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The cohomology of $S(n, k)$ relevant to Morava stabilizer algebra

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The cohomology of $S(n,k)$ relevant to Morava stabilizer algebra

1 Generalized homology theories

Generalized homology theories; Let E be a nice ring spectrum, then there is the generalized homology theory $E_*(-)$. For any space (spectrum) X

$$E_*(X) = \pi_*(E \wedge X).$$

Hopf algebroid; For the ring spectrum E

$$(E_*,\ E_*E) = (E_*(S^0),\ E_*(E))$$

is a Hopf algebroid. i.e. E_* acts on E_*E on both side and there are structure maps

$$\begin{aligned} \varphi :&E_*E \otimes_{E_*} E_*E \longrightarrow E_*E, \\ \Delta :&E_*E \longrightarrow E_*E \otimes_{E_*} E_*E. \end{aligned}$$

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Examples;

- The mod p **homology** theory with respect to Eilenberg-MacLane spectrum KZ/p
$$KZ/p_*KZ/p = A$$
is the dual of Steenrod algebra.
- **BP**$_*$ homology with respect to Brown-Peterson spectrum BP.
$$BP_* = Z_{(p)}[v_1, v_2, \cdots], \ BP_*BP = BP_*[t_1, t_2, \cdots]$$
where $Z_{(p)} = \{\frac{m}{n} | p \not| n\}$ and $|v_n| = |t_n| = 2(p^n - 1)$.

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- **E(n)$_*$** theory with respect to Johnson-Wilson spectrum $E(n)$

$$\begin{aligned} E(n)_* =& Z_{(p)}[v_1, \cdots, v_{n-1}, v_n, v_n^{-1}] \\ E(n)_*E(n) =& E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_* \end{aligned}$$

  where BP_* acts on $E(n)_*$ by sending v_i to 0 for $i > n$.

- **Morava K-theory**

$$\begin{aligned} K(n)_* =& Z/p[v_n,\ v_n^{-1}] = E(n)_*/I_n \\ \Sigma(n) =& K(n)_*K(n) = K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_* \\ =& K(n)_*[t_1, t_2, \cdots]/(v_n t_i^{p^n} - v_n^{p^i} t_n) \end{aligned}$$

- **Lubin-Tate (or Morava) theory $\mathbf{E_n}$**

$$(E_n)_* = W(\mathbb{F}_{p^n})[[u_1, \cdots, u_{n-1}]][u^{\pm 1}]$$

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Localization; Let E be a spectrum (homology theory).

- A spectrum X is E-acyclic if $E \wedge X \simeq *$.
- A spectrum X is E-local if every may from an E-acyclic spectrum to it is null.
- A map $X \to Y$ is E-equivalence if

  $$E_*(X) \longrightarrow E_*(Y)$$

  is an isomorphism.

79 Bousfield; There is a functor L_E from spectrum to E-local spectrum and natural E-equivalence

$$X \longrightarrow L_E X$$

Problem; To understand $L_E X$ especially

$$\mathbf{L_{E(n)}X} = \mathbf{L_n X} \qquad \text{and} \qquad \mathbf{L_{K(n)}X}.$$

- The homotopy groups of $L_n T(k)$, where $T(k)$ is the Ravenel spectrum characterized by

  $$BP_* T(k) = BP_*[t_1, t_2, \cdots, t_k].$$

  There is sequence

  $$S^0 = T(0) \to T(1) \to \cdots \to T(k) \to \cdots \to BP$$

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2 Adams-Novikov spectral sequence

60 Adams;
67 Novikov; For any homology theory E and any speutrum X, one has the E-based Adams resolution

$$\begin{tikzcd}
X_0 \arrow[dr] & & X_1 \arrow[ll] \arrow[dr] & & X_2 \arrow[ll] \arrow[dr] & \cdots \arrow[l] & X_s \arrow[l] \arrow[dr] & & X_{s+1} \arrow[ll] \arrow[dr] & \cdots \arrow[l] \\
& E \wedge X_0 \arrow[ur, dotted] & & E \wedge X_1 \arrow[ur, dotted] & & E \wedge X_2 & & E \wedge X_s \arrow[ur, dotted] & & E \wedge X_{s+1}
\end{tikzcd}$$

where $X_0 = X$, $X_{s+1} \to X_s \to E \wedge X_s$ is a cofibration for each s.

