Generalized moment-angle complexes, the polyhedral product functor, their properties and applications

joint work with Tony Bahri, Martin Bendersky, and Sam Gitler

plus engineering applications with Dan Koditschek, and Clark Lynch

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This report addresses joint work with Tony Bahri, Martin Bendersky, and Sam Gitler in addition to a an engineering question in joint work with Dan Koditschek, and Clark Lynch. Much of this report gives a picture of 'how and where' some mathematical structures fit together and covers the following:

(1) Generalized moment-angle complexes (polyhedral product functors), definitions and basic properties,

(2) connections to questions in engineering robotics, generalized moment-angle complexes as "spaces of robot legs".

Ingredients

(1) Let $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)_{i=1}^m\}$ denote a set of triples of CW–complexes with base-point x_i in A_i.

(2) Let K denote an abstract simplicial complex with m vertices labeled by the set $[m] = \{1, 2, \ldots, m\}$. Thus, a $(k-1)$-simplex σ of K is given by an ordered sequence

$$\sigma = (i_1, \cdots, i_k)$$

with $1 \leq i_1 < \cdots < i_k \leq m$ such that if $\tau \subset \sigma$, then τ is required to be a simplex of K. In particular the empty set ϕ is a subset of σ and so it is in K. Define define the length of I by the formula $|I| = k$.

Definition 0.1. As above, let (i) $(\underline{X}, \underline{A})$ denote the collection $\{(X_i, A_i, x_i)\}_{i=1}^m$ and (ii) K denote a simplicial complex.

The *generalized moment-angle complex or polyhedral product functor* determined by $(\underline{X}, \underline{A})$ and K denoted

$$Z(K; (\underline{X}, \underline{A}))$$

is defined as follows: For every σ in K, let

$$D(\sigma) = \prod_{i=1}^{m} Y_i, \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \in [m] - \sigma. \end{cases}$$

with $D(\emptyset) = A_1 \times \cdots \times A_m$.

The generalized moment-angle complex is

$$Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} D(\sigma) = \text{colim } D(\sigma).$$

In the special case where $X_i = X$ and $A_i = A$ for all $1 \leq i \leq m$, it is convenient to denote the generalized moment-angle complex by $Z(K;(X,A))$ to coincide with the notation in work of Graham Denham, and Alex Suciu who inspired much of the work here.

Examples

(1) Let K denote the 2-point complex $\{1,2\}$ with $(X,A)=(D^2,S^1)$. Then

$$Z(K;(D^2,S^1))=(D^2\times S^1)\cup(S^1\times D^2)=S^3.$$

(2) More generally, $Z(K;(D^2,S^1))$ has the homotopy type of the complement of unions of certain coordinate planes in $\mathbb{C}^m$ corresponding to 'coordinate subspace arrangements' as described next.

Given a simplicial complex K with m vertices, and a simplex $\omega \in \Delta[m-1]$, define

$$L_\omega = \{(z_1, \cdots, z_m) \in \mathbb{C}^m |\, z_{i_1} = \cdots = z_{i_k} = 0\}$$

for

$$(i_1, \cdots, i_k) \in \omega.$$

Definition 0.2. Define

$$U(K) = \cup_{\omega \notin K} \mathbb{C}^m - L_\omega$$

The following is an elegant result is due to Taras Panov.

Proposition 0.3. *The natural inclusion*

$$Z(K; (D^2, S^1)) \to U(K)$$

is

(a) *a homotopy equivalence, and*

(b) $(S^1)^m$*-equivariant.*

In addition, $(S^1)^m = T^m$ acts naturally on the product $(D^2)^m$ and on $Z(K;(D^2,S^1))$. The Davis-Januszkiewicz space is the associated Borel construction

$$\mathcal{DJ}(K) = ET^m \times_{T^m} Z(K;(D^2,S^1))$$

which, as a special case of a beautiful theorem of Denham-Suciu, is homeomorphic to

$$Z(K;(\mathbb{CP}^\infty,*)).$$

(3) Configuration spaces of certain singular spaces are sometimes homotopy equivalent to generalized moment-angle complexes as discovered by Sun Qiang in his thesis. Let

$$X = \mathbb{R}^n \vee \mathbb{R}^m.$$

Then the configuration space $\mathrm{Conf}(X, k)$ is homotopy equivalent to a certain choice of $Z(K; (X, A))$. These results at the nascent stages represent ways to measure 'choke points' or 'bottlenecks' and are currently being developed with an eye toward the next example.

