Finite Chevalley groups and loop groups

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  Preprint is available at arXiv.(arXiv:0810.1678)

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There exists a reductive complex linear algebraic group $G(\mathbb{C})$ associated with G, called the complexification of G. One may consider $G(\mathbb{C})$ as $\mathbb{C}$-rational points of a group scheme over $\mathbb{C}$ obtained by the base-change of a reductive integral affine group scheme $G_{\mathbb{Z}}$, so-called Chevalley group scheme, with the complex analytic topology.

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Replacing $\mathbb{C}$ by another field k, we have Chevalley group $G(k)$.

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Denote by $\overline{\mathbb{F}}_p$ the algebraic closure of the finite field $\mathbb{F}_q$. We may consider the finite Chevalley group $G(\mathbb{F}_q)$ as the fixed point set $G(\overline{\mathbb{F}}_p)^{\phi^q}$ where

$$\phi^q : G(\overline{\mathbb{F}}_p) \longrightarrow G(\overline{\mathbb{F}}_p)$$

is the Frobenius map induced by the Frobenius homomorphism $\phi^q : \overline{\mathbb{F}}_p \longrightarrow \overline{\mathbb{F}}_p$ sending

$$x \mapsto x^q.$$

In

> *On the cohomology and K-theory of the general linear groups over a finite field,*
> *Ann. of Math. (2)* **96** *(1972), 552–586,*

Quillen computed the mod ℓ cohomology of a finite general linear group $GL_n(\mathbb{F}_q)$.

The finite general linear group $GL_n(\mathbb{F}_q)$ is a finite Chevalley group associated with the unitary group $U(n)$.

We recall Quillen's computation from the viewpoint of the the following Theorem due to Friedlander (1982), Friedlander-Mislin (1984).

Let $BG^\wedge$ be the Bousfield-Kan $\mathbb{Z}/\ell$-completion of the classifying space BG of the connected compact Lie group G.

We write $H^*(X)$, $\tilde{H}^*(X)$ for the mod ℓ cohomology, reduced mod ℓ cohomology of a space X, respectively.

We denote by $\mathrm{fib}(\alpha)$, $\pi_0 : P_\alpha \to X$ the homotopy fibre, mapping track of a map $\alpha : A \to X$. That is,

$$P_\alpha = \{(a, \lambda) \in A \times X^I \mid \alpha(a) = \lambda(1)\},$$

$\pi_0((a, \lambda)) = \lambda(0)$ and $\mathrm{fib}(\alpha) = \pi_0^{-1}(*)$, where $I = [0, 1]$ is the unit interval, X^I is the set of continuous maps from I to X, $*$ is the base-point of X.

Theorem (Friedlander-Mislin)

There exist maps

$$D : BG(\overline{\mathbb{F}}_p) \to BG^\wedge$$

and

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(2) $\phi^q \circ D = D \circ \phi^q$ where $\phi^q : BG(\overline{\mathbb{F}}_p) \to BG(\overline{\mathbb{F}}_p)$ is the Frobenius map induced by the Frobenius homomorphism $\phi^q : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p$.

(3) There exists a map

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that induces an isomorphism $H^*(\mathrm{fib}(\Delta)) \to H^*(\mathrm{fib}(D_q))$. The above map is obtained from the following homotopy commutative diagram by choosing a suitable homotopy:

$$\begin{array}{ccc} BG(\mathbb{F}_q) & \longrightarrow & BG^\wedge \\ \downarrow{\scriptstyle D_q} & & \downarrow{\scriptstyle \Delta} \\ BG^\wedge & \xrightarrow{(1\times\phi^q)\circ\Delta} & BG^\wedge \times BG^\wedge. \end{array}$$

The first part of Quillen's computation is the homotopy theoretical interpretation of the problem.

$$\mathcal{L}_f X = \{\lambda \in X^I \mid \lambda(1) = f(\lambda(0))\}.$$

We call this space the *twisted loop space* of f following the terminology of Kishimoto's preprint.

