Homology of local systems on configuratin spaces and conformal field theory

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Plan

Two approaches for braid group representations

- KZ connection

related to conformal field theory, quantum groups ...

- LKB representation

action of braid groups on local system homology

The Aim : The space of conformal blocks is expressed as homology of local system on configuration spaces.

1 KZ connection

$\mathfrak{g}$: semi-simple Lie algebra.

$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.

$\Omega = \sum_\mu I_\mu \otimes I_\mu$

$r_i : \mathfrak{g} \to End(V_i)$, $1 \le i \le n$ representations.

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Ω_{ij} : the action of Ω on the i-th and j-th components of $V_1 \otimes \cdots \otimes V_n$.

$$\omega = \frac{1}{\kappa} \sum_{i,j} \Omega_{ij} d \log(z_i - z_j), \quad \kappa \in \mathbf{C} \setminus \{0\}$$

ω defines a flat connection for a trivial vector bundle over the configuration space

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j, \ i \neq j\}$$

with fiber $V_1 \otimes \cdots \otimes V_n$ since we have

$$\boxed{\ \omega \wedge \omega = 0\ }$$

As the holonomy we have representations

$$\theta_\kappa : P_n \longrightarrow GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

$$\theta_\kappa : B_n \to GL(V^{n\otimes}).$$

We shall express the horizontal sections of the KZ connection : $d\varphi = \omega\varphi$ in terms of homology with coefficients in local system homology on the fiber of the projection map

$$\pi : X_{m+n} \longrightarrow X_n.$$

2 Relating to local system

Consider the case $\mathfrak{g} = sl_2(\mathbf{C})$ with basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

V_λ : highest weight representation of $sl_2(\mathbf{C})$ with highest weight vector v :

$$Hv = \lambda v, \ Ev = 0$$

$V_{\lambda_1} \otimes \cdots V_{\lambda_n}$: tensor product of highest weight representations of $sl_2(\mathbf{C})$

Set $\lambda = \lambda_1 + \cdots + \lambda_n$.
For a non-negative integer ℓ put

$$W[\lambda - 2\ell] = \{x \in V_{\lambda_1} \otimes \cdots V_{\lambda_n} \ ; \ Hx = (\lambda - 2\ell)x\},$$

$$N[\lambda - 2\ell] = \{x \in W[\lambda - 2\ell] \ ; \ Ex = 0\}.$$

The KZ connection ω commutes with the diagonal action of $\mathfrak{g}$ on $V_{\lambda_1} \otimes \cdots V_{\lambda_n}$, hence it leaves invariant the space of null vectors $N[\lambda - 2\ell]$.

$\pi : X_{n+m} \longrightarrow X_n$: projection defined by $(z_1, \cdots, z_n, t_1, \cdots, t_m) \mapsto (z_1, \cdots, z_n)$.

$X_{n,m}$: fiber of π.

$$\Phi = \prod_{1\le i<j\le n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{\kappa}} \prod_{1\le i\le m, 1\le \ell \le n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \times \prod_{1\le i<j\le m} (t_i - t_j)^{\frac{2}{\kappa}}$$

(multi-valued function on X_{n+m}).

Construct horizontal sections of KZ connection with values in $N[\lambda - 2\ell]$.

Example. (the case $\ell = 1$)
$W[\lambda - 2]$ is spanned by

$$v_1 \otimes \cdots \otimes Fv_j \otimes \cdots \otimes v_n, \quad 1 \leq j \leq m$$

with highest weight vectors $v_1, \cdots, v_n$ for $V_{\lambda_1}, \cdots V_{\lambda_n}$. $N[\lambda - 2\ell]$ is a codimension one linear subspace.

The solutions are

$$\varphi = \sum I_j v_1 \otimes \cdots \otimes F v_j \otimes \cdots \otimes v_m$$

with

$$I_j = \int_\Delta \eta_j , \quad \eta_j = \Phi \frac{dt}{t - z_j}$$

$E\varphi = 0$ implies $\lambda_1 I_1 + \cdots + \lambda_n I_n = 0$ which is a relation among de Rham cohomology classes. The action of P_n on $N[\lambda - 2\ell]$ is identified with Gassner representation.

3 Hypergeometric pairing

Put $Y_{n,m} = X_{n,m}/S_m$.
$\mathcal{L}$: local system on $Y_{n,m}$ associated with the multi-valued function Φ.
The twisted de Rham complex $(\Omega^*(Y_{n,m}), \bigtriangledown)$ is defined by

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Hypergeometric pairing:

$$H_m(Y_{n,m}, \mathcal{L}^*) \times H^m(\Omega^*(Y_{n,m}), \bigtriangledown) \to \mathbf{C}$$

defined by

$$(c, w) \mapsto \int_c \Phi \omega.$$

There is a map

$$\rho : N[\lambda - 2m]^* \to \Omega^m(Y_{n,m})$$

with $\rho(w) = R_w(t, z) dt_1 \wedge \cdots \wedge dt_m$ a rational form so that the following theorem holds.

Let w_J be a basis of $N[\lambda - 2\ell]^*$.

Theorem [Schechtman-Varchenko...].

$$\sum_J \int_\Delta \Phi\rho(w_J)$$

is a solution of the KZ equation, where Δ is a cycle in $H_m(Y_{n,m}, \mathcal{L}^*)$.

