Brunnian Braids on Surfaces

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Motivation

Configurations and braids

Brunnian Braids

Generating Set for Brunnian Braids

Braid Equations

Combinatorial Descriptions of Homotopy Groups

- **Try possible** new approach for homotopy groups.
- **Combinatorial Description of Homotopy Groups** [W-, 2001]:
- Let $G(n)$ be the group with generators $x_1, \ldots, x_n$ and defining relations:
  1. $x_1 \cdots x_n = 1$.
  2. the all possible iterated commutators

     $$[x_{i_1}^{\epsilon_1}, \ldots, x_{i_t}^{\epsilon_t}] = 1$$

     with $\{i_1, \ldots, i_t\} = \{1, \ldots, n\}$.

  Then $\pi_n(S^2)$ is the center of the group $G(n)$.
- There is a generalization of the above result for $\pi_*(S^k)$ for general k by **Hao Zhao and Xiangjun Wang.**

Homotopy Groups and Braid Groups

- **[W, 2002]:** The braid group B_n acts on $G(n)$. For $n \geq 4$, $\pi_n(S^2)$ is the **fixed set** of the **pure braid group** action on $G(n)$.

- **[Berrick-Cohen-Wong-W-, 2006]:**

  The n-th homotopy group of the sphere is given by the quotient of the $(n+1)$-strand **Brunnian braid group** over the **sphere** modulo the $(n+1)$-strand Brunnian braid group over the **disk** for $n \geq 4$.

Motivations

- **Hope** to get combinatorial descriptions of $\pi_*(S^k)$ for general k.
- **Observation**[Gray-Hilton-Milnor Decomposition:] $\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y)$.
- Let $X = T$ be a torus and let $Y = S^2$. Then $\Omega(T \vee S^2) \simeq \Omega T \times \Omega S^2 \times \Omega\Sigma(\Omega T \wedge \Omega S^2)$.
- If X is a surface with $X \neq D^2$, then $\pi_*(X \vee S^2)$ contains of $\pi_*(S^k)$ as its summands for all $k \geq 2$.

- In this project, we **only look** at Brunnian braids on general surfaces. It is still **not clear** how to obtain a good braided description of $\pi_*(S^k)$ for general k yet.

Remarks Brunnian Braids

- A classical question proposed by Makanin in 1980 is to determine a set of generators for Brunnian braids over the disk. (Brunnian braids were called *smooth braids* by Makanin.)
- This question was answered by D. L. Johnson, *Towards a characterization of smooth braids*, Math. Proc. Cambridge Philos. Soc. **92** (1982), 425–427.
- A different approach to this question can be found in W-, *Combinatorial descriptions of the homotopy groups of certain spaces*, Math. Proc. Camb. Philos. Soc. **130** (2001), 489–513.
- In Birman's book [Question 23, p. 219], she asked to determine the free basis for Brunnian braids over the sphere. Birman's question remains open.

Configuration Spaces

Let M be a space and let M^n be the n-fold Cartesian product of M. The **n-th ordered configuration space** $F(M,n)$ is defined by

$$F(M,n) = \{(x_1,\ldots,x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

with subspace topology of M^n. The symmetric group Σ_n acts on $F(M,n)$ by permuting coordinates. The orbit space

$$B(M,n) = F(M,n)/\Sigma_n$$

is called the **n-th unordered configuration space.**

- The *braid group* $B_n(M)$ is defined to be the fundamental group $\pi_1(F(M,n)/\Sigma_n)$.
- The *pure braid group* $P_n(M)$ is defined to be the fundamental group $\pi_1(F(M,n))$.

