Uniformed treatment to periodic points and fixed points

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Outline

The object

Given a self map $f : X \to X$, we consider

- the set of fixed point of f:

$$Fix(f) = \{x \in X \mid f(x) = x\};$$

- the set of periodic point of f of least period n:

$$P_n(f) = \{x \in X \mid f^n(x) = x, f^k(x) \neq x \text{ for } 0 < k < n\}.$$

n-times symmetric product

The n-times symmetric product SP_nX is the the quotient space

$$\prod^n X/\Sigma_n$$

of the product space $\prod^n X$ under the natural action of n-symmetric group Σ_n.

Any self map $f: X \to X$ induces a self map

$$SP_nf: SP_nX \to SP_nX,$$

which is given by

$$SP_nf([x_1, x_2, \ldots, x_n]) = [f(x_1), f(x_2), \ldots, f(x_n)]$$

Some observations

Example

Let $f\colon X \to X$. Then it induce a map $SP_4f\colon SP_4X \to SP_4X$ given by

$$SP_4f([x_1, x_2, x_3, x_4]) = [f(x_1), f(x_2), f(x_3), f(x_4)].$$

If $[x_1, x_2, x_3, x_4]$ is a fixed point of SP_4f, then we have $(f(x_1), f(x_2), f(x_3), f(x_4)) = \sigma(x_1, x_2, x_3, x_4)$ for a $\sigma \in \Sigma_4$.
(1) If $\sigma = (2, 3, 4, 1)$, then $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x_3) = x_4$ and $f(x_4) = x_1$. There are three possibilities:
(1.1) $x_1 = x_2 = x_3 = x_4$ is a fixed point of f;
(1.2) $x_1 = x_3 \neq x_2 = x_4$, two same periodic orbits of period 2;
(1.3) x_i's are different, consisting of a periodic orbit of period 4.

Some observations

(2) If $\sigma = (1, 3, 4, 2)$, then $f(x_1) = x_1$, $f(x_2) = x_3$, $f(x_3) = x_4$ and $f(x_4) = x_2$. So, x_1 is a fixed point of f. There are two possibilities:

(2.1) $x_2 = x_3 = x_4$ is a fixed point of f

(2.2) x_2, x_3, x_4 is a periodic orbit of period 3.

A conclusion

Any fixed point of SP_nf consists of periodic point of f, the sum of all periods is n.

By the Lefschetz fixed point theorem, we have

Proposition

If the Lefschetz number $L(SP_nf)$ is non-zero, then f itself has a periodic point with period $\leq n$.

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Fixed point classes

Given a self map $f: X \to X$, we have

$$Fix(f) = \bigcup_{\tilde{f}} p(Fix(\tilde{f})),$$

where ranges all the liftings of f.

Proposition

Let $\tilde{f}$ and $\tilde{f}'$ be two liftings of a self map $f: X \to X$, then
(1) $p(Fix(\tilde{f}')) = p(Fix(\tilde{f}))$, if $\tilde{f}' = \gamma\tilde{f}\gamma^{-1}$ for some $\gamma \in \mathcal{D}(\tilde{X})$,
(2) $p(Fix(\tilde{f}')) \cap p(Fix(\tilde{f})) = \emptyset$, if $\tilde{f}' \neq \gamma\tilde{f}\gamma^{-1}$ for any $\gamma \in \mathcal{D}(\tilde{X})$.

Thus, the fixed point set $Fix(f)$ of a self map f is splited into a disjoint union of some isolated subsets, where $p(Fix(\tilde{f}))$ is said to be a fixed point class determined by $\tilde{f}$.

Estimation for fixed points

Each fixed point class F has a well-defined index, denoted $ind(f, F)$. The sum of the indices of all fixed point classes of f is just the Lefschetz number $L(f)$.

Any homotopy between two maps gives rise a 1-1 index preserving correspondence between this two maps.

Proposition

Any map in the homotopy class of f has least $N(f)$ fixed points, where the Nielsen number $N(f)$ is defined to be the number of fixed point classes with non-zero indices.

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How to detect periodic points

If $m|n$, there is a correspondence

$$\iota_{m,n} : FPC(f^m) \to FPC(f^n)$$

from the set of fixed point classes of f^m to that of f^n, which is given by

$$[\tilde{f}_1\tilde{f}_2 \cdots \tilde{f}_m] \mapsto [(\tilde{f}_1\tilde{f}_2 \cdots \tilde{f}_m)^{n/m}]$$

Proposition

If F is a fixed point class of f^n, which is not in the image of $\iota_{m,n}$ for all m with $m|n$, then any periodic point in F has period n.

Such a class is said to be irreducible.

product of fixed point classes

Let $F_1, F_2, \dots, F_k$ be fixed point classes of $f^{m_1}, f^{m_2}, \dots, f^{m_k}$, where $\sum_{j=1}^{k} m_j = n$. Then

$$(F_1, F_2, \dots, F_k) = \{(x_1, f(x_1), \dots, f^{m_1-1}(x_1), x_2, \dots, \dots, \dots) | x_i \in F_i\}$$

is a fixed point class of

$$(\prod^{n} f) \circ \sigma_{\{m_1, m_2, \dots, m_k\}} : \prod^{n} X \to \prod^{n} X,$$

where $\sigma_{\{m_1, m_2, \dots, m_k\}}$ is the element in Σ_n:

$$(m_1, 1, 2, \dots, m_1 - 1, m_1 + m_2, m_1 + 1, m_1 + 2, \dots, m_1 + m_2 - 1, \dots, \dots, \sum_{j=1}^{k-1} m_j + m_k, \sum_{j=1}^{k-1} m_j + 1, \sum_{j=1}^{k-1} m_j + m_k - 1).$$

