Spaces of homomorphisms, and spaces of representations

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A report on joint work with Alex Adem, Enrique Torres-Giese, and José Gomez.

The setting and the problems

Let π and G be groups. Define

$$Hom(\pi, G) = \{f : \pi \to G \mid f \text{is a homomorphism}\}.$$

Further, define

$$Rep(\pi, G) = Hom(\pi, G)/G^{\mathrm{ad}}$$

the quotient of the natural action of G on

$$Hom(\pi, G)$$

via conjugation.

The problems here concern features of the spaces $Hom(\pi, G)$ and $Rep(\pi, G)$. These problems as well as some of the results below address

- fundamental groups,

- cohomology,

- decompositions, some of which arise after suspending spaces which then informs on cohomology and other properties, and

- whether certain spaces below are $K(\pi, 1)$'s.

Motivation

The original motivation was to try to understand features of subgroups of the pure braid groups by considering topological properties of functions out of the pure braid groups.

Why braid groups ?

This question will be addressed in the talk on joint work with Jon Berrick, Yan Loi Wong, and Jie Wu.

One goal was to understand whether stable real vector bundles measured interesting properties of certain subgroups of braid groups.

The main feature in joint work with Alex Adem and Dan Cohen arose from bundles obtained from real, orthogonal representations of the pure braid group on n-strands P_n given by homomorphisms

$$f : P_n \to G$$

with $G = O$ the stable, real orthogonal group.

The main result was roughly that all such stable bundles were 'detected' by bundles over a product of circles obtained from an element in $Hom(\oplus_n \mathbb{Z}, G)$ for which $\oplus_n \mathbb{Z}$ is a subgroup of P_n. This feature led to a different direction described here.

The space of representations

$$Rep(\oplus_n \mathbb{Z}, G) = Hom(\oplus_n \mathbb{Z}, G)/G^{ad}$$

is the moduli space of isomorphism classes of flat G-bundles over the n-torus $(S^1)^n$. These appear in other interesting contexts such as work A. Borel, R. Friedman, J. Morgan as well as many others.

Examples, and definitions

Example 1: Let

$$\pi = F_n = F[x_1, \cdots, x_n]$$

denote a free group of rank n. Consider the map

$$e : Hom(F_n, G) \to G^n$$

which evaluates a homomorphism on the choice of 'basis' $x_1, \cdots, x_n$. This map is a bijection of sets. Thus topologize $Hom(F_n, G)$ with the topology of G^n.

In what follows, identify $Hom(F_n, G)$ as the topological space G^n.

Example 2: Let

$$\pi = \oplus_n \mathbb{Z}$$

a quotient of F_n given by $(F_n/[F_n, F_n]) = \oplus_n \mathbb{Z}$ where $[F_n, F_n]$ is the commutator subgroup of F_n. Consider the natural quotient map

$$F_n \to F_n/[F_n, F_n]$$

together with the induced map

$$Hom(\oplus_n \mathbb{Z}, G) \to Hom(F_n, G).$$

Since this last map is a monomorphism of sets, topologize $Hom(\oplus_n \mathbb{Z}, G)$ as a subspace of G^n.

Example 3: Let Γ^q denote the q-th stage of the descending central series of F_n, the subgroup generated by commutators of the form

$$[\cdots[v_1, v_2], v_3]\cdots], v_t]$$

where $v_i \in F_n$ with $t \geq q$, and

$$[v, w] = vwyv^{-1}w^{-1}.$$

Consider

$$Hom(F_n/\Gamma^q, G)$$

to obtain a non-decreasing family of spaces

$$Hom(F_n/\Gamma^2, G) \subset Hom(F_n/\Gamma^3, G) \subset \cdots \subset G^n.$$

Other filtrations

Example 4:

