LECTURES ON ELEMENTS OF TRANSFORMATION GROUPS AND ORBIFOLDS

ZHI LÜ

Abstract. The first three sections include basic notations, examples and basic facts of transformation groups and orbifolds, such as topological groups, G-spaces etc. Sections 4 and 5 are on "homogeneous spaces and orbit types" and "twisted product and slice", respectively, which are also fundamental. Section 6 is on equivariant cohomology. This involves principal bundles, the Borel construction, localization theorem, etc. Most of the above material written in lectures is mainly based upon three standard textbooks by Glen E. Bredon [B], Wu-Yi Hsiang [H1] and Dale Husemoller [H2]. In Section 7, we introduce the Davis-Januszkiewicz theory [DJ], which establishes a direct link between equivariant topology and polytopes of combinatorics.

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1. Topological groups and Lie groups

1.1. Topological groups.

Definition 1.1. Let G be a Hausdorff space and also a group. If the topological structure and the group structure on G satisfy the compatibility condition: the maps $G \times G \longrightarrow G$ given by $(g, h) \longmapsto gh$ and $G \longrightarrow G$ given by $g \longmapsto g^{-1}$ are continuous, then G is said to be a *topological group*.

Example 1.1. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then $M(n, \mathbb{F})$ consisting of all $n \times n$ square matrices forms an abelian group under addition. On the other hand, a metric d on $M(n, \mathbb{F})$ can be defined as follows: for $A = (a_{ij}), B = (b_{ij}) \in M(n, \mathbb{F})$

$$d(A, B) = \max_{i,j}\{|a_{ij} - b_{ij}|\}$$

so that $M(n, \mathbb{F})$ becomes an open manifold with $\dim_{\mathbb{F}} M(n, \mathbb{F}) = n^2$. Obviously, $(A, B) \longmapsto A + B$ and $A \longmapsto -A$ are continuous with respect to the topology on $M(n, \mathbb{F})$. Thus, $M(n, \mathbb{F})$ is a topological group.

Example 1.2. The map

$$M(n, \mathbb{F}) \times M(n, \mathbb{F}) \longrightarrow M(n, \mathbb{F})$$

by $(A, B) \longmapsto AB$ is continuous with respect to the topology of $M(n, \mathbb{F})$. Furthermore, the following groups under multiplication

$$GL(n, \mathbb{F}) = \{A \in M(n, \mathbb{F}) | \det A \neq 0\} \text{ (general linear group)}$$

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$$SL(n,\mathbb{F}) = \{A \in M(n,\mathbb{F}) | \det A = 1\} \text{ (special linear group)}$$
$$O(n,\mathbb{F}) = \{A \in M(n,\mathbb{F}) | AA^\top = I_n\} \text{ (orthogonal group)}$$
$$SO(n,\mathbb{F}) = O(n,\mathbb{F}) \cap SL(n,\mathbb{F})$$
$$U(n) = \{A \in GL(n,\mathbb{C}) | A\bar{A}^\top = I_n\} \text{ (unitary group)}$$
$$SU(n) = U(n) \cap SL(n,\mathbb{C})$$

are topological groups with the relative topology from $M(n,\mathbb{F})$. Note that $A \longmapsto A^{-1}$ is continuous on $GL(n,\mathbb{F})$.

1.2. **Lie groups.**

Definition 1.2. Let G be a topological group and let e be its identity. G is called a *Lie group* if there is a chart (U,ψ) near e with $\psi(e) = 0 \in \mathbb{R}^n$ such that the group operations are real analytic near e in the local coordinates of (U,ψ). More precisely speaking, write $\psi(g) = (\psi_1(g),...,\psi_n(g))$ for $g \in U$. Then for all g,h in some open neighborhood $V \subset U$ of e, each $\psi_i(gh)$ is a real analytic function $u_i(\psi_1(g),...,\psi_n(g),\psi_1(h),...,\psi_n(h))$ on some neighborhood of $0 \in \mathbb{R}^{2n}$, and for g near in U, each $\psi_i(g^{-1})$ is also a real analytic function $v_i(\psi_1(g),...,\psi_n(g))$ defined near $0 \in \mathbb{R}^n$.

Example 1.3. All $GL(n,\mathbb{F}), SL(n,\mathbb{F}), O(n,\mathbb{F}), SO(n,\mathbb{F}), U(n), SU(n)$ are Lie groups.

Proposition 1.1. *A Lie group G is a smooth manifold.*

Proof. Obviously, a chart (U,ψ) near e can give a chart near any point $g \in G$ since U is homeomorphic to gU, so that one can obtain a smooth structure on G. ☐

Exercise 1. Show that $\dim GL(n,\mathbb{C}) = 2n^2 = 2\dim GL(n,\mathbb{R})$, $\dim SL(n,\mathbb{C}) = 2(n^2-1) = 2\dim SL(n,\mathbb{R})$, $\dim O(n,\mathbb{C}) = n(n-1) = 2\dim O(n,\mathbb{R})$, $\dim SO(n,\mathbb{C}) = n(n-1) = 2\dim SO(n,\mathbb{R})$, $\dim U(n) = n^2$, and $\dim SU(n) = n^2-1$.

Remark 1. It is known from [B, Chapter 0, Theorems 5.1 and 5.4] that

- A compact topological group is a Lie group if and only if it is isomorphic to a closed subgroup of $O(n)$ for some n.
- A connected abelian Lie group is isomorphic to $T^k \times \mathbb{R}^{n-k}$ for some n,k, where $T^k = S^1 \underbrace{\times \cdots \times}_{k} S^1$ is the torus of rank k.

2. G-ACTIONS (OR TRANSFORMATION GROUPS) ON TOPOLOGICAL SPACES

Let G be a topological group and let X be a Hausdorff topological space.

Definition 2.1. A *G-action* on X means a continuous map $\phi : G \times X \longrightarrow X$ satisfying

(1) $\phi(g,\phi(h,x)) = \phi(gh,x)$ for all $g,h \in G$ and $x \in X$;
(2) $\phi(e,x) = x$ for all $x \in X$, where e is the identity of G.

A G-action ϕ on X is often denoted by $G \curvearrowright^{\phi} X$.
The topological space X with a given G-action is also called a *G-space*.

Lemma 2.1. *Let ϕ be a G-action on X. For each $g \in G$, $\phi(g, \cdot) : X \longrightarrow X$ is a homeomorphism of X to itself.*

Proof. By (1) and (2) of Definition 2.1, one has that $\phi(g, \cdot)\phi(h, \cdot) = \phi(gh, \cdot)$ and $\phi(e, \cdot) = \mathrm{id}_X$, so

$$\phi(g, \cdot)\phi(g^{-1}, \cdot) = \phi(e, \cdot) = \mathrm{id}_X = \phi(g^{-1}, \cdot)\phi(g, \cdot).$$

☐

Let $\mathcal{H}(X)$ denote the set of all homeomorphisms of X to itself. Then $\mathcal{H}(X)$ forms a group under composition.

Theorem 2.1. *Let G be a compact topological group. Then there is a one-to-one correspondence between all G-actions on X and all continuous homomorphisms from G to $\mathcal{H}(X)$.*

Proof. By Lemma 2.1, each G-action ϕ on X defines a continuous homomorphism $\Phi : G \longrightarrow \mathcal{H}(X)$ by $g \longmapsto \phi(g, \cdot)$. Conversely, each continuous homomorphism $\Phi : G \longrightarrow \mathcal{H}(X)$ determines a G-action ϕ on X by $\phi(g, x) = \Phi(g)(x)$ for all $x \in X$. ☐

Given a G-action ϕ on X, let $\Phi : G \longrightarrow \mathcal{H}(X)$ be the corresponding continuous homomorphism of the action ϕ.

$$\ker \Phi = \{g \in G | \Phi(g)(x) = x \text{ for all } x \in X\}$$

is called the *kernel* of the action ϕ, also denoted by $\ker \phi$.

If $\ker \Phi$ is trivial (i.e., Φ is a monomorphism), then the action ϕ is called *effective*. If $\ker \Phi$ is a discrete subgroup of G, then the action ϕ is called *almost effective*.

Exercise 2. Show that $\ker \Phi$ is a normal subgroup of G and is closed in G.

The meaning of the following theorem is that generally one needs only to consider effective actions.

Theorem 2.2 (cf. [B]). *Let $G \curvearrowright^{\phi} X$ and $N = \ker \phi$. Then there is a canonical action $\phi / \ker \phi$ of G/N on X that is effective.*

Proof. Define $\phi/\ker\phi : (G/N) \times X \longrightarrow X$ by $(gN, x) \longmapsto \phi(g, x)$. To complete the proof, it only needs to be shown that $\phi/\ker\phi$ is continuous. For any open subset $U \subset X$, consider the following commutative diagram

$$\begin{tikzcd}
G \times X \arrow[r, "\phi"] \arrow[d, "\pi \times \mathrm{id}_X"'] & X \\
G/N \times X \arrow[ur, "\phi/\ker\phi"'] &
\end{tikzcd}$$

where $\pi : G \longrightarrow G/N$ is the natural projection, since π is open, one has that

$$(\phi/\ker\phi)^{-1}(U) = (\pi \times \mathrm{id}_X)\phi^{-1}(U)$$

is open in $G/N \times X$. ☐

Remark 2. The G-actions mentioned above are all left actions. A right action of G on X is a continuous $\phi : X \times G \longrightarrow X$ satisfying

(1) $\phi(\phi(h, x), g) = \phi(x, hg)$ for all $g, h \in G$ and $x \in X$;
(2) $\phi(x, e) = x$ for all $x \in X$, where e is the identity of G.

A right action ϕ of G on X is denoted by $X \curvearrowleft^{\phi} G$. Obviously, there is a one-to-one correspondence between left G-actions and right G-actions, so generally one only needs to deal with left G-actions.

Remark 3. Given $G \curvearrowright^{\phi} X$, for convenience, one often replaces $\phi(g, x)$ by gx or $g(x)$.

Definition 2.2. Let X and Y be two G-spaces. A continuous map $f : X \longrightarrow Y$ is called a *G-equivariant map* (or *G-map*) if $f(g(x)) = g(f(x))$ for all $x \in X, g \in G$, i.e., the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

In particular, if f is also a homeomorphism, then f is said to be an *G-equivariant homeomorphism*.

Remark 4. Suppose that $f : X \longrightarrow Y$ is an G-equivariant homeomorphism. Then the inverse f^{-1} is also G-equivariant. In fact, for any $y \in Y$, there is a $x \in X$ such that $f(x) = y$. Furthermore, for any $g \in G$, $f^{-1}(g(y)) = f^{-1}(g(f(x))) = (f^{-1}f)(g(x)) = g(x) = g(f^{-1}(y))$.

Remark 5. There is also a notion of a weakly G-equivariant map, which is stated as follows: a continuous map $f : X \longrightarrow Y$ is called a *weakly G-equivariant map* if $f(g(x)) = \sigma(g)(f(x))$ for all $x \in X, g \in G$ and an automorphism σ of G, i.e., the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "\sigma(g)"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Similarly, if f is also a homeomorphism, then f is said to be a *weakly G-equivariant homeomorphism*

2.1. **Orbit, fixed point set, isotropy subgroup.**

Definition 2.3. Let $G \curvearrowright^{\phi} X$. Given a $x \in X$, $G(x)$ (or Gx) $= \{g(x) | g \in G\}$ is called the *orbit* of x.

Lemma 2.2. *Let $G \curvearrowright^{\phi} X$. For any two points $x, y \in X$, one has that either $G(x) = G(y)$ or $G(x) \cap G(y) = \emptyset$.*

Proof. Suppose that $G(x) \cap G(y) \neq \emptyset$. Then there exist $g, h \in G$ such that $g(x) = h(y)$. For any $a \in G$, $a(x) = ag^{-1}h(y) \in G(y)$ so $G(x) \subset G(y)$. In a similar way, one has that $G(y) \subset G(x)$. Thus $G(x) = G(y)$. $\square$

Remark 6. By Lemma 4.1, the action $G \curvearrowright^{\phi} X$ gives an equivalence relation:

$$x \sim_{\phi} y \Longleftrightarrow G(x) = G(y).$$

Let $G \curvearrowright^{\phi} X$. Set

$$X/G := \{G(x) | x \in X\}.$$

Then there is a natural map $\pi : X \longrightarrow X/G$ by $x \longmapsto G(x)$. Furthermore, X/G can be endowed with the quotient topology:

$$U \subset X/G \text{ is open if and only if } \pi^{-1}(U) \text{ is open in } X.$$

X/G with the quotient topology is called the *orbit space* of $G \curvearrowright^{\phi} X$.

Exercise 3. Let $G \curvearrowright^{\phi} X$. Show that $\pi : X \longrightarrow X/G$ is an open map and X/G is Hausdorff.

Definition 2.4. Let $G \curvearrowright^{\phi} X$. Then $X^G := \{x \in X | g(x) = x \text{ for all } g \in G\}$ is called the *fixed point set* of $G \curvearrowright^{\phi} X$. It is a subspace of X.

More generally, let $H < G$ be a subgroup of G. Then $G \curvearrowright^{\phi} X$ naturally induces $H \curvearrowright X$. It is the restriction of ϕ to H. Furthermore, $X^H := \{x \in X | h(x) = x \text{ for all } h \in H\}$ is the fixed point set of $H \curvearrowright X$.

Definition 2.5. Let $G \curvearrowright^{\phi} X$. Given a $x \in X$, $G_x = \{g \in G | g(x) = x\}$ is called the *isotropy subgroup* of G at x.

Obviously G_x is a closed subgroup of G.

Lemma 2.3. *Let $G \curvearrowright^{\phi} X$. For each $x \in X$ and each $g \in G$, $G_{g(x)} = gG_xg^{-1}$ (i.e., $G_{g(x)}$ and G_x are conjugate).*

Proof. In fact, $gG_xg^{-1}(g(x)) = gG_x(x) = g(x)$ so $gG_xg^{-1} \subset G_{g(x)}$. Conversely, $g^{-1}G_{g(x)}g \subset G_{g^{-1}g(x)} = G_x$. ☐

$G \curvearrowright X$ is said to be *free* if G_x is trivial for each $x \in X$. For example, the antipodal map $a : S^n \longrightarrow S^n$ gives a $\mathbb{Z}_2$-action on S^n, which is free.

$G \curvearrowright X$ is said to be *semi-free* if G_x is either trivial or all of G for each $x \in X$.

Exercise 4. Let $G \curvearrowright^{\phi} X$. Show that $\ker \phi = \bigcap_{x \in X} G_x$.

Example 2.1. Let $G = \mathbb{Z}_2 = \{\pm 1\}$ and $X = S^1 = \{x = (x_0, x_1) \in \mathbb{R}^2 | x_0^2 + x_1^2 = 1\}$. Let $\mathbb{Z}_2 \curvearrowright S^1$ defined by $(g, (x_0, x_1)) \longmapsto (gx_0, x_1)$. Then, $X^G = (S^1)^{\mathbb{Z}_2} = \{(0, \pm 1)\}$. Given a $x \in X$, if $x = (0, 1)$ or $(0, -1)$, then $G(x) = \{x\}$ and $G_x = \mathbb{Z}_2$, and if $x = (x_0, x_1) \neq (0, \pm 1)$, then $G(x) = \{(\pm x_0, x_1)\}$ and $G_x = \{1\}$. Furthermore, $X/G = S^1/\mathbb{Z}_2$ is an interval.