We have the following fibre square:

$$\begin{array}{ccc} \mathcal{L}_f X & \xrightarrow{g} & P_\Delta \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\ X & \xrightarrow{(1\times f)\circ\Delta} & X \times X, \end{array}$$

Theorem(Mislin-Friedlander)(3) implies that $H^*(\mathcal{L}_f X)$ is isomorphic to $H^*(BG(\mathbb{F}_q))$ for $X = BG^\wedge$, $f = \phi^q$.

The second part of Quillen's computation is the computation using the Eilenberg-Moore spectral sequence.

Let us write A for $H^*(X)$. The Eilenberg-Moore spectral sequence is given by

$$\mathrm{Tor}_{A\otimes A}(A, A) \Rightarrow \mathrm{gr}\, H^*(\mathcal{L}_f X)$$

If the induced homomorphism $f^* : A \to A$ is the identity homomorphism and if A is a polynomial algebra, then the above E_2-term is a polynomial tensor exterior algebra

$$A \otimes V$$

where $V = \mathrm{Tor}_A(\mathbb{Z}/\ell, \mathbb{Z}/\ell)$.

Since as an algebra over $\mathbb{Z}/\ell$, it is generated by $\mathrm{Tor}^0_{A\otimes A}(A, A)$ and $\mathrm{Tor}^{-1}_{A\otimes A}(A, A)$, the spectral sequences collapses at the E_2-level.

There exists a homotopy equivalence between the classifying space of the loop group $\mathcal{L}G$ and the free loop space $\mathcal{L}BG$, where

$$\mathcal{L}X = \{\lambda \in X^I \mid \lambda(1) = \lambda(0)\}.$$

For the free loop space $\mathcal{L}BG$, we have the following fibre square:

$$\begin{array}{ccc} \mathcal{L}X & \longrightarrow & P_\Delta \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\ X & \xrightarrow{\Delta} & X \times X, \end{array}$$

As in the case of finite Chevalley groups, there exists the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{A \otimes A}(A, A) \Rightarrow \mathrm{gr}\, H^*(\mathcal{L}BG).$$

Therefore, if $H^*(BG)$ is a polynomial algebra, and if $\ell | q-1$,

$$H^*(\mathcal{L}BG) = H^*(BG(\mathbb{F}_q))$$

as a graded $\mathbb{Z}/\ell$-module.

Even if $H^*(BG)$ is not a polynomial algebra over $\mathbb{Z}/\ell$, if the induced homomorphism $\phi^{q*} : A \to A$ is the identity homomorphism, E_2-terms of the above Eilenberg-Moore spectral sequences are the same.

Conjecture (Tezuka)

If $\ell | q-1$ (resp. $4|q-1$) when ℓ is odd (resp. even), there exists a ring isomorphism between $H^*(BG(\mathbb{F}_q))$ and $H^*(\mathcal{L}BG)$.

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Theorem

There exists an integer b such that, for $q = p^{ab}$ where a is an arbitrary positive integer, there exists an isomorphism of graded $\mathbb{Z}/\ell$-modules

$$H^*(BG(\mathbb{F}_q)) = H^*(\mathcal{L}BG).$$

If $H^*(BG)$ is not a polynomial algebra, it is not easy to compute the mod ℓ cohomology of $BG(\mathbb{F}_q)$ and $\mathcal{L}BG$. The only computational results in the literature are the computation of

- the mod 2 cohomology of $B\mathrm{Spin}_{10}(\mathbb{F}_q)$ and $\mathcal{L}B\mathrm{Spin}(10)$ in (Kleinerman,1982) and (Kuribayasi-Mimura-Nishimoto,2006) for $\ell = 2$ and
- the mod 3 cohomology of $\mathcal{L}BPU(3)$ for $\ell = 3$ in (Kuribayasi-Mimura-Nishimoto,2006).

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For example, Theorem (Friedlander-Mislin) is proved by this method.

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If there exists a chain of maps such as above, then they also induce isomorphisms of Bockstein spectral sequences. This contradicts the above observation on the rational (and integral) cohomology of $BG(\mathbb{F}_q)$ and $\mathcal{L}BG$.

