International Conference on Algebraic Topology
Chinese Academy of Science, Beijing,May 25-29, 2009

Bordism Methods in Transformation Groups

Yanying Wang

Hebei Normal University

Home Page
Title Page
Page 1 of 20
Go Back
Full Screen
Close
Quit

- Introduction
- Notes on Bordism
- Application of Bordism to G-actions

1 Introduction

In topology, one studies such objects, X, as topological spaces,topological manifolds,differentiable manifolds,and so on.

In the theory of transformation groups,one studies the homeomorphisms of such objects,or generally subgroups of the full group,$Homeo(X)$, of homeomorphisms. The subgroups naturally act on X.

Let G be a group. If $\eta : G \to Homeo(X)$ is a homomorphism , then G act on X. Now We are concerned with smooth actions of compact Lie groups on smooth manifold. The following problems are fundamental and important in study of transformation groups:

Home Page
Title Page
Page 3 of 20
Go Back
Full Screen
Close
Quit

- Given a closed manifold M, is there any non-trivial $G-$action on M?
- For a fixed group G, try to give a classification of $G-$manifolds up to some equivalence relation.
- Let $F = \{x \mid g(x) = x, x \in M, \ g \in G\}$ be the fixed point set of $G-$action on M. By dimF we mean the dimension of the highest dimensional non-empty component of the fixed point set.What could we say about dimF?
- From the algebraic or geometric point of view, what is the relationship among the $G-$manifold, the fixed point set F and the orbit space M/G?

The objective of this talk is to show that bordism techniques can be applied to the study of transformation groups on closed manifolds.

Other tools are available in the field of transformation groups on smooth manifolds,such as P.A.Smith Theory,Equivariant Cohomology,Equivariant K-Theory and Equivariant Surgery. Of course these paths cross frequently.

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 4 of 20
Go Back
Full Screen
Close
Quit

2 Notes on Bordism

A closed manifold is said to bound if and only if the manifold is the boundary of some compact manifold. Two closed manifolds are called bordant if their disjoint union bounds.We refer to this as the bordism relation. Bordism relation is an equivalence relation on the diffeomorphism classes of closed $n-$manifolds. The equivalence class(bordism class)to which M^n belongs is denoted by $[M^n]$. In bordism theory, key points are:

- To specify some kind of objects(manifolds)
- To determine and calculate the bordism classes of given objects
- To establish bordism invariants

Home Page
Title Page
Page 5 of 20
Go Back
Full Screen
Close
Quit

Example 1: Unoriented bordism: $MO_\star$
Objects: All unoriented closed manifolds
Calculation: The collection of all classes of specified $n-$manifolds is denoted by MO_n. The plus is given on MO_n by disjoint union,that is, $[M_1^n]+[M_2^n]=[M_1^n\cup M_2^n]$. The product is put on $MO_\star=\sum MO_n$ by $[M_1^n]\times[M_2^m]=[M_1^n\times M_2^m]$. $MO_\star$ is the polynomial ring over Z_2 on classes x_i of dimension $i(i\neq 2^s-1)$[Thom,1954].
Invariants: Z_2 cohomology characteristic numbers give complete invariants[Thom,1954].All relations among these numbers are given by Wu formulae.

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 6 of 20
Go Back
Full Screen
Close
Quit

Example 2: Oriented bordism: $MSO_\star$

Objects: All oriented closed manifolds

Calculation: The collection of all classes of specified $n-$manifolds is denoted by MSO_n. The plus is given on MSO_n by disjoint union,that is,$[M_1^n]+[M_2^n]=[M_1^n\cup M_2^n]$.The product is put on $MSO_\star=\sum MSO_n$ by $[M_1^n]\times[M_2^m]=[M_1^n\times M_2^m]$. $MSO_\star\otimes Q$ is the rational polynomial ring on classes x_{4i} of the complex projective space[Thom,1954]. $MSO_\star$ has no odd torsion and $MSO_\star$/Torsion is the polynomial ring over Z on $4i$ dimensional generators[Milnor,1958].

Invariants: Bordism is determined by Z and Z_2 cohomology, all relations among the Z_2 numbers being given by the relations of Wu together with the vanishing of the first Stiefel-Whitney class[Wall,1960] . All relations among the Z numbers are given by the Riemann-Roch theorem [Stong,1966].