Let Γ_p^q denote the q-th stage of the mod-p descending central series of F_n, the subgroup generated by commutators of the form

$$[\cdots[v_1, v_2], v_3]\cdots], v_t]^{p^r}$$

where $v_i \in F_n$ with $t \cdot p^r \geq q$. Consider

$$Hom(F_n/\Gamma^q, G)$$

to give a non-decreasing family of spaces

$$Hom(F_n/\Gamma^2, G) \subset Hom(F_n/\Gamma^3, G) \subset \cdots \subset G^n.$$

Example 5:

Let $\widehat{F_n}$ denote the pro-finite completion of F_n filtered by the associated descending central series. There is an induced filtration with useful properties

$$Hom(\widehat{F_n}/\Gamma^2, G) \subset \cdots \subset Hom(\widehat{F_n}, G).$$

Simplicial spaces

Fix an integer $q \geq 2$. This section is a description of how the spaces

$$Hom(F_n/\Gamma^q, G)$$

assemble into a **simplicial space** for each fixed q and for all non-negative integers n.

There are $(n+1)$ natural maps

$$d_i : Hom(F_n, G) \to Hom(F_{n-1}, G)$$

as well as

$$s_j : Hom(F_n, G) \to Hom(F_{n+1}, G)$$

defined as follows.

Regard a homomorphism

$$f : F_n \longrightarrow G$$

as an ordered n-tuple of points $g_1, \cdots, g_n \in G$. Define functions (face operations for a simplicial space)

$$d_i : Hom(F_n, G) \longrightarrow Hom(F_{n-1}, G)$$

for which

$$d_i((g_1, \cdots, g_n))$$

is given by the formula

(1) $(g_2, \cdots, g_n)$ if $i = 0$,

(2) $(g_1, \cdots, g_{i-1}, g_i g_{i+1}, \cdots, g_n)$ if $0 < i < n$, and

(3) $(g_1, \cdots, g_{n-1})$ if $i = n$.

There are additional functions (degeneracies)

$$s_i : Hom(F_n, G) \to Hom(F_{n+1}, G)$$

defined by the formula

$$s_i(g_1, \cdots, g_n)$$

given by

$$(g_1, \cdots, g_{i-1}, g_i, e, g_{i+1}, \cdots, g_n),$$

if $0 \leq i \leq n$.

The functions d_i and s_j satisfy the simplicial identities, as well as restrict to analogous functions on the level of

$$Hom(F_n/\Gamma^q, G).$$

Geometric realization

Given any simplicial space, there are associated topological spaces given by the

geometric realization

described roughly by considering

(1) the disjoint union of n-simplices, one for each point in $Hom(F_n, G)$, and

(2) making identifications according to the face and degeneracies.

A formal definition is given next.

Definition 0.1. The geometric realization of a simplicial space Z_* is the following topological space

$$|Z_*| := \coprod_{n\geq 0} Z_n \times \Delta[n]/\sim$$

where $\Delta[n]$ denotes the n–simplex.

The equivalence relation $\sim$ is defined as follows.

Identify
$$(x, \delta_i t) \in X_n \times \Delta[n]$$
with
$$(d_i x, t) \in X_{n-1} \times \Delta[n-1]$$
for any $x \in X_n$, $t \in \Delta[n-1]$ and
$$(x, \sigma_j t) \in X_n \times \Delta[n]$$
with
$$(s_j x, t) \in X_{n+1} \times \Delta[n+1]$$
for any $x \in X_{n-1}$ and $t \in \Delta[n+1]$.

Give $|Z_*|$ is the quotient topology.

The geometric realization is also filtered by the images of

$$F_k|Z_*| := \text{image of} \coprod_{k \geq n \geq 0} Z_n \times \Delta[n].$$

Record this information in the following lemma.

Lemma 0.2. *The inclusions*

$$Hom(F_n/\Gamma^2, G) \subset Hom(F_n/\Gamma^3, G) \subset \cdots \subset G^n$$

induce morphisms of simplicial spaces. Furthermore, the following properties are satisfied.

