT)$$

Note

$$H_2(BT) = [B\mathbb{C}^*, BT] = \mathrm{Hom}(\mathbb{C}^*, T) = \mathbb{Z}^n$$

It turns out that

$v_i \in \mathrm{Hom}(\mathbb{C}^*, T)$ fixes X_i, and

v_i's are the edge vectors of the fan $\Delta(X)$. $\square$

Description of $H^*(X)$.

From the fibration

$$X \to ET \times_T X \xrightarrow{\pi} BT$$

we have

$$H^{>0}(BT) \xrightarrow{\pi^*} H_T^*(X) \to H^*(X) \to 0$$

Thm (Danilov-Jurkiewicz)**.**

$$H^*(X) = \mathbb{Z}[\tau_1, \ldots, \tau_m]/(\mathcal{I} + \mathcal{J})$$

where $\mathcal{I}$ and $\mathcal{J}$ are ideals respectively generated by

(1) $\prod_{i\in I} \tau_i$ *for $I \notin K_X$ (i.e. $\cap_{i\in I} X_i = \emptyset$).*

(2) $\sum_{i=1}^{m} \langle u, v_i\rangle \tau_i$ *for $u \in H^2(BT)$.*

Part II. Cohomological rigidity problem

We have seen that equivariant cohomology distinguishes toric manifolds as varieties.

<u>Cohomological Rigidity Problem (CRP)</u> for Toric Manifolds. Let X, X' be toric manifolds.

$H^*(X) \cong H^*(X')$ as graded rings
$\Longrightarrow$
$X \cong X'$ diffeo (or homeo) ?

Note. Surgery Theory $\Longrightarrow$

For a toric manifold X of $\dim_{\mathbb{C}} X \geq 3$,
$\exists\ \infty$ many diffeomorphism classes in closed smooth manifolds $\simeq X$, and they are distinguished by their Pontrjagin classes (up to finite ambiguity)

Exam. Recall $H_a = P(\gamma^a \oplus \mathbb{C}) \xrightarrow{\mathbb{C}P^1} \mathbb{C}P^1$ $(a \in \mathbb{Z})$.

- $H_a \cong H_{a'}$ as varieties $\Longleftrightarrow |a| = |a'|$.
- $H_a \cong H_{a'}$ diffeo $\Longleftrightarrow a \equiv a' \pmod 2$
  $\Longleftrightarrow H^*(H_a) \cong H^*(H_{a'})$

CRP is affirmative for Hirzebruch surfaces H_a's.

Exam (with Choi and Suh)**.**
CRP is affirmative for $P(E \oplus \mathbb{C})$'s where E is a sum of line bundles over $\mathbb{C}P^{d_1}$.

$$P(E \oplus \mathbb{C}) \xrightarrow{\mathbb{C}P^{d_2}} \mathbb{C}P^{d_1}$$

Generalized Bott (or Dobrinskaya) tower of height n

$$X_n \xrightarrow{\mathbb{C}P^{d_n}} X_{n-1} \xrightarrow{\mathbb{C}P^{d_{n-1}}} \cdots \xrightarrow{\mathbb{C}P^{d_3}} X_2 \xrightarrow{\mathbb{C}P^{d_2}} X_1 \xrightarrow{\mathbb{C}P^{d_1}} \{*\}$$

where $X_{k+1} = P(E_k \oplus \mathbb{C}) \to X_k$ and E_k is a sum of line bundles.

The tower is a Bott tower when $d_i = 1$ for $\forall i$.

Thm (with Panov, with Choi and Suh)**.**

Let X be a toric manifold. Then

$H^*(X) \cong H^*(\prod_{i=1}^n \mathbb{C}P^{d_i})$ *as graded rings*

$\Longrightarrow$

$X \cong \prod_{i=1}^n \mathbb{C}P^{d_i}$ *(diffeo)*

Exam (with Choi and Suh)**.** CRP is affirmative for Bott manifolds of $\dim_{\mathbb{C}} 3$.

$$X_3 \xrightarrow{\mathbb{C}P^1} X_2 \xrightarrow{\mathbb{C}P^1} X_1 \xrightarrow{\mathbb{C}P^1} \{*\}$$

Sketch of Proof. Let X, X' be Bott manifolds of $\dim_{\mathbb{C}} 3$. Then one can show any isomorphism

$$\psi\colon H^*(X) \to H^*(X')$$

preserves their Pontrjagin classes (and Stiefel-Whitney classes as well). So the classification result on 6-manifolds by Wall (spin case) and Jupp (non-spin case) implies the the desired fact. $\square$

This leads us to ask

Question. Let X, X' be toric mfds. If $\psi\colon H^*(X) \to H^*(X')$ is an isomorphism, then $\psi(p(X)) = p(X')$?

Real toric manifolds

Toric manifold X admits a "complex conjugation" and its fixed point set $X(\mathbb{R})$ is called a real toric manifold.

Exam. (1) $X = \mathbb{C}P^n$, $X(\mathbb{R}) = \mathbb{R}P^n$.

(2) $X = \mathbb{C}P^1 \times \mathbb{C}P^1$, $X(\mathbb{R}) = \mathbb{R}P^1 \times \mathbb{R}P^1 (= T^2)$.