Example 2.2. Let $G = S^1 = \{g \in \mathbb{C} | |g| = 1\}$ and $X = S^2 = \{(z, y) \in \mathbb{C} \times \mathbb{R} | |z|^2 + y^2 = 1\}$. Let $S^1 \curvearrowright S^2$ defined by $(g, (z, y)) \longmapsto (gz, y)$. Then, $X^G = (S^2)^{S^1} = \{(0, \pm 1)\}$. For $x \in X$, if $x = (z, y) = (0, 1)$ or $(0, -1)$, then $G(x) = \{x\}$ and $G_x = S^1$, and if $x \neq (0, \pm 1)$, then $G(x)$ is homeomorphic to S^1 and $G_x = \{1\}$. Furthermore, $X/G = S^2/S^1$ is an interval.

Exercise 5. Given a $G \curvearrowright X$ with compact G, show that

(a) The natural projection $\pi : X \longrightarrow X/G$ is proper (i.e., for any compact subset A of X/G, $\pi^{-1}(A)$ is compact) and closed.
(b) X is compact if and only if X/G is compact
(c) X is locally compact if and only if X/G is locally compact.

Exercise 6. Let $f : X \longrightarrow Y$ be a G-equivariant map between G-spaces. Then for all $x \in X$, $G_x < G_{f(x)}$.

2.2. **G-representations.** Let $X = V$ be a vector space over a field $\mathbb{F}$ and let $GL(V, \mathbb{F})$ denote the group of all automorphisms of V. Then a homomorphism $\lambda : G \longrightarrow GL(V, \mathbb{F})$ gives an action of G on V as follows: $\phi : G \times V \longrightarrow V$ defined by $(g, v) \longmapsto \lambda(g)(v)$. The homomorphism λ (or $G \curvearrowright^{\phi} V$) is called a *$G$-representation* on V. If V is finite-dimensional, then $\lambda : G \longrightarrow GL(V, \mathbb{F})$ is called a finite-dimensional G-representation.

Example 2.3. The $\mathbb{Z}_2^n$-representation on $\mathbb{R}^n$ defined by

$$((g_1, ..., g_n), (x_1, ..., x_n)) \longmapsto ((-1)^{g_1}x_1, ..., (-1)^{g_n}x_n)$$

fixes only the origin of $\mathbb{R}^n$, and its orbit space is the positive cone $\mathbb{R}^n_{\geq 0}$ of $\mathbb{R}^n$. Such $\mathbb{Z}_2^n \curvearrowright \mathbb{R}^n$ is called the *standard* $\mathbb{Z}_2^n$-representation on $\mathbb{R}^n$. Similarly, $T^n \curvearrowright \mathbb{C}^n$ defined by

$$((e^{\theta_1 i}, ..., e^{\theta_n i}), (z_1, ..., z_n)) \longmapsto (e^{\theta_1 i}z_1, ..., e^{\theta_n i}z_n)$$

also fixes only the origin of $\mathbb{C}^n$, and its orbit space is $\mathbb{R}^n_{\geq 0}$. Such $T^n \curvearrowright \mathbb{C}^n$ is called the *standard* T^n-representation on $\mathbb{C}^n$.

A G-representation λ is said to be *orthogonal* (resp. *unitary*) if it is a homomorphism from G to $O(n)$ (resp. $U(n)$).

Two G-representations $\lambda, \eta : G \longrightarrow GL(V, \mathbb{F})$ are said to be *equivalent* if there exists a $\sigma \in GL(V, \mathbb{F})$ such that $\lambda(g) = \sigma^{-1}\eta(g)\sigma$ for all $g \in G$.

Exercise 7. Let V be an n-dimensional vector space over $\mathbb{R}$. Then a G-representation $\phi : G \longrightarrow GL(V, \mathbb{R})$ is equivalent to an orthogonal G-representation if and only if there exists a positive definite inner product $< \cdot, \cdot >$ on V such that for all $g \in G$ and $u, v \in V$, $< \phi(g)u, \phi(g)v >=< u, v >$.

Let G be a compact group. It is well-known (cf. [B, Chapter 0, Theorem 3.1]) that there is a unique real-valued function I defined on $C(G, \mathbb{R})$ (i.e., all real-valued continuous functions) such that

(1) For $f_1, f_2 \in C(G, \mathbb{R})$, $I(f_1 + f_2) = I(f_1) + I(f_2)$.
(2) For $f \in C(G, \mathbb{R})$, $I(cf) = cI(f)$ where $c \in \mathbb{R}$.
(3) For $f \in C(G, \mathbb{R})$, if $f(g) \geq 0$ for all $g \in G$, then $I(f) \geq 0$.
(4) $I(1) = 1$.
(5) For $f \in C(G, \mathbb{R})$ and $h \in G$, define f_h^r and f_h^l by $f_h^r(g) = f(gh)$ and $f_h^l(g) = f(h^{-1}g)$. Then $I(f_h^r) = I(f) = I(f_h^l)$ for all $h \in G$.

Such real-valued function I defined on $C(G, \mathbb{R})$ is called the *Haar integral*.

Theorem 2.3 (cf. [B]). *Let G be a compact group and V an n-dimensional vector space over $\mathbb{R}$. Then each G-representation on V is equivalent to an orthogonal G-representation on $\mathbb{R}^n$.*

Proof. With no loss, assume that $V = \mathbb{R}^n$. Then $GL(V, \mathbb{R}) = GL(n, \mathbb{R})$. Let $\phi : G \longrightarrow GL(n, \mathbb{R})$ be a G-representation on $\mathbb{R}^n$. By Exercise 7 it suffices to show that there exists a positive definite inner product $< \cdot, \cdot >: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that for all $g \in G$ and $x, y \in \mathbb{R}^n$, $< \phi(g)x, \phi(g)y >=< x, y >$.

Using the Haar integral, define $< \cdot, \cdot >: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$< x, y >= I(f_{x,y})$$

where $f_{x,y} \in C(G, \mathbb{R})$ is defined by $f_{x,y}(g') = (\phi(g')x) \cdot (\phi(g')y)$ and $(\phi(g')x) \cdot (\phi(g')y)$ denotes the usual euclidean inner product. Obviously, $< \cdot, \cdot >$ is bilinear and symmetric. For $x \neq 0$, since $f_{x,x}(g') = (\phi(g')x) \cdot (\phi(g')x) > 0$, one has that $I(f_{x,x}) > 0$ so $< \cdot, \cdot >$ is positive definite. Now, for any $g \in G$, one has that

$$< \phi(g)x, \phi(g)y >= I((f_{x,y})^l_g) = I(f_{x,y}) =< x, y > .$$

□

Remark 7. In Theorem 2.3, if V is a complex n-dimensional vector space over $\mathbb{C}$, then each G-representation on V is equivalent to a unitary G-representation on $\mathbb{C}^n$ (see [H1, Corollary I.1.1]).

Let $\lambda : G \longrightarrow GL(V, \mathbb{F})$ and $\eta : G \longrightarrow GL(W, \mathbb{F})$ be two G-representations. Then $\lambda \oplus \eta : G \longrightarrow GL(V \oplus W, \mathbb{F})$ defined by $(\lambda \oplus \eta)(g)(v, w) = (\lambda(g)(v), \eta(g)(w))$ is called the *sum of the G-representations λ, η.*

A G-representation $\lambda : G \longrightarrow GL(V, \mathbb{F})$ is said to be *reducible* if there exists a non-trivial proper subspace W of V such that the action of G on W is invariant (i.e., $\lambda(g)(W) = W$ for all $g \in G$); otherwise, it is called *irreducible.*

A G-representation $\lambda : G \longrightarrow GL(V, \mathbb{F})$ is said to be *completely reducible* if it is equivalent to the direct sum $\lambda_1 \oplus \cdots \oplus \lambda_k$ of irreducible G-representations.

Theorem 2.4 (cf. [B]). *Let G be a compact group and let V be a finite-dimensional vector space over $\mathbb{R}$. Then every real G-representation $G \longrightarrow GL(V, \mathbb{R})$ is completely reducible.*

Proof. By Theorem 2.3, with no loss one may assume that $V = \mathbb{R}^n$ and the G-representation on $\mathbb{R}^n$ is an orthogonal real representation $\lambda : G \longrightarrow O(n)$. Suppose that $U \subset \mathbb{R}^n$ is invariant under λ. For $x, y \in \mathbb{R}^n$ and $g \in G$, since

$$(\lambda(g)x) \cdot y = x \cdot (\lambda(g))^\top y) = x \cdot (\lambda(g^{-1})y)$$

one has that $U^\perp = \{x \in \mathbb{R}^n | x \cdot y = 0 \text{ for all } y \in U\}$ is also invariant. So $\lambda \approx (\lambda|_U) \oplus (\lambda|_{U^\perp})$, and the result then holds since λ is finite-dimensional. □

Remark 8. Every finite-dimensional complex G-representation $G \longrightarrow GL(V, \mathbb{C})$ is completely reducible where G is a compact group (see [H1, Corollary I.1.2]).

Remark 9. Let $\lambda : G \longrightarrow GL(n+1,\mathbb{R})$ be a G-representation. Then it is easy to see that the fixed point set $(\mathbb{R}^{n+1})^G$ of this representation is a linear subspace of $\mathbb{R}^n$. Furthermore, for an orthogonal G-action on an n-sphere S^n, its fixed point set is a sphere S^l.

The Schur Lemma below is useful and fundamental in the study of irreducible representations.

Theorem 2.5 (Schur Lemma, cf. [H1]). *Let V_1, V_2 be two vector spaces, and let λ_1, λ_2 be two irreducible G-representations on V_1, V_2 respectively, where G is a compact group. Suppose that $f : V_1 \longrightarrow V_2$ is a linear map such that for any $g \in G$, $f \circ \lambda_1(g) = \lambda_2(g) \circ f$. Then either $f = 0$ or f is invertible. In particular, if $V_1 = V_2 = V$ (so $\lambda_1 = \lambda_2 = \lambda$) is a vector space over $\mathbb{C}$, then there is some $c \in \mathbb{C}$ such that $f = cI_V$, where I_V is the identity.*

Proof. Obviously, $\ker f$ and $\mathrm{Im} f$ are G-invariant subspaces of V_1 and V_2 respectively. Since λ_1 and λ_2 are irreducible, $\ker f$ and $\mathrm{Im} f$ cannot be proper sub-representations. If $\ker f = V_1$, then $f = 0$. If $\ker f = 0$, then $\mathrm{Im} f = V_2$. This implies that f is invertible.

Since $\mathbb{C}$ is algebraically closed, f has an eigenvalue $c \in \mathbb{C}$. The eigenspace V_c associated with c is G-invariant, so there is a G-representation on V_c. Since λ is irreducible, one must have $V_c = V$. □

As a direct consequence, one has

Corollary 2.1. *Let G be a compact commutative group. Then each complex irreducible G-representation is complex 1-dimensional.*

Example 2.4. If $G = T^n = (S^1)^n$, then each complex irreducible T^n-representation has the form $\kappa_\phi : T^n \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by $\kappa_\phi(g, z) = \phi(g)z$, where $\phi \in \mathrm{Hom}(T^n, S^1)$. Similarly, if $G = \mathbb{Z}_2^n$, then each real irreducible $\mathbb{Z}_2^n$-representation is 1-dimensional, and can be defined as follows: $\lambda_\rho : \mathbb{Z}_2^n \times \mathbb{R} \longrightarrow \mathbb{R}$ by $\lambda_\rho(g, x) = (-1)^{\rho(g)}x$, where $\rho \in \mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$.

The characters play the essential role in the study of irreducible representations

Definition 2.6. Let $\lambda : G \longrightarrow GL(V)$ be a G-representation on a vector space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, where G is a compact group. For each $g \in G$, $\lambda(g)$ is a linear map. Then the map

$$\chi_\lambda : G \longrightarrow \mathbb{F}$$

by $g \longmapsto \mathrm{trace}\lambda(g)$ is called the *character* of the G-representation λ.

Remark 10. The algebraic properties of trace can directly induce the following results (cf. [H1] and [S1]):

1. Given a G-representation $G \curvearrowright^{\lambda} V$ on an n-dimensional vector space over $\mathbb{F}$. Then
   - $\chi_\lambda(1) = n$.
   - $\chi_\lambda(h^{-1}gh) = \chi_\lambda(g)$ for $g, h \in G$.

(2) Given two G-representations $G \curvearrowright^{\lambda_i} V_i$ on vector spaces $V_i, i = 1, 2$ over $\mathbb{F}$. Then
   - $\chi_{\lambda_1 \oplus \lambda_2} = \chi_{\lambda_1} + \chi_{\lambda_2}$, where $\chi_{\lambda_1 \oplus \lambda_2} : G \longrightarrow \mathbb{F}$ is defined by $\chi_{\lambda_1 \oplus \lambda_2}(g) = \chi_{\lambda_1}(g) + \chi_{\lambda_2}(g)$.
   - $\chi_{\lambda_1 \otimes \lambda_2} = \chi_{\lambda_1} \chi_{\lambda_2}$, where $\chi_{\lambda_1 \otimes \lambda_2} : G \longrightarrow \mathbb{F}$ is defined by $\chi_{\lambda_1 \otimes \lambda_2}(g) = \chi_{\lambda_1}(g) \chi_{\lambda_2}(g)$.

For simplicity, now one assumes that $\mathbb{F} = \mathbb{C}$ and G is a finite group. For the general case, we would like to refer to [H1] for more details.

Let $C(G, \mathbb{C})$ denote the set of all functions from G to $\mathbb{C}$, and let $|G|$ denote the order of G. Define the inner product on $C(G, \mathbb{C})$ as follows: for $\eta_1, \eta_2 \in C(G, \mathbb{C})$,

$$< \eta_1, \eta_2 >= \frac{1}{|G|} \sum_{g \in G} \eta_1(g) \overline{\eta_2(g)}$$

Together with this inner product, the characters can completely classify all irreducible G-representations.

Theorem 2.6. (Classification Theorem, cf. [H1])

(1) *If* $G \curvearrowright^{\lambda} V$ *is an irreducible* G*-representation, then* $< \chi_\lambda, \chi_\lambda >= 1$.
(2) *If* $G \curvearrowright^{\lambda_i} V_i, i = 1, 2$ *are two non-equivalent irreducible* G*-representations, then* χ_{λ_1} *and* χ_{λ_2} *are orthogonal, i.e.,* $< \chi_{\lambda_1}, \chi_{\lambda_2} >= 0$.

Proof. (1) By Remark 7, λ is equivalent to a unitary G-representation. Then, for each $g \in G$, $\lambda(g) = (\lambda_{ij}(g)) \in U(n)$ where $\dim_{\mathbb{C}} V = n$. Each λ_{ij} is a function from G to $\mathbb{C}$, so $\lambda = (\lambda_{ij})$ can be regarded as a matrix function. The action $G \curvearrowright^{\lambda} V$ naturally induces an action $G \curvearrowright^{\Phi} M(n, \mathbb{C})$ by

$$\Phi(g, \sigma) = \lambda(g) \sigma \lambda(g)^{-1} = \lambda(g) \sigma \overline{\lambda(g)}^{\top}$$

where $\sigma \in M(n, \mathbb{C})$ and $g \in G$. Given a $\sigma \in M(n, \mathbb{C})$, set $\widetilde{\sigma} = \lambda \sigma \overline{\lambda}^{\top} = (\lambda_{ij}) \sigma (\overline{\lambda_{ij}})^{\top}$, where each entry $\widetilde{\sigma}_{ij}$ in $\widetilde{\sigma}$ is a linear combination formed by $\lambda_{st} \cdot \overline{\lambda}_{kl} =< \lambda_{st}, \lambda_{kl} >$. It is easy to see that $\widetilde{\sigma}$ is a fixed point of $G \curvearrowright^{\Phi} M(n, \mathbb{C})$. This implies that for all $g \in G$, $\lambda(g) \widetilde{\sigma} = \widetilde{\sigma} \lambda(g)$. By Schur's Lemma, there is a $c_\sigma \in \mathbb{C}$ such that $\widetilde{\sigma} = c_\sigma I$. Obviously, $\mathrm{trace}(\sigma) = \mathrm{trace}(\widetilde{\sigma}) = c_\sigma n$. Take $\sigma = E_{ij}$ where the ij-th entry in E_{ij} is 1 and other entries are all zero. Then one has that $\widetilde{E_{ij}} = \frac{\delta_{ij}}{n} I = \lambda E_{ij} \widetilde{\lambda}^{\top}$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Furthermore, one has that for all $1 \leq i, j, k, l \leq n$,

$$< \lambda_{li}, \lambda_{kj} >= \frac{\delta_{ij} \delta_{lk}}{n}.$$

Therefore

$$< \chi_\lambda, \chi_\lambda >=< \sum_{i=1}^{n} \lambda_{ii}, \sum_{i=1}^{n} \lambda_{ii} >= \sum_{i=1}^{n} < \lambda_{ii}, \lambda_{ii} >= 1.$$

(2) In a similar way to (1), one may give the proof of Theorem 2.6(2). ☐

Exercise 8. Show that Theorem 2.6(2) holds.

