Some Stable and Unstable Homotopy of Cell Complexes

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1. Exponents (at odd primes)

- Instead of answering the very difficult question "what are the homotopy groups of a space?", we ask for something weaker like "approximately how fast do the homotopy groups grow?".

- The p-exponent $exp_p(G)$ of an abelian group G is the smallest power p^t such that
$$p^t \times (p\text{-torsion in } G) = 0.$$

- We measure the growth of homotopy groups of X by finding bounds for the growth of $exp_p(\pi_i(X))$.

- The largest value of $exp_p(\pi_i(X))$ for $i > 1$ (if it exists) is called the p-exponent of X, and is denoted $exp_p(X)$. In this case we can also say $exp_p(\pi_*(X))$ is bounded.

2. Cases of Bounded Exponents

2.1. Finite p-local H-spaces

- Building on work of Toda, for odd primes p Cohen, Moore, and Neisendorfer showed
$$exp_p(S^{2n+1}) = p^n.$$
- Selick proved the special case
$$exp_p(S^3) = p.$$
- For every $n > 0$ the image of the suspension map $X \longrightarrow \Omega^n \Sigma^n X$ on π_* has bounded p-exponent given X is a finite connected CW-complex [Don Stanley] .
- A consequence is that every finite connected H-space (or more generally an H-space that is the p-localization of a finite connect CW-complex) has a bounded p-exponent [Stanley, Long] .

2.2. Rationally trivial spaces

- The *Moore space* $P^n(p^r)$ has p-exponent p^{r+1} [Cohen, Moore, and Neisendorfer].
- Any finite wedge of Moore spaces has a bounded p-exponent.
- A simply connected finite CW-complex X is rationally trivial if and only if $\pi_*^s(X)$ has bounded p-exponent [Stanley].
- A simply connected finite CW-complex X s.t. $\pi_*(X) \otimes \mathbb{Q}$ is a finite dimensional vector space has a bounded p-exponent at *almost all* primes p [McGibbon,Wilkerson].

2.3. Conjectured Examples

- <u>Moore conjecture:</u> A simply connected finite CW-complex X has $\pi_*(X) \otimes \mathbb{Q}$ as a finite dimensional vector space if and only if it has bounded p-exponent at all primes p.
- <u>Barrats conjecture (weak form):</u> If the identity map on $\Sigma^2 X$ has order p^r then $exp_p(\Sigma^2 X) = p^{r+1}$.

3. Unbounded Exponents

- Any wedge of spheres does not have a bounded p-exponent (by Hilton-Milnor Theorem).
- More generally, ΣX is rationally non-trivial implies $S^m \vee \Sigma X$ does not have a p-exponent at any prime p and all $m > 1$ [Stanley].

  note: That ΣX is rationally nontrivial is necessary since $S^{2k+1} \vee P^n(p^r)$ has p-exponent $\max\{p^{r+1}, p^k\}$.

- Any 2-cell complex $X = S^m \cup_\alpha e^n$ such that the attaching map $S^{n-1} \xrightarrow{\alpha} S^m$ is of finite order has no bounded p-exponent when $n, m \geq 2$ [Selick, Neisendorfer].

Theorem 3.1 (Hilton-Milnor) *For any spaces X and Y, there is a functorial decomposition*

$$\Omega\Sigma(X \vee Y) \simeq \Omega\Sigma X \times \Omega\Sigma(Y \vee \bigvee_{i\geq 1}(X^{(i)} \wedge Y)).$$

□

Iterating the Hilton-Milnor Theorem implies $\Omega(S^n \vee S^m)$ is the loopspace of a certain infinite weak product of spheres of increasing dimension. Then using the known exponents of spheres, an upper bound for the growth of $exp_p(\pi_i(S^n \vee S^m))$ can be found.

<u>PROBLEM:</u> For other less trivial spaces, try to obtain bounds for growth of exponents...

<u>MAIN ASSUMPTIONS:</u> From now on assume all spaces are CW-complexes localized at an odd prime p. Take some such p-localized finite cell complex X, let

$$V = \widetilde{H}_*(X; \mathbb{Z}_p),$$

and let M be the sum of the degrees of the generators in V.
Let $\mathbb{Z}_p[S_k]$ be the group ring of the symmetric group S_k on k letters.