For the given b,

$$\{B\mathbb{Z}/(q-1)^{\wedge} \mid q = ab, a \geq 1\}$$

contains infinitely many homotopy types.

Let X be a space and let $f : X \to X$ be a self-map of X with a non-empty fixed point set. Let $\alpha : A \to X$ be a map such that

$$f \circ \alpha = \alpha.$$

We choose a base-point $*$ in A, X, so that both f, α are base-point preserving.

Firstly, we define a map

$$\varphi : \mathcal{L}_f X \times_X \mathcal{L}_f X \to \mathcal{L}X,$$

where

$$\mathcal{L}_f X \times_X \mathcal{L}_f X = \{(\lambda_1, \lambda_2) \in \mathcal{L}_f X \times \mathcal{L}_f X \mid \lambda_1(0) = \lambda_2(0)\}.$$

The map φ is defined by

$$\varphi(\lambda_1, \lambda_2)(t) = \begin{cases} \lambda_1(2t) & \text{for } 0 \leq t \leq \dfrac{1}{2}, \\ \lambda_2(2 - 2t) & \text{for } \dfrac{1}{2} \leq t \leq 1. \end{cases}$$

Since $\lambda_1(1) = f(\lambda_1(0))$, $\lambda_2(1) = f(\lambda_2(0))$ and $\lambda_1(0) = \lambda_2(0)$, this map is well-defined.

Next, we define a map from P_α to $\mathcal{L}_f X$, say $\psi : P_\alpha \to \mathcal{L}_f X$, by

$$\psi((a,\lambda))(t) = \begin{cases} \lambda(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ f(\lambda(2-2t)) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $\lambda(1) = f(\lambda(1))$, this map is also well-defined.

Now, we consider the following diagram:

$$\begin{array}{ccccc} \mathcal{L}_f X & \xleftarrow{p_1} & \mathcal{L}_f X \times_X \mathcal{L}_f X & \xrightarrow{\varphi} & \mathcal{L}X \\ & & \Big\uparrow {\scriptstyle 1\times\psi} & & \\ & & \mathcal{L}_f X \times_X P_\alpha. & & \end{array}$$

where

$$\mathcal{L}_f X \times_X P_\alpha = \{(\lambda_1, (a, \lambda_2)) \in \mathcal{L}_f X \times P_\alpha \mid \lambda_1(0) = \lambda_2(0),\ \alpha(a) = \lambda_2(1)\},$$

p_1 is the projection onto the first factor and $\pi_0 \circ p_1 = \pi_0$, $\pi_0 \circ \varphi = \pi_0$, $\pi_0 \circ (1 \times \psi) = \pi_0$. Let us denote by $E_r(Y)$ the Leray-Serre spectral sequence associated with a fibration $\xi : Y \to X$.

Then we have the following diagram of spectral sequences:

$$\begin{array}{ccccc} E_r(\mathcal{L}_f X) & \xrightarrow{p_1^*} & E_r(\mathcal{L}_f X \times_X \mathcal{L}_f X) & \xleftarrow{\varphi^*} & E_r(\mathcal{L}X) \\ & & \Big\downarrow {\scriptstyle 1\times\psi^*} & & \\ & & E_r(\mathcal{L}_f X \times_X P_\alpha). & & \end{array}$$

By abuse of notation, we denote by $\psi : \mathrm{fib}(\alpha) \to \Omega X$ the restriction of $\psi : P_\alpha \to \mathcal{L}_f X$ to fibres.

Lemma (3.2)

Suppose that X is simply connected, that $H^i(\mathrm{fib}(\alpha)) = 0$ for $i > k$ and that there exists a sequence of maps

$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \longrightarrow \cdots \longrightarrow A_k \xrightarrow{i_k} A \xrightarrow{\alpha} X$$

such that the induced homomorphism $\tilde{H}^*(\mathrm{fib}(\alpha_j)) \to \tilde{H}^*(\mathrm{fib}(\alpha_{j-1}))$ is zero for $j = 1, 2, \ldots, k$. Then the projection on the first factor $p_1 : Y \times_X P_\alpha \to Y$ induces a monomorphism $p_1^* : E_r(Y) \to E_r(Y \times_X P_\alpha)$ of Leray-Serre spectral sequences for arbitrary fibration $\xi : Y \to X$.