2.3. G-vector bundles.

Definition 2.7. A G-vector bundle over a G-space X is a vector bundle $\xi = (E, \pi, X)$ such that

1. E is a G-space;
2. $\pi : E \longrightarrow X$ is a G-map;
3. For each $g \in G$, the restriction to each fiber of $g : E \longrightarrow E$ is a linear map.

Example 2.5. Let M be a smooth manifold, and let $G \curvearrowright M$ smoothly. Then the tangent bundle TM is a G-vector bundle over M.

In fact, $G \curvearrowright M$ can induce a G-action on TM as follows: First, for each $v \in TM$, there is a $x \in M$ such that $v \in T_xM$. Then, for each $g \in G$, the derivative at x of $g : M \longrightarrow M$ is $(Tg)_x : T_xM \longrightarrow T_{g(x)}M$, which is linear. Furthermore, $G \curvearrowright TM$ is defined by $(g, v) \longmapsto (Tg)_x(v) := g(v)$. Obviously, the natural projection $TM \longrightarrow M$ is a G-map. Thus, $G \curvearrowright M$ gives TM the structure of a smooth G-vector bundle over M.

Given $x \in M^G$, since $g(x) = x$ for any $g \in G$, one has that

$$(Tg)_x : T_xM \longrightarrow T_{g(x)}M = T_xM$$

so T_xM inherits an action from $G \curvearrowright M$, and then T_xM is a G-space. Since T_xM is a vector space and $g \longmapsto (Tg)_x$ gives a homomorphism $G \longrightarrow GL(T_xM)$, T_xM is a G-representation, which is often called *an isotropy G-representation.*

Generally, at any point $x \in M$, one has that $x \in M^{G_x}$. For $g \in G_x$, one has that $g(x) = x$, so $g \longmapsto (Tg)_x$ gives a homomorphism $G_x \longrightarrow GL(T_xM)$. Thus, T_xM is a G_x-representation.

3. ORBIFOLDS

Simply speaking, an orbifold is locally modeled on quotients of open subsets of $\mathbb{R}^n$ by finite group actions.

Definition 3.1. An n-dimensional *orbifold* $\mathcal{O}$ is a Hausdorff space, called the underlying space, with an open covering $\mathfrak{U} = \{U_i\}$ (which is closed under finite intersection) such that

1. Each open set U_i is associated with an orbifold chart (V_i, ϕ_i, U_i, G_i), where G_i is a finite group, $V_i \subset \mathbb{R}^n$ is an open subset that is invariant under an effective linear action of G_i on $\mathbb{R}^n$, and $\phi_i : V_i \longrightarrow U_i$ is surjective and induces a homeomorphism $V_i/G_i \longrightarrow U_i$.
2. The collection of all orbifold charts $\{(V_i, \phi_i, U_i, G_i)\}$, called the orbifold atlas, satisfies the condition that for each inclusion $U_i \hookrightarrow U_j$, there exists an injective group homomorphism $g_{ij} : G_i \longrightarrow G_j$ and there exists a G_i-equivariant embedding φ_{ij} of V_i onto an open subset of V_j such that $\phi_j\varphi_{ij} = \phi_i$ and φ_{ij} is unique up to composition with elements of G_j (i.e., any other possible map from V_i to V_j has the form $g\varphi_{ij}$ for a unique $g \in G_j$).

Remark 11. The orbifold atlas determines the orbifold structure. As in the case of manifolds, a differentiability condition can be imposed on the gluing maps φ_{ij} to give a definition of a differentiable orbifold.

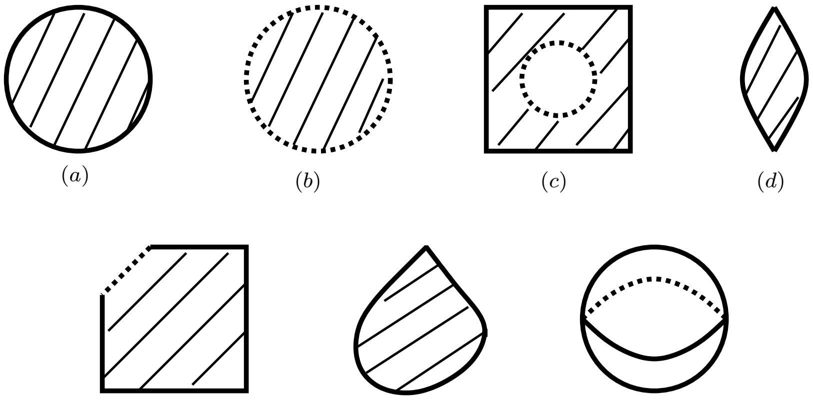
Example 3.1. Let N be a compact manifold with boundary. Then its double $M = N \cup_\partial N$ admits a natural reflection (i.e., a $\mathbb{Z}_2$-action) such that the orbit space is exactly N. Thus N has a natural orbifold structure.

Example 3.2. Let Σ be a closed surface. Then a new orbifold structures on Σ can be defined by removing finitely many disjoint closed discs from Σ and gluing back copies of discs D^2/Γ_i where D^2 is the closed unit disc and Γ_i is a finite cyclic group of rotations of rank i.

3.1. **Manifolds with corners.** All n-dimensional manifolds with corners form a special kind of orbifolds, each of which locally looks like the open subsets of the orbit (a positive cone $\mathbb{R}^n_{\geq 0}$) of the standard $\mathbb{Z}_2^n$-representation on $\mathbb{R}^n$ (see, Example 2.3).

Definition 3.2 (cf. [D], [LY2])**.** An *n-manifold with corners Q* is a Hausdorff space together with a maximal atlas of local charts onto open subsets of $\mathbb{R}^n_{\geq 0}$ such that the overlap maps are homeomorphisms preserving codimension, where the codimension $c(x)$ of a point $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n_{\geq 0}$ is the number of x_i that is 0.

Some examples of manifolds with corners are shown in the following figure.



(a) (b) (c) (d)

(e) (f) (g)

tear-drop

Generally, a manifold Q with corners may be either compact or non-compact. If Q is non-compact, then Q is said to be an *open* manifold with corners. For example, given a polygon P^2, if we remove a closed 2-disk in its interior or cut out a vertex, then the resulting space is an open 2-manifold with corners (see Figures (c) and (e)).

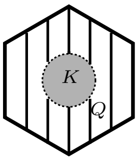
Let Q be a manifold with corners. The *boundary* ∂Q of Q is defined as the boundary of Q as a topological manifold. Obviously, the boundary of Q is empty if and only if the codimension of each point in Q is always zero. Now suppose that Q has non-empty boundary. An *open pre-face* of Q of codimension m is a connected component of $c^{-1}(m)$. A *closed pre-face* is the closure of an open pre-face. A closed pre-face of codimension 1 is called a *facet* of Q. For any $x \in Q$, let $\Sigma(x)$ be the set of facets which contain x.

A manifold Q with corners is said to be *nice* if either ∂Q is empty or ∂Q is non-empty but $\mathrm{Card}(\Sigma(x)) = 2$ for any x with $c(x) = 2$. It is easy to see that if an n-dimensional nice manifold Q with corners has empty boundary, then it is a closed or open topological manifold; otherwise, it has facets, and any l-dimensional closed pre-face F^l is a component of the intersection of $n-l$ facets in Q, and ∂Q is the union of all facets. Figure (f) is not nice.

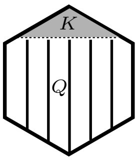
Remark 12. An n-dimensional convex polytope P^n is said to be *simple* if exactly n facets (i.e., $(n-1)$-dimensional faces) meet at each of its vertices. Each point of a simple convex polytope P^n has a neighborhood which is affine isomorphic to an open subset of the positive cone $\mathbb{R}^n_{\geq 0}$, so P^n is an n-dimensional manifold with corners.

A subset K of an n-manifold Q with corners is said to be an n-dimensional *open submanifold with corners* of Q if the restriction to K of the atlas of Q makes K become an open n-manifold with corners in its own right. Obviously, the intersection of $Q-K$ and the closure $\bar{K}$ is also an $(n-1)$-manifold with corners, which is called the *section* of $Q-K$ and is denoted by $S(K)$.

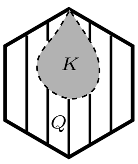
Let Q be a nice manifold with corners, and let K be an open submanifold of Q with corners. If either $S(K)$ is a nice manifold with corners or the boundary of $S(K)$ is empty, then K is said to be a *nice* open submanifold of Q with corners (see the following figure for examples).



nice



nice



not nice

A continuous map between two manifolds with corners $f : Q_1 \to Q_2$ is called a *facial map* if it preserves codimension of each point, i.e., $c(f(x)) = c(x)$ for $\forall\, x \in Q_1$. In particular, if f is a homeomorphism, we call it a *facial homeomorphism.*

4. HOMOGENEOUS SPACES AND ORBIT TYPES

Let G be a topological group and let X be a Hausdorff topological space.

Definition 4.1. An action $G \curvearrowright X$ is said to be *transitive* if $G(x) = X$ for any $x \in X$. In this case, X is also called a *homogeneous space* for G.

Example 4.1. Let $H < G$ be closed in G. Then $G \curvearrowright G/H$ defined by $(g, kH) \longmapsto gkH$ is transitive, so G/H is a homogeneous space for G.

Let $G \curvearrowright X$ and $x \in X$. Then there is a natural map $\theta_x : G/G_x \longrightarrow G(x)$ defined by $gG_x \longmapsto g(x)$. Obviously θ_x is continuous. Since G_x is closed in G, one has that G/G_x admits a G-action $G \curvearrowright G/G_x$ defined by $(g, kG_x) \longmapsto gkG_x$, which is transitive. Also, $G(x)$ admits a natural G-action induced by $G \curvearrowright X$. Then $\theta_x : G/G_x \longrightarrow G(x)$ is a G-map since $\theta_x(g(kG_x)) = \theta_x(gkG_x) = (gk)(x) = g(k(x)) = g\theta_x(kG_x)$.

Lemma 4.1 (cf. [B])**.** *If G is compact, then $\theta_x : G/G_x \longrightarrow G(x)$ is a G-equivariant homeomorphism.*

Proof. It suffices to show that θ_x is a homeomorphism. Suppose that $\theta_x(g_1G_x) = g_1(x) = g_2(x) = \theta_x(g_2G_x)$. Then $g_2^{-1}g_1(x) = x$ so $g_2^{-1}g_1 \in G_x$. Thus $g_1G_x = g_2G_x$. This means that θ_x is injective. For any $y \in G(x)$, there exists a $g \in G$ such that $y = g(x)$, so $\theta_x(gG_x) = g(x) = y$, and θ_x is surjective. Since G is a compact topological group, G/G_x is compact and $G(x)$ is Hausdorff. Hence, θ_x is a homeomorphism. $\square$

Remark 13. When G is compact, by Lemma 4.1 each orbit $G(x)$ of $G \curvearrowright X$ is homogeneous for G and can be regarded as a coset space G/G_x of G.

Given a compact topological group G, let us consider the category $\mathcal{C}_{G-\text{orbits}}$ such that all objects consist of coset spaces G/H for closed subgroups $H < G$, and all morphisms are G-equivariant maps among coset spaces. $\mathcal{C}_{G-\text{orbits}}$ is often called the *category of G-orbits.*

The following result characterizes the morphisms in $\mathcal{C}_{G-\text{orbits}}$.

Proposition 4.1 (cf. [B]). *Let G/H and G/K be two objects in $\mathcal{C}_{G-orbits}$. Then there exists an equivariant map $G/H \longrightarrow G/K$ if and only if H is conjugate to a subgroup of K.*

Proof. If there exists an equivariant map $f : G/H \longrightarrow G/K$, then $f(H) = g_0^{-1}K$ for some $g_0 \in G$, and further one has that $f(gH) = gg_0^{-1}K$ for all $g \in G$. Since $hH = H$ for all $h \in H$, $f(ghH) = f(gH)$ so $ghg_0^{-1}K = gg_0^{-1}K$. Furthermore, $g_0hg_0^{-1} \in K$ for all $h \in H$. Thus, H is conjugate to a subgroup of K.

Conversely, suppose that H is conjugate to a subgroup of K. Then there exists an element $g_0 \in G$ such that $g_0Hg_0^{-1} < K$. Define $f : G/H \longrightarrow G/K$ by $f(gH) = gg_0^{-1}K$ for all $g \in G$. Obviously, f is well-defined and equivariant. $\square$

Corollary 4.1 (cf. [B]). *Let G/H and G/K be two objects in $\mathcal{C}_{G-orbits}$. Then G/H and G/K are equivariantly homeomorphic if and only if H is conjugate to K.*

Let $\mathcal{T}_G$ denote the set of equivariant homeomorphism classes of all objects in $\mathcal{C}_{G-\text{orbits}}$. Then $\mathcal{T}_G$ gives all possible types of G-orbits. $\mathcal{T}_G$ admits a natural poset structure, whose partial ordering relation $\succ$ is defined as follows: for two classes $[G/H]$ and $[G/K]$ in $\mathcal{T}_G$,

$$[G/H] \succ [G/K] \Longleftrightarrow H \text{ is conjugate to a subgroup of } K.$$

Obviously, the poset $(\mathcal{T}_G, \prec)$ has a minimum $[G/G]$ and a maximum $[G]$.

Given a $G \curvearrowright X$ with G compact, let $\mathcal{I}(X) = \{[G_x] | x \in X\}$ denote the set of conjugacy classes of all isotropy subgroups of $G \curvearrowright X$. Then $\mathcal{I}(X)$ corresponds to the types of orbits $\mathcal{O}(X) = \{[G(x)] | x \in X\}$ in X, which is a subset of $\mathcal{T}_G$. Of course, $\mathcal{O}(X)$ is also a poset with respect to the partial ordering relation $\succ$. However, generally $[G/G]$ and $[G]$ are not the minimum and the maximum of $(\mathcal{O}(X), \succ)$, respectively.

Theorem 4.1 (Montgomery-Samelson-Yang, cf. [H1]). *Let $G \curvearrowright M$ be a smooth action where G is a compact Lie group and M is a connected smooth manifold. Then there exists a unique maximum in $(\mathcal{O}(X), \succ)$ (i.e., there is a point $x_0 \in M$*

such that $[G(x_0)]$ *is a unique maximal orbit type, called the principal orbit type) such that*

1. $M_{[G_{x_0}]} = \{x \in M | G_x \in [G_{x_0}]\}$ *is open dense in* M *and* $M - M_{[G_{x_0}]}$ *has dimension at most* $\dim M - 1$.
2. *The fixed point set* $M^{G_{x_0}}$ *intersects each orbit.*
3. $M_{[G_{x_0}]}/G$ *is connected.*

The proof of Theorem 4.1 will be postponed to the next section.