Presentation of $B_n(S_{g,p})$

Let $M = S_{g,p}$ be the oriented surface of genus g with $p \geq 0$ boundary components. [P. Bellingeri, *On presentation of surface braid groups* J. Algebra **274** (2004), 543–563.] The group $B_n(M)$ admits a presentation with generators:

$$\sigma_1, \ldots, \sigma_{n-1}, x_1, \ldots, x_{2g}, z_1, \ldots, z_{p-1},$$

and defining relations

1. Braid relations for σ_i as above.
2. Mixed relations given as follows:
   - (R1) $x_r\sigma_i = \sigma_i x_r,\ i \neq 1,\ 1 \leq r \leq 2g;$
   - (R2) $(\sigma_1^{-1}x_r\sigma_1^{-1})x_r = x_r(\sigma_1^{-1}x_r\sigma_1^{-1}),\ 1 \leq r \leq 2g;$
   - (R3) $(\sigma_1^{-1}x_s\sigma_1)x_r = x_r(\sigma_1^{-1}x_s\sigma_1),\ 1 \leq s < r \leq 2g,\ (s, r) \neq (2m-1, 2m);$
   - (R4) $(\sigma_1^{-1}x_{2m-1}\sigma_1^{-1})x_{2m} = x_{2m}(\sigma_1^{-1}x_{2m-1}\sigma_1),\ 1 \leq m \leq g;$
   - (R5) $z_j\sigma_i = \sigma_i z_j,\ i \neq 1,\ 1 \leq j \leq p-1;$
   - (R6) $(\sigma_1^{-1}z_j\sigma_1)x_r = x_r(\sigma_1^{-1}z_j\sigma_1),\ 1 \leq r \leq 2g,\ 1 \leq j < p-1;$
   - (R7) $(\sigma_1^{-1}z_j\sigma_1)z_l = z_l(\sigma_1^{-1}z_j\sigma_1),\ 1 \leq j < l \leq p-1;$
   - (R8) $(\sigma_1^{-1}z_j\sigma_1^{-1})z_j = z_j(\sigma_1^{-1}z_j\sigma_1),\ 1 \leq j \leq p-1.$

$B_n(N_{g,p})$ by P. Bellingeri, J. Algebra 2004

Let $M = N_{g,p}$ be a non-oriented surface of genus $g \geq 1$ with $p > 0$ boundary components. The group $B_n(M)$ admits a presentation with generators:

$$\sigma_1, \ldots, \sigma_{n-1}, a_1, \ldots, a_g, z_1, \ldots, z_{p-1},$$

and defining relations

1. Braid relations for σ_i as above.
2. Mixed relations given as follows:
   - (R1) $a_r \sigma_i = \sigma_i x a r, \; i \neq 1, \; 1 \leq r \leq g;$
   - (R2) $(\sigma_1^{-1} a_r \sigma_1^{-1}) a_r = a_r (\sigma_1^{-1} a_r \sigma_1), \; 1 \leq r \leq g;$
   - (R3) $(\sigma_1^{-1} a_s \sigma_1) a_r = a_r (\sigma_1^{-1} a_s \sigma_1), \; 1 \leq s < r \leq g;$
   - (R4) $z_j \sigma_i = \sigma_i z_j, \; i \neq 1, \; 1 \leq j \leq p-1;$
   - (R5) $(\sigma_1^{-1} z_i \sigma_1) a_r = a_r (\sigma_1^{-1} z_i \sigma_1), \; 1 \leq r \leq g, \; 1 \leq j \leq p-1, \; n > 1;$
   - (R6) $(\sigma_1^{-1} z_j \sigma_1) z_l = z_l (\sigma_1^{-1} z_j \sigma_1), \; 1 \leq j < l \leq p-1;$
   - (R7) $(\sigma_1^{-1} z_j \sigma_1^{-1}) z_j = z_j (\sigma_1^{-1} z_j \sigma_1^{-1}), \; 1 \leq j \leq p-1.$

Removing Strands and Brunnian Braids

A *simple (half-open) curve* in a space M means a continuous injection $\theta\colon \mathbb{R}^+ = [0,\infty] \to M$. The basepoints of $F(M,n)$ are given by the ordered points on the curve θ.

Proposition. Let M be a space with a simple curve. Then the operations

$$d_i\colon B_n(M) \to B_{n-1}(M), \text{ removing } i \text{ th strand}, 1 \le i \le n,$$

satisfy the following identities:

1) $d_i d_j = d_j d_{i+1}$ for $i \ge j$;
2) $d_i(\beta\beta') = d_i(\beta) d_{i\cdot\beta}(\beta')$. □

Definition. Let M be a space with a simple curve. A braid $\beta \in B_n(M)$ is called *Brunnian* if $d_i(\beta) = 1$ for each $1 \le i \le n$. The set of n-strand Brunnian braids is denoted by $\mathrm{Brun}_n(M)$. For convention, any 1-strand braid is regarded as a Brunnian braid.