(1) *The geometric realization obtained from the simplicial space $Hom(F_n, G)$ is precisely Milgram's construction of the classifying space BG.*
(2) *The geometric realization obtained from the simplicial space for any fixed integer $q \geq 1$, denoted $B(q, G)$, is a subspace of BG.*

(3) *The spaces* $B(q,G)$ *give a filtration of* BG

$$B(2,G) \subset B(2,G) \subset \cdots \subset BG.$$

(4) *If* $q \geq 2$, *there are natural morphisms of principle* G*-bundles*

$$\begin{array}{ccccc} G & \xrightarrow{1} & G & \xrightarrow{1} & G \\ \downarrow & & \downarrow & & \downarrow \\ E(2,G) & \xrightarrow{i} & E(q,G) & \xrightarrow{i} & EG \\ \downarrow p & & \downarrow p & & \downarrow p \\ B(2,G) & \xrightarrow{i} & B(q,G) & \xrightarrow{i} & BG \end{array}$$

(5) *Analogous properties are satisfied for the 'other filtrations' above.*

$E(q, G)$?

What is $E(q, G)$? The space $E(q, G)$ is defined by the pull-back:

$$\begin{array}{ccc} E(q,G) & \xrightarrow{i} & EG \\ \Big\downarrow p & & \Big\downarrow p \\ B(q,G) & \xrightarrow{i} & BG \end{array}$$

Alternatively, $E(q, G)$ can be defined as the geometric realization of a second natural simplicial space.

The first homework problem as an introduction to $B(2, G)$

Give informative properties of the fundamental group $\pi_1(B(2, G))$ and the induced map

$$\pi_1(E(2, G)) \to \pi_1(B(2, G)).$$

In the case where G is discrete, there is an associated regular G-cover

$$E(2, G) \to B(2, G)$$

with

$$E(2, G)/G = B(2, G).$$

Show that if G is finite of odd order, then the map

$$H_1(E(2,G)) \to H_1(B(2,G))$$

is not an epimorphism.

One unacceptable partial solution:

"The dog ate my homework."

A second unacceptable solution follows.

Proposition 0.3. *Let G be a discrete, finite group of odd order. The following two statements are equivalent.*

(1) *The map*

$$H_1(E(2,G)) \to H_1(B(2,G))$$

is not an epimorphism.

(2) *The group G is solvable (the odd order Theorem of Feit-Thompson).*

An unacceptable solution is to quote the odd order Theorem.

An acceptable solution is to see whether the topology of the covering space informs on this, as yet open, problem. **Caution: It is far from clear whether this approach is informative.**

Observe that the regular covering space

$$E(2, G) \to B(2, G)$$

with

$$B(2, G) = E(2, G)/G$$

gives an induced homomorphism

$$\rho : G \longrightarrow Out(\pi_1(E(2, G)).$$

What features of this homomorphism inform on qualitative features of $H_1(B(2, G))$?

Questions which we have been considering are as follows.

(1) Identify natural properties of $Hom(F_n/\Gamma^q, G)$ as well as $B(q, G)$ such as cohomology and stable structure developed below.
(2) Identify the fundamental group $\pi_1(E(2, G))$ in case G is finite.
(3) Give conditions which force $\pi_1(E(2, G))$ to be a free group.
(4) The group G acts on $\pi_1(E(2, G))$. In case $\pi_1(E(2, G))$ is free, identify this action as a homomorphism with values in the automorphism group of a free group.
(5) Give conditions which force $E(2, G)$ to be a $K(\pi, 1)$.

Second homework problem

Other filtrations of F_n arise by filtering via principal congruence subgroups of level p^r in $PSL(2,\mathbb{Z})$.

Does anything sensible "work" with simplicial spaces for these filtrations ?

The thesis of Jon Lopez gives the structure of associated graded Lie algebras for these as well as for $SL(n, \mathbb{Z}[t_1^{\pm 1}, \cdots, t_k^{\pm 1}])$, and $SL(n, \mathbb{Z}[G])$ for discrete groups G.