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ANSS; The E-based Adams resolution induces a spectral sequence (Adams-Novikov spectral sequence or ANSS)

$$\{E_r^{s,t}X,\ d_r\}$$

- $E_1^{s,t}X = \pi_{t+s}(E \wedge X_s) = E_{t+s}(X_s)$ is the cobar complex of $E_*(X)$
- The E_2-term $E_2^{s,t} = Ext_{E_*E}^{s,t}(E_*,\ E_*(X))$.
- Adams differential $d_r : E_r^{s,t}X \longrightarrow E_r^{s+r,t+r-1}X$.

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Convergence; $\{E_r^{s,t}X,\ d_r\} \Longrightarrow \pi_*Y$ if there is a filtration of π_*Y

$$F^0\pi_*Y \hookrightarrow F^1\pi_*Y \hookrightarrow \cdots \hookrightarrow F^{s-1}\pi_*Y \hookrightarrow F^s\pi_*Y \hookrightarrow \cdots \hookrightarrow \pi_*Y$$

such that $E_\infty^{s,t}X = F^s\pi_{t-s}Y/F^{s-1}\pi_{t-s}Y$.

Example;

- $K(n)-$based ANSS; $K(n)_r^{s,t}X = E_r^{s,t}X$

$$\{E_r^{s,t}X,\ d_r\} \Longrightarrow \pi_*L_nV(n-1)\wedge X$$

  if the $E(n)$-localized Smith-Toda spectrum $L_nV(n-1)$ exists.

- $E(n)-$based ANSS; $E(n)_r^{s,t}X = E_r^{s,t}X$

$$\{E_r^{s,t}X,\ d_r\} \Longrightarrow \pi_*L_nX$$

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77, Miller, Ravenel and Wilson; (Chromatic spectral sequence and Bockstein spectral sequence)
To compute the homotopy groups $\pi_* L_n X$, one has the following cofiber sequences

$$L_n X_0^k \longrightarrow L_k X_0^k \longrightarrow L_n X_0^{k+1},$$

$$L_n X_{n-k+1}^{k-1} \longrightarrow L_n X_{n-k}^k \longrightarrow L_n X_{n-k}^k.$$

Start from the homotopy groups

$$\pi_* L_k X_k^0 = \pi_* L_k V(k-1) \wedge X$$

we deduce the homotopy groups $\pi_* L_k X_0^k$. Next from $\pi_* L_k X_0^k$, one can deduce that of $L_n X$.

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Start point; To compute the E_2-term of $K(n)$-based ANSS

$$Ext^{s,t}_{\Sigma(n)}(K(n)_*,\ K(n)_*X)$$

and then to compute the homotopy groups $\pi_*L_nV(n-1)\wedge X$

- To compute $Ext^{s,t}_{\Sigma(n)}(K(n)_*,\ K(n)_*T(k-1))$ $\Longrightarrow \pi_*L_nV(n-1)\wedge T(k-1)$.

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3 The Hopf algebra $S(n,k)$

Change of rings theorem; Set

$$\Sigma(n,k)=\Sigma(n)/(t_i|i<k)=K(n)_*[t_k,t_{k+1},\cdots]/(v_nt_i^{p^n}-v_n^{p^i}t_n)$$

and

$$S(n,k)=Z/p\otimes_{K(n)_*}\Sigma(n,k)\otimes_{K(n)_*}Z/p=Z/p[t_k,t_{k+1},\cdots]/(t_i^{p^n}-t_n)$$

where $K(n)_*$ acts on Z/p by sending $v_n^{\pm 1}$ to 1. Then by change of rings theorem we have

$$Ext^{s,t}_{\Sigma(n)}(K(n)_*,\ K(n)_*T(k-1))$$
$$=Ext^{*,*}_{S(n,k)}(Z/p,Z/p)\otimes K(n)_*[v_{n+1},\cdots v_{n+k-1}]$$