(4) The special case of $Z(K;(X,A))$ for

$$(X,A) = (S^1, E_-)$$

where E_- is the closed lower hemisphere in S^1 and K is the q-skeleton of the $(m-1)$-simplex homeomorphic to the space of

'robot legs with at q off the ground'

(joint work with D. Koditschek, and Clark Lynch). Rather than defining these spaces formally, an illustrative slide will be given together with the remark that additional mathematical structure has now been used to build a working model. The next slides are that of Rhex pronking.

Preparation for some Theorems

Definition 0.4. Let K denote a simplicial complex with m vertices. Given a sequence

$$I = (i_1, \cdots, i_k)$$

with $1 \leq i_1 < \cdots < i_k \leq m$, define $K_I \subseteq K$ to be the *full sub-complex* of K consisting of all simplices of K which have all of their vertices in I, that is $K_I = \{\sigma \cap I | \sigma \in K\}$.

Definition 0.5. Given $(\underline{X}, \underline{A}, \underline{*})$, and a simplicial complex K with m vertices, define the *generalized smash moment-angle complex*

$$\widehat{Z}(K; (\underline{X}, \underline{A}))$$

to be the image of

$Z(K; (\underline{X}, \underline{A}))$ in the smash product

$$X_1 \wedge X_2 \wedge \cdots \wedge X_m.$$

Language required to state the main theorems is given next.

Definition 0.6. There is a partially ordered set (poset) $\bar{K}$ associated to any simplicial complex K as follows. A point σ in $\bar{K}$ corresponds to a simplex $\sigma \in K$ with order given by *reverse* inclusion of simplices. Thus $\sigma_1 \leq \sigma_2$ in $\bar{K}$ if and only if $\sigma_2 \subseteq \sigma_1$ in K. The empty simplex ϕ is the unique maximal element of $\bar{K}$. Let P be a poset with $p \in P$. There are further posets given by

$$P_{\leq p} = \{q \in P | q \leq p\}$$

as well as

$$P_{<p} = \{q \in P | q < p\}.$$

Thus

$$\bar{K}_{<\sigma} = \{\tau \in \bar{K} | \tau < \sigma\} = \{\tau \in K | \tau \supset \sigma\}.$$

Given a poset P, there is an associated simplicial complex $\Delta(P)$ called the order complex of P which is defined as follows.

Definition 0.7. Given a poset P, the *order complex* $\Delta(P)$ is the simplicial complex with vertices given by the set of points of P and k-simplices given by the ordered $(k+1)$-tuples $(p_1, p_2, \ldots, p_{k+1})$ in P with $p_1 < p_2 < \ldots < p_{k+1}$.

Stable decompositions

Theorem 0.8. *Given*

$$(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^{m}$$

where (X_i, A_i, x_i) *are connected, pointed CW-pairs, there are natural, pointed homotopy equivalence*

$$\Sigma(Z(K; (\underline{X}, \underline{A}))) \to \Sigma(\bigvee_{I \leq [m]} \widehat{Z}(K_I; (\underline{X_I}, \underline{A_I}))).$$

Theorem 0.9. *Let K be an abstract simplicial complex with m vertices, and let*

$$(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^{m}$$

denote m choices of connected, pointed pairs of CW-complexes with the inclusion $A_i \subset X_i$ null-homotopic for all i. Then there is a homotopy equivalence

$$\Sigma(Z(K; (\underline{X}, \underline{A}))) \to \Sigma \bigvee_{I} \bigvee_{\sigma \in K_I} |\Delta((\overline{K}_I)_{<\sigma}| * \widehat{D}(\sigma).$$

Corollary 0.10. *Let (X_i, A_i, x_i) denote the triple $(D^{n+1}, S^n, *)$ for all i. Then there are homotopy equivalences*

$$\Sigma(Z(K; (D^{n+1}, S^n))) \to \bigvee_{I \notin K} \Sigma^{2+n|I|}|K_I|.$$

When specialized to $(X, A, *) = (D^2, S^1, *)$, this result implies earlier results of Hochster, Goresky/MacPherson, Davis/Januskiewicz, Buchstaber/Panov and others concerning singular cohomology of these complexes. This decomposition also implies analogous results for any cohomology theory.