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 7 of 20
Go Back
Full Screen
Close
Quit

Example 3: Unoriented singular bordism:$MO_\star(X,A)$,where (X,A) is topological pair

Objects: Unoriented singular manifolds (B^n, f) in (X,A) consisting of a compact unoriented manifold and a map $f : (B^n, \partial B^n) \to (X,A)$. A unoriented singular manifold is said to bound if and only if there is a compact unoriented manifold C^{n+1} and a map $F : C^{n+1} \to X$ for which

(1) B^n is contained in ∂C^{n+1} as a regular submanifold

(2)$F/B^n = f$ and $F(\partial C^{n+1} - B^n) \in A$

Two unoriented singular manifolds are called bordant if their disjoint union bounds.

Calculation: The collection of all classes is denoted by $MO_n(X,A)$. The plus is given by disjoint union.The $MO_\star-$ module structure on $MO_\star(X,A) = \sum MO_n(X,A)$ by $[B^n, f][M^m] = [B^n \times M^m, g]$. For any CW-pair,$MO_\star(X,A)$ is a free graded $MO_\star-$ module isomorphic to $H_\star(X,A;Z_2) \otimes MO_\star$.

Invariants: Bordism is determined by Z_2 cohomology.

Remark:If $A = \emptyset, X = BO(k)$,then (B^n, f) can be considered as a vector bundle$\eta^k \to B^n$.

There is an oriented analogue.

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 8 of 20
Go Back
Full Screen
Close
Quit

3 Application of Bordism to G-actions

Let G be a compact Lie group.(G, M^n) is a smooth action of G on a closed unoriented $n-$manifold. We say (G, M^n) bounds if and only if there is an action (G, B^{n+1}) on a compact unoriented $(n+1)-$ manifold B^{n+1} for which the induced action $(G, \partial B^{n+1})$ is equivariantly diffeomorphic to (G, M^n). We say (G, M_1^n) is bordant to (G, M_2^n) if their disjoint union$(G, M_1^n \cup M_2^n)$ bounds. This is shown to be an equivalence relation. The resulting set of equivalence classes is denoted by $I_n(G)$.By disjoint union this becomes an abelian group. We set$[G, M^n][G, V^m] = [G, M^n \times V^m]$,so$I_\star(G) = \sum I_n(G)$ is a graded commutative algebra with identity over $MO_\star$,called the unoriented $G-$bordism algebra.

There is an oriented analogue of course.

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 9 of 20
Go Back
Full Screen
Close
Quit

Now let G be finite and the action free. Under this ristriction we may proceed to define a bordism group $MO_n(G)$ and $MO_\star-$ module $MO_\star(G)$.

Theorem: *For any finite group there is a natural isomophism of $MO_\star-$ module*

$$MO_\star(G) = MO_\star(BG)$$

which preserves dimension.

For $G = Z_2$,a $G-$ action is called an involution. A lot of results on bordism of involutions could be found in [Conner, Differentiable Periodic Maps,1979].We list some of theorems:

Theorem 1(Conner): *Let (A, S^n) be the antipodal involution on the $n-$sphere.Then $[A, S^n]$ is a $MO_\star-$ module basis for$MO_\star(Z_2)$.*

Theorem 2(Conner): *Let (T, M^n) denote a smooth involution on a closed manifold with $\eta \to F$ the normal bundle to the fixed point set. Then the non-equivariant class $[M^n] = [RP(\eta \oplus R)]$.*

The structure of $I_\star(Z_2)$ is very complicated. It appears impossible to determine $I_\star(Z_2)$ completely. Let us consider the problems posed in section 1.

- What about the existence of an non-trivial involution ?

  Theorem 3(Stong): *Let $\{(F^{n-r}, \nu^r)\}(0 \leq r \leq n)$ be bundles over manifolds. A necessary and sufficient condition that the collection be the fixed data of an involution (T, M^n) is that*

$$\sum_r \frac{f(1+y,z)}{\prod(1+y)}[F^{n-r}] = 0.$$

  for all f of degree less than n.

- To determine the bordism class of involutions with a prescribed fixed point set .

