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Toric Topology-homotopy theoretical aspects

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COMBINATORICS

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ALGEBRA

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-(stable and unstable) decompositions of $\mathcal{Z}_K$

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Combinatorics

SIMPLICIAL COMPLEXES

$V = \{v_1, \dots, v_m\} = [m]$ set of vertices

$K := \{\sigma_1, \dots, \sigma_s \mid \sigma_i \subset V\}$ $(\varnothing \in K)$ – abstract simplicial complex closed under formation of subsets

$\sigma \in K$ – simplex $\quad \dim \sigma = |\sigma| - 1 \quad \dim(K) = \max_{\sigma \in K}\{\dim \sigma\}$

Algebra

STANLEY-REISNER FACE RING

$\mathbf{k}$ – commutative ring with unit (interested in $\mathbb{Z}$, $\mathbb{Q}$, or finite field);

$\deg(v_i) = 2$ – topological grading

$\mathbf{k}[V] = \mathbf{k}[v_1, \dots, v_m]$ graded polynomial algebra on V over $\mathbf{k}$

For $\sigma = \{i_1, \dots, i_r\} \subset [m]$, set

$$v^\sigma = v_{i_1} \dots v_{i_r} \quad \text{–square free monomial}$$

The Stanley-Reisner algebra (or face ring) of K is

$$\mathbf{k}[K] := \mathbf{k}[v_1, \dots, v_m]/I_K$$

where $I_K = (v^\sigma \mid \sigma \notin K)$ -Stanley-Reisner ideal of K.

Definition. Let K_1, K_2 be simplicial complexes on $[m_1]$ and $[m_2]$, and $\phi\colon K_1 \longrightarrow K_2$ a simplicial map.

Define $\phi^*\colon \mathbf{k}[v_1, \dots, v_{m_2}] \longrightarrow \mathbf{k}[w_1, \dots, w_{m_1}]$ by

$$\phi^*(v_j) := \sum_{w_i \in \phi^{-1}(v_j)} w_i.$$

Then ϕ^* induces a homomorphism $\phi^*\colon \mathbf{k}[K_2] \longrightarrow \mathbf{k}[K_1]$.

RESOLUTIONS AND TOR-ALGEBRA

Let M be a finitely generated $\mathbf{k}[v_1,\ldots,v_m]$-module. The free resolution of M over $\mathbf{k}[v_1,\ldots,v_m]$ is an exact sequence

$$\ldots \longrightarrow R^{-i} \longrightarrow \ldots \xrightarrow{d} R^{-1} \xrightarrow{d} R^0 \longrightarrow M \longrightarrow 0$$

where R^{-i} are finitely generated free $\mathbf{k}[v_1,\ldots,v_m]$-modules.

Consider it as a bigraded differential $\mathbf{k}[v_1,\ldots,v_m]$-module $[R,d]$, where $R = \bigoplus R^{-i,j}, \quad R^{-i,j} = (R^{-i})^j \quad d\colon R^{-i,j} \longrightarrow R^{-i+1,j}$.

If $[M,0]$ is the bigraded module $M^{-i,k} = 0$ for $i > 0$, $M^{0,k} = M^k$ with the trivial differential, then $[R,d] \longrightarrow [M,0]$ is a bigraded quasi-isomorphism.

Let N be another module, applying the functor $\otimes_{\mathbf{k}[v_1,\ldots,v_m]} N$ to $[R,d]$, we get a homomorphism of differential modules

$$[R \otimes_{\mathbf{k}[v_1,\ldots,v_m]} N, d] \longrightarrow [M \otimes_{\mathbf{k}[v_1,\ldots,v_m]} N, d].$$

$$\begin{aligned} \mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\ldots,v_m]}(M,N) &= H^{-i}([R \otimes_{\mathbf{k}[v_1,\ldots,v_m]} N, d]) \\ &= \frac{\mathrm{Ker}[d\colon R^i \otimes_{\mathbf{k}[v_1,\ldots,v_m]} N \longrightarrow R^{i+1} \otimes_{\mathbf{k}[v_1,\ldots,v_m]} N]}{d(R^{i-1} \otimes_{\mathbf{k}[v_1,\ldots,v_m]} N)} \;. \end{aligned}$$

Since all R^{-i} and N are graded modules

$$\mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(M,N) = \bigoplus_{i,j} \mathrm{Tor}^{-i,j}_{\mathbf{k}[v_1,\ldots,v_m]}(M,N)$$

is a bigraded $\mathbf{k}$-module.