Since these filtrations converge like "a bat out of hell", it is natural to wonder whether there are associated free simplicial groups.

Further properties of $B(q, G)$, and $Hom(F_n/\Gamma^q, G)$

Some theorems and definitions follow.

Definition 0.4. Define a subspace of

$$Hom(F_n/\Gamma^q, G)$$

given by

$$S_n(q, G) = \cup_{0 \leq i \leq n} s_i(Hom(F_{n-1}/\Gamma^q, G)).$$

The following result describes the stable structure of the spaces of homomorphisms in terms of more recognizable pieces (sometimes !).

Theorem 0.5. *If G is a closed subgroup of $GL(m,\mathbb{R})$, then the spaces*

$$\Sigma Hom(F_n/\Gamma^q, G),$$

and

$$\bigvee_{1\leq k\leq n} \Sigma \bigvee^{\binom{n}{k}} Hom(F_k/\Gamma^q, G)/S_k(q,G)$$

are naturally homotopy equivalent.

Theorem 0.6. *If G is a closed subgroup of $GL(m,\mathbb{R})$, then the natural filtration quotients*

$$E^0_k(B(q,G)) = F_kB(q,G)/F_{k-1}B(q,G)$$

of the geometric realization $B(q,G)$ are stably homotopy equivalent to

$$\Sigma^k(Hom(F_k/\Gamma^q,G)/S_k(q,G)).$$

Thus the following spaces are naturally stably homotopy equivalent.

(1)

$$Hom(F_n/\Gamma^q,G)$$

(2)

$$\bigvee_{1\leq k\leq n} \bigvee^{\binom{n}{k}} \Sigma^{-k}(E^0_k(B(q,G)))$$

Remark: These results are consequences of more general results concerning simplicial spaces in joint work with Alex, Tony Bahri, Martin Bendersky, and Sam Gitler. Furthermore, these results are inspired by earlier work of Dan Kan, Norman Steenrod, Jack Milnor, Jim Milgram, Graeme Segal, and Peter May.

However, to make the precise identifications here, it is necessary to use the fact that G is a Lie group. Proofs involve the geometry of Lie groups with the natural geodesic flow on a neighborhood of the identity to verify requisite topological conditions (e.g. Is a map a cofibration ?).

Similar splitting theorems apply to spaces of representations, yet another simplicial space. In many cases, the summands can be identified in terms of Thom spaces associated to natural fibre bundles.

This direction is still in a process of development in joint work with Alex, Enriques, and José as well as further work of Tom Baird, Michael Crabb, Negumi Harada, and Paul Selick.

Counting the cardinality of $Hom(F_n/\Gamma^q, G)$

These spaces are also "counting" naive features of a finite group G as illustrated next.

Definition 0.7. Let G denote a finite group. The integer $\lambda_n(q, G)$ is defined as the cardinality of $Hom(F_n/\Gamma^q, G)$, and the integer $\mu_k(q, G)$ is defined as the rank of $H_k(E_k^0(B(q, G)); \mathbb{Z})$.

An immediate consequence of earlier features is that the cardinality of the set of homomorphisms $Hom(F_n/\Gamma^q, G)$ is given in terms of homology.

Corollary 0.8. *If* G *is a finite group, then*

$$\lambda_n(q, G) = 1 + \sum_{1 \leq k \leq n} \binom{n}{k} \mu_k(q, G).$$

The special case $q = 2$ gives information on the cardinality of the set of commuting elements in a finite group addressed in work of Hopkins, Kuhn, and Ravenel. Some calculations for abelian and transitively commutative groups are listed at the end (and are not part of the oral lecture notes).

Sundry basic properties of $B(q, G)$

The functor $B(q, G)$ satisfies further properties:

(1) If $q \geq 2$ and H is a topological group of nilpotency class less than q, then
$$B(q, H) = BH,$$
the usual classifying space.

