Splitting of gauge groups

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Introduction

- G : topological group
- $P \to B$: principal G-bundle

Definition : The gauge group of P, denoted $\mathcal{G}(P)$, is the group of automorphisms of P covering 1_B.

Example : $P = B \times G \Rightarrow \mathcal{G}(P) \cong \mathrm{map}(B, G)$

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Example : $P = B \times G \Rightarrow \mathcal{G}(P) \cong \mathrm{map}(B, G)$

- $b_0 \in B$: basepoint

$\Rightarrow b_0 \hookrightarrow B$ induces an epimorphism:

$$\mathcal{G}(P) \xrightarrow{\pi} \mathcal{G}(P|_{b_0}) \cong G$$

- $\mathcal{G}_0(P) = \mathbf{Ker}\pi$: based gauge group of P

Problem : Study splitting of the exact sequence

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Plan :

1. Formulate splitting of group extensions in the category of A_n-spaces (A_n-splitting).
2. See what A_n-splitting of ($\spadesuit$) means to P.
   $\rightsquigarrow$ triviality of $\mathrm{ad}P = P \times_{\mathrm{ad}} G$
3. Give criteria for A_n-splitting of ($\spadesuit$).
   $\rightsquigarrow$ higher homotopy commutativity of G

A_n-splitting

Definition : An A_n-splitting of an exact sequence of topological groups

$$1 \to H \to G \xrightarrow{\pi} K \to 1 \qquad (\clubsuit)$$

is an A_n-space structure on $H \times K$ such that:

1. It restricts to the group structure on $H \times 1, 1 \times K$.
2. There is an A_n-map $\theta : H \times K \to G$ satisfying the homotopy commutative diagram:

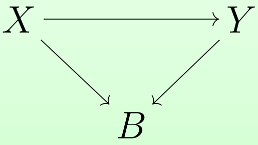
$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & H \times K & \longrightarrow & K & \longrightarrow & 1 \\ \| & & \| & & \downarrow \theta & & \| & & \| \\ 1 & \longrightarrow & H & \longrightarrow & G & \xrightarrow{\pi} & K & \longrightarrow & 1 \end{array}$$

Proposition : There is an A_n-splitting for (♣) if and only if there is a section for π which is an A_n-map (A_n-section).

Fibrewise topological monoid

Definition : Fix a space B.

- A fibrewise space over B is a map $X \to B$.
- A fibrewise map is a commutative triangle:



Example : $\underline{B} = B$ is a fibrewise space over B by 1_B.

Example : $X_B = X \times B$ is a fibrewise space over B by the second projection.

- X, Y : fibrewise space over B

$\Rightarrow$ The fibrewise product $X \times_B Y$ is the pullback of $X \to B \leftarrow Y$.

$\Rightarrow$ We have a fibrewise diagonal $\Delta_B : X \to X \times_B X$.

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Definition : Fibrewise maps $f, g : X \to Y$ are fibrewise homotopic if there is a fibrewise map $h_t : X \times_B I_B \to Y$ such that $h_0 = f, h_1 = g$.

$\Rightarrow$ We have a notion of fibrewise homotopy equivalence.

Definition : A fibrewise space $X \xrightarrow{\pi} B$ is a fibrewise topological monoid if it is equipped with fibrewise maps $\mu : X \times_B X \to X$ and $\epsilon : \underline{B} \to X$ satisfying:

- $\mu \circ (\mu \times_B 1) = \mu \circ (1 \times_B \mu)$
- $\mu \circ (1 \times_B \epsilon \circ \pi) \circ \Delta_B = 1 = \mu \circ (\epsilon \circ \pi \times_B 1) \circ \Delta_B$

X is a fibrewise topological group if, in addition, it is equipped with a fibrewise map $\iota : X \to X$ satisfying:

- $\mu \circ (1 \times_B \iota) \circ \Delta_B = \epsilon \circ \pi = \mu \circ (\iota \times_B 1) \circ \Delta_B$

$\Rightarrow$ The set of sections $\Gamma(X)$ is a topological monoid (group).

Example : The adjoint bundle $\mathrm{ad}P$ is a fibrewise topological group over B by:

- $\mu : \mathrm{ad}P \times_B \mathrm{ad}P \longrightarrow \mathrm{ad}P$, $([p,g],[p,h]) \mapsto [p,gh]$
- $\epsilon : \underline{B} \longrightarrow \mathrm{ad}P$, $b \mapsto [\pi^{-1}(b), 1]$
- $\iota : \mathrm{ad}P \longrightarrow \mathrm{ad}P$, $[p,g] \mapsto [p,g^{-1}]$

$(p \in P, b \in B, g, h \in G)$

$\Rightarrow \Gamma(\mathrm{ad}P)$ is a topological group.