Basic Properties of Brunnian Braids

Proposition. Let M be a space with a simple curve. Then

1. The subgroup $\mathrm{Brun}_n(M) \cap P_n(M)$ is normal in $B_n(M)$ for each $n \geq 1$.
2. $\mathrm{Brun}_n(M) \leq P_n(M)$ for $n \geq 3$
3. $\mathrm{Brun}_n(M)$ is a normal subgroup of $B_n(M)$ for $n \geq 3$.
4. Let M be a path-connected 2-manifold. Then $\mathrm{Brun}_2(M)$ is a normal subgroup of $B_2(M)$ if and only if $\pi_1(M) = \{1\}$.

Main Computational Tool

Fadell-Neuwirth Theorem.[*Configuration spaces*, Math. Scand. **10** (1962), 111–118.]
Let M be a path-connected manifold. The coordinate projection

$$d_i\colon F(M,n) \to F(M,n{-}1) \quad (x_1,\ldots,x_n) \mapsto (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$$

is a fibre bundle with fibre $M \smallsetminus Q_{n-1}$, where Q_{n-1} is a set of $(n-1)$ distinct points in M.

Notations

Let D^2 be a small disk in $M \smallsetminus \partial M$. The basepoints $\{p_1, p_2, \ldots\}$ for the braids on M are chosen inside $D^2 \smallsetminus \partial D^2$. The embedding $j\colon D^2 \rightarrowtail M$ induces a map

$$j^n\colon F(D^2, n)/\Sigma_n \rightarrowtail F(M, n)/\Sigma_n$$

and so a group homomorphism

$$j^n_*\colon B_n(D^2) = \pi_1(F(D^2, n)/\Sigma_n) \longrightarrow B_n(M) = \pi_1(F(M, n)/\Sigma_n)$$

with a commutative diagram

$$\begin{tikzcd}
B_n(D^2) \arrow[r, "j^n_*"] \arrow[d] & B_n(M) \arrow[d] \\
B_n(D^2)/P_n(D^2) = \Sigma_n \arrow[r, equal] & \Sigma_n = B_n(M)/P_n(M).
\end{tikzcd}$$

For any braid $\beta \in B_n(D^2)$, we write $\beta[M]$ (or simply β if there are no confusions) for the braid $j^n_*(\beta)$ on M.

2-strand Brunnian Braids

Let M be any path-connected 2-manifold. Then the 2-strand Brunnian braids are determined as follows

1) $\mathrm{Brun}_2(M) \cap P_2(M)$ is the normal closure of the element $A_{1,2}$ in $B_2(M)$.
2) $\mathrm{Brun}_2(M) = \langle \mathrm{Brun}_2(M) \cap P_2(M), \sigma_1 \rangle$ is the subgroup of $B_2(M)$ generated by $\mathrm{Brun}_2(M) \cap P_2(M)$ and σ_1.

Example.

- $P_2(\mathbb{R}\mathrm{P}^2) = Q_8$.
- $\mathrm{Brun}_2(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}/4$ generated by σ_1.
- $\mathrm{Brun}_2(\mathbb{R}\mathrm{P}^2) \cap P_2(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}/2$ generated by $A_{1,2}$.

3-strand Brunnian Braids

Let M be a path-connected 2-manifold. Then the 3-strand Brunnian braids on M are determined as follows:

1) $\mathrm{Brun}_3(S^2) = P_3(S^2) = \mathbb{Z}/2$.
2) For $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$,

$$\mathrm{Brun}_3(M) = [\langle\langle A_{1,3}\rangle\rangle^P, \langle\langle A_{2,3}\rangle\rangle^P]$$

   the commutator subgroup of the normal closures in $P_3(M)$ generated by $A_{1,3}$ and $A_{2,3}$, respectively.
3) $\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$ is a free group of rank 9.