Letting $M = \mathbf{k}[K]$ and $N = \mathbf{k}$, and recalling $|v_i| = 2$,

$$\mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k}) = \bigoplus_{i,j=0}^{m} \mathrm{Tor}^{-i,2j}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k}).$$

Definition. The bigraded Betti numbers of $\mathbf{k}[K]$ are

$$\beta^{-i,2j}(\mathbf{k}[K]) = \dim_{\mathbf{k}} \mathrm{Tor}^{-i,2j}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k}) \quad 0 \le i,j \le m$$

$$\beta^{-i}(\mathbf{k}[K]) = \dim_{\mathbf{k}} \mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k}) = \sum_j \beta^{-i,2j}(\mathbf{k}[K]).$$

Example. Let K be the boundary of a square. Then $\mathbf{k}[K] = \frac{\mathbf{k}[v_1,\ldots,v_m]}{(v_1v_3,v_2v_4)}$.

R^0 has 1 generator of degree 0

$R^0 \xrightarrow{d} \mathbf{k}[K]$ is the quotient projection and $\mathrm{Ker}\, d = (v_1v_3, v_2v_4)$

R^{-1} is the free module on 2 generators of degree 4, denoted by $v_{13},\ v_{24}$.

$R^{-1} \xrightarrow{d} R^0$ is given by $v_{13} \mapsto v_1v_3,\ v_{24} \mapsto v_2v_4$, and therefore $\mathrm{Ker}\, d = (v_2v_4v_{13} - v_1v_3v_{24})$

R^{-2} is the free module on 1 generator of degree 8, denoted by a

$R^{-2} \xrightarrow{d} R^{-1}$ is given by $a \mapsto v_2v_4v_{13} - v_1v_3v_{24}$ and thus it is injective.

We obtain a resolution of $\mathbf{k}[K]$

$$0 \longrightarrow R^{-2} \longrightarrow R^{-1} \longrightarrow R^0 \longrightarrow \mathbf{k}[K]$$

and $\beta^{0,0}(\mathbf{k}[K]) = 1,\ \beta^{-1,4}(\mathbf{k}[K]) = 2,\ \beta^{-2,8}(\mathbf{k}[K]) = 1.$

HOCHSTER THEOREM

For $\omega \subset [m]$, the full subcomplex $K_\omega = \{\sigma \in K \mid \sigma \subset \omega\}$ of K.

Theorem. *(Hochster)*

$$\beta^{-i,2j}(\mathbf{k}[K]) = \sum_{\omega \subset [m] \ |\omega|=j} \dim_{\mathbf{k}} \widetilde{H}^{j-i-1}(K_\omega; \mathbf{k})$$

assuming $\widetilde{H}^{-1}(\emptyset) = \mathbf{k}$.

KOSZUL RESOLUTION

is the free differential bigraded algebra

$$R = \Lambda(u_1, \dots, u_m) \otimes_{\mathbf{k}} \mathbf{k}[v_1, \dots, v_m]$$

where $\operatorname{bideg} u_i = (-1, 2),\ \operatorname{bideg} v_j = (0, 2),\ du_i = v_i,\ dv_i = 0$ considered as a free resolution of $\mathbf{k}$ over $\mathbf{k}[v_1, \dots, v_m]$.

Proposition. *There is an isomorphism of bigraded modules*

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}[K], \mathbf{k}) \cong H(\Lambda(v_1, \dots, v_m] \otimes \mathbf{k}[K], d)$$

which endows $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}[K], \mathbf{k})$ *with a bigraded algebra structure in a canonical way.*

Definition. The bigraded algebra $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}[K], \mathbf{k})$ is called the Tor-algebra of a simplicial complex K.

Lemma. *A simplicial map* $\phi\colon K_1 \longrightarrow K_2$ *induces a homomorphism*

$$\phi^*\colon \operatorname{Tor}_{\mathbf{k}[w_1,\dots,w_n]}(\mathbf{k}[K_2], \mathbf{k}) \longrightarrow \operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}[K_1], \mathbf{k})$$

of Tor-algebras.