- For the k-fold self smash $X^{(k)}$, we have $\widetilde{H}_*(X^{(k)}; \mathbb{Z}_p) \cong V^{\otimes k}$.
- $\mathbb{Z}_p[S_k]$ acts on $V^{\otimes k}$ sending elements to sums of their permutations.
- For each $\sigma \in \mathbb{Z}_p[S_k]$, the action of σ on $V^{\otimes k}$ determines a self map

$$V^{\otimes k} \xrightarrow{\sigma} V^{\otimes k}.$$

- When X is a suspension, one can construct a self-map $X^{(k)} \xrightarrow{\bar{\sigma}} X^{(k)}$ inducing $V^{\otimes k} \xrightarrow{\sigma} V^{\otimes k}$ on homology.
- If $\sigma \circ \sigma = \sigma$ (i.e. σ is an idempotent), the mapping telescope $T(\sigma)$ of the sequence

$$X^{(k)} \xrightarrow{\bar{\sigma}} X^{(k)} \xrightarrow{\bar{\sigma}} \cdots$$

is a retract of $X^{(k)}$ with homology isomorphic to the image of σ.

By the Poincare-Birkhoff-Witt Theorem, there is an isomorphism of $\mathbb{Z}_{(p)}$-modules

$$T(V) \cong \bigotimes_{i=1}^{\infty} S(L_i(V)),$$

where $L_n(V)$ is the submodule of length n Lie brackets in $V^{\otimes n}$. Part of this decomposition can be geometrically realized:

Theorem 3.2 (Jie Wu) *Let X be a suspension. There is a decomposition*

$$\Omega\Sigma X \simeq \prod_j \Omega\Sigma L_{k_j}(X) \times (\textit{Some other space}).$$

for each sequence $1 < k_1 < k_2 < \cdots$ such that

1. *k_i is prime to p;*
2. *k_i is not a multiple of k_j whenever $i \neq j$.*

□

Each space $L_n(X)$ is a retract of $X^{(n)}$, and is isomorphic to $L_n(V)$ on homology. They are geometrically realized as telescopes $T(\beta_n)$ using the *Dynkin-Specht-Wever element*

$$\beta_n \in \mathbb{Z}_p[S_n].$$

FURTHER RESTRICTIONS: From now on assume X is a suspension, and has either only cells of even dimension or cells of odd dimension.

Then the basis of $V = \widetilde{H}_*(X;\mathbb{Z}_p)$ has generators in degrees corresponding to the dimensions of the cells. Also $\widetilde{H}_*(X;\mathbb{Z}_{(p)}) \cong V$ and $\widetilde{H}_*(X^{(i)};\mathbb{Z}_{(p)}) \cong V^{\otimes i}$ as $\mathbb{Z}_{(p)}$-modules.

For the odd case, let $s_k \in \mathbb{Z}_{(p)}[S_k]$ be

$$s_k = \sum_{\sigma \in S_k} \sigma,$$

and for the even case

$$s_k = \sum_{\sigma \in S_k} sgn(\sigma)\sigma.$$

In either case it is well known that $s_k s_k = k! s_k$.

We obtain the self map $V^{\otimes k} \xrightarrow{s_\ell} V^{\otimes k}$ and its geometric realization $X^{(k)} \xrightarrow{\bar{s}_\ell} X^{(k)}$.

Proposition 3.3 *When V has $\ell < p$ generators,*

(i) *the telescope $T(s_\ell)$ is a retract of $X^{(\ell)}$ and*

(ii) *$T(s_\ell) = S^M$ where M is the sum of the degrees for each generator in V.*

□

Corollary 3.4 *When V has $\ell < p$ generators,*

(i) *$\Sigma^M X$ is a retract of $X^{(\ell+1)}$;*

(ii) *as a submodule of $\widetilde{H}_*(X^{(\ell+1)};\mathbb{Z}_p) \cong V^{\otimes \ell+1}$, $\widetilde{H}_*(\Sigma^M X;\mathbb{Z}_p)$ is the image of $V^{\otimes \ell+1} \xrightarrow{s_\ell \otimes \mathbb{1}} V^{\otimes \ell+1}$.*

□

Take the self maps

$$V^{\otimes k+1} \xrightarrow{\beta_{k+1}} V^{\otimes k+1}$$

and

$$V^{\otimes k} \xrightarrow{s_k} V^{\otimes k}.$$

Proposition 3.5 *Suppose a free $\mathbb{Z}_{(p)}$-module V has basis of dimension $k > 1$ consisting either of only odd degree generators or of even degree generators. Then*

$$(s_k \otimes \mathbb{1}) \circ \beta_{k+1} \circ (s_k \otimes \mathbb{1}) = (k! + (k-1)!)(s_k \otimes \mathbb{1}).$$

□

Therefore the composition

$$\Sigma^M X \longrightarrow X^{(\ell+1)} \longrightarrow L_{\ell+1}(X) \longrightarrow X^{(\ell+1)} \longrightarrow \Sigma^M X$$

is a p-local homotopy equivalence when $1 < \ell < p-1$.

Notice $\beta_{k+1} \circ (s_k \otimes \mathbb{1})$ is an idempotent modulo p when $k = 2, 3, ..., p-2, p$ (excluding $p-1$).