(2) If G is a finite group then there exists an N that depends on G such that
$$B(q, G) = B(N, G)$$
for all $q \geq N$.

(3) If $q \geq 2$ and $\iota\colon H \to G$ is a homomorphism where H is a group of nilpotency class less than q, then there is a commutative diagram

$$\begin{array}{ccc} BH & \xrightarrow{1} & BH \\ \big\downarrow & & \big\downarrow B(\iota) \\ B(q,G) & \longrightarrow & BG. \end{array}$$

Thus if BG is a stable retract of BH, then BG is a stable retract of $B(q,G)$.

(4) If G is a connected Lie group G, there is a homotopy equivalence for all $q \geq 2$:

$$G \times \Omega(E(q, G)) \to \Omega B(q, G).$$

(5) If G is a finite group, then for any $q \geq 2$, the map on mod-p cohomology

$$\phi^* : H^*(BG; \mathbb{F}_p) \to H^*(B(q,G); \mathbb{F}_p)$$

has a nilpotent kernel.

(6) If G is a finite group with mod p cohomology detected by subgroups of nilpotence class less than q, then

$$H^*(BG; \mathbb{F}_p) \to H^*(B(q,G); \mathbb{F}_p)$$

is a monomorphism.

(7) Let G denote a compact, 1-connected Lie group with maximal torus $T \subset G$ such that every abelian subgroup of G is conjugate to a subgroup of T. Then the spaces $Hom(\mathbb{Z}^n, G)$ are all path-connected. Furthermore, there is an isomorphism

$$H^*(B(2,G);\mathbb{Q}) \to H^*(G/T \times BT;\mathbb{Q})^{W(G)}$$

which is compatible with the well-known isomorphism $H^*(BG;\mathbb{Q}) \to H^*(BT;\mathbb{Q})^{W(G)}$, where $W(G)$ denotes the Weyl group of T in G.

On representation spaces

This section gives two examples of some associated spaces of representations spaces

$$Rep(\pi, G) = Hom(\pi, G)/G^{\mathrm{ad}}$$

where $\pi = \oplus_q \mathbb{Z}$, and $U(m),$ or $Sp(m)$. Let

$$Sym^m(X) = X^m/\Sigma_m$$

denote the m-fold symmetric product of a space X, the quotient space of X^m obtained from the natural action of the symmetric group Σ_m.

Example 0.9. Assume Let

$$X_q = (S^1)^q / H$$

where H is the group $\mathbb{Z}/2\mathbb{Z}$ acting by complex conjugation of each factor of S^1.

(1) There are homeomorphisms

$$Rep(\oplus_q \mathbb{Z}, U(m)) \to Sym^m((S^1)^q).$$

(2) There are homeomorphisms

$$Rep(\oplus_q \mathbb{Z}, Sp(m)) \to Sym^m(X_q).$$

On colimits and homotopy colimits

Let $\mathcal{N}_q(G)$ denote the family of all subgroups of G of nilpotence class less than q and let

$$G(q) = \varinjlim_{A \in \mathcal{N}_q(G)} A.$$

Theorem 0.10. *If G is a finite group, then for any $q \geq 2$, there is a homotopy equivalence*

$$hocolim_{A \in \mathcal{N}_q(G)} BA \to B(q, G).$$

Furthermore, there is a natural fibration

$$B(q, G) \to BG(q)$$

with homotopy theoretic fiber a simply–connected finite dimensional complex K_q.

Complex characters

By appealing to a theorem of Geoff Robinson, these spaces are also ‘keeping track’ of the degrees of the complex irreducible characters of a finite group G.

Problems

Assume that G is a finite discrete group.

(1) Is the space

$$B(q, G)$$

a $K(\pi, 1)$?

(2) Describe the action of G on the first homology group of $E(q.G)$.