Lemma (3.3)

Let

$$E'_r \xrightarrow{\rho'_r} E_r \xleftarrow{\rho''_r} E''_r$$

be homomorphisms of spectral sequences. Suppose that

(1) $\operatorname{Im} \rho'_2 = \operatorname{Im} \rho''_2$,

(2) ρ'_r *is a monomorphism for* $r \geq 2$.

Then, there exists a unique homomorphism of spectral sequences

$$\{\tau_r : E''_r \to E'_r \mid r \geq 2\}$$

such that $\rho'_r \circ \tau_r = \rho''_r$ *for* $r \geq 2$. *In particular, if* ρ''_2 *is also a monomorphism, then* ρ''_r *is a monomorphism and* τ_r *is an isomorphism for* $r \geq 2$.

Proof of Theorem

- Let $k = \dim G$. Let $V = H^*(G)$ as a finite dimensional vector space.
- Let $q_j = p^{e_1^j}$ for $j = 0, \ldots, k$.
- Let $q = p^{ab} = {q_k}^{ae_2}$ $(a \geq 1)$.
- Let $X = BG^\wedge$, $A = BG(\mathbb{F}_{q_k})$, $\alpha = D_{q_k}$, $f = \phi^q$, $A_j = BG(\mathbb{F}_{q_j})$ and $\alpha_i = D_{q_j}$ for $j = 0, 1, \ldots, k$.
  In order to prove Theorem, we consider the Leray-Serre spectral sequence $E_r(\mathcal{L}_f X)$, $E_r(\mathcal{L}X)$ and establish an isomorphism of spectral sequences $\tau : E_r(\mathcal{L}X) \to E_r(\mathcal{L}_f X)$.
- If we take e_1 to be $(\ell \times |GL(V)|)^k$,

$$\tilde{H}^*(\mathrm{fib}(\alpha_j)) \to \tilde{H}^*(\mathrm{fib}(\alpha_{j-1}))$$

  is zero for $j = 1, \ldots, k$.
  By Lemma (3.2), we have a monomorphism

$$(1 \times \psi)^* \circ p_1^* : E_r(\mathcal{L}_f X) \longrightarrow E_r(\mathcal{L}_f X \times_X P_\alpha).$$

Proof of Theorem (continue)

Identifying the E_2-terms $E_2(\mathcal{L}_f X)$, $E_2(\mathcal{L}X)$, $E_2(\mathcal{L}_f X \times_X P_\alpha)$ of Leray-Serre spectral sequences with $H^*(X) \otimes H^*(\Omega X)$, $H^*(X) \otimes H^*(\Omega X)$, $H^*(X) \otimes H^*(\Omega X) \otimes H^*(\mathrm{fib}(\alpha))$, respectively,

we have

$$(1 \times \psi)^*(p_1^*(x \otimes y)) = x \otimes y \otimes 1 \quad \text{and}$$
$$(1 \times \psi)^*(\varphi^*(x \otimes y)) = \sum x \otimes y' \otimes \psi^*(\chi(y'')),$$

where $\varphi^*(y) = \sum y' \otimes \chi(y'')$ and $\chi : H^*(\Omega X) \to H^*(\Omega X)$ is the canonical anti-automorphism of connected Hopf algebra over $\mathbb{Z}/\ell$.

Proof of Theorem (continue)

- If we take e_2 to be $|GL(V \otimes V)|$, we obtain

$$\mathrm{Im}\,(1 \times \psi)^* \circ p_1^* = \mathrm{Im}\,(1 \times \psi)^* \circ \varphi^*$$

in the E_2-term
$E_2(\mathcal{L}_f X \times_X P_\alpha) = H^*(X) \otimes H^*(\Omega X) \otimes H^*(\mathrm{fib}(\alpha))$.
Therefore, using Lemma (3.3), we obtain an isomorphism between Leray-Serre spectral sequences $E_r(\mathcal{L}_f X)$ and $E_r(\mathcal{L}X)$.

Thank you.