Generators for $\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$

As a subgroup of $B_3(\mathbb{R}\mathrm{P}^2)$, $\mathrm{Brun}_3(\mathbb{R}\mathrm{P}^2)$ has a free basis given by

$$\begin{array}{ccc} A_{1,3}^2, & \omega^4, & [A_{1,3},\omega], \\ [A_{1,3},\omega^2], & [A_{1,3},\omega^3], & [A_{1,3},\omega^4], \\ [[A_{1,3},\omega],A_{1,3}], & [[A_{1,3},\omega^2],A_{1,3}], & [[A_{1,3},\omega^3],A_{1,3}], \end{array}$$

where ω is a braid in $B_3(\mathbb{R}\mathrm{P}^2)$ with $\omega^2 = A_{1,3}A_{2,3}$. From the monomorphism

$$\pi_1(\mathbb{R}\mathrm{P}^2 \smallsetminus \{p_1,p_2\}) \hookrightarrow P_3(\mathbb{R}\mathrm{P}^2) = \pi_1(F(\mathbb{R}\mathrm{P}^2,3)) \hookrightarrow B_3(\mathbb{R}\mathrm{P}^2),$$

ω is represented by the half-circle around the punctured points p_1 and p_2.

Key Steps for determining $\mathrm{Brun}_3(M)$ with $M \neq S^2, \mathbb{R}P^2$

- From the fibration $M \smallsetminus \{p_1, p_2\} \to F(M,3) \xrightarrow{d_3} F(M,2)$ with $\pi_2(F(M,2)) = 0$, we have $\mathrm{Ker}(d_3\colon P_3(M) \to P_2(M)) = \pi_1(M \smallsetminus \{p_1, p_2\}) = G$.
- $\mathrm{Brun}_3(M) = \langle\langle A_{1,3}\rangle\rangle^G \cap \langle\langle A_{2,3}\rangle\rangle^G$.
- By Brown-Loday Theorem,

$$\frac{\langle\langle A_{1,3}\rangle\rangle^G \cap \langle\langle A_{2,3}\rangle\rangle^G}{[\langle\langle A_{1,3}\rangle\rangle^G, \langle\langle A_{2,3}\rangle\rangle^G]} \cong \pi_2(M) = 0.$$

- $\Longrightarrow \mathrm{Brun}_3(M) = [\langle\langle A_{1,3}\rangle\rangle^G, \langle\langle A_{2,3}\rangle\rangle^G]$. Then check that $\mathrm{Brun}_3(M) = [\langle\langle A_{1,3}\rangle\rangle^P, \langle\langle A_{2,3}\rangle\rangle^P]$.

Symmetric Iterated Commutator Subgroups

Given a group G, and a set of its normal subgroups $R_1, \ldots, R_n,\ (n \geq 2)$ denote

$$[R_1, \ldots, R_n]_S := \prod_{\sigma \in \Sigma_n} [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}],$$

where Σ_n is the nth symmetric group.

- If $n = 2$, $[R_1, R_2]_S = [R_1, R_2] \cdot [R_2, R_1] = [R_1, R_2]$ because $[x, y] = [y, x]^{-1}$.
- If $n > 2$, $[R_1, \ldots, R_n]_S$ seems bigger than $[[R_1, R_2], \ldots, R_n]$.

n-strand Brunnian Braids

Let M be a path-connected 2-manifold and let $n \geq 4$. Let

$$R_n(M) = [\langle\langle A_{1,n}[M]\rangle\rangle^P, \langle\langle A_{2,n}[M]\rangle\rangle^P, \ldots, \langle\langle A_{n-1,n}[M]\rangle\rangle^P]_S$$

be the symmetric commutator subgroup.