Remark 14. As pointed out in [H1], Theorem 4.1 can further be generalized into the following cases:

- Topological G-action on cohomology manifold over $\mathbb{Z}$.
- Connected orbit types (i.e., orbit types corresponding to connected isotropy subgroups) of topological G-actions on cohomology manifolds over $\mathbb{Q}$.

5. Twisted product and slice

5.1. **Twisted product.** Twisted product provides a fundamental way to the study of transformation groups.

Definition 5.1. Given $G \curvearrowright X$. Let $H < G$ be a closed subgroup of G and $A \subset X$ a subspace. Suppose $H \curvearrowright A$. Then $G \times A$ admits a H-action defined by $(h, (g, a)) \longmapsto (gh^{-1}, ha)$. The orbit space denoted by $G \times_H A$ is called a *twisted product.*

Remark 15. $G \times_H A$ naturally admits a G-action defined by $(g', [(g, a)]) \longmapsto [(g'g, a)]$. Then the projection $G \times A \longrightarrow G$ induces a G-map $p : G \times_H A \longrightarrow G/H$ by mapping $[(g, a)] \longmapsto gH$.

Exercise 9. Show that $p : G \times_H A \longrightarrow G/H$ is a fibration with fibre A.

Given $G \curvearrowright X$. Let $H < G$ be a closed subgroup of G. Suppose that $f : X \longrightarrow G/H$ is a G-equivariant map. Set $A = f^{-1}(eH)$. Then A is invariant under H-action. Actually, for any $a \in A$ and any $h \in H$, since f is G-equivariant, one has that $f(h(a)) = hf(a) = h(eH) = eH$, so $h(a) \in A$.

Proposition 5.1 (cf. [B]). $\Phi : G \times_H A \longrightarrow X$ *defined by* $[(g, a)] \longmapsto g(a)$ *is a* G*-equivariant homeomorphism.*

Proof. First, from the following commutative diagram

$$\begin{tikzcd}
G \times A \arrow[r] \arrow[d] & X \\
G \times_H A \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

it is easy to check that Φ is well-defined and a G-equivariant map. Next, it suffices to show that Φ is a bijection. Take $x \in X$ and let $f(x) = gH$, one then has that $f(g^{-1}(x)) = eH$ so $g^{-1}(x) \in A$ and $x = g(g^{-1}(x)) = \Phi([(g, g^{-1}(x))])$. Thus, Φ is surjective. To see that Φ is also injective, suppose that $\Phi([(g,a)]) = \Phi([(g',a')])$. Then one has that $g(a) = g'(a')$ so

$$gH = g(f(a)) = f(g(a)) = f(g'(a')) = g'(f(a')) = g'H.$$

Furthermore, $g^{-1}g' \in H$ so $g(a) = g'(a') = g(g^{-1}g')(a)$. Thus $a = g^{-1}g'(a)$ and $[(g',a')] = [(g'(g^{-1}g')^{-1}, g^{-1}g'(a'))] = [(g,a)]$. $\square$

5.2. **Slice.**

Definition 5.2. Given $G \curvearrowright X$, let $x \in S \subset X$ be such that $G_x(S) = S$. S is called a *slice* at x if $G \times_{G_x} S$ is G-embedded in X as an open invariant neighborhood of $G(x)$.

Proposition 5.2 (cf. [B]). *Given $G \curvearrowright X$, let $x \in S \subset X$ be such that $G_x(S) = S$. If G is compact, then S is a slice at x if and only if $G(S)$ is an open neighborhood of $G(x)$ and there is an equivariant retract $r : G(S) \longrightarrow G(x)$ such that $r^{-1}(x) = S$.*

Proof. The map $f : G(S) \longrightarrow G/G_x$ defined by $g(s) \longmapsto gG_x$ is clearly equivariant. Since $G_x(S) = S$, one has that $f^{-1}(eG_x) = S$. Furthermore, by Proposition 5.1 one has that $G(S) = G \times_{G_x} S$. If G is compact, by Lemma 4.1 G/G_x and $G(x)$ are equivariantly homeomorphic. Also, the natural projection $p : G \times_{G_x} S \longrightarrow G/G_x \approx G(x)$ satisfies $p^{-1}(G_x) = [e, S]$. Furthermore, Proposition 5.1 follows from these facts. $\square$

Exercise 10. If S is a slice at x, then for each $g \in G$, $g(S)$ is a slice at $g(x)$.

Now let us consider the existence of a slice.

Lemma 5.1 (Mostow-Palais, cf. [M2] and [P]). *Let G be a compact Lie group and $H < G$ is closed. Then there exists a G-representation on a finite-dimensional vector space V and a vector $v \in V$ such that H is the isotropy subgroup G_v at v.*

Proof. We would like to refer to the original papers ([M2], [P]) for a detail proof. $\square$

Lemma 5.2 (Gleason, cf. [H1]). *Given an action $G \curvearrowright X$ with G a compact Lie group and a G-representation on a finite-dimensional vector space V. Suppose that $A \subset X$ is a compact G-invariant subspace and there is an equivariant map $f : A \longrightarrow V$. Then f admits an equivariant extension $\widetilde{f} : X \longrightarrow V$.*

Proof. First, by Tietze's extension theorem, one can extend f to a continuous $f' : X \longrightarrow V$. Next, by using Haar integral, define $\widetilde{f}(x) = I(f''_x)$, where $f''_x \in C(G,V)$ is defined by $f''_x(g) = g^{-1}f'(gx)$ for all $g \in G$. $\square$

Theorem 5.1 (cf. [H1]). *Let G be a compact Lie group, and let $G \curvearrowright X$ and $x \in X$. Then there always exists a slice S at x.*

Proof. First, by Lemma 5.1 there exists a $G \curvearrowright V$ where V is a finite-dimensional vector space V such that $G(x)$ may be equivariant embedded into V by the following way

$$f : G(x) \xrightarrow{\approx} G/G_x \xrightarrow{\cong} G(v) \hookrightarrow V.$$

Next, by Lemma 5.2 f can be extended to an equivariant map $\widetilde{f} : X \longrightarrow V$. Since V is linear, there is a slice S' at v. Furthermore, $S = \widetilde{f}^{-1}(S')$ is the desired slice at x. $\square$

In smooth category, one has the following result.

Theorem 5.2 (Differentiable Slice Theorem, cf. [B]). *Suppose that $G \curvearrowright M$ is a smooth action where G is a compact Lie group and M is a smooth manifold. Then for each point $x \in M$, $G(x)$ has a G-invariant tubular neighborhood of the form $G \times_{G_x} S_x$, where $S_x = T_xM/T_xG(x)$ and its image in M is the slice at x.*

Now let us complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Consider the map $\Phi : M/G \longrightarrow \mathcal{O}(M)$ defined by $G(x) \longmapsto [G(x)]$, where $\mathcal{O}(M)$ is endowed with discrete topology on it. Note that $\mathcal{O}(M)$ has a poset structure. By Theorem 5.2, it is easy to see that $G(x) \in M/G$ is locally maximal if and only if $G_x \curvearrowright S_x$ is trivial. Thus, Φ is locally constant in a neighborhood of those locally maximal points in M/G.

Now suppose that $G(x')$ is not locally maximal. Then there are the following two possible cases:

1. The case $\text{codim} S_{x'}^{G_{x'}} \geq 2$. In this case, $S_{x'} - S_{x'}^{G_{x'}}$ is still connected, so if those orbits which have the same as orbit type as $G(x')$ are removed from the neighborhood of $G(x')$, then $(S_{x'} - S_{x'}^{G_{x'}})/G_{x'}$ is also connected.
2. The case $\text{codim} S_{x'}^{G_{x'}} = 1$. In this case, $G_{x'} \curvearrowright S_{x'}$ is exactly a reflection with respect to the hyperplane $S_{x'}^{G_{x'}}$. Thus, $S_{x'}/G_{x'}$ is a half plane with $S_{x'}^{G_{x'}}$ as boundary. Furthermore, $(S_{x'} - S_{x'}^{G_{x'}})/G_{x'}$ is also connected.

Since M is connected, M/G is connected, too. We see from (1) and (2) that if all those orbits which are not locally maximal are removed from M/G, the connectedness of M/G is still unchanged. This means that the subspace X formed by all locally maximal points in M/G is still connected. However, Φ is locally constant in X, so Φ is constant on X. Therefore, there exists a unique maximal orbit type $[G(x_0)]$ such that $X = M_{[G_{x_0}]}/G$ is connected, open dense in M/G. Theorem 4.1 then follows from this. $\square$

6. Equivariant cohomology

6.1. **Principal G-bundle.** Let G be a topological group.

Definition 6.1. A bundle $\xi = (X, \pi, B)$ is called a *G-bundle* if X admits an effective right (or left) G-action such that the following diagram is commutative

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d] & X \arrow[d, "\pi"] \\
X/G \arrow[r, "\approx"'] & B
\end{tikzcd}$$

Let $\xi_1 = (X_1, \pi_1, B)$ and $\xi_2 = (X_2, \pi_2, B)$ be two G-bundles over B. If there is a G-map $f : X_1 \longrightarrow X_2$ such that $\pi_1 = \pi_2 f$, i.e., the following diagram is commutative

$$\begin{tikzcd}
X_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\
& B &
\end{tikzcd}$$

then f is called a morphism between ξ_1 and ξ_2. In particular, if f is an equivariant homeomorphism, then ξ_1 and ξ_2 are said to be *isomorphic*, denoted by $\xi_1 \cong \xi_2$.

Following [H2], given an effective right action $X \curvearrowleft G$, let $X^* = \{(x, xg) \in X \times X | x \in X, g \in G\}$. A continuous map $\tau : X^* \longrightarrow G$ with $x\tau(x, x') = x'$ for all $(x, x') \in X^*$ is called a *translation function* if it has the following properties:

(1) $\tau(x, x) = e$;
(2) $\tau(x, x')\tau(x', x'') = \tau(x, x'')$;
(3) $\tau(x', x) = \tau(x, x')^{-1}$.

An action $X \curvearrowleft G$ is said to be *principal* if X admits a translation function $\tau : X^* \longrightarrow G$.

Definition 6.2. A G-bundle $\xi = (X, \pi, B)$ is said to be *principal* if the action on X is principal.

Example 6.1. The action $B \times G \curvearrowleft G$ defined by $((b, g'), g) \longmapsto (b, g'g)$ is principal. Actually,

$$((b, g), (b', g')) \in (B \times G)^* \Longleftrightarrow b = b' \quad \text{(since } (b, g')g = (b, g'g))$$

and the translation function τ has the form

$$\tau((b, g), (b, g')) = g^{-1}g'.$$

Then the corresponding principal G-bundle is the product bundle $(B \times G, \pi, B)$ where $\pi(b, g) = b$.

Proposition 6.1 (cf. [H2])**.** *Let $\xi = (X, \pi, B)$ be a principal G-bundle. Then ξ is a bundle with fiber G.*

Proof. Since the following diagram is commutative

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d] & X \arrow[d, "\pi"] \\
X/G \arrow[r, "\approx"'] & B
\end{tikzcd}$$

for each $x \in \pi^{-1}(b)$, define a map $f : G \longrightarrow \pi^{-1}(b) \approx G(x)$ by $g \longmapsto xg$. Then, the inverse map of f is $\pi^{-1}(b) \longrightarrow G$ defined by $xg \longmapsto \tau(x, xg)$. $\square$

Remark 16. In some sense (e.g., G is a compact Lie group and X is regular), the notions of a principal G-bundle and of a free G-action $G \curvearrowright X$ are equivalent (see, [B]).

Theorem 6.1 (cf. [H2])**.** *Every morphism between two principal G-bundles over B is an isomorphism.*

Proof. Suppose $f : (X_1, \pi_1, B) \longrightarrow (X_2, \pi_2, B)$ is a morphism of two principal G-bundles. Take $x_1, x_1' \in X_1$, let $f(x_1) = f(x_1')$. Then $\pi_1(x_1) = \pi_2 f(x_1') = \pi_1(x_1')$, so $(x_1, x_1') \in X_1^*$ and $x_1 g = x_1'$ for some $g \in G$. Further, one has that $f(x_1) = f(x_1') = f(x_1)g$, so $g = 1$ and $x_1 = x_1'$. Thus, f is injective. Now, let $x_2 \in X_2$. Then there is a $x_1 \in X_1$ such that $\pi_1(x_1) = \pi_2(x_2)$. Further, $\pi_2(x_2) = \pi_1(x_1) = \pi_2 f(x_1)$ and so $(f(x_1), x_2) \in X_2^*$. Since $f(x_1)g = x_2$ for some $g \in G$, one has $f(x_1 g) = x_2$. Thus, f is surjective.

Next, it needs to show that f^{-1} is continuous. Let $f(x_1) = x_2$ in X_2, and let U be an open neighborhood of x_1 in X_1. Then there are open neighborhoods V of x_1 and H of 1 in G such that $VH = \{xh | x \in V, h \in H\} \subset U$ since $X_1 \curvearrowleft G$ is continuous, and there is also an open neighborhood W of x_2 in X_2 such that $\tau_2((W \times W) \cap X_2^*) \subset H$, where τ_2 is the translation function of X_2^*. Further, one may replace V by $V \cap f^{-1}(W)$ such that $f(V) \subset W$. Since π_1 is open, one has $\pi_1(V)$ is an open neighborhood of $\pi_1(x_1) = \pi_2(x_2)$ in B, so one may replace W by $W \cap \pi_2^{-1}(\pi_1(V))$ such that $\pi_2(W) = \pi_1(V)$. Now, for each $x_2' \in W$, take $x_1' \in V$ such that $\pi_1(x_1') = \pi_2(x_2')$. Then $f(x_1'), x_2' \in W$ and $f(x_1')h = x_2'$ for some $h \in H$. Since $x_2' = f(x_1')h = f(x_1' h)$ and $x_1' h \in VH \subset U$, one has $f^{-1}(W) \subset U$. Therefore, f^{-1} is continuous at each point in X_2. □

Definition 6.3. A principal G-bundle $\xi = (X, \pi, B)$ is *numerable* if there is a numerable open cover $\{U_i\}$ of B such that $\xi|_{U_i}$ is trivial for each U_i. Note that a numerable open cover $\{U_i\}$ of B means that there exists a locally finite partition of unity $\{\lambda_i\}$ such that the closure $\overline{\lambda_i^{-1}((0,1])} \subset U_i$.

Lemma 6.1 (cf. [H2])**.** *Let $\xi = (X, \pi, B)$ be a numerable principal G-bundle over B and let $f : B' \longrightarrow B$ be a continuous map. Then $f^*(\xi)$ is also a numerable principal G-bundle over B', where $f^*(\xi) = (X', \pi', B')$ is the pullback of ξ via f, which has as base space B', as total space which is the subspace $\{(b', x) \in B' \times X | f(b') = \pi(x)\}$ of $B' \times X$, and as projection $\pi'(b', x) = b'$.*

Proof. This directly follows from the definitions of a numerable principal G-bundle and $f^*(\xi)$. □

The following theorem characterizes the homotopy property of numerable principal G-bundles. We would like to refer to [H2, Section 9, Chapter 4] for a detail proof.

Theorem 6.2. *Let $\xi = (X, \pi, B)$ be a numerable principal G-bundle over B and $f_1 \simeq f_2 : B' \longrightarrow B$ be two homotopic maps. Then $f_1^*(\xi)$ and $f_2^*(\xi)$ are isomorphic as numerable principal G-bundles.*

It is well-known that a Hausdorff space is paracompact iff each open cover is numerable. Now let $\mathcal{H}$ denote the category of all paracompact Hausdorff spaces and homotopy classes of maps. Given a space B in $\mathcal{H}$, let $P_G(B)$ denote the set of all isomorphism classes of numerable principal G-bundles over B. For a homotopy class

$[f] : B' \longrightarrow B$, one may define a map $P_G([f]) : P_G(B) \longrightarrow P_G(B')$ by $P_G([f])(\{\xi\}) = \{f^*(\xi)\}$ for each class $\{\xi\} \in P_G(B)$. By Theorem 6.2, $P_G([f])$ is well-defined. Let $\mathcal{P}_G$ denote the category consisting of all $P_G(B)$ and $P_G(B)([f])$.