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$\mathbf{S(n,k)}$; $\qquad S(n,k) = Z/p[t_k, t_{k+1}, \cdots]/(t_i^{p^n} - t_i)$.
The structure map

$$\Delta : S(n,k) \longrightarrow S(n,k) \otimes S(n,k)$$

acts on t_s as

$$\Delta(t_s) = 1 \otimes t_s + \sum_{k \leqslant i \leqslant s-k} t_i \otimes t_{s-i}^{p^i} + t_s \otimes 1 \qquad \text{for } s \leqslant n+k-1$$
$$\Delta(t_s) = 1 \otimes t_s + \sum_{k \leqslant i \leqslant s-k} t_i \otimes t_{s-i}^{p^i} + t_s \otimes 1 - b_{s-n,n-1} \qquad \text{for } s \geqslant n+k$$

where $b_{i,j} = \sum_{0<m<p} \binom{p}{m}/p \cdot t_i^{mp^j} \otimes t_i^{(p-m)p^j}$ at odd primes and $b_{i,j} = t_i^{2^j} \otimes t_i^{2^j}$ at the prime 2

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May filtration; We define May filtration M in $S(n,k)$ as follows:

- For $k \leqslant s \leqslant n+k-1$, set the May filtration of $t_s^{p^j}$ as $M(t_s^{p^j}) = 2s-1$.
- For $n+k \leqslant s$, inductively set the May filtration of $t_s^{p^j}$ as

$$M(t_s^{p^j}) = max\{2s-1, pM(t_{s-n}^{p^j})+1\}.$$

- Let $s_0 = max\left\{\left[\frac{2pn+p-2}{2(p-1)}\right], n+k-1\right\}$. Then the May filtration satisfies:

Generators	$t_k,$	$t_{k+1},$	$\cdots,$	$t_{s_0},$	$t_{s_0+1},$	$\cdots$
Filtration	$2k-1,$	$2k+1,$	$\cdots,$	$2s_0-1,$	$pM(t_{s_0-n+1})+1,$	

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$\mathbf{E^{*,M}S(n,k)}$; Let $F^{*,M}S(n,k)$ be the sub-module of $S(n,k)$ with May filtration $\leqslant M$. Set

$$E^{*,M}S(n,k) = F^{*,M}S(n,k)/F^{*,M-1}S(n,k)$$

One can see that

$$E^{*,*}S(n,k) \cong \otimes_{k\leqslant s} T[t_s^{p^j} | j \in Z/n] \qquad (1)$$

is a bigraded Hopf algebra, where $T[\quad]$ denote the truncated polynomial algebra of height p on the indicated generators. The structure map

$$\Delta : E^{*,*}(n,k) \longrightarrow E^{*,*}(n,k) \otimes E^{*,*}(n,k)$$

acts the the generators $t_s^{p^j}$ as

$$\Delta(t_s^{p^j}) = 1 \otimes t_s^{p^j} + t_s^{p^j} \otimes 1. \qquad (2)$$

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4 May spectral sequence

We use $H^{*,*}S(n,k)$ to denote $Ext^{*,*}_{S(n,k)}(Z/p,Z/p)$ for short.

Cobar complex; Let $C^{s,t}S(n,k) = \otimes^s \bar{S}(n,k)$ denote the cobar construction of $S(n,k)$. The differential $d : C^{s,t}S(n,k) \longrightarrow C^{s+1,t}S(n,k)$ is given on the generators as

$$\begin{aligned} & d(\alpha_1 \otimes \cdots \otimes \alpha_s) \\ = & \sum_{1 \leqslant i \leqslant s} (-1)^i \alpha_1 \otimes \cdots \otimes (\Delta(\alpha_i) - \alpha_i \otimes 1 - 1 \otimes \alpha_i) \otimes \cdots \otimes \alpha_s . \end{aligned}$$

Then $H^{s,t}(C^{*,t}S(n,k),d) = H^{*,*}S(n,k)$.

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May spectral sequence; Let $F^{*,*,M}S(n,k)$ denote the sub-complex of $C^{*,*}S(n,k)$ with May filtration $\leqslant M$. Then we get a short exact sequence for each M

$$0 \longrightarrow F^{*,*,M-1}S(n,k) \longrightarrow F^{*,*,M}S(n,k) \longrightarrow E_0^{*,*,M}S(n,k) \longrightarrow 0$$

of cochain complexes.