Theorem 4(Stong): *If (T, M^n) is a non-trivial involution on a closed manifold whose fixed point set is $RP(2r)$,then $n = 4r$ and $[M^n] = [RP(2r)]^2$.*

Theorem 5(Wang and Chen,2008): *Every vector bundle ξ over $CP(j) \times HP(k)$ has the total Stiefel-Whitney class of the form*

$$u = (1+\alpha)^a(1+\beta)^b(1+\alpha^2+\beta)^d(1+\alpha^i\beta^{\frac{2^s-2i}{4}})^\varepsilon,$$

where $\alpha \in H^2(CP(j); Z_2)$, $\beta \in H^4(HP(k); Z_2)$ are nonzero classes, with $\varepsilon = 0$ or 1. In order that the exotic class $\alpha^i\beta^{\frac{2^s-2i}{4}}$ occurs, we must have

$$\begin{cases} i = 2^t(2p+1), & t \geq 1, \\ j = 2^t(2p+1) + x, & 0 \leq x < 2^t, \\ 4k = 2^s - 2^{t+1}(2p+1) + y, & 0 \leq y < 2^{t+1}. \end{cases}$$

As an application of Theorem 5, we prove

Home Page
Title Page
Page 12 of 20
Go Back
Full Screen
Close
Quit

Theorem 6(Wang and Chen,2008): *There is no non-bounding involution fixing* $CP(2m+1)\times HP(k)$.

Corollary (Wang and Chen,2008): *Every involution fixing* $CP(2m+1)\times HP(k)$ *bounds.*

- Try to determine the bordism class of involutions on a prescribed manifold.

  In 1999, Meng and Wang determined the bordism classes of involutions on even dimensional homology projective space[Chinese Quarterly J. Math , 14(2),1999.06,33-37].

- What could we say about dimF ?

  Theorem 7(Boardman) : *If* (T, M^n) *is a smooth involution on a closed manifold for which* $[T, M^n]\neq 0$ *and* F *is the fixed point set,then* $dimF \geq 2n/5$.

Home Page
Title Page
Page 13 of 20
Go Back
Full Screen
Close
Quit

- What is the relationship among the $G-$manifold, the fixed point set F and the orbit space M/G?

Theorem 8(Classical): *If (T, M^n) is a smooth involution on a closed manifold ,then $\chi(M^n) = \chi(F)(mod2)$.*

Theorem 9(Stong): *Let (T, M^n) be a smooth involution on a closed n-dimensional manifold with the fixed point data $(F, \nu) = \coprod_r (F^{n-r}, \nu^r)$. If $f(x_1, \cdots, x_n)$ is a symmetric polynomial over Z_2 in n variables of degree at most n, then*

$$f(x_1, \cdots, x_n)[M^n] = \sum_r \frac{f(1+y_1, \cdots, 1+y_r, z_1, \cdots, z_{n-r})}{\prod_1^r (1+y_i)}[F^{n-r}],$$

where the expressions are evaluated by replacing the elementary symmetric functions $\sigma_i(x), \sigma_i(y)$, and $\sigma_i(z)$ by the Stiefel-Whitney classes $w_i(M), w_i(\nu^r)$, and $w_i(F)$ respectively and taking the value of the resulting cohomology class on the fundamental homology class of M or F.

Home Page
Title Page
Page 14 of 20
Go Back
Full Screen
Close
Quit

Set $M_n = \sum_{m=0}^{n} MO_m(BO(n-m))$.Then $M_\star = \sum M_n$ has the structure of graded commutative algebra over $MO_\star$.

Theorem 10(Conner): *The sequence*

$$0 \longrightarrow I_\star(Z_2) \longrightarrow M_\star \longrightarrow MO_\star(Z_2) \longrightarrow 0$$

is split exact , where $j^\star : I_\star(Z_2) \longrightarrow M_\star$ is a homomorphism of $MO_\star$ algebra and $\partial : M_\star \longrightarrow MO_\star(Z_2)$ is an $MO_\star-$ module homomorphism .

This theorem shows

- The bordism class of (T, M) is determined by the bordism class of the bundle (F, ν), where ν denote the normal bundle of F in M.
- The real projective space bundle $RP(\nu)$ bounds in the bordism of RP^∞, where the map into RP^∞ classifies the double cover of $RP(\nu)$ by the sphere bundle $S(\nu)$.

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 15 of 20
Go Back
Full Screen
Close
Quit

For $G = (Z_2)^k or (Z_p)^k, p$ odd prime ,see [Conner, Differentiable Periodic Maps,1979].