Theorem 6.3 (cf. [H2]). *P_G is a cofunctor from $\mathcal{H}$ to $\mathcal{P}_G$.*

Proof. Let $[f] : B' \longrightarrow B$ and $[g] : B'' \longrightarrow B'$ be two homotopy classes in $\mathcal{H}$, and let $\xi = (X, \pi, B)$ be a numerable principal G-bundle over B. Then one has that $\{(gf)^*(\xi)\} = \{f^*(g^*(\xi))\}$, so $P_G([g][f]) = P_G([f])P_G([g])$. Obviously, $P_G([\mathrm{id}_B])$ is the identity on $P_G(B)$. □

Exercise 11. If $f : B' \xrightarrow{\simeq} B$ is a homotopy equivalence, then $P_G([f]) : P_G(B) \longrightarrow P_G(B')$ is bijective.

Exercise 12. Suppose that B is contractible. Then each numerable principal G-bundle over B is trivial.

Now let $\xi_0 = (X_0, \pi_0, B_0)$ be a given numerable principal G-bundle. For each B in $\mathcal{H}$, one may define $\Phi_{\xi_0}(B) : [B, B_0] \longrightarrow P_G(B)$ by $\Phi_{\xi_0}(B)([f]) = \{f^*(\xi_0)\}$, where $[B', B]$ denotes the set of homotopy classes of all maps from B' to B. Obviously, $\Phi_{\xi_0}(B)$ is well-defined.

Definition 6.4. A numerable principal G-bundle $\xi_0 = (X_0, \pi_0, B_0)$ is said to be *universal* if $\Phi_{\xi_0}(B) : [B, B_0] \longrightarrow P_G(B)$ is a bijection for each B in $\mathcal{H}$.

The above definition implies that

Proposition 6.2. *A necessary and sufficient condition that a numerable principal G-bundle $\xi_0 = (X_0, \pi_0, B_0)$ is universal is that the following conditions are satisfied*

(a) *For any numerable principal G-bundle $\xi = (X, \pi, B)$, there must be a map $f : B \longrightarrow B_0$ such that ξ is exactly the pullback from (X_0, π_0, B_0) via f.*

(b) *If $f_1, f_2 : B \longrightarrow B_0$ are two continuous maps such that $f_1^*(\xi_0) \cong f_2^*(\xi_0)$, then $f_1 \simeq f_2$.*

Remark 17.

(1) Generally, the universal principal G-bundle is denoted by (EG, π, BG), and BG is called the *classifying space* of G.

(2) For a numerable principal G-bundle (X, π, B), if X is contractible and admits a free G-action, then (X, π, B) is universal.

Given a topological group G, one can obtain the associated universal principal G-bundle by using Milnor construction as follows. First, EG is defined as an infinite join

$$EG = G * G * \cdots * G * \cdots ,$$

each element of which is denoted as $< g, t >$ and written as

$$< g, t >= (t_0 g_0, t_1 g_1, ..., t_i g_i, ...), \quad g_i \in G, \ t_i \in [0, 1]$$

such that only a finite number of $t_i \neq 0$ and $\sum t_i = 1$. A right G-action on EG is given by $< g, t > g' =< gg', t >$. Then a topology on EG such that EG is a G-space is given in the following way: Consider maps $t_i : EG \longrightarrow [0, 1]$ by mapping $(t_0 g_0, t_1 g_1, ..., t_i g_i, ...)$ to t_i, and $g_i : t_i^{-1}((0, 1]) \longrightarrow G$ by mapping

$(t_0g_0, t_1g_1, ..., t_ig_i, ...)$ to g_i. Associating with the G-action, one can find the following relations:

$$g_i(xg) = g_i(x)g, \quad t_i(xg) = t_i(x)$$

where $x \in EG$ and $g \in G$. Furthermore, one may choose the smallest topology on EG such that t_i and g_i are continuous.

Now EG becomes a topological space and admits a free G-action. Then $EG \longrightarrow EG/G = BG$ is the desired universal principal G-bundle, which is called the Milnor construction.

Remark 18. EG and BG can be filtered by subspaces

$$\cdots \subset EG(n) \subset EG(n+1) \subset \cdots \subset EG$$
$$\cdots \subset BG(n) \subset BG(n+1) \subset \cdots \subset BG$$

where $(t_0g_0, t_1g_1, ..., t_ig_i, ...) \in EG(n)$ if $t_i = 0$ for $i > n$, and $BG(n) = EG(n)/G$.

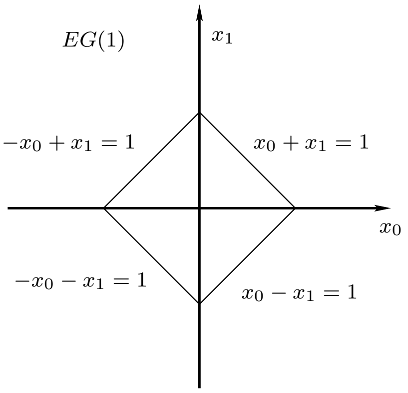
Example 6.2. (see [H2, Example 11.3]) When $G = \mathbb{Z}_2 = \{\pm 1\}$, by the Milnor construction, one has that

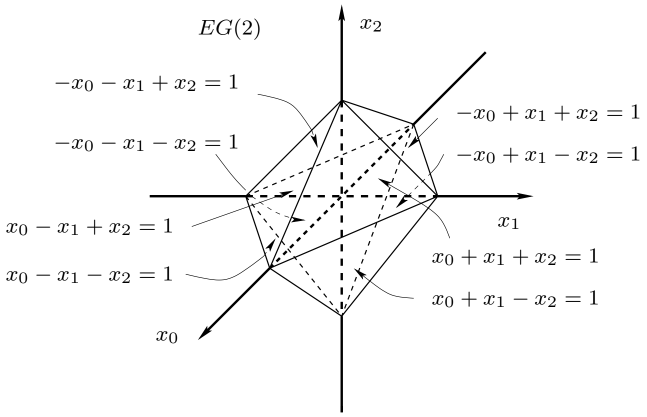
$$EG(n) = \{(t_0g_0, t_1g_1, ..., t_ng_n, 0, 0, ...) \in EG | t_0 + t_1 + \cdots + t_n = 1, t_i \in [0,1], g_i \in G\}.$$

Then it is easy to see that $EG(n)$ is homeomorphic to an n-sphere S^n. In fact, let $x_i = t_ig_i$, one then has that

$$EG(n) = \{(x_0, x_1, ..., x_n, 0, 0, ...) \in \mathbb{R}^\infty | x_0g_0 + x_1g_1 + \cdots + x_ng_n = 1, |x_i| \in [0,1], g_i \in \mathbb{Z}_2\}$$

so $EG(n)$ is exactly the boundary of the dual of an n-cube I^n (as a simple convex polytope) in $\mathbb{R}^\infty$. Thus, $EG(n)$ is homeomorphic to S^n. For example, when $n = 1, 2$, $EG(n)$ can be explicitly shown as follows:





Furthermore, the right action of G on $EG(n)$ induces the antipodal action of G on S^n, so $BG(n)$ is homeomorphic to $\mathbb{R}P^n$. Therefore, one has that up to homeomorphism

$$\cdots \subset EG(n) = S^n \subset EG(n+1) = S^{n+1} \subset \cdots \subset EG = S^\infty$$
$$\cdots \subset BG(n) = \mathbb{R}P^n \subset BG(n+1) = \mathbb{R}P^{n+1} \subset \cdots \subset BG = \mathbb{R}P^\infty.$$

Similarly, one has that

Example 6.3. (see [H2, Example 11.4]). When $G = S^1$, one has that up to homeomorphism

$$\cdots \subset EG(n) = S^{2n+1} \subset EG(n+1) = S^{2n+3} \subset \cdots \subset EG = S^\infty$$
$$\cdots \subset BG(n) = \mathbb{C}P^n \subset BG(n+1) = \mathbb{C}P^{n+1} \subset \cdots \subset BG = \mathbb{C}P^\infty$$

where the S^1-action on $EG(n) = S^{2n+1}$ is given by

$$(z_0, z_1, ..., z_n)e^{i\theta} = (e^{i\theta}z_0, e^{i\theta}z_1, ..., e^{i\theta}z_n)$$

for each $e^{i\theta} \in S^1$.

Remark 19. More generally, when $G = \mathbb{Z}_2^k$ or $(S^1)^k$ with $k > 1$, by directly using the filtration of the Milnor construction of EG in Remark 18, it is not easy to determine what EG is. For example, if $G = \mathbb{Z}_2^2$, then $EG(1)$ is only a graph $K_{4,4}$ rather than a closed manifold.

Proposition 6.3. (*see* [H2, Exercise 10, Chapter 4]). *Suppose that $\xi_i = (X_i, \pi_i, B_i)$ is a numerable principal G_i-bundle over B_i, $i = 1, 2$. Then $\xi_1 \times \xi_2 = (X_1 \times X_2, \pi_1 \times \pi_2, B_1 \times B_2)$ has the structure of a numerable principal $G_1 \times G_2$-bundle. Moreover, if ξ_i is a numerable universal principal G_i-bundle, $i = 1, 2$, then $\xi_1 \times \xi_2$ is a numerable universal principal $G_1 \times G_2$-bundle.*

Proof. It is immediate by Definitions 6.2-6.3 that $\xi_1 \times \xi_2$ has the structure of a numerable principal $G_1 \times G_2$-bundle.

Suppose that ξ_i is a universal principal G_i-bundle, $i = 1, 2$. To show that $\xi_1 \times \xi_2$ is a universal principal $G_1 \times G_2$-bundle, it suffices to prove that $\Phi_{\xi_1\times\xi_2}(B) : [B, B_1 \times B_2] \longrightarrow P_{G_1\times G_2}(B)$ is a bijection for each B in $\mathcal{H}$. Let $\xi = (X, \pi, B)$ be a numerable principal $G_1 \times G_2$-bundle. Then there is a numerable principal $G_{2^{2-i}}$-bundle $\xi|_{G_i} = (X/G_i, \pi_{G_i}, B)$, $i = 1, 2$, where π_{G_i} is the result of factoring π by the orbit map $X \longrightarrow X/G_i$. Since ξ_i is a numerable universal principal G_i-bundle, $i = 1, 2$, one has that there is a continuous map $f_i : B \longrightarrow B_i$ ($i = 1, 2$) such that $\xi|_{G_1} \cong f_2^*(\xi_2)$ and $\xi|_{G_2} \cong f_1^*(\xi_1)$. Further, the map $f : B \longrightarrow B_1 \times B_2$ defined by $f(b) = (f_1(b), f_2(b))$ determines a numerable universal principal $G_1 \times G_2$-bundle $f^*(\xi_1 \times \xi_2)$ over B. It is easy to see that one may define a morphism between ξ and $f^*(\xi_1 \times \xi_2)$, so $\xi \cong f^*(\xi_1 \times \xi_2)$ by Theorem 6.1. Thus, $\Phi_{\xi_1\times\xi_2}(B)$ is surjective.

To prove $\Phi_{\xi_1\times\xi_2}(B)$ is injective, let $f = (f_1, f_2), g = (g_1, g_2) : B \longrightarrow B_1 \times B_2$ be two continuous maps such that $f^*(\xi_1 \times \xi_2) \cong g^*(\xi_1 \times \xi_2)$. Now let h be an isomorphism from $f^*(\xi_1 \times \xi_2)$ to $g^*(\xi_1 \times \xi_2)$. Then one has the following commutative diagram

$$\begin{tikzcd}
E(f^*(\xi_1 \times \xi_2)) \arrow[rr, "h"] \arrow[dr] \arrow[dd] & & E(g^*(\xi_1 \times \xi_2)) \arrow[dl] \arrow[dd] \\
& B & \\
E(f_i^*(\xi_i)) \arrow[ur] \arrow[rr, dotted, "\bar{h}_i"] & & E(g_i^*(\xi_i)) \arrow[ul]
\end{tikzcd}$$

where $E(\eta)$ denotes the total space of a bundle η. It is easy to see that one may define a morphism $\bar{h}_i$ from $f_i^*(\xi_i)$ to $g_i^*(\xi_i)$ by h in the above commutative diagram.

By Theorem 6.1, $\bar{h}_i$ is an isomorphism from $f_i^*(\xi_i)$ to $g_i^*(\xi_i)$, so $f_i \simeq g_i, i = 1,2$ because ξ_i is a numerable universal principal G_i-bundle, $i = 1,2$. Thus $f \simeq g$, and so $\Phi_{\xi_1\times\xi_2}(B)$ is injective. $\square$

As a consequence, one has

Corollary 6.1. *There is a universal principal $\mathbb{Z}_2^k$-bundle $(E\mathbb{Z}_2^k, \pi, B\mathbb{Z}_2^k)$ such that $E\mathbb{Z}_2^k = (S^\infty)^k$ and $B\mathbb{Z}_2^k = (\mathbb{R}P^\infty)^k$. Similarly, there is a universal principal $(S^1)^k$-bundle $(E(S^1)^k, \pi, B(S^1)^k)$ such that $E(S^1)^k = (S^\infty)^k$ and $B(S^1)^k = (\mathbb{C}P^\infty)^k$.*

6.2. **The Borel construction.** Given a topological group G, let $G \longrightarrow EG \longrightarrow BG$ be the universal principal G-bundle of G. Let $G \curvearrowright X$ be a G-action on the topological space X. Define the action of G on the product $EG \times X$ by $(u,x) \longmapsto (ug^{-1}, gx)$. Then the orbit space of the action, denoted by

$$X_G = EG \times_G X,$$

is called the *Borel construction.*

There is a natural projection $p : X_G \longrightarrow BG$ defined by $[u,x] \longmapsto [u]$. It is easy to see that $p : X_G \longrightarrow BG$ is a fibration with the typical fiber X.

Remark 20.
1. If X is contractible, then $X_G \simeq BG$.
2. Generally, $\sigma : X_G \longrightarrow X/G$ defined by $[u,x] \longmapsto [x]$ is not a fibration. This is because $\sigma^{-1}([x]) = EG/G_x$ depends upon the choice of $x \in X$. However, if the action $G \curvearrowright X$ is free, then each $G_x = \{e\}$, so $\sigma : X_G \longrightarrow X/G$ is a fibration with the typical fiber EG, and $X_G \simeq X/G$.

6.3. **Equivariant cohomology.** Equivariant cohomology was introduced by Borel in 1960's, and it has become one of main tools in the study of transformation groups.

Given $G \curvearrowright X$, applying ordinary cohomology to the Borel construction X_G gives the *equivariant cohomology*

$$H_G^*(X) := H^*(X_G)$$

The fibration $p : X_G \longrightarrow BG$ with fiber X induces the ring homomorphism

$$p^* : H^*(BG) \longrightarrow H_G^*(X),$$

so that $H_G^*(X)$ has not only a $H^*(BG)$-module structure, but also an algebraic structure over $H^*(BG)$. Actually, the scalar multiplication can given by $r\cdot x = p^*(r)x$ where $r \in H^*(BG)$ and $x \in H_G^*(X)$.