1. If $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$, then
$$\mathrm{Brun}_n(M) = R_n(M).$$
2. If $M = S^2$ and $n \geq 5$, then there is a short exact sequence
$$R_n(S^2) \hookrightarrow \mathrm{Brun}_n(S^2) \twoheadrightarrow \pi_{n-1}(S^2).$$
3. If $M = \mathbb{R}\mathrm{P}^2$, then there is a short exact sequence
$$R_n(\mathbb{R}\mathrm{P}^2) \hookrightarrow \mathrm{Brun}_n(\mathbb{R}\mathrm{P}^2) \twoheadrightarrow \pi_{n-1}(S^2).$$

Ideas of the Proof

- From $G = \pi_1(M \smallsetminus \{p_1, \ldots, p_{n-1}\}) \to P_n(M) \xrightarrow{d_n} P_{n-1}$,
$$\mathrm{Brun}_n(M) = \bigcap_{i=1}^{n-1} \langle\langle A_{i,n} \rangle\rangle^G.$$
- By using a generalization of Brown-Loday Theorem or simplicial group model,
$$\frac{\bigcap_{i=1}^{n-1} \langle\langle A_{i,n} \rangle\rangle^G}{[[\langle\langle A_{1,n} \rangle\rangle^G, \ldots, \langle\langle A_{n-1,n} \rangle\rangle^G]]} = \pi_n(M),$$
where $[[\langle\langle A_{1,n} \rangle\rangle^G, \ldots, \langle\langle A_{n-1,n} \rangle\rangle^G]]$ is generated by all possible iterated commutators $[x_1, \ldots, x_t]$ with $t \geq n-1$ such that the elements from **each** $\langle\langle A_{i,n} \rangle\rangle^G$ occurs at least once in the bracket.
- By using Witt-Hall identity, the brackets $[x_1, \ldots, x_t]$ can be written as in the form of left normal commutator $[[x_1, x_2], \ldots, x_n]$.

The Groups $B_n(M)/\operatorname{Brun}_n(M)$ and $P_n(M)/\operatorname{Brun}_n(M)$

Theorem. Let M be a path-connected compact 2-manifold. Then the factor groups $P_n(M)/\operatorname{Brun}_n(M)$ and $B_n(M)/\operatorname{Brun}_n(M)$ are finitely presented for each $n \geq 3$.

- We can determine a set of normal generators for $R_n(M) = [\langle\langle A_{1,n}[M]\rangle\rangle^P, \langle\langle A_{2,n}[M]\rangle\rangle^P, \ldots, \langle\langle A_{n-1,n}[M]\rangle\rangle^P]_S$ in $P_n(M)$.
- Let $\bar{A}_{i,j} = A_{i,j}$ if $i < j$ and $\bar{A}_{i,j} = A_{j,i}$ if $i > j$.
- $R_n(M)$ is the normal closure of the elements

$$[[A_{\sigma(1),j_1}, A_{\sigma(2),j_2}], \ldots, A_{\sigma(n-1),j_{n-1}}]$$

for $\sigma \in S_{n-1}$, $1 \leq j_i \leq n$ with $j_i \neq \sigma(i)$ and $1 \leq i \leq n-1$ in $P_n(M)$.

Sketch of Proof

- $R_n(M) = \prod_{\sigma \in S_{n-1}} [[\langle\langle A_{\sigma(1),n}\rangle\rangle^P, \langle\langle A_{\sigma(2),n}\rangle\rangle^P], \ldots, \langle\langle A_{\sigma(n-1),n}\rangle\rangle^P]$.
- $R_n(M) = \prod_{i=1}^{n-1} \left[\bigcap_{j \neq i} \mathrm{Ker}(d_j), \langle\langle A_{i,n}\rangle\rangle^P\right]$.
- The subgroup $\langle\langle A_{i,n}\rangle\rangle^P = \mathrm{Ker}(d_i)$ is generated by $\bar{A}_{i,j}$ for $1 \leq j \neq i \leq n$.
- If T_i is a set of normal generators for $\bigcap_{j \neq i} \mathrm{Ker}(d_j)$, then $\left[\bigcap_{j \neq i} \mathrm{Ker}(d_j), \langle\langle A_{i,n}\rangle\rangle^P\right]$ has a set of normal generators $[x, \bar{A}_{i,j}]$ for $x \in T_i$ and $1 \leq j \neq i \leq n$.
- Prove by induction.

Cohen Group

Let M be any path-connected 2-manifold. Define

$$\mathfrak{H}_n^B(M) = \{\beta \in B_n(M) \mid d_1\beta = d_2\beta = \cdots = d_n\beta\}$$

Namely $\mathfrak{H}_n^B(M)$ consists of n-strand pure braids such that it stays the same braid after removing any one of its strands.