TOPOLOGY

MOMENT–ANGLE COMPLEX $\mathcal{Z}_K$

Torus $T^m \subset (D^2)^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_i| \le 1, \forall i\}$

For $\sigma \subset [m]$, define

$B_\sigma := \{(z_1, \dots, z_m) \in (D^2)^m \mid |z_i| = 1 \quad i \notin \sigma\}$.

Notice

$$B_\sigma \cong (D^2)^{|\sigma|} \times T^{m-|\sigma|}$$

Given a simplicial complex K on $[m]$, define the moment–angle complex $\mathcal{Z}_K$ by

$$\mathcal{Z}_K := \bigcup_{\sigma \in K} B_\sigma \subset (D^2)^m.$$

T^m acts on $(D^2)^m$ coordinatewise, thus B_σ is invariant under the action of T^m $\rightsquigarrow$ T^m acts on $\mathcal{Z}_K$

$(D^2)^m/T^m \cong I^m = \{(y_1, \dots, y_m) \in \mathbb{R}^m \mid 0 \le y_i \le 1 \quad i = 1, \dots, m\}$

thus

$B_\sigma/T^m = C_\sigma := \{(y_1, \dots, y_m) \in I^m \mid y_i = 1 \quad i \notin \sigma\}$ $|\sigma|$-dim face of I^m

Lemma.

$\mathcal{Z}_K/T^m \cong Cone K'$ $\qquad$ *K'- barycentric subdivision of K.*

Proposition. *If K is a triangulation of an $(n-1)$-sphere, then $\mathcal{Z}_K$ is a closed $(m+n)$-manifold.*

Example. Let $K = \partial\Delta^{m-1}$, then $\mathcal{Z}_K = \partial((D^2)^m) \cong S^{2m-1}$

If $m = 2$, and $K = \{v_1, v_2\}$, then

$$\mathcal{Z}_K \cong S^3 \cong D^2 \times S^1 \cup S^1 \times D^2 \subset D^2 \times D^2.$$

Using the isotropy subgroups $T(x)$ of points in the orbit space $ConeK'$

$$\mathcal{Z}_K = T^m \times |ConeK'|/\sim \qquad \begin{array}{ccc} (t_1, x) \sim (t_2, y) & & \text{iff} \\ x = y & & t_1 t_2 \in T(x). \end{array}$$

(Thus $\mathcal{Z}_K$ coincides with the Davis-Januszkiewicz T^m-manifold when $K = P^*$, P-simple polytope.)

DAVIS–JANUSZKIEWICZ SPACE DJ_K

– homotopy quotient of $\mathcal{Z}_K$ by the T^m-action.

Davis–Januszkiewicz $\quad DJ_K = ET^m \times_{T^m} \mathcal{Z}_K = ET^m \times \mathcal{Z}_K/\sim$ where $(e, z) \sim (et^{-1}, tz)$.

Thus there is a fibration

$$\mathcal{Z}_K \longrightarrow DJ_K \longrightarrow BT^m.$$

Theorem. *The Davis-Januszkiewicz space is a topological realisation of the Stanley–Reisner ring* $\mathbf{k}[K]$*, that is,*

$$H^*(DJ_K; \mathbf{k}) = \mathbf{k}[K] \quad (\textit{for } \mathbf{k} = \mathbb{Z} \textit{ or } \mathbf{k} = \mathbb{Z}/2).$$

Buchstaber–Panov through a simple colimit of nice blocks

Assume $\mathbf{k} = \mathbb{Z}$. Denote $\mathbb{C}P^\infty = BS^1,$ thus $BT^n = (\mathbb{C}P^\infty)^m$.

For $\omega \subset [m]$, define

$$BT^\omega := \big\{(x_1, \ldots, x_m) \in BT^m \mid x_i = * \text{ if } i \notin \omega\big\}.$$

For K on $[m]$, the Davis-Januszkiewicz space of K is given by

$$DJ_K := \bigcup_{\sigma \in K} BT^\sigma \subset BT^m.$$

The inclusion of the cellular complexes $i\colon \bigcup_{\sigma\in K} BT^\sigma \longrightarrow BT^m$ induces the quotient epimorphism

$$i^*\colon \mathbb{Z}[v_1, \ldots, v_m] \longrightarrow \mathbb{Z}[K] = \mathbb{Z}[v_1, \ldots, v_m]/I_K$$

in cohomology.