Remark 21.
1. Generally, the restriction to a typical fiber
$$H_G^*(X) \longrightarrow H^*(X)$$
is not an epimorphism.
2. If X is contractible, then $H_G^*(X) \cong H^*(BG)$. In particular, $H_G^*(\{\text{pt}\}) = H^*(BG)$.
3. If $G \curvearrowright X$ is free, then $H_G^*(X) \cong H^*(X/G)$.
4. Let $H < G$ be closed. Then there is a fibration $\pi : X_H \longrightarrow X_G$ with fiber G/H, which induces a ring homomorphism $\pi^* : H_G^*(X) \longrightarrow H_H^*(X)$.

(5) Since $G \curvearrowright X^G$ is trivial, $H^*_G(X^G)$ is a free $H^*(BG)$-module.

Example 6.4. The computation of $H^*_G(\{\text{pt}\}) = H^*(BG)$ (cf. [CF]).

a) When $G = \mathbb{Z}_2$, $H^*(BG; \mathbb{Z}_2) = \mathbb{Z}_2[t]$ where $t \in H^1(BG; \mathbb{Z}_2)$ is the generator. More generally, when $G = \mathbb{Z}_2^k$, $H^*(BG; \mathbb{Z}_2) = \mathbb{Z}_2[t_1, ..., t_k]$ where the t_i's are generators in $H^1(BG; \mathbb{Z}_2)$.

b) When $G = S^1$, $H^*(BG; \mathbb{Z}) = \mathbb{Z}[t]$ where $t \in H^2(BG; \mathbb{Z})$ is the generator. More generally, when $G = (S^1)^k$, $H^*(BG; \mathbb{Z}) = \mathbb{Z}[t_1, ..., t_k]$ where the t_i's are generators in $H^2(BG; \mathbb{Z})$.

c) When $G = (\mathbb{Z}_p)^k$ (p-torus of rank k) where p is an odd prime,

$$H^*(BG; \mathbb{F}_p) = \Lambda(a_1, ..., a_k) \otimes \mathbb{F}_p[b_1, ..., b_k]$$

where $\Lambda(a_1, ..., a_k)$ denotes the exterior algebra over $\mathbb{F}_p$ generated by elements $a_1, ..., a_k$ of degree 1, and $\mathbb{F}_p[b_1, ..., b_k]$ the polynomial algebra generated by elements $b_1, ..., b_k$ of degree 2, and $\mathbb{F}_p$ is a field of characteristic p.

Example 6.5. Let $G = S^1 = \{g \in \mathbb{C} | |g| = 1\}$ and $X = S^2 = \{(z, y) \in \mathbb{C} \times \mathbb{R} | |z|^2 + y^2 = 1\}$. Consider $S^1 \curvearrowright S^2$ defined by $(g, (z, y)) \longmapsto (gz, y)$. This is an rotation of S^2 along the axis joining the south pole and the north pole. Then $X^G = (S^2)^{S^1} = \{(0, \pm 1)\}$, and $X/G = S^2/S^1$ is an interval.

Let $X_+ = \{(z, y) \in \mathbb{C} \times \mathbb{R} | |z|^2 + y^2 = 1, y > 0\}$ and $X_- = \{(z, y) \in \mathbb{C} \times \mathbb{R} | |z|^2 + y^2 = 1, y < 0\}$. Then $\{X_+, X_-\}$ forms an open covering of X. Furthermore,

$$X_G = ES^1 \times_{S^1} X = ES^1 \times_{S^1} (X_+ \cup X_-) = (ES^1 \times_{S^1} X_+) \cup (ES^1 \times_{S^1} X_-).$$

Obviously, $ES^1 \times_{S^1} X_\pm \simeq (\{\text{pt}\})_G$ and $(ES^1 \times_{S^1} X_+) \cap (ES^1 \times_{S^1} X_-) \simeq \{\text{pt}\}$. By Mayer-Vietoris sequence, it is easy to see that

$$H^i_{S^1}(S^2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i \text{ is even} \\ 0 & i \text{ is odd} \end{cases}$$

so $H^*_{S^1}(S^2; \mathbb{Z}) = \mathbb{Z}[a_1, a_2]/(a_1 a_2)$ with $\deg a_i = 2$.

Exercise 13. Let X and Y be two G-spaces. If X and Y are G-homotopically equivalent, then $H^*_G(X)$ and $H^*_G(Y)$ are isomorphic.

6.4. **Localization theorem.**

Definition 6.5. Let R be a commutative ring with identity 1. Let S be a multiplicatively closed subset of R such that $1 \in S$. Then

$$S^{-1}R = \left\{ \{\frac{r}{s}\} | r \in R, s \in S \right\}$$

is called the *localization* of R with respect to S, where $\left\{\frac{r}{s}\right\}$ is an equivalence class of fractions defined as follows:

$$\frac{r_1}{s_1} \sim \frac{r_2}{s_2} \Longleftrightarrow (r_1 s_2 - r_2 s_1)s = 0 \text{ for some } s \in S.$$

Clearly, $S^{-1}R$ is also a ring.

Example 6.6. Let $R = \mathbb{Z}$. When $S = \mathbb{Z} - (0)$, $S^{-1}R = \mathbb{Q}$. When $S = \mathbb{Z} - (p)$ where p is a prime, $S^{-1}R = \mathbb{Z}_{(p)}$.

Definition 6.6. Let $\mathcal{M}$ be a R-module where R is a commutative ring with identity 1. Let S be a multiplicatively closed subset of R such that $1 \in S$. Then

$$S^{-1}\mathcal{M} = \Big\{\{\frac{m}{s}\} | m \in \mathcal{M}, s \in S\Big\}$$

is called the *localization* of $\mathcal{M}$ with respect to S, where $\{\frac{m}{s}\}$ is an equivalence class of fractions defined as follows:

$$\frac{m_1}{s_1} \sim \frac{m_2}{s_2} \Longleftrightarrow (m_1s_2 - m_2s_1)s = 0 \text{ for some } s \in S.$$

Clearly, $S^{-1}\mathcal{M}$ is a $S^{-1}R$-module.

Now consider $G \curvearrowright X$ where G is a compact Lie group and X is compact. Then we have a fibration $p : X_G \longrightarrow BG$ with fiber X. Let S be a multiplicatively closed subset of $H^*(BG)$ with $1 \in S$. Set

$$X^S = \{x \in X | \text{ for any } s \in S,\ i_x^*(s) \neq 0 \text{ in } H^*(BG_x)\}$$

where $i_x^* : H^*(BG) \longrightarrow H^*(BG_x)$ is induced by the natural inclusion $i_x : BG_x \hookrightarrow BG$.

Theorem 6.4 (Localization theorem, cf. [H1])**.** *The localized restriction homomorphism*

$$S^{-1}H_G^*(X) \longrightarrow S^{-1}H_G^*(X^S)$$

is an isomorphism.

Outline of Proof.

- The special case $X^S = \emptyset$. It suffices to show that $\exists s \in S$ such that $p^*(s) = 0$ in $H_G^*(X)$. By Theorem 5.1, for each point $x \in X$, $G(x)$ has invariant open neighborhood U in X such that $G(x)$ is an equivariant retract of U. Since X is compact, there are finitely many such neighborhoods $U_1, ..., U_l$ covering X, with U_i retracting to $G(x_i)$. Furthermore, $X^S = \emptyset$ implies that there is one $s_i \in S$ such that $i_{x_i}^*(s_i) = 0$ in $H^*(BG_{x_i}) = H_G^*(G(x_i))$, so is also in $H_G^*(U_i)$. Taking $s = s_1 \cdots s_l$, one then has $p^*(s) = 0$ in $H_G^*(X)$.
- For the general case, it suffices to show that $S^{-1}H_G^*(X, X^S) = 0$. Namely, one needs to prove that for each $x \in H_G^*(X, X^G)$ there is one $s \in S$ such that $s \cdot x = 0$. With no loss assume that $x \in H_G^k(X, X^S) = H^n(X_G, X_G^S) = H^n(p^{-1}(BG(k)), X_G^S \cap p^{-1}(BG(k)))$ for $k > n$. Furthermore, since $p^{-1}(BG(k))$ is compact, there is an invariant neighborhood W of X^S such that

$$x \in \mathrm{Im}\{H_G^*(X, V) \longrightarrow H_G^*(X, X^S)\}.$$

  On the other hand, one may find an invariant compact subspace $Y \subset X - X^S$ such that $W \cup \mathrm{int}Y = X$. Since $Y^S = \emptyset$, there is a $s \in S$ such that $p^*(s) \in \mathrm{Im}\{H_G^*(X, Y) \longrightarrow H_G^*(X, \mathrm{int}Y) \longrightarrow H_G^*(X)\}$. Moreover, $s \cdot x$ lies in the image of $0 = H_G^*(X, W \cup \mathrm{int}Y) \longrightarrow H_G^*(X, X^S)$. Thus, $s \cdot x = 0$. $\square$

Remark 22. More generally, if X is assumed to be paracompact and has finite cohomological dimension, then there is the following localization theorem (see, [H1, Theorem III.1′]: Let $s \in S$. Then the localized restriction homomorphism

$$S^{-1}H_G^*(X) \longrightarrow S^{-1}H_G^*(X^s)$$

is an isomorphism. If X consists of only finite orbit types, then

$$S^{-1}H_G^*(X) \longrightarrow S^{-1}H_G^*(X^S)$$

is also an isomorphism.

Now let $G = T^k$ or $\mathbb{Z}_p^k$ where p is a prime. Take $S = H^*(BG; \mathbf{k}) - \{0\}$ where

$$\mathbf{k} = \begin{cases} \mathbb{Q} & \text{if } G = T^k \\ \mathbb{Z}_p & \text{if } G = \mathbb{Z}_p^k. \end{cases}$$

For any proper subgroup $H < G$, it is easy to see that $S \cap \ker\{H^*(BG; \mathbf{k}) \longrightarrow H^*(BH; \mathbf{k})\}$ is nonempty, so $X^S = X^G$. As a consequence, one has

Corollary 6.2 (Borel)**.** *Suppose that X is a paracompact G-space with finite cohomological dimension. Then the localized restriction homomorphism*

$$S^{-1}H_G^*(X; \mathbf{k}) \longrightarrow S^{-1}H_G^*(X^G; \mathbf{k})$$

is an isomorphism.

7. DAVIS-JANUSZKIEWICZ THEORY

During the last two decades, the theory of toric varieties has produced a strong penetration among many mathematical areas, such as symplectic geometry, commutative algebra, toric topology etc., because toric varieties admit equivalent descriptions arising naturally in those areas (e.g., see [BP], [Si]). In 1991, Davis and Januszkiewicz [DJ] introduced and studied two kinds of topological versions of toric varieties: small covers and (quasi-) toric manifolds.

Davis-Januszkiewicz theory gives a direct link between equivariant topology and polytopes of combinatorics. From the viewpoint of topology, Davis-Januszkiewicz theory contains two key points as follows:

- One point is on geometric topology. Each small cover or quasi-toric manifold can be reconstructed from certain data.
- The other point is on algebraic topology. The most essential information of algebraic topology for small covers and quasi-toric manifolds, such as equivariant cohomology, betti numbers etc., can be explicitly expressed in terms of combinatorics.

Following the notions of [DJ], let $G_d^n = \begin{cases} \mathbb{Z}_2^n & \text{if } d = 1 \\ T^n & \text{if } d = 2 \end{cases}$, $\mathbb{F}_d^n = \begin{cases} \mathbb{R}^n & \text{if } d = 1 \\ \mathbb{C}^n & \text{if } d = 2 \end{cases}$, and $R_d = \begin{cases} \mathbb{Z}_2 & \text{if } d = 1 \\ \mathbb{Z} & \text{if } d = 2 \end{cases}$

Definition 7.1. A closed manifold M^{dn} of dimension $dn(d=1,2)$ is called a *locally standard G_d^n-manifold* if M^{dn} admits a locally standard G_d^n-action, i.e., for each point $x \in M^{dn}$, there is an invariant neighborhood V_x of x such that up to an automorphism of G_d^n, V_x is weakly equivariantly homeomorphic to an invariant open subset of the standard G_d^n-representation space $\mathbb{F}_d^n$ (for the definition of the standard G_d^n-representation, see Example 2.3).

Lemma 7.1. *Let M^{dn} be a locally standard G_d^n-manifold. Then the orbit space $Q^n = M^{dn}/G_d^n$ is an n-dimensional compact nice manifold with corners.*

Proof. Obviously, Q is a compact manifold with corners. Let $\pi_d : M^{dn} \longrightarrow Q^n$ be the orbit map. Given $x \in M^{dn}$, it is easy to see that the rank of the isotropy subgroup G_x at x is equal to the codimension of $\pi_d(x)$ in Q^n. Further, it easily follows that if the codimension of $\pi_d(x)$ is 2, then $\pi_d(x)$ is exactly in the relative interior of the intersection of two facets of Q^n. Thus, Q^n is nice. $\square$

Definition 7.2. Let M^{dn} be a locally standard G_d^n-manifold. If its orbit space Q is a simple convex polytope, then M^{dn} is called a *small cover* if $d=1$ and a *quasi-toric manifold* if $d=2$.

The following are typical examples.

Example 7.1. The action $\mathbb{Z}_2^n \curvearrowright \mathbb{R}P^n$ defined by

$$((g_1,...,g_n),[x_0,x_1,...,x_n]) \longmapsto [x_0,(-1)^{g_1}x_1,...,(-1)^{g_n}x_n]$$

is locally standard and its orbit space is an n-simplex Δ^n. So $\mathbb{R}P^n$ is a small cover over Δ^n.

Example 7.2. The action $T^n \curvearrowright \mathbb{C}P^n$ defined by

$$((e^{\theta_1 i},...,e^{\theta_n i}),[z_0,z_1,...,z_n]) \longmapsto [z_0,e^{\theta_1 i}z_1,...,e^{\theta_n i}z_n]$$

is locally standard and its orbit space is an n-simplex Δ^n. So $\mathbb{C}P^n$ is a quasi-toric manifold over Δ^n.

Exercise 14. Show that $(S^1)^n$ is a small cover over an n-cube, and $(S^2)^n$ is a quasi-toric manifold over an n-cube.

Example 7.3. The sphere S^n with the standard $\mathbb{Z}_2^n$-action by

$$((g_1,...,g_n),(x_0,x_1,...,x_n)) \longmapsto (x_0,(-1)^{g_1}x_1,...,(-1)^{g_n}x_n)$$

is a locally standard $\mathbb{Z}_2^n$-manifold, and it is not a small cover because the quotient space is not a polytope except for $n=1$.

The sphere $S^{2n} = \{(x,z_1,...,z_n) \in \mathbb{R}\times\mathbb{C}^n | x^2 + \sum_{i=1}^n |z_i|^2 = 1\}$ with the standard T^n-action by

$$((e^{\theta_1 i},...,e^{\theta_n i}),(x,z_1,...,z_n)) \longmapsto (x,e^{\theta_1 i}z_1,...,e^{\theta_n i}z_n)$$

is also a locally standard T^n-manifold, but it is not a quasi-toric manifold because the quotient space is not a polytope except for $n=1$.

The following is an example of non-locally standard action.

Example 7.4 (cf. [LY2]). Consider the $(\mathbb{Z}_2)^2$-action on the unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ given by two commutative involutions τ_1 and τ_2, where τ_1 is the reflection of S^2 about the xy-plane, τ_2 is the antipodal map. Obviously, this $\mathbb{Z}_2^2$-action in a small neighborhood of the north pole and the south pole is not locally standard.

7.1. **Characteristic function and reconstruction.** Let $\pi : M^{dn} \longrightarrow P^n$ be a small cover or a quasi-toric manifold over a simple convex n-polytope P^n where $d=1$ or 2.