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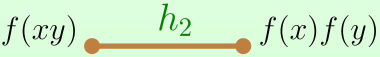
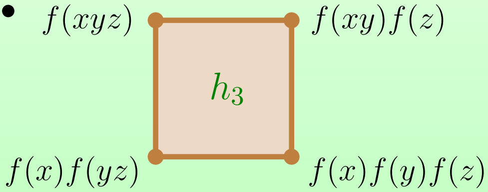
Proposition : $\mathcal{G}(P) \cong \Gamma(\mathrm{ad}P)$ as topological groups.

Fibrewise A_n-map

Definition : Let X, Y be fibrewise topological monoids over B. A fibrewise map $f : X \to Y$ is a fibrewise A_n-map if there is a sequence of fibrewise maps $\{h_i : I_B^{i-1} \times_B X^i \to Y\}_{i=1}^{n}$ such that:

- $h_1 = f$
- 

- 

and so on...

Proposition :

1. A fibrewise map which is fibrewise homotopic to a fibrewise A_n-map is a fibrewise A_n-map.
2. The composite of fibrewise A_n-maps is a fibrewise A_n-map.
3. If a fibrewise homotopy equivalence is a fibrewise A_n-map, then so is the fibrewise homotopy inverse.

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Definition : A fibrewise homotopy equivalence which is a fibrewise A_n-map is called a fibrewise A_n-equivalence.

$\Rightarrow$ Fibrewise A_n-equivalences yield an equivalence relation among fibrewise topological monoids.

Triviality of adjoint bundles

Theorem : There is an A_n-splitting for (♠) if and only if $\mathrm{ad}P$ is fibrewise A_n-equivalent to $B \times G$.

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Outline of the proof : ($\Leftarrow$) Easy.

($\Rightarrow$) The case $n = 0$: Identify $\mathcal{G}(P)$ with $\Gamma(\mathrm{ad}P)$.

$\Rightarrow \pi : \mathcal{G}(P) \to G$ is the evaluation:

$$\hat{\pi} : \Gamma(\mathrm{ad}P) \to G,\ s \mapsto s(b_0)$$

Given a homotopy section $\theta : G \to \Gamma(\mathrm{ad}P)$ for $\hat{\pi}$, define

$$\bar{\theta} : B \times G \to \mathrm{ad}P,\ (b, g) \mapsto \theta(g)(b).$$

$\Rightarrow \bar{\theta}|_{b_0 \times G} = \hat{\pi}(\theta) \simeq 1_G$

$\Rightarrow$ The Dold theorem completes the proof.

The case $n > 0$:

- X : fibrewise topological monoid over B

1. Construct a fibrewise projective space $P^n_B X$ by the fibrewise Dold-Lashof construction.
2. Characterize fibrewise A_n-maps by $P^n_B X$ following Stasheff.
3. Construct a natural map $\rho : P^n\Gamma(X) \to \Gamma(P^n_B X)$.
4. Generalize the proof for $n = 0$ by using the map ρ.

$\square$

Evaluation and gauge groups

- $X \xrightarrow{f} Y$: based map between based spaces X, Y
- $\mathrm{map}(X, Y; f)$: component of unbased $X \to Y$ including f
- $\mathrm{map}_0(X, Y; f) = \{g \in \mathrm{map}(X, Y; f) \mid g$ is based$\}$

$\Rightarrow$ We have the evaluation fibration:

$$\mathrm{map}_0(X, Y; f) \to \mathrm{map}(X, Y; f) \xrightarrow{\omega} Y$$

Theorem (D.H. Gottlieb '72, M.F. Atiyah-R. Bott '83):
For a classifying map α of P, we have a homotopy commutative diagram:

$$\begin{array}{ccccc}
B\mathcal{G}_0(P) & \longrightarrow & B\mathcal{G}(P) & \xrightarrow{B\pi} & BG \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
\mathrm{map}_0(B, BG; \alpha) & \longrightarrow & \mathrm{map}(B, BG; \alpha) & \xrightarrow{\omega} & BG
\end{array}$$

$H(k,l)$-space

Definition : A space X is an $H(k,l)$-space if there is a homotopy commutative diagram

$$\begin{array}{ccc} P^k\Omega X \vee P^l\Omega X & \xrightarrow{i_k \vee i_l} & X \\ \downarrow & \nearrow & \\ P^k\Omega X \times P^l\Omega X & & \end{array}$$

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Remark : If we can patch together the structure map of $H(k,l)$-spaces for $k+l=n$, we obtain an $H(n)$-space of Félix and Tanré.