COORDINATE SUBSPACE ARRANGEMENT

is a finite set $\mathcal{CA} = \{L_1, \dots, L_r\} \subset \mathbb{C}^m$ of coordinate subspaces, that is,

$$L_\omega = \big\{(z_1, \dots, z_n) \in \mathbb{C}^m : z_{i_1} = \dots = z_{i_k} = 0\big\}$$

where $\omega = \{i_1, \dots, i_k\} \subset [m]$ and its complement $\mathsf{U}(\mathcal{CA})$ is defined as

$$\mathsf{U}(\mathcal{CA}) := \mathbb{C}^m \backslash \bigcup_{i=1}^{r} L_i.$$

There are one-to-one correspondences

simp. complex		coord. subspace arrang.		complement
K	$\Leftrightarrow$	$\{L_\omega \mid \omega \notin K\}$	$\Leftrightarrow$	$\mathsf{U}(K) = \mathbb{C}^m \backslash \bigcup_{\omega \notin K} L_\omega$

If $L \subset K$ is a subcomplex, then $\mathsf{U}(L) \subset \mathsf{U}(K)$.

Proposition. *There is a T^m equivariant deformation retraction*

$$\mathsf{U}(K) \xrightarrow{\simeq} \mathcal{Z}_K$$

Proof. $\mathsf{U}(K) = \cup_{\sigma \in K} U_\sigma$, $U_\sigma = \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid z_i \neq 0 \text{ for } i \notin \sigma\}$

Thus $U_\sigma \cong \mathbb{C}^\sigma \times (\mathbb{C}\backslash\{0\})^{m-|\sigma|} \xrightarrow{\simeq} B_\sigma = (D^2)^\sigma \times T^{m-|\sigma}$ □

Example. i) $K = \partial\Delta^{m-1}$, $\mathcal{Z}_K = S^{2m-1}$, $\mathsf{U}(K) = \mathbb{C}\backslash\{0\}$

ii) $K = \{v_1, \dots, v_m\}$, $\mathsf{U}(K) = \mathbb{C}^m \setminus \bigcup_{1 \leq i < j \leq m} \{z_i = z_j = 0\}$
– complement of the codim 2 coordinate subspaces.

COHOMOLOGY OF $\mathcal{Z}_K$

Studying the Eilenberg-Moore ss of the fibration $\mathcal{Z}_K \longrightarrow DJ_K \longrightarrow BT^m$, we get

$$H^*(\mathcal{Z}_K;\mathbb{Q}) \cong \mathrm{Tor}_{\mathbb{Q}[v_1,\dots,v_m]}(\mathbb{Q}[K],\mathbb{Q}).$$

For integral calculation, we will exploit a cellular decomposition of $\mathcal{Z}_K$ and a special *cellular approximation* of the diagonal map

$$\Delta : \mathcal{Z}_K \longrightarrow \mathcal{Z}_K \times \mathcal{Z}_K.$$

Recall: Cellular chains do not admit a functional associative multiplication as a proper cellular diagonal approximation does not exist in general.

Consider the moment–angle complex as a functor

$$\mathcal{Z}: \left\{ \begin{array}{c} K - \text{simpl complexes} \\ \text{simplicial maps} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \mathcal{Z}_K - \text{toric spaces} \\ \text{equivariant maps} \end{array} \right\}$$

The cellular approximation of the diagonal is functorial with respect to maps of moment–angle complexes which are induced by simplicial maps.

CELLULAR DECOMPOSITION OF $\mathcal{Z}_K$

D^2 consists of one 0-dim cell 1, one 1-dim cell T, and one 2-dim cell D.

There is the following correspondence

$$\text{cell in } (D^2)^m \quad \Longleftrightarrow \quad \tau \in \{D, T, 1\}^m$$

Notation: For $\sigma, \omega \subset [m]$ such that $\sigma \cap \omega = \emptyset$, the word $\tau(\sigma, \omega)$ has the letter D on the positions indexed by σ, T on the positions indexed by ω.

Lemma. *$\mathcal{Z}_K$ is a cellular subcomplex of $(D^2)^m$.*

$\tau(\sigma, \omega) \subset (D^2)^m$ belongs to $\mathcal{Z}_K$ iff $\sigma \in K$. □

The cellular cochain complex $C^*(\mathcal{Z}_K)$, which has an additive basis consisting of the cochains $\tau(\sigma, \omega)^*$, has natural bigrading defined by

$$\operatorname{bideg} \tau(\sigma, \omega)^* = (-|\omega|, 2|\sigma| + 2|\omega|).$$

Note: $C^*(\mathcal{Z}_K) = \oplus_{j=1}^m C^{*,2j}(\mathcal{Z}_K)$

The cohomology of $\mathcal{Z}_K$ thereby acquires an additional grading.