Take a k-face F^k, an easy observation shows (see also [DJ, Lemma 1.3]) that

$$\pi^{-1}(F^k) \longrightarrow F^k$$

is still a k-dimensional small cover or a $2k$-dimensional quasi-toric manifold. In particular, for any $x \in \pi^{-1}(\mathrm{int}F^k)$, G_x is independent of the choice of x, denoted by G_F. Note that G_F is isomorphic to G_d^{n-k}, and G_F fixes $\pi^{-1}(F^k)$ in M^{dn}. In the case $k=n-1$, F^k is a facet and G_F has rank 1. For $d=1$, G_F uniquely corresponds to a nonzero vector v^1 in $\mathbb{Z}_2^n$. For $d=2$, G_F uniquely corresponds to a primitive vector $v^2=(v_1,...,v_n) \in \mathbb{Z}^n$ (i.e., v^2 is only unique up to multiplication by ± 1) such that G_F can be written as

$$G_F = \{(e^{2\pi i v_1 \phi}, ..., e^{2\pi i v_n \phi}) \in T^n | \phi \in \mathbb{R}\}.$$

Let $\mathcal{F}(P) = \{F_1, ..., F_m\}$ be the set of facets of P^n. Then there is a natural map (called *characteristic function*)

$$\lambda : \mathcal{F}(P) \longrightarrow R_d^n$$

by mapping each facet to a nonzero vector in $\mathbb{Z}_2^n$ (or a primitive vector in $\mathbb{Z}^n$) with the following property:

($\star$) For each vertex v in P^n, there are n facets, say $F_1, ..., F_n$ with $v = F_1 \cap \cdots \cap F_n$ such that

$$\det(\lambda(F_1), ..., \lambda(F_n)) = \begin{cases} 1 & \text{if } d=1 \\ \pm 1 & \text{if } d=2. \end{cases}$$

Remark 23. Since P^n is simple, for each k-face F, there are $n-k$ facets, say $F_1, ..., F_{n-k}$ such that $F = F_1 \cap \cdots \cap F_{n-k}$ and $\pi^{-1}(F)$ is a transverse intersection of $\pi^{-1}(F_1)$, ..., $\pi^{-1}(F_{n-k})$. Then the group G_F determined by F is actually generated by $\lambda(F_1)$, ..., $\lambda(F_{n-k})$.

An interesting thing is that M^{dn} can be recovered from the pair (P^n, λ). Now let us state this reconstruction process as follows: define an equivalence relation $\sim$ on product $G_d^n \times P^n$ by

$$(g,p) \sim (h,q) \Longleftrightarrow \begin{cases} p=q \text{ and } g=h & \text{if } p=q \in \mathrm{int}P^n \\ p=q \text{ and } g^{-1}h \in G_F & \text{if } p=q \in F \subset \partial P^n. \end{cases}$$

Note that for each point $p \in \partial P^n$, there must be a unique face F that contains p in its relative interior. Then the quotient space $G_d^n \times P^n / \sim$ denoted by $M^{dn}(\lambda)$ becomes a closed manifold and admits a natural G_d^n-action by $(g, [h,p]) \longmapsto [gh, p]$.

Obviously, the orbit space of this action $G_d^n \curvearrowright M^{dn}$ is exactly P^n. On the other hand, the standard G_d^n-representation on $\mathbb{F}_d^n$ can be recovered in the following way: the natural projection $G_d^n \times \mathbb{R}_{\geq 0}^n \longrightarrow \mathbb{F}_d^n$ by $(g, x) \longmapsto gi(x)$ identifies $G_d^n \times \mathbb{R}_{\geq 0}^n/\sim$ with $\mathbb{F}_d^n$, where $i : \mathbb{R}_{\geq 0}^n \hookrightarrow \mathbb{F}_d^n$ is the natural inclusion, and the equivalence relation $\sim$ on $G_d^n \times \mathbb{R}_{\geq 0}^n$ is defined by

$$(g, x) \sim (h, y) \Longleftrightarrow x = y \text{ and } g^{-1}h \in G_{i(x)}$$

where $G_{i(x)}$ is the isotropy subgroup at $i(x)$ for the standard representation $G_d^n \curvearrowright \mathbb{F}_d^n$. Now it is easy to see from the fact above that the action $G_d^n \curvearrowright M^{dn}(\lambda)$ is locally standard. Therefore, $M^{dn}(\lambda)$ is a small cover or a quasi-toric manifold.

A simple argument shows (see also [DJ, Lemma 1.4]) that there is a continuous map $f : G_d^n \times P^n \longrightarrow M^{dn}$ such that for each $p \in P^n$, f maps $G_d^n \times \{p\}$ onto $\pi^{-1}(p)$. Furthermore, f descends to a G_d^n-equivariant homeomorphism $M^{dn}(\lambda) \longrightarrow M^{dn}$ covering the identity on P^n.

Theorem 7.1 (cf. [DJ])**.** *Let P^n be a simple convex polytope. Then all possible locally standard G_d^n-manifolds over P^n bijectively correspond to all possible characteristic functions $\lambda : \mathcal{F}(P^n) \longrightarrow R_d^n$.*

Remark 24. When $n \leq 3$, any simple convex polytope P^n admits a characteristic function. However, when $n \geq 4$, there is a simple convex polytope which admits no characteristic function, see [DJ, Nonexamples 1.22].

Remark 25. In the case $d = 1$, geometrically $M^n(\lambda)$ is exactly obtained by gluing 2^n copies of P^n along their boundaries via λ. This reconstruction of small covers provides a way of studying closed manifolds from pairs (P^n, λ). In [I], Izmestiev studied a class of 3-dimensional small covers such that each λ of characteristic functions on their orbit spaces is 3-colorable (i.e., the image of λ contains only three linearly independent elements of $\mathbb{Z}_2^3$), and showed that each such small cover can be formed from finitely many 3-dimensional tori with the canonical $\mathbb{Z}_2^3$-action under the operations of the equivariant connected sum and the equivariant Dehn surgery. In the general case, a description of topological types of all 3-dimensional small covers has been given in [LY1] by using six kinds of cut-and-paste operations.

Actually, the above reconstruction can be still carried out in the setting of locally standard G_d^n-manifolds. Suppose that M^{dn} is a locally standard G_d^n-manifold. Let $\pi : M^{dn} \to Q^n$ be the orbit map.

Lemma 7.2 (cf. [LY2])**.** (1) *∂Q^n is empty if and only if $G_d^n \curvearrowright M^{dn}$ is free.* (2) *If ∂Q^n is non-empty, then Q^n admits a characteristic function λ on its facets.*

Proof. (1) If the boundary of Q^n is empty, then Q^n is a closed manifold, so each open neighborhood of any point x in Q^n is identified with one in the interior of $\mathbb{R}_{\geq 0}^n$. Since the action $G_d^n \curvearrowright M^{dn}$ is locally standard, one has that the action $G_d^n \curvearrowright M^{dn}$ must be free. Conversely, if $G_d^n \curvearrowright M^{dn}$ is free, then it is well-known that Q^n is also a closed manifold, so ∂Q^n is empty.

(2) Suppose that Q^n has non-empty boundary. Then ∂Q^n is the union of its all facets. Similarly to the cases of small covers and quasi-toric manifolds, each facet F

of Q^n corresponds to a primitive vector $v_F \in R_d^n$, such that $\pi^{-1}(F)$ is fixed by the rank-one subgroup determined by v_F. So a characteristic function λ on Q^n can be defined as

$$\begin{aligned} \lambda : \{\text{facets of } Q^n\} &\longrightarrow R_d^n \\ F \quad &\longmapsto v_F \end{aligned}$$

satisfying the condition that whenever the intersection $\bigcap_i F_i \neq \varnothing$ of some facets is non-empty, all elements of the set $\{\lambda(F_i)\}$ are linearly independent in R_d^n. $\square$

Remark 26. If $G_d^n \curvearrowright M^{dn}$ is not free, then it is easy to see that the boundary of Q^n together with the λ on its facets gives the information of the non-free orbits, while the interior of Q^n corresponds to all free orbits.

If the boundary of Q is empty, then by Lemma 7.2, $G_d^n \curvearrowright M^{dn}$ is free, so $\pi : M^{dn} \longrightarrow Q^n$ is actually a principal G_d^n-bundle over Q^n.

If $\partial Q^n \neq \emptyset$, then by Lemma 7.2 there is a characteristic function λ on Q^n. However, generally the pair (Q^n, λ) is not sufficient to recover M^{dn}. To do that, one needs another data $\xi : E \longrightarrow Q^n$, which is a principal G_d^n-bundle over Q^n. This bundle is directly associated with M^{dn} and can be produced in the following way: Take a facet F of Q^n, one can obtain a closed submanifold $\pi^{-1}(F)$ of M^{dn}. Then the desired bundle is given by removing the union of small invariant tubular neighborhoods of all these $\pi^{-1}(F)$ in M^{dn}. One knows from [D] and [J] that such bundle is unique up to isomorphism.

Now let us use the λ and the principal bundle ξ over Q to reconstruct M^{dn} (also see [LM]). First, define an equivalence $\sim$ on E: for $x_1, x_2 \in E$,

$$x_1 \sim x_2 \Longleftrightarrow \begin{cases} \xi(x_1) = \xi(x_2) \in \mathrm{Int}(Q^n) = Q^n - \partial Q^n \text{ or} \\ \xi(x_1) = \xi(x_2) \in \partial Q^n \text{ and } x_1 = gx_2 \text{ for some } g \in G_F \end{cases}$$

where F is the closed pre-face of Q^n such that $\xi(x_1) = \xi(x_2)$ is contained in the relative interior of F (note that there must be some facets $F_{i_1}, ..., F_{i_r}$ of Q^n such that F is a component of the intersection $F_{i_1} \cap \cdots \cap F_{i_r}$), and G_F is the subgroup determined by $\lambda(F_{i_1}), ..., \lambda(F_{i_r})$. Then, up to equivariant homeomorphism the quotient space $E/\sim = M^{dn}(\lambda, \xi)$ reproduces M^{dn}. It should be pointed out that all possible locally standard G_d^n-manifolds with Q^n as orbit space can be constructed in the above way.

This reconstructions of locally standard G_d^n-manifolds lead to the following classification result.

Theorem 7.2 (cf. [LM]). *Let $M^{dn}(\lambda_1, \xi_1)$ and $M^{dn}(\lambda_2, \xi_2)$ be two locally standard G_d^n-manifolds over a compact nice manifold Q^n with corners and boundary. Then $M^{dn}(\lambda_1, \xi_1)$ and $M^{dn}(\lambda_2, \xi_2)$ are equivariantly homeomorphic if and only if there is an automorphism h of Q^n such that*

(1) *$\lambda_1 = \lambda_2 \circ \bar{h}$, where $\bar{h}$ induced by h is an automorphism of all facets of Q^n.*
(2) *$\xi_1 = h^*(\xi_2)$, i.e., ξ_1 is the pullback of ξ_2 via h.*

Proof. If $M^{dn}(\lambda_1, \xi_1)$ is equivariantly homeomorphic to $M^{dn}(\lambda_2, \xi_2)$, then there is an equivariant homeomorphism $H\colon M^{dn}(\lambda_1, \xi_1) \to M^{dn}(\lambda_2, \xi_2)$ and it is easy to see that the automorphism of Q induced from H is the desired h in the theorem.

Conversely, suppose that there is an automorphism h of Q^n such that $\lambda_1 = \lambda_2 \circ \bar{h}$ and $\xi_1 = h^*(\xi_2)$. Then there is a bundle map $\hat{h}\colon \xi_1 \to \xi_1$ which covers h, and $\hat{h}$ descends to a map H from $M^{dn}(\lambda_1, \xi_1)$ to $M^{dn}(\lambda_2, \xi_2)$ because $\lambda_1 = \lambda_2 \circ \bar{h}$. It is not difficult to see that H is an equivariant homeomorphism. $\square$

7.2. **Betti numbers and h-vector.** The notion of the h-vector play essential important roles in the combinatorial theory of polytopes, while the notion of Betti numbers is also so in the topology of manifolds. Davis-Januszkiewicz theory indicates that the Dehn-Sommerville relations for the h-vectors and the Poincaré duality for the Betti numbers are essentially consistent in the setting of small covers and quasi-toric manifolds.

Let P^n be an n-dimensional simple convex polytope. Then its dual P^* is a simplicial polytope, and in particular, the boundary ∂P^* denoted by K_P is a finite simplicial complex of dimension $n-1$. For $0 \leq i \leq n-1$, by f_i one denotes the number of all i-faces in K_P. Then the vector $(f_0, f_1, ..., f_{n-1})$ is called the *f-vector* of P^n, denoted by $\mathbf{f}(P^n)$. Then the *h-vector* (denoted by $\mathbf{h}(P^n)$) of P^n is an integer vector $(h_0, h_1, ..., h_n)$ defined from the equation

$$h_0 t^n + \cdots + h_{n-1} t + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \cdots + f_{n-1}$$

with the following relations (cf. [BP]):

- $h_i = \sum_{k=0}^{i} (-1)^{i-k} \binom{n-k}{n-i} f_{k-1}$, $i = 0, 1, ..., n$
- $f_{n-1-i} = \sum_{k=i}^{n} \binom{k}{i} h_{n-k}$, $i = 0, ..., n$ where $f_{-1} = 1$.

The following are famous Dehn-Sommerville relations for h-vector.

Theorem 7.3 (Dehn-Sommerville relations)**.** *The h-vector $\mathbf{h}(P^n)$ of P^n is symmetric, i.e.,*

$$h_i = h_{n-i}, i = 0, 1, ..., n.$$

Proof. Here is a proof of Morse-theoretical argument (cf. [BP, Theorem 1.20]). Regard P^n as a polytope in $\mathbb{R}^n$. Choose a linear function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which is generic in the sense that the values of ϕ on the vertices of P^n are different. Associating with ϕ, there is a vector $w \in \mathbb{R}^n$ such that $\phi(x) =< w, x >$, the standard inner product on $\mathbb{R}^n$. Since ϕ is generic, w is parallel to no edge of P^n, so that ϕ can be regarded as a height function on P^n. Using ϕ, one may make the 1-skeleton of P^n a directed graph by orienting each edge in such a way that ϕ increases along it. For each vertex v of P^n, the number of incident edges that point towards v is defined as its index, denoted by $\mathrm{ind}_w(v)$.

Now let $N_w(i)$ denote the number of those vertices in P^n of index i. We shall prove that $N_w(i) = h_{n-i}$. Actually, for a k-face F^k of P^n, the maximum of ϕ on F^k determines a unique top vertex v_F of F^k (of course, the minimum of ϕ determines also a unique bottom vertex of F^k). Since P^n is simple, there are exactly k edges of F^i which meet at v_F , so $\mathrm{ind}_w(v_F) \geq k$. On the other hand, it is easy to see that the number of faces in P^n with top vertex of index $i(\geq k)$ is exactly $\binom{i}{k}$. Thus, the number f_{n-1-k} of all k-faces can be calculated as

$$f_{n-1-k} = \sum_{i=k}^{n} \binom{i}{k} N_w(i).$$

However, $f_{n-1-k} = \sum_{i=k}^{n} \binom{i}{k} h_{n-i}$. Thus $N_w(i) = h_{n-i}$. Consider another height function $\phi' : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\phi'(x) =< -w, x >$. Then, it is easy to see that for any vertex v in P^n, $\mathrm{ind}_w(v) = n - \mathrm{ind}_{-w}(v)$. Therefore,

$$h_i = N_{-w}(n-i) = N_w(i) = h_{n-i}.$$

□

Theorem 7.4 (Davis-Januszkiewicz)**.** *Let $\pi : M^{dn} \longrightarrow P^n$ be a locally standard G_d^n-manifold over a simple convex polytope P^n. Then*

(1) *When $d = 1$, write $b_i = \dim H^i(M^n; \mathbb{Z}_2)$. Then*

$$\mathbf{h}(P^n) = (h_0, ..., h_n) = (b_0, ..., b_n).$$

(2) *When $d = 2$, $H^{\mathrm{odd}}(M^{2n}; \mathbb{Z}) = 0$ and $H^{\mathrm{even}}(M^{2n}; \mathbb{Z})$ is free abelian. Let $b_{2i} = \mathrm{rank} H^{2i}(M^{2n}; \mathbb{Z})$. Then*

$$\mathbf{h}(P^n) = (b_0, b_2, ..., b_{2n}).$$

Outline of Proof. It suffices to show that M^{dn} has a cell structure which is perfect in the sense of Morse theory, with one cell for each vertex of P^n and with exactly h_i cells of dimension di.