Theorem 3.6 *Let X be a suspension and M be the sum of degrees of generators in $V = \widetilde{H}_*(X;\mathbb{Z}_p)$. When V has $1 < \ell < p-1$ generators, and all generators are either of even or of odd degree, then*

(i) *$\Omega\Sigma^{M+1}X$ is a retract of $\Omega\Sigma X$;*

(ii) *Therefore $\Omega\Sigma^{c_k+1}X$ is a retract of $\Omega\Sigma X$ for all $k \geq 0$, where (recursively)*

$$c_0 = 0$$

and

$$c_k = (\ell+1)c_{k-1} + M.$$

(iii) *Therefore $\pi_i^s(\Sigma X)$ is a retract of $\pi_{i+c_k}(\Sigma X)$ for all k large enough s.t. $i < c_k$.*

□

When $\ell = p$, Proposition 3.5 implies $\beta_{\ell+1} \circ (s_\ell \otimes \mathbb{1})$ is an idempotent modulo p. Then the telescope $T(\beta_{\ell+1} \circ (s_\ell \otimes \mathbb{1}))$ splits off from $L_{\ell+1}(X)$, and its mod-p homology is isomorphic to that of $\Sigma^M X$.

QUESTION: is $T(\beta_{\ell+1} \circ (s_\ell \otimes \mathbb{1}))$ homotopy equivalent to $\Sigma^M X$ when $\ell = p$? FURTHER WORK: What happens when V has *both* even and odd degree generators.

Possibly we can still say something when $\ell \geq p-1$:

Lemma 3.7 *Let Y be any connected finite cell complex, and suppose a self map $Y \xrightarrow{f} Y$ is trivial on integral homology. Then $f^{2^{k-1}} \simeq *$, where k is the number of dimensions Y has cells inside.* □

Let c be the number of dimension $i < M$ in which $V^{\otimes \ell}$ is nonzero. Let d be the number of dimensions $i > M$ in which $V^{\otimes \ell}$ is nonzero. Let ν be the number of times p divides $\ell!$.

Corollary 3.8

(i) *There is a map $S^M \longrightarrow X^{(\ell)}$ on homology taking the generator of $\widetilde{H}_*(S^M; \mathbb{Z}_{(p)})$ to $p^{\nu(2^{c-1}-1)}$ times the generator of $s_\ell(V^{\otimes \ell}) \subseteq V^{\otimes \ell}$.*

(ii) *There is a map $\Sigma X^{(\ell)} \longrightarrow S^{M+1}$ on homology taking the generator of $\Sigma s_\ell(V^{\otimes \ell})$ to $p^{\nu(2^{d-1}-1)}$ times the generator of $\widetilde{H}_*(S^{M+1}; \mathbb{Z}_{(p)})$.*

□

So in general there exists a factorization $S^{M+1} \longrightarrow \Sigma X^{(\ell)} \longrightarrow S^{M+1}$ of degree $p^{\nu(2^{c-1}-1)} p^{\nu(2^{d-1}-1)}$.

EXAMPLE: When $\ell = 3$ and $p = 3$, we have (in general) a factorization

$$S^{M+1} \longrightarrow \Sigma X^{(3)} \longrightarrow S^{M+1}$$

of degree 3^{22}, and therefore a degree 3^{22} factorization

$$\Sigma^{M+1} X \longrightarrow \Sigma X^{(4)} \longrightarrow \Sigma^{M+1} X$$

Since

$$(s_3 \otimes \mathbb{1}) \circ \beta_4 \circ (s_3 \otimes \mathbb{1}) = 8(s_3 \otimes \mathbb{1})$$

by Proposition 3.5, we obtain a factorization

$$\Sigma^{M+1} X \longrightarrow \Sigma X^{(4)} \longrightarrow \Sigma L_4(X) \longrightarrow \Sigma X^{(4)} \longrightarrow \Sigma^{M+1} X$$

of degree $8 \times 3^{22} \mod p = 3^{22}$ on $\mathbb{Z}_{(p)}$-homology. Is this actually a degree 3^{22} map geometrically?

Given it is, on similar lines as Theorem 3.6 we could obtain looped degree 3^{22} maps

$$\Omega\Sigma^{M+1} X \longrightarrow \Omega\Sigma X \longrightarrow \Omega\Sigma^{M+1} X.$$

QUESTION: In general, can we retract a space ΩY from $\Omega\Sigma X$ such that Y has less cells than X?

4. Exponent Growth

4.1. Lower bounds

An immediate consequence of Theorem 3.6:

Corollary 4.1 *If X has $1 < \ell < p$ cells,*

$$exp_p(\Sigma X) \geq exp_p(\Sigma^{M+1} X).$$

□

QUESTION: When is it true that $exp_p(Y) \geq exp_p(\Sigma Y)$?