Define the bigraded Betti numbers by

$$b^{-i,2j}(\mathcal{Z}_K) := \operatorname{rank} H^{-i,2j}(\mathcal{Z}_K), \quad b^k(\mathcal{Z}_K) = \sum_{2j-1=k} b^{-i,2j}(\mathcal{Z}_K)$$

Theorem. *There is an isomorphism, functorial in K, of bigraded algebras*

$$H^{*,*}(\mathcal{Z}_K; \mathbb{Z}) \cong \operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[K]; \mathbb{Z}) \cong H[\Lambda(u_1, \ldots, u_m) \otimes \mathbb{Z}[K], d].$$

Example. 1) Let $K = \partial\Delta^{m-1}$.

Then $\mathbb{Z}[K] = \mathbb{Z}[v_1, \ldots, v_m]/(v_1 \ldots v_m)$.

The fundamental class of $\mathcal{Z}_K = S^{2m-1}$ is represented by the bidegree $(-1, 2m)$ cocycle $u_1 v_2 \ldots v_m \in \Lambda(u_1, \ldots, u_m) \otimes \mathbb{Z}$.

2) Let $K = \{v_1, \ldots, v_m\}$.

Then $\mathbb{Z}[K] = \mathbb{Z}[v_1, \ldots, v_m]/(v_i v_j, \mid i \neq j)$.

The subspace of cocycles is generated by

$$v_{i_1} u_{i_2} \ldots u_{i_k} \text{ for } k \geq 2 \text{ and } i_p \neq i_q \text{ for } p \neq q$$

and has dimension $m\binom{m-1}{k-1}$.

The subspace of coboundaries is generated by the elements of the form $d(u_{i_1} \ldots u_{i_k})$ and is $\binom{m}{k}$-dimensional.

Thus

$\dim H^0(\mathcal{Z}_K) = 1$,

$\dim H^1(\mathcal{Z}_K) = H^2(\mathcal{Z}_K) = 0$,

$\dim H^{k+1}(\mathcal{Z}_K) = (k-1)\binom{m}{k}$ for $2 \leq k \leq m$.

A MULTIPLICATIVE VERSION OF HOCHSTER'S THEOREM

The bigraded structure in the cellular cochains of $\mathcal{Z}_K$ can be further refined to

$$C^*(\mathcal{Z}_K) = \bigoplus_{\omega\subset[m]} C^{*,2\omega}(\mathcal{Z}_K)$$

where $C^{*,2\omega}(\mathcal{Z}_K)$ is the subcomplex generated by the cochains $\tau(\sigma, \omega\setminus\sigma)^*$ with $\sigma\subset\omega$, $\sigma\in K$.

Thus $C^*(\mathcal{Z}_K)$ now becomes $\mathbb{Z}\oplus\mathbb{Z}^m$-graded module and the bi-graded cohomology groups decompose accordingly as

$$H^{-i,2j}(\mathcal{Z}_K) = \bigoplus_{\substack{\omega\subset[m]\\ |\omega|=j}} H^{-i,2\omega}(\mathcal{Z}_K)$$

Theorem. *(Baskakov) There are isomorphisms*

$$\widetilde{H}^p(K_\omega) \xrightarrow{\cong} H^{p+1-|\omega|,2\omega}(\mathcal{Z}_K)$$

which are functorial with respect to simplicial maps and induce a ring isomorphism

$$r\colon \bigoplus_{\substack{p\ge -1\\ \omega\subset[m]}} H^p(K_\omega) \xrightarrow{\cong} H^*(\mathcal{Z}_K).$$

Corollary. *There is an iso* $H^{-i,2j}(\mathcal{Z}_K)\cong\bigoplus_{\substack{\omega\subset[m]\\ |\omega|=j}}\widetilde{H}^{j-i-1}(K_\omega)$.

Corollary. $\mathrm{Tor}^{-i,*}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[K],\mathbb{Z})\cong\bigoplus_{\omega\subset[m]}\widetilde{H}^{|\omega|-i-1}(K_\omega)$