- A typical element in $\mathfrak{H}_n^B(M)$ is the half-twist braid

$$\Delta_n = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\sigma_2\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)\sigma_1.$$

Properties of $\mathfrak{H}_n^B(M)$

- **Proposition.** Let M be any path-connected 2-manifold. Then $\mathfrak{H}_n^B(M)$ is subgroup of $B_n(M)$. Moreover $d_i(\mathfrak{H}_n^B(M)) \subseteq \mathfrak{H}_{n-1}^B(M)$ and the function

  $$d_1 : \mathfrak{H}_n^B(M) \to \mathfrak{H}_{n-1}^B(M)$$

  is a group homomorphism.
- **Proposition.** Let M be any path-connected 2-manifold. Let $n \geq 2$. Then $\mathfrak{H}_n^B(M) \cap P_n(M)$ is a subgroup of $\mathfrak{H}_n^B(M)$ of index 2.
- Let $\mathfrak{H}_n(M) = \mathfrak{H}_n^B(M) \cap P_n(M)$. Then $d_1(\mathfrak{H}_n(M)) \leq \mathfrak{H}_{n-1}(M)$. This gives a tower of groups

  $$\cdots \xrightarrow{d_1} \mathfrak{H}_n(M) \xrightarrow{d_1} \mathfrak{H}_{n-1}(M) \xrightarrow{d_1} \cdots.$$

  Let $\mathfrak{H}(M) = \lim_{n} \mathfrak{H}_n(M)$ be the inverse limit of the tower of groups.

Solution to the Braid Equations

- **Proposition.** Let M be any path-connected 2-manifold such that $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$. Then

  $$d_1 \colon \mathfrak{H}_n(M) \to \mathfrak{H}_{n-1}(M)$$

  is an epimorphism for each $n \geq 2$.
- **Corollary.** Let M be any path-connected 2-manifold such that $M \neq S^2$ or $\mathbb{R}\mathrm{P}^2$ and let $\alpha \in B_n(M)$. Then the equation

  $$d_1\beta = \cdots = d_{n+1}\beta = \alpha$$

  for $(n+1)$-strand braids β has a solution if and only if α satisfies the condition that

  $$d_1\alpha = \ldots = d_n\alpha.$$

The James-Hopf Invariants

Let M be a path-connected 2-manifold with nonempty boundary. For $n \geq k$, the *James-Hopf operation* is a function

$$H_{k,n}\colon \mathrm{Brun}_k(M) \longrightarrow \mathfrak{H}_n(M)$$

defined by setting $H_{k,k}(\beta) = \beta$ with

$$H_{k,n}(\beta) = \prod_{1\leq i_1<i_2<\cdots<i_{n-k}\leq n} d^{i_{n-k}} d^{i_{n-k-1}} \cdots d^{i_1}(\beta)$$

with lexicographic order from right for $\beta \in \mathrm{Brun}_k(M)$, where $d^j\colon B_m \to B_{m+1}$ is the group homomorphism by adding the trivial strand on position j for $1 \leq j \leq m+1$.

Distributivity Law and Semi-direct product Decompositions

Let M be a path-connected 2-manifold with nonempty boundary.

- **Proposition.**[Distributivity Law] Let $\alpha \in \mathfrak{H}_n(M)$ with $1 \leq n \leq \infty$. Then there exists an unique element $\delta_k(\alpha) \in \mathrm{Brun}_k(M)$ for $1 \leq k \leq n$ such that the equation
$$\alpha = \prod_{k=1}^{n} H_{k,n}(\delta_k(\alpha))$$
holds.
- **Proposition.** The group $P_n(M)$ is the (iterated) semi-direct product of the subgroups
$$d^{i_k} d^{i_{k-1}} \cdots d^{i_i}(\mathrm{Brun}_{n-k}(M)),$$
$1 \leq i_1 < i_2 < \cdots < i_k \leq n$, $0 \leq k \leq n-1$, with lexicographic from right.

Thank You!