Proposition :

1. X is an $H(1,1)$-space if and only if ΩX is homotopy commutative.
2. X is an $H(\infty,\infty)$-space if and only if it is an H-space.
3. An $H(n)$-space of Félix and Tanré is an $H(k,l)$-space for $k+l=n$.
4. X is an $H(n,\infty)$-space if and only if X is Aguadé's T_n-space.

$\Rightarrow$ The loop space of an $H(k,l)$-space is highly homotopy commutative.

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Proof : Let $\mu : P^k G \to \mathrm{map}(P^l G, BG; i_l)$ be the adjoint of the structure map $P^k G \times P^l G \to BG$.

$\Rightarrow \omega \circ \mu \simeq i_k$

$\Rightarrow \omega \circ (\mu|_{\Sigma G}) \simeq i_1$

$\Rightarrow$ The adjoint of $\mu|_{\Sigma G}$,

$$G \to \Omega\mathrm{map}(P^l G, BG; i_l) \simeq \mathcal{G}(E^l G),$$

yields an A_k-splitting for $\mathcal{G}(P) \xrightarrow{\pi} G$. □

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Corollary : If $\mathrm{cat} B \leq l$ and BG is an $H(k,l)$-space, then there is an A_n-splitting for (♠).

Characterizing $H(1,n)$-spaces

- $\mathrm{ad} : G \to \mathrm{Aut}(G)$: adjoint action

$\Rightarrow$ There is an induced map:

$$\overline{\mathrm{ad}} : G \to \mathrm{map}_0(BG, BG; 1), \; g \mapsto [B\mathrm{ad}(g) : BG \to BG]$$

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Theorem : The map δ in the fibre sequence

$$G \xrightarrow{\delta} \mathrm{map}_0(X, BG; f) \to \mathrm{map}(X, BG; f) \xrightarrow{\omega} BG$$

is given by

$$\delta = f^*(\overline{\mathrm{ad}}).$$

Corollary : BG is an $H(1,n)$-space $\Leftrightarrow$ $i_n^*(\overline{\mathrm{ad}})$ is based null-homotopic.

- Y : countable connected simplicial complex

$\Rightarrow$ We can regard ΩY as a topological group.

$\Rightarrow$ The above theorem gives a description of δ in:

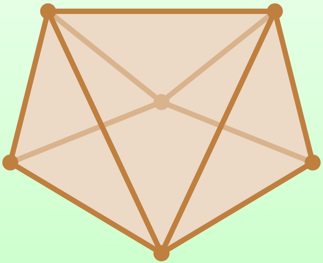
$$\Omega Y \xrightarrow{\delta} \mathrm{map}_0(X, Y; f) \to \mathrm{map}(X, Y; f) \xrightarrow{\omega} Y$$

$\Rightarrow$ When $X = \Sigma Z$, δ is expressed by a Samelson product $\leftrightsquigarrow$ a Whitehead product (G.W. Whitehead '46 and G.E. Lang Jr. '73).

$C(k,l)$-space

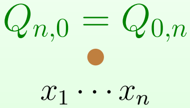
- $N_{k,l}$: resultohedron ($(k+l-1)^{\text{dim}}$ convex polytope)

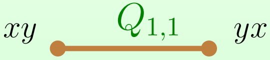
Example : $N_{k,0} = N_{0,k} = *$, $N_{k,1} = N_{1,k} = \Delta^k$

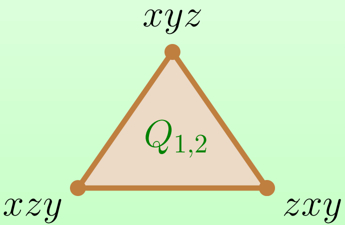
$N_{2,2} =$ 

Definition : A topological monoid X is an $C(k,l)$-space if there is a sequence of maps

$\{Q_{r,s} : N_{r,s} \times X^{r+s} \to X\}_{0 \le r \le k, 0 \le s \le l}$ such that:

- 

- 

- 

and so on...

Proposition :

1. A topological monoid is a $C(1,1)$-space if and only if it is homotopy commutative.
2. If a topological monoid is a $C(1,n-1)$-space, it is a C_n-space in the sense of Williams.

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Theorem (cf. Y. Hemmi and Y. Kawamoto '08) : A connected topological monoid X is a $C(k,l)$-space if and only if BX is an $H(k,l)$-space.

Corollary : If $\mathrm{cat} B \leq l$ and G is a $C(k,l)$-space, there is an A_k-splitting for (♠).