Cohomological consequences

Consider the cohomology ring $H^*(X;R)$ which satisfies the natural strong form of the Künneth theorem for the cohomology of X. Thus the natural map

$$H^*(X;R)^{\otimes m} \to H^*(X^m;R)$$

is an isomorphism.

Definition 0.11. Define the

generalized Stanley-Reisner ideal

$$I(K) \subset H^*(X; R)^{\otimes m}$$

as the two-sided ideal generated by all elements

$$x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_r}$$

for which $x_{j_t} \in \bar{H}^*(X_{j_t}; R)$ and the sequence $J = (j_1, \ldots, j_r)$ is not a simplex of K.

Theorem 0.12. *Let K be an abstract simplicial complex with m vertices and let*

$$(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^{m}$$

be m pointed, connected CW-pairs. If all of the A_i are contractible and coefficients are taken in a ring R for which either

(1) *R is a field, or*
(2) *the cohomology of X with coefficients in R satisfies the strong form of the Künneth Theorem,*

then there is an isomorphism of algebras

$$(\bigotimes_{i=1}^{m} H^*(X_i; R)/I(K) \to H^*(Z(K; (\underline{X}, \underline{A})); R).$$

Furthermore, there are isomorphisms of underlying abelian groups given by

$$E^*(Z(K;(\underline{X},\underline{A})) \to \bigoplus_{\sigma\in K} E^*(\widehat{D}(\sigma))$$

for any reduced cohomology theory E^*.

These results extend earlier results for the special case of $(X, A) = (D^2, S^1)$ in work of Hochster, Goersky-MacPherson (via stratified Morse theory), Davis-Januszkiewicz, Buchstaber-Panov, and Denham-Suciu in the special case of

$$(X, A) = (D^2, S^1)$$

for singular cohomology.

These methods also give the cohomology ring of the Davis-Januszkiewicz space

$$\mathcal{DJ}(K)$$

given by the Stanley-Reisner ring of K and are closely tied to work of Santiago Lopez de Medrano (on simultaneous quadratic equations), Charney-Davis, Jewell and others.

For example, the Davis-Januszkiewicz space $\mathcal{DJ}(K)$ admits a stable decomposition in which the summands are forcing the structure of the integral cohomology ring.

Other applications of

$$Z(K;(X,A))$$

arise with

$$(X,A) = (S^1, E_-)$$

where E_- denotes the lower hemisphere. The integral cohomology of $Z(K;(S^1,E_-))$ is used.

These spaces $Z(K;(S^1,E_-))$ were analyzed in elegant work of Charney-Davis.

Robotic motion

This section, ongoing joint initial work with Dan Koditschek and Clark Lynch (electrical engineers), is an extension of the above, but in a mildly different direction. Let

$$Legs(m, q)$$

denote the space of ordered m-tuples in a circle S^1 with at most q "off of the ground". That means at most q of the coordinates are in the open upper hemisphere U^+ of the circle, the complement of the closed lower hemisphere E_-.

The feature of 6 legs with at most 2 "off of the ground" is illustrated in the next slide.

The starting point is

Proposition 0.13. *If* $q = 2$,

$$Legs(m, 2) = Z(K; (S^1, E_-))$$

where K *is the complete graph with* m *vertices. If* $q \geq 2$ *and* $K = \Delta[m-1]_q$, *the* q*-skeleton of the* $(m-1)$*-simplex, then*

$$Legs(m, q) = Z(K; (S^1, E_-)).$$

One naive goal is to try to construct flows on the space of legs $Legs(m, q)$. The motivation is that flows would give instructions to a robot as to how to alter its' gait.

However, useful flows do not exist on the entire polyhedral product functor $Z(K; (S^1, E_-))$ **by inspecting the cohomology of this polyhedral product functor.**

In fact, the cup-product structure informs on the minimal number of required regions which support useful flows. The arguments are the analogue of classical arguments due to Lusternik, and Schnirelmann.

One partial solution is to give a dictionary for 'gait states'. A dictionary for 'gait states' for robotic motion is then given by certain choices of Young diagrams. Flows on subspaces are constructed by elementary means.

Basic Property

There is a $1-1$ correspondence between 'gait states' and Young diagrams.

Some pictures are given next.

Final remarks

The constructions and results here are analogous to properties involving configuration spaces, and their connections to homotopy groups as arising in joint work with J. Berrick, Y. L.Wong, and J. Wu, the subject of my last lecture.

Thank you very much !