Theorem. For generic λ, κ, there is an isomophism

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \cong N[\lambda - 2m]$$

where ϕ is defined by

$$\langle \phi(c), w \rangle = \int_c \Phi \rho(w).$$

This gives a basis of the horizontal sections of KZ connection with values in $N[\lambda - 2m]$.

Moreover, the following two representations of pure braid groups are equivalent:

(1) Action of P_n on the twisted homology $H_m(Y_{n,m}, \mathcal{L}^*)$.

(2) Holonomy representation of the KZ equation with valued in $N[\lambda - 2m]$.

Theorem. In the case $\lambda_1 = \cdots = \lambda_n$ and $m = 2$
LKB representation

$$B_n \to Aut\ H_m(Y_{n,m}, \mathcal{L}^*)$$

is identified with the action of B_n obtained as the holonomy of the KZ connection

$$B_n \to Aut\ N[\lambda - 2m].$$

Remark. For generic λ, κ,

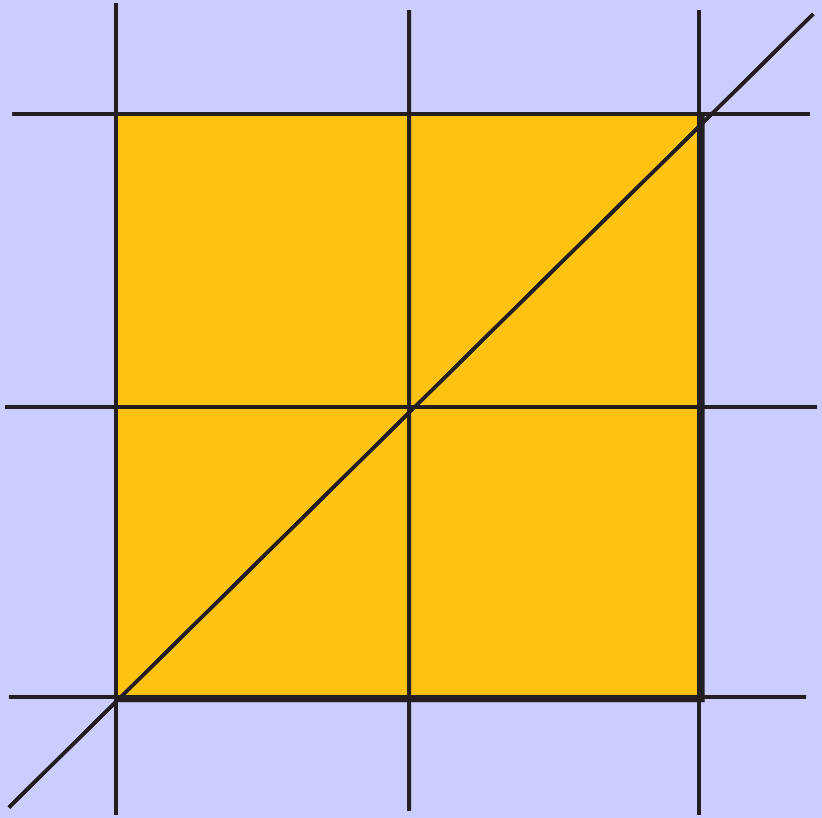
$$H_j(Y_{n,m},\mathcal{L}^*) \cong 0, \quad j \neq m$$

and we have an isomorphism

$$H_m(Y_{n,m},\mathcal{L}^*) \cong H_m^{lf}(Y_{n,m},\mathcal{L}^*)$$

(homology with locally finite chains)

The above homology is spanned by bounded chambers.



bounded chambers : basis of twisted homology (the case $n = 3, m = 2$).

4 Space of conformal blocks

Put $\kappa = K + 2$ (K a positive integer). Suppose $0 \leq \lambda_1, \cdots, \lambda_{n+1} \leq K$.

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$: affine Lie algebra.

$p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$ with $p_{n+1} = \infty$

Assign highest weights $\lambda_1, \cdots, \lambda_{n+1} \in \mathbf{Z}$ to $p_1, \cdots, p_{n+1}$.

$\mathcal{H}_j$: irreducible representations of $\widehat{\mathfrak{g}}$ with highest weight λ_j at level K.

Ref. [T. Kohno] Conformal Field Theory and Topology, Monograph, AMS 2002

The space of conformal blocks is defined as

$$\mathcal{H}(p,\lambda) = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_{n+1}} / (\mathfrak{g} \otimes \mathcal{M}_p)$$

where $\mathcal{M}_p$ is the set of meromorphic functions on $\mathbf{C}P^1$ with poles at most at $p_1, \cdots, p_{n+1}$.

$\mathcal{H}(p,\lambda)$ is identified with a quotient space of $N[\lambda_{n+1}]$ and there is a map

$$\rho : \mathcal{H}(p,\lambda) \to H^m(\Omega^*(Y_{n,m}), \nabla).$$

so that the map

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \longrightarrow \mathcal{H}(p, \lambda)^*$$

defined by

$$\langle \phi(c), w \rangle = \int_c \rho(w)$$

is surjective.

Consider the natural map

$$\alpha : H_m(Y_{n,m}, \mathcal{L}^*) \to H_m^{lf}(Y_{n,m}, \mathcal{L}^*)$$

and put $\mathrm{Im}(\alpha) = H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg}$
(the set of regularizable cycles).

Theorem. ϕ induces an isomorphism

$$H_m^{lf}(Y_{n,m}, \mathcal{L}^*)_{reg} \cong \mathcal{H}(p, \lambda)^*$$

equivariant under the action of braids.