Let $p\colon \tilde{X} \to X$ be the universal covering of X with covering translation group $\mathcal{D}(\tilde{X})$. We have a semi-product $\bigtimes^n \mathcal{D}(\tilde{X}) \rtimes \Sigma_n$. Its group structure is given by

$$(\alpha, \sigma)(\alpha', \sigma') = (\alpha\sigma(\alpha'), \sigma\sigma'), \quad \alpha, \alpha \in \bigtimes^n \mathcal{D}(\tilde{X}), \ \sigma, \sigma' \in \Sigma_n.$$

We write $\mathcal{D}(\tilde{X}, n)$ for this group. It has a well-defined (left) action on the product space $\bigtimes^n \tilde{X}$ defined by:

$$((\bigtimes_{j=1}^n \alpha_j, \sigma), \bigtimes_{j=1}^n \tilde{x}_j) \mapsto \bigtimes_{j=1}^n \alpha_j \tilde{x}_{\sigma^{-1}(j)},$$

where $\sigma \in \Sigma$, $\tilde{x}_j \in \tilde{X}$, $\alpha_j \in \mathcal{D}(\tilde{X})$.

Suppose that N is the smallest normal subgroup of $\mathcal{D}(\tilde{X}, n)$ containing all elements with action fixed points. Then there is a commutative diagram consisting of quotient maps

$$\begin{array}{ccccc}
\times^n \tilde{X} & \xrightarrow{\tilde{q}_n} & \times^n \tilde{X}/N & = & \widetilde{SP_n X} \\
{\scriptstyle \times^n p}\downarrow & \searrow^{/\mathcal{D}(\tilde{X},n)} & \downarrow{\scriptstyle p_{SP_n}} & & \\
\times^n X & \longrightarrow & \times^n \tilde{X}/\mathcal{D}(\tilde{X}, n) & = & SP_n X
\end{array}$$

Furthermore
(1) the orbit space $\times^n \tilde{X}/\mathcal{D}(\tilde{X}, n)$ of its action is $SP_n X$;
(2) the orbit space $\times^n \tilde{X}/N$ is the universal covering of $SP_n X$;
(3) the fundamental group of $SP_n X$ is isomorphic to the quotient group $\mathcal{D}(\tilde{X}, n)/N$;

Fixed point data and related functor

If we are given a communicative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & X \end{array}$$

then there is a well-defined correspondence

$$h_{FPC} : FPC(g) \to FPC(f).$$

In fact, if x lies in a fixed point class G of g, then $h(x)$ lies in the fixed point class $h_{FPC}(G)$ of f.

Consider commutative diagram

$$\begin{array}{ccc} \times^n X & \xrightarrow{\times^n f} & \times^n X \\ {\scriptstyle q_n}\downarrow & & \downarrow{\scriptstyle q_n} \\ SP_n X & \xrightarrow{SP_n f} & SP_n X \end{array}$$

Definition

A fixed point class F of $SP_n f : SP_n X \to SP_n X$ is said to be composed by $F_1, F_2, \ldots, F_k$ if $F = q_{n,FPC}(F_1, F_2, \ldots, F_k)$.
In this case, the set $\{F_1, F_2, \ldots, F_k\}$ is said to be a *composition* of F, and each F_i is said to be a *member* of F. We shall write $F = [F_1, F_2, \ldots, F_k]$.

Basic property of composition

Proposition

The fixed point class $[F_1, F_2, \ldots, F_k]$ is independent of the choice of the order of these F_j's.

Recall that if F is a fixed point class of f^s, $f_{FPC}(F)$ is also a fixed point class.

Proposition

For any $j_i, i = 1, 2, \ldots, k$, we have

$$[f_{FPC}^{j_1}(F_1), f_{FPC}^{j_2}(F_2), \ldots, f_{FPC}^{j_k}(F_k)] = [F_1, F_2, \ldots, F_k].$$

Consider the natural filtration

$$(\prod^n X)^k = \{(x_1, x_2, \ldots, x_n) \in \prod^n X : |\{x_1, x_2, \ldots, x_n\}| \leq k\}$$

Clearly, $SP_n f$ keeps filtration.

$$\begin{array}{ccccccc}
X \cong SP_n X^1 & \longrightarrow & SP_n X^2 & \longrightarrow \cdots \longrightarrow & SP_n X^n = SP_n X \\
\downarrow SP_n f & & \downarrow SP_n f & & \downarrow SP_n f \\
X \cong SP_n X^1 & \longrightarrow & SP_n X^2 & \longrightarrow \cdots \longrightarrow & SP_n X^n = SP_n X
\end{array}$$

A fixed point class of $SP_n f|SP_n X^m$ is said to be related to $SP_n X^k$ if it lies in the image of

$$j_{FPC} : FPC(SP_n f|SP_n X^k) \to FPC(SP_n f|SP_n X^m).$$

Formalize irreducibilities

Proposition

Let F be a fixed point class of $SP_n f$. If there is a composition of F such that two members are the same fixed point class of f^s, then F is related to $SP_n X^{n-s}$, i.e. contains a fixed point class of $SP_n f | SP_n X^{n-s}$.

Theorem

Let F be a fixed point class of $SP_n f : SP_n X \to SP_n X$. If F is not related to $SP_n X^{n-1}$, then the members of a composition of F are distinct, and each of them is an irreducible periodic point class.

History remark

Borsuk-Ulam (1931): define symmetric product, and observe that $2^X = \overline{\cup_n SP_n X}$

Dold-Thom theorem(1958): The homotopy of the infinite symmetric product $\mathrm{SP}(X)$ of a based connected CW complex is isomorphic to the integral homology of X.

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