- As in the proof of Theorem 7.3, choose a linear function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\phi(x) =< w, x >$ such that ϕ is a generic height function on P^n. For each vertex v of P^n, let F_v be the smallest face of P^n which contains the inward pointing edges incident to v. It is easy to see that $\dim F_v = \mathrm{ind}_w(v)$. Let $\widehat{F}_v$ denote the union of the relative interiors of those faces F, whose top vertex is v. Then $\widehat{F}_v$ is exactly F_v with some faces not incident to v deleted, and it is diffeomorphic to the positive cone $\mathbb{R}_{\geq 0}^{\mathrm{ind}_w(v)}$. Note that the number of those vertices of index i is h_i.
- For each vertex v of P^n, set $e_v = \pi^{-1}(\widehat{F}_v)$ and $M_v = \pi^{-1}(F_v)$. Since $\widehat{F}_v$ is diffeomorphic to $\mathbb{R}_{\geq 0}^{\mathrm{ind}_w(v)}$, e_v is equivariantly homeomorphic to $\mathbb{F}_d^{\mathrm{ind}_w(v)}$ so e_v is a cell of dimension $d\ \mathrm{ind}_w(v)$. Clearly, the closure of e_v is M_v. As mentioned before, $M_v = \pi^{-1}(F_v)$ is also a locally standard manifold over F_v. Thus, the cell structure is perfect. Note when $d = 2$, clearly all the cells are of even dimension. □

Remark 27. For a small cover $\pi : M^n \longrightarrow P^n$, the induced action $\mathbb{Z}_2^n \curvearrowright H^*(M^n; \mathbb{Z}_2)$ is trivial. This is because each cell e_v in the cell structure of M^n above is $\mathbb{Z}_2^n$-stable and its closure is a mod 2 cycle.

Example 7.5. We know from Examples 7.1 and 7.2 that $\mathbb{R}P^n$ is a small cover over an n-simplex Δ^n and $\mathbb{C}P^n$ is a quasi-toric manifold over an n-simplex Δ^n. Since $\mathbf{h}(\Delta^n) = (1, ..., 1)$, one has that $b_i(\mathbb{R}P^n) = \dim H^i(\mathbb{R}P^n; \mathbb{Z}_2) = 1$ for $0 \leq i \leq n$, and

$$b_i(\mathbb{C}P^n) = \text{rank} H^i(\mathbb{C}P^n; \mathbb{Z}) = \begin{cases} 1 & \text{if } i \text{ is even with } 0 \leq i \leq 2n \\ 0 & \text{otherwise.} \end{cases}$$

7.3. **Stanley-Reisner face ring and equivariant cohomology.** Stanley-Reisner face ring is a basic combinatorial invariant, and equivariant cohomology is an essential invariant in the theory of transformation groups. Davis-Januszkiewicz theory indicates that these two kinds of invariants are also essentially consistent in the setting of small covers and quasi-toric manifolds.

Definition 7.3. Let K be a simplicial complex with vertex set $\{v_1, ..., v_m\}$, and let R be a commutative ring. The *Stanley-Reisner face ring* of K, denoted by $R(K)$, is defined as

$$R[v_1, ..., v_m]/I$$

where $R[v_1, ..., v_m]$ is a polynomial ring over R with the v_i's as indeterminates, and I is a homogenous ideal generated by all sequence free monomials of the form $v_{i_1} \cdots v_{i_s}$ with $(v_{i_1}, ..., v_{i_s}) \notin K$.

Remark 28. Let P^n be a simple convex polytope with m facets $F_1, ..., F_m$. Then K_P (the boundary complex of the dual P^* of P^n) has m vertices which correspond to $F_1, ..., F_m$. The Stanley-Reisner face ring of P^n, denoted by $R(P^n)$, is defined as $R(K_P)$. Furthermore, we can write $R(P^n)$ as follows:

$$R(P^n) = R[F_1, ..., F_m]/I$$

where $I = (F_{i_1} \cdots F_{i_s} | F_{i_1} \cap \cdots \cap F_{i_s} = \emptyset)$.

Example 7.6. Let P^n be an n-simplex Δ^n with $n+1$ facets $F_1, ..., F_{n+1}$. Then

$$R(\Delta^n) = R[F_1, ..., F_{n+1}]/(F_1 \cdots F_{n+1}).$$

Example 7.7. Let $F_1, ..., F_{2n}$ be $2n$ facets of an n-cube I^n with $F_i \cap F_{i+n} = \emptyset, i = 1, ..., n$. Then

$$R(I^n) = R[F_1, ..., F_{2n}]/(F_i F_{i+n} | i = 1, ..., n).$$

Theorem 7.5 (Davis-Januszkiewicz). *Let $\pi : M^{dn} \longrightarrow P^n$ be a locally standard G_d^n-manifold over a simple convex polytope P^n. Then*

$$H^*_{G_d^n}(M^{dn}; R_d) \cong R_d(P^n).$$

We would like to refer to the original paper of Davis-Januszkiewicz[DJ] for a detail proof of Theorem 7.5. Here we only give a outline of proof as follows:

- The Borel construction $M^{dn}_{G_d^n}$ is independent of the existence of G_d^n-manifold, and its homotopy type only depends upon P^n. Based on this, denote $M^{dn}_{G_d^n}$ by $B_d P$.

- P^n is a cubical complex, and it is decomposed into cubes indexed by the simplices of a simplicial complex K. Regarding the k-cube as the orbit space of a G_d^k-action on the dk-disk $D^{dk} = \{(x_1, ..., x_k) \in \mathbb{F}_d^k | |x_i| \leq 1\}$. For each $(k-1)$-simplex $\sigma \in K$, letI_σ be the indexed k-cube in P, and G_σ the dk-disk with G_d^k-action, and let $B_dI_\sigma = EG_d^k \times_{G_d^k} D_\sigma$. For a face τ of σ, BI_σ is identified with a subset of B_dI_σ. Thus, B_dP is formed by the B_dI_σ's. Note that an easy observation shows that if $d = 2$, then B_dP is simply connected.
- Consider the spacial case in which P is dual to a simplex or to a boundary of the simplex, and show that Theorem 7.5 holds in this case.
- Use the induction on the dimension of K. If $\dim K = 0$, then K is a disjoint union of vertices $v_1, ..., v_m$ and P is the cone on K, so $M_{G_d^n}^{dn}$ is a bouquet of m copies of $\mathbb{F}_dP^\infty$. Thus, $H_{G_d^n}^*(M^{dn}; R_d)$ is isomorphic to $R_d[v_1, ..., v_m]/I$ where I is the ideal generated by all square free monomials in more than one variable, and so the theorem holds in this case. Suppose inductively that the theorem holds if $\dim K \leq n-1$. Consider the case $\dim K = n$. Note the theorem holds for the $(n-1)$-skeleton of K by inductive hypothesis. Then the theorem follows by adding n-simplices one at a time to the $(n-1)$-skeleton and using the Mayer-Vietoris sequence.

Corollary 7.1. *$H_{G_d^n}^*(M^{dn}; R_d)$ and $R_d(P^n)$ have the same Poincaré series. Furthermore, by* [S2, Theorem II.1.4]

$$P(H_{G_d^n}^*(M^{dn}; R_d), t) = P(R_d(P^n), t) = \frac{h_0 + h_1t^d + \cdots + h_nt^{dn}}{(1-t^d)^n}.$$

Example 7.8. Let us return to Example 6.5. The S^2 with the S^1-action defined by $(g, (z, y)) \longmapsto (gz, y)$ is a quasi-toric manifold over a 1-simplex Δ^1. Let F_1, F_2 be two facets of Δ^1. Then

$$H_{S^1}^*(S^2; \mathbb{Z}) = \mathbb{Z}[F_1, F_2]/(F_1F_2).$$

Example 7.9. We know from Examples 7.1 and 7.2 that $\mathbb{R}P^n$ is a small cover over an n-simplex Δ^n and $\mathbb{C}P^n$ is a quasi-toric manifold over an n-simplex Δ^n. Thus,

$$H_{\mathbb{Z}_2^n}^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2(\Delta^n) = \mathbb{Z}_2[F_1, ..., F_{n+1}]/(F_1 \cdots F_{n+1}).$$

and

$$H_{T^n}^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}(\Delta^n) = \mathbb{Z}[F_1, ..., F_{n+1}]/(F_1 \cdots F_{n+1}).$$

In 2008, Masuda [M1] showed that for a locally standard G_d^n-manifold over a simple convex polytope P^n, its equivariant cohomology $H_{G_d^n}^{dn}(M^{dn}; R_d)$ as a $H^*(BG_d^n; R_d)$-algebra is a complete invariant of the equivariant homeomorphism type of M^{dn}.

Theorem 7.6 (Masuda, cf. [M1])**.** *Suppose that M_1^{dn} and M_2^{dn} are two locally standard G_d^n-manifolds over a simple convex polytope P^n. Then M_1^{dn} and M_2^{dn} are equivariantly homeomorphic if and only if $H_{G_d^n}^*(M_1^{dn}; R_d)$ and $H_{G_d^n}^*(M_1^{dn}; R_d)$ are isomorphic as $H^*(BG_d^n; R_d)$-algebras.*

We would like to refer to the Masuda's paper for a detail proof of Theorem 7.6.

7.4. **Ordinary cohomology.** Let $\pi : M^{dn} \longrightarrow P^n$ be a locally standard G_d^n-manifold over a simple convex polytope P^n with m facets $F_1, ..., F_m$. Now let us look at its ordinary cohomology $H^*(M^{dn}; R_d)$.

The projection $p : M^{dn}_{G_d^n} = B_dP \longrightarrow BG_d^n$ is a Serre fibration with fiber $EG_d^n \times M^{dn} \simeq M^{dn}$. Then the Serre spectral sequence of this fibration has E_2-term

$$E_2^{p,q} = H^p(BG_d^n; H^q(M^{dn}; R_d)).$$

It is well-known that BG_d^n is simply connected if $d = 2$ and $\pi_1(BG_d^n) = \mathbb{Z}_2^n$ if $d = 1$, and we know from Remark 27 that $\mathbb{Z}_2^n$ trivially acts on $H^*(M^n; \mathbb{Z}_2)$. Thus

$$E_2^{p,q} = H^p(BG_d^n; R_d) \otimes H^q(M^{dn}; R_d).$$

Furthermore, the Poincaré series $P(E_2, t)$ of E_2 is

$$P(E_2, t) = P(H^*(BG_d^n; R_d), t) \cdot P(H^*(M^{dn}; R_d)) = \frac{h_0 + h_1t^d + \cdots + h_nt^{dn}}{(1 - t^d)^n}.$$

By Corollary 7.1, $P(E_2, t) = P(H^*_{G_d^n}(M^{dn}; R_d), t)$ so $P(E_2, t) = P(E_\infty, t)$. Therefore, one has that $E_2 = E_\infty$ since E_∞ is generally an iterated subquotient of E_2. This means that M^{dn} is totally non-homologous to zero in the Borel construction $M^{dn}_{G_d^n}$. In other words, $j^* : H^*_{G_d^n}(M^{dn}; R_d) \longrightarrow H^*(M^{dn}; R_d)$ is surjective where $j : M^{dn} \hookrightarrow M^{dn}_{G_d^n}$ is the inclusion of fiber, so that one may obtain the following short exact sequence

$$0 \longrightarrow H^d(BG_d^n; R_d) \xrightarrow{p^*} H^d_{G_d^n}(M^{dn}; R_d) \xrightarrow{j^*} H^d(M^{dn}; R_d) \longrightarrow 0.$$

Now let us extend $\lambda : \mathcal{F}(P^n) = \{F_1, ..., F_m\} \longrightarrow R_d^n$ to $\widetilde{\lambda} : R_d^m \longrightarrow R_d^n$ by regarding $\{F_1, ..., F_m\}$ as the basis $\{e_1, ..., e_m\}$. Then $\widetilde{\lambda} : R_d^m \longrightarrow R_d^n$ is linear and surjective, so $\widetilde{\lambda}$ can be regarded as an $n \times m$-matrix (λ_{ij}). Since $H_d(BG_d^n; R_d) = H_d(B_dP; R_d) = R_d^m$ and $H_d(BG_d; R_d) = R_d^n$, one has that $p_* : H_d(B_dP; R_d) \longrightarrow H_d(BG_d^n; R_d)$ can be identified with $\widetilde{\lambda} : R_d^m \longrightarrow R_d^n$. Then $p^* : H^d(BG_d^n; R_d) \longrightarrow H^d(B_dP; R_d)$ is identified with the dual map $\widetilde{\lambda}^* : R_d^{n*} \longrightarrow R_d^{m*}$, where $\widetilde{\lambda}^* = \widetilde{\lambda}^\top$ as matrix. Therefore, column vectors of $\widetilde{\lambda}^*$ can be understood as linear combinations of $F_1, ..., F_m$ in $R_d(P^n) = R_d[F_1, ..., F_m]/I$. Write

$$\lambda_i = \lambda_{i1}F_1 + \cdots + \lambda_{im}F_m.$$

Let J be the homogeneous ideal $(\lambda_1, ..., \lambda_n)$ in $R_d[F_1, ..., F_m]$ and let $\bar{J}$ be its image in the face ring $R_d(P^n)$. One sees from the short exact sequence above that $\bar{J}$ is in $\ker j^*$. Since j^* is surjective, j^* induces a surjection $R_d(P^n)/\bar{J} \longrightarrow H^*(M^{dn}; R_d)$.

Since $E_2 = E_\infty$ (i.e., the spectral sequence degenerates),

$$H^*(B_dP; R_d) \cong H^*(BG_d^n; R_d) \otimes H^*(M^{dn}; R_d)$$

so $p^* : H^*(BG_d^n; R_d) \longrightarrow H^*(B_dP; R_d)$ is a monomorphism and then $\bar{J}$ is identified with the image $p^*(\bar{J})$. This argument gives the following

Theorem 7.7 (Davis-Januszkiewicz). *Let* $\pi : M^{dn} \longrightarrow P^n$ *be a locally standard* G_d^n*-manifold over a simple convex polytope* P^n*. Then*

$$H^*(M^{dn}; R_d) \cong R_d[F_1, ..., F_m]/I + J.$$

Example 7.10. We know from Examples 7.1 and 7.2 that $\mathbb{R}P^n$ is a small cover over an n-simplex Δ^n and $\mathbb{C}P^n$ is a quasi-toric manifold over an n-simplex Δ^n. Let $F_1, ..., F_{n+1}$ be $n+1$ facets of Δ^n. It is easy to check that $J = (F_1 = F_{n+1}, ..., F_n = F_{n+1})$. Thus,

$$H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[F_1, ..., F_{n+1}]/(F_1 \cdots F_{n+1}) + J \cong \mathbb{Z}_2[a]/(a^{n+1})$$

with $\deg a = 1$. Similarly, $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[b]/(b^{n+1})$ with $\deg b = 2$.

Exercise 15. Calculate $H^*((S^1)^n; \mathbb{Z}_2)$ and $H^*((S^2)^n; \mathbb{Z})$ (cf. Exercise 14).

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Institute of Mathematics, School of Mathematical Sciences, Fudan University, Shanghai, 200433, P.R.China.
E-mail address: `zlu@fudan.edu.cn`