Zhao and Wang have discussed the problem of commuting involutions with fixed point set $RP(2m+1) \cup RP(2n+1)$. Let $\Phi : (Z_2)^2 \times M \to M$ be a smooth action of the group $(Z_2)^2 = \{T_1, T_2 | T_i^2 = 1, T_1T_2 = T_2T_1\}$ on a smooth closed manifold M. Let $T_3 = T_1T_2$. The fixed point data of Φ is $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $F_\Phi = \{x \in M | T_i(x) = x, i = 1, 2, 3\}$ is a closed manifold, ε_i is the normal bundle of F_Φ in $F_{T_i} = \{x \in M | T_i(x) = x\}, i = 1, 2, 3$. we have proved the following

Theorem 11(Zhao and Wang,2009): *Let (Φ, M) be a smooth $(Z_2)^2$-action on a closed and smooth manifold M whose fixed point set is $F_\Phi = RP(2m+1) \cup RP(2n+1)$, where $RP(2m+1)$ and $RP(2n+1)$ are projective spaces with dimensions $2m+1$ and $2n+1$ respectively. Let*

$$(RP(2m+1); \mu_1, \mu_2, \mu_3) \cup (RP(2n+1); \nu_1, \nu_2, \nu_3)$$

be the fixed point data of Φ. If at least two $\mu_{i's}$ have dimension greater than $2m+1$, and at least one ν_i has dimension greater than $2n+1$, where $m \geq n$, then (Φ, M) bounds equivariantly.

Home Page
Title Page
◀◀ ▶▶
◀ ▶

Go Back
Full Screen
Close
Quit

This theorem is the generalized result about that of [Royster,1980]. The idea of the proof is: In some cases we directly construct a closed manifold with a $(Z_2)^2$-action whose fixed point set is $RP(2m+1) \cup RP(2n+1)$.In the other cases we find characteristic classes to calculate invariants, getting to contradiction,and so prove the non-existence of such actions.

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 17 of 20
Go Back
Full Screen
Close
Quit

4 References

1. R.E. Stong,*Vector bundles over Dold manifolds,* Fund. Math.,2001,**169:** 85-95.
2. R.Li,*Vector bundles over* $RP(m) \times CP(n)$, PhD Thesis, Hebei normal University, China, 2005.
3. J.Li and Y.Wang,*Characteristic classes of vector bundles over* $RP(h) \times HP(k)$, Topology Appl., 2007,**154:**1778-1793.
4. P.E. Conner, *Differentiable Periodic Maps*, Second ed., Lecture Notes in Math.,vol.738, Springer-Verlag, Berlin, Heidelberg, New York, 1979.
5. A.Gathmann,*Algebraic Geometry*,Notes for a class,taught at the University of Kaiserslautern 2002/2003.
6. R.E. Stong, *Involutions fixing projective spaces*, Michigan Math.J.,1966,**13:** 445-447.
7. D.C. Royster, *Involutions fixing the disjoint union of two projective spaces*, Indiana Univ. Math. J., 1980,**29:** 267-276.
8. D.Hou, B.Torrence, *Involutions fixing the disjoint union of odd-dimensional projective spaces*, Can. Math. Bull.,1994,**37:**66-74.

Home Page
Title Page
Page 18 of 20
Go Back
Full Screen
Close
Quit

9. Z.Lü, *Involutions fixing* $RP^{odd} \cup P(h,i)$,*I*, Trans. Amer. Math. Soc.,2002,**354:** 4539-4570.
10. Z.Lü, *Involutions fixing* $RP^{odd} \cup P(h,i)$,*II*, Trans. Amer. Math. Soc., 2003,**356:** 1291-1314.
11. S.M. Kelton, *Involutions fixing* $RP^j \cup F^n$, Topology Appl.,2004,**142:**, 197-203.
12. R.E.Stong, *Involutions fixing product of circles*, Proc.Amer. Math. Soc., 1993, **119:**1005-1008.
13. Y.Ding,*Involutions fixing product of projective spaces*, Northeast Math. J.,1999,**15:**97-102.
14. P.E. Conner, *The bordism class of a bundle space*, Michigan Math.J., 1967,**14:**289-303.
15. P.L.Q. Pergher, R.E.Stong, *Involutions fixing* $\{point\} \cup F^n$, Transformation Groups, 2001,**6:**78-85.
16. C. Kosniowski, R.E.Stong, *Involutions and characteristic numbers*, Topology, 1978,**17:**309-330.

Introduction to...
Notes on (co)Bordism
Application to...
References

Home Page
Title Page
Page 19 of 20
Go Back
Full Screen
Close
Quit

Thanks!