(Similarity to the complex case)

- $H^*(X(\mathbb{R}); \mathbb{Z}/2) \cong H^{2*}(X; \mathbb{Z}) \otimes \mathbb{Z}/2$.

(Non-similarity to the complex case)

- $\pi_1(X) = \{1\}$, but $\pi_1(X(\mathbb{R})) \neq \{1\}$ and $X(\mathbb{R})$'s provide many examples of aspherical manifolds.

Cohomological Rigidity Problem (CRP) for <u>Real</u> Toric Manifolds.

Let M, M' be real toric manifolds.

$H^*(M; \mathbb{Z}/2) \cong H^*(M'; \mathbb{Z}/2)$ as graded rings

$\Longrightarrow$

$M \cong M'$ diffeo (or homeo) ?

Exam. Real Bott tower of height n

$$M_n \xrightarrow{\mathbb{R}P^1} M_{n-1} \xrightarrow{\mathbb{R}P^1} \cdots \xrightarrow{\mathbb{R}P^1} M_2 \xrightarrow{\mathbb{R}P^1} M_1 \xrightarrow{\mathbb{R}P^1} \{*\}$$

Here $M_{k+1} = P(L_k \oplus \mathbb{R}) \to M_k$ and L_k is a real line bundle. We call M_n a real Bott manifold.

Choices of $L_k \longleftrightarrow H^1(M_k;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^k$.

An upper triangular $(0,1)$ matrix $A = (A^i_j)$ is associated to the real Bott tower, and

$$M_n = \mathbb{R}^n/\Gamma(A)\ (= M(A))$$

where $\Gamma(A)$ is generated by $s_1, \ldots, s_n$ and

$$s_i(u_1, \ldots, u_n) = (u_1, \ldots, u_{i-1}, u_i+\frac{1}{2}, (-1)^{A^i_{i+1}}u_{i+1}, \ldots, (-1)^{A^i_n}u_n).$$

So

$$s_i^2(u_1, \ldots, u_n) = (u_1, \ldots, u_i + 1, \ldots, u_n)$$

$$1 \to \langle s_1^2, \ldots, s_n^2\rangle = \mathbb{Z}^n \to \Gamma(A) \to (\mathbb{Z}_2)^n \to 1$$

Thm (with Y. Kamishima)**.**
$H^*(M(A);\mathbb{Z}/2) \cong H^*(M(B);\mathbb{Z}/2)$ *as graded rings*
$\Longrightarrow$
$M(A) \cong M(B)$ *(diffeo)*

<u>Idea of Proof</u>. Remember

$$M(A) = \mathbb{R}^n/\Gamma(A) \text{ and } \Gamma(A) = \pi_1(M(A)).$$

We show

$$H^*(M(A);\mathbb{Z}/2) \cong H^*(M(B);\mathbb{Z}/2) \Longrightarrow \Gamma(A) \cong \Gamma(B)$$

Since $M(A)$ and $M(B)$ are flat riemannian manifolds,

$$\Gamma(A) \cong \Gamma(B) \overset{Bieberbach}{\Longrightarrow} M(A) \cong M(B) \quad \square$$

In fact, we can say more.

Thm. *Any iso.* $H^*(M(A);\mathbb{Z}/2) \to H^*(M(B);\mathbb{Z}/2)$ *is induced by an affine diffeomorphism.*

Exam. $D_n = \#$ of diffeomorphism classes of real Bott manifolds of dim n.

$$D_2 = 2, \quad D_3 = 4, \quad D_4 = 12, \quad D_5 = 54,$$
$$D_6 = 472, \quad D_7 = 8512, \quad D_8 = 328416, \quad \ldots$$

One can also prove

Thm. *Decomposition of a real Bott manifold into a product of indecomposable real Bott manifolds is unique (up to a permutation of factors).*

In particular, if $S^1 \times M \cong S^1 \times M'$ *where* M, M' *are real Bott manifolds, then* $M \cong M'$ *(i.e. cancellation property holds).*

Note. Not true for general riemannian flat manifolds (Charlap, 1965)

There are more affirmative solutions to CRP for real toric manifolds, e.g. for some 3-dim ones (Lü-Yu).

Counterexamples to CRP for *real* toric manifolds.

Let γ be the Hopf line bundle over $\mathbb{R}P^{a-1}$ and

$$M(q) := P(q\gamma \oplus (b-q)\mathbb{R}) \to \mathbb{R}P^{a-1}$$

for $0 \leqq q \leqq b$.

- $H^*(M(q)) \cong H^*(M(q')) \Longleftrightarrow q' \equiv q \text{ or } b-q \pmod{2^{h(a)}}$

where $h(a) = \min\{n \in \mathbb{Z} \mid 2^n \geqq a\}$.

- $M(q) \cong M(q') \Longleftrightarrow q' \equiv q \text{ or } b-q \pmod{2^{k(a)}}$

where

$k(a) = \#\{n \in \mathbb{Z} \mid 0 < n < a \text{ and } n \equiv 0, 1, 2, 4 \pmod 8\}$.

Since $h(a) < k(a)$ for $a \geqq 10$, this provides a counterexample to CRP.

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However, it is true that

$$M(q) \simeq M(q') \Longrightarrow M(q) \cong M(q').$$