Application of stable homotopy to unstable homotopy:

Recall simply connected finite CW-complex is rationally trivial if and only if its stable homotopy has bounded p-exponent.

Since our space ΣX has either even or odd cells, it is *not* rationally trivial, and so has unbounded p-exponent on stable homotopy.

Since the stable homotopy of ΣX retracts off from its unstable homotopy (by Theorem 3.6), ΣX has unbounded p-exponent.

Theorem 4.2 (Cohen, Neisendorfer) *Let X be any p-localized cell complex consisting of $\ell < p-1$ odd cells, $V = \widetilde{H}_*(X)$. Then there exists a functorial decomposition*

$$\Omega\Sigma X \simeq A(X) \times \Omega Q(X)$$

such that $A(X)$ is a finite H-space with

$$H_*(A(X)) \cong S(V)$$

as primitively generated algebras, and

$$H_*(\Omega Q(X)) \cong S([L(V), L(V)]) = \bigotimes_{i=2}^{\infty}(S(L_i(V))).$$

□

Given X consists only of $1 < \ell < p-1$ odd cells, Theorem 3.6 and 4.2 imply $A(\Sigma^{c_k} X)$ is a retract of $\Omega\Sigma X$ for all $k \geq 1$ such that c_k is even. Since

$$T(L_{\ell+1}(V)) \cong \bigotimes_{i=1}^{\infty} S(L_i(L_{\ell+1}(V))) \subseteq S([L(V), L(V)]),$$

this result can be strengthened:

Theorem 4.3 *If V consists of $1 < \ell < p-1$ odd dimensional generators and ℓ is even:*

$$\Omega\Sigma X \simeq \prod_{i=0}^{\infty} A(\Sigma^{c_i} X) \times (\textit{Some other space}).$$

□

If N is the dimension of the top cell(s) in X, and $\bar{X}$ is X minus an N-cell, there exist homotopy fibrations

$$A(\Sigma^{c_i}\bar{X}) \longrightarrow A(\Sigma^{c_i}X) \xrightarrow{q} S^{c_i+N}.$$

If the order of the attaching map for the N-cell that we removed (from X to get $\bar{X}$) is known (say p^t), there exist degree p^t factorizations

$$S^{c_i+N} \longrightarrow A(\Sigma^{c_i}X) \xrightarrow{q} S^{c_i+N}.$$

In this case, by a result of Theriault we get homotopy fibrations

$$\Omega S^{c_i+N} \times \Omega A(\Sigma^{c_i}\bar{X}) \longrightarrow \Omega A(\Sigma^{c_i}X) \longrightarrow S^{c_i+N}\{p^t\}$$

where $exp_p(S^{c_i+N}\{p^t\}) = p^t$.

4.2. The curious case of rank 2

When X is a suspension with only $\ell = 2$ even dimensional cells (M is the sum of their dimensions), from Proposition 3.5

$$L_2(X) = S^M,$$

and

$$L_3(X) = \Sigma^M X.$$

Recall $c_1 = M$, $c_k = (\ell + 1)c_{k-1} + M = 3c_{k-1} + M$.

By iteration we have:

Theorem 4.4 *For any suspended 2-cell complex X of even cells*

$$\Omega\Sigma X \simeq \prod_{i=1}^{\infty} \Omega(S^{c_i+1}) \times (\textit{Some other space}).$$

□

<u>QUESTION:</u> In general, do loop spaces of spheres retract off from $\Omega\Sigma X$ when $\ell > 2$.

5. Upper Bounds

Theorem 5.1 (Barratt) *If an* $(n-1)$*-connected space* ΣY *has finite* $\mathbb{Z}_{(p)}$*-homology characteristic* p^m *(equivalently the identity map on* ΣY *has order* p^m *or* Y *is rationally trivial) then*

$$exp_p(\pi_i(\Sigma Y)) \leq p^{mk}$$

for all $i \leq 2^k n$*.*
If Y *itself is a suspension, then*

$$exp_p(\pi_i(\Sigma Y)) \leq p^{m+k}$$

for all $i \leq p^{k+1} n$*.*

Let

$$Z = \prod_{i=1}^{\infty} S^i$$

and

$$e_i = exp_p(\pi_i(Z)) \leq p^i.$$

Let Y be an $(n-1)$-connected 2-cell complex with attaching map of finite order p^t, and Y itself being a suspension. Then (very roughly)

$$\exp_p(\pi_i(\Sigma Y)) \leq e_i p^{(k+g_i)t} \leq p^{(k+g_i)t+i}$$

for all $i < 2nk - 1$, where g_i is the largest integer such that $2ng_i - 1 < i$.

6. THANKS!