- The short exact sequences give rise to a spectral sequence (so called the May spectral sequence)

$$\{E_r^{s,t,M}S(n,k), d_r\}$$

  that converges to $H^{*,*}S(n,k)$.
- The cochain complex

$$E_0^{*,t,M}S(n,k) = F^{*,t,M}S(n,k)/F^{*,t,M-1}S(n,k)$$

  is isomorphic to the cobar complex of $E^{*,*}S(n,k)$ as in (1)

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64 May (revised); *There is the spectral sequence*

$$\{E_r^{s,t,M}S(n,k), d_r\} \Longrightarrow H^{s,*}S(n,k)$$

the E_1*-term* $E_1^{s,t,M}S(n,k)$ *is isomorphic to*

$$E[h_{i,j}|k \leqslant i, j \in Z/n] \otimes P[b_{i,j}|k \leqslant i, j \in Z/n].$$

Here $h_{i,j}$ *corresponds to* $t_i^{p^j}$ *and* $b_{i,j}$ *corresponds to* $\sum \binom{p}{m}/pt_i^{mp^j} \otimes t_i^{(p-m)p^j}$. *One has*

$$d_r : E_r^{s,t,M}S(n,k) \longrightarrow E_r^{s+1,t,M-r}S(n,k)$$

and if $x \in E_r^{s,*,*}S(n,k)$ *then*

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y).$$

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Reduced E_1**-term;** Because of the May filtration, one has

Theorem 1; *The* E_2-*term of the May spectral sequence is isomorphic to the cohomology of*

$$\widetilde{E}_1^{*,*,*}S(n,k) = E[h_{i,j}|k \leqslant i \leqslant s_0, j \in Z/n] \otimes P[b_{i,j}|k \leqslant s \leqslant s_0 - n, j \in Z/n].$$

where $s_0 = max\left\{\left[\frac{2pn+p-2}{2(p-1)}\right], n+k-1\right\}.$

Theorem 2; *If* $\frac{2pn+p-2}{2(p-1)} \leqslant n+k-1$ *i.e.* $s_0 = n+k-1$, *then the cohomology of* $S(n,k)$ *is of dimensional* n^2.

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5 Some computations

77 D. Ravenel;

- $H^{*,*}S(1)$ at all primes.
- $H^{*,*}S(2)$ at all primes.
- $H^{*,*}S(n,n)$ at an odd prime.
- $H^{*,*}S(n,n+1)$ at the primes 2

where $S(n) = S(n,1)$.

08 Shimomura and Tokashiki; $H^{*,*}S(n,n-1)$ at primes $p > 3$.

08 Ichigi and ∼; $H^{*,*}S(n,n)$ at the prime 2.

08 Xiaoying Zhou and ∼;

- $H^{*,*}S(3,2)$ at the prime 3.
- $H^{s,*}S(4,2)$ at the primes $p > 3$ up to $s \leqslant 3$.
  In this case, $s_0 = 5$ and the reduced May E_1-term is
  $$\widetilde{E}_1^{*,*,*}S(4,2) = E[h_{2,j}, h_{3,j}, h_{4,j}, h_{5,j} | j \in Z/4].$$
  $H^{*,*}S(4,2)$ is of dimensional 16 and has Poincare series
  $$1 + 10x + 48x^2 + 171x^3 + 428x^4(?) + \cdots$$

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Some problems;

- **$\mathbf{H^{*,*}S(3,2)}$ at the prime 2.**
  In this case, $s_0 = 6$ and the reduced May E_1-term is

  $$\widetilde{E}_1^{*,*,*}S(n,k) = E[h_{4,j},\ h_{5,j},\ h_{6,j} | j \in Z/3] \otimes P[h_{2,j},\ h_{3,j} | j \in Z/3].$$

- **$\mathbf{H^{*,*}S(3)}$ at the prime 3**
  In this case, $s_0 = 4$ and the reduced May E_1-term is

  $$\widetilde{E}_1^{*,*,*}S(3) = E[h_{1,j}, h_{2,j}, h_{3,j}, h_{4,j} | j \in Z/3] \otimes P[b_{1,j} | j \in Z/3].$$

  But because that $2 \nmid 3$, $H^{*,*}S(3)$ is of dimensional $3^2 = 9$.
- **$\mathbf{H^{*,}S(4,2)}$ at the primes $p > 3$**
- **$\mathbf{H^{3,}S(n)}$ for large n and primes p**

THANK YOU VERY MUCH!!!