(3) Use this action to show that

$$H_1E(q.G) \to H_1B(2, G)$$

is not a surjection in case G is a finite, discrete group of odd order..

Concrete examples for TC groups

The structure of the spaces $B(2, G)$ reflect underlying properties of the group G as well as centralizers of elements. One further particular class of finite groups for which $B(2, G)$ has been studied is known as transitively commutative groups as described next.

Definition 0.11. A finite group G is said to be transitively commutative (TC) if $[a, b] = 1$ and $[b, c] = 1$, implies that $[a, c] = 1$ whenever b is not in the center of G.

Definition 0.12. Let $C_G(a)$ denote the centralizer of a in G with $Z(G)$ the center of G. Let $a_1, \ldots, a_k \in G - Z(G)$ be a set of representatives of their centralizers so that

$$G = \bigcup_{1 \le i \le k} C_G(a_i)$$

and no smaller number of centralizers covers G.

Corollary 0.13. *If G is a TC group with trivial center, then there is a homotopy equivalence*

$$B(2, G) \to \bigvee_{1 \le i \le k} \left(\prod_{p \| C_G(a_i) |} BP \right)$$

where $P \in Syl_p(G)$.

On the cardinality of $Hom(F_n/\Gamma^q, G)$ for abelian and TC groups G

Example 0.14. Let A denote a finite abelian group, then $Hom(\mathbb{Z}^n, A) \cong A^n$. The preceding formula expresses this quantity in a particular form.

Recall that $S_k(2, A)$ denotes the elements in A^n where at least one coordinate is equal to 1. Thus the cardinality of $Hom(\mathbb{Z}^k, A)/S_k(2, A)$ is given by $1 + (|A| - 1)^k$. Now $B(2, A) = BA$ and the rank of $H_k(E_k^0(B(2, A); \mathbb{Z})$ is $(|A|-1)^k$ (recall that the k-th homology group of a k–fold suspension of a finite set has rank one less than the cardinality of the set).

Therefore $\mu_k(2, A) = (|A| - 1)^k$ and the formula becomes

$$|A|^n = 1 + \sum_{1 \le k \le n} \binom{n}{k} (|A| - 1)^k.$$

Example 0.15. Consider the case when G is a finite transitively commutative group with trivial center. Now the space $B_*(2,G)$ is the one–point union of the simplicial spaces $B_*C_G(a_i)$, $1 \leq i \leq N$ where each $C_G(a_i)$ is a maximal abelian subgroup and so in this case

$$\mu_k(2,G) = \sum_{1 \leq i \leq N} (|C_G(a_i)| - 1)^k$$

and

$$\lambda_n(2,G) = 1 + \sum_{1 \leq k \leq n} \binom{n}{k} \sum_{1 \leq i \leq N} (|C_G(a_i)| - 1)^k$$

If $G = A_5$, the alternating group, then there are three isomorphism classes of centralizers: $\mathbb{Z}/2 \times \mathbb{Z}/2$ (five copies), $\mathbb{Z}/3$ (ten copies) and $\mathbb{Z}/5$ (six copies). Using this yields

$$\mu_k(2, A_5) = 5 \cdot 3^k + 10 \cdot 2^k + 6 \cdot 4^k$$

and

$$\lambda_n(2, A_5) = 1 + \sum_{1 \leq k \leq n} \binom{n}{k} [5 \cdot 3^k + 10 \cdot 2^k + 6 \cdot 4^k]$$

On the other hand these formulae can be rearranged to express

$$\lambda_n(2, G) = 1 + \sum_{1 \leq i \leq N} [\lambda_n(2, C_G(a_i)) - 1]$$

which can also be deduced directly from the structure of the group.

Summary

Thus the homology of the spaces $Hom(\pi, G)$ are encoding information about whether ceratin finite discrete groups are solvable, the numbers of commuting n-tuples in G as well as numbers of irreducible complex characters.

Thank you very much for this opportunity.