On Uniform Recurrence of a Direct Product

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Abstract

A direct product of two words is a naturally defined word on the alphabet of pairs of symbols. We introduce the class URP of uniformly recurrent words such that a direct product of any its member and each uniformly recurrent word is also uniformly recurrent. This class is proved to contain all fixed points of expanding binary symmetric morphisms. In particular, the Thue-Morse word is in URP.

1 Introduction

Let $\Sigma = \Sigma_s = \{0, 1, \dots, s-1\}$ be a finite alphabet, and $x = x_0x_1 \cdots \in \Sigma^{\mathbb{N}_0}$ be an infinite word on Σ with indices in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

A word u is called a *factor* or a *subword* of a word $v = v_0v_1\dots$ if $u = v_iv_{i+1}\dots v_{i+n}$ for some i and n. In this case we say the word u *occur* in v at the position i. A subword occurring in a word at position 0 is called a *prefix* of that word. The *length* of a word $u = u_0u_1\dots u_{n-1}$ is n and is denoted by $|u|$.

An infinite word x is called *uniformly recurrent* if for every n there exists $m = R_x(n)$ such that all subwords of x of length n occur in each subword of x of length m.

A notion of the uniform recurrence (see [4]) originally came from dynamic. Later, it was widely adopted in other areas of mathematics like combinatorial number theory, factorial language theory and computer science.

Let us define a *direct product* of two infinite words (or words of the same length) $x = x_0x_1\dots$ on Σ and $y = y_0, y_1\dots$ on Δ as the word $x \otimes y = \langle x_0, y_0\rangle\langle x_1, y_1\rangle\dots$ on the alphabet $\Sigma \times \Delta$.

An interesting question is: When a product of uniformly recurrent words is also uniformly recurrent?

This paper is devoted to another question close to the previous one. It is: When a product of the word x and each uniformly recurrent word also is uniformly recurrent?

Let us define the class URP as a class of uniformly recurrent words satisfying this property.

The class URP contains all periodic words (see Proposition 2) and is closed under direct multiplication (see Proposition 1). We prove that it contains some non-periodic words.

Let us denote the set of all finite words on Σ by Σ^*, the set of all non-empty finite words by Σ^+ and the set of all words of length n by Σ^n.

A *morphism* $\varphi : \Sigma^* \to \Delta^*$ is a map that obeys the identity $\varphi(xy) = \varphi(x)\varphi(y)$ for all words $x, y \in \Sigma^*$. A *binary* morphism is a $\Sigma_2^* \to \Sigma_2^*$ morphism. A morphism $\varphi : \Sigma^* \to \Delta^*$ is an *expanding* one if for each $a \in \Sigma$ the inequality $|\varphi(a)| > 1$ takes place.

A morphism is called *uniform* (*m-uniform*) if images of all letters are of the same length (equal to m). A uniform morphism $\varphi : \Sigma^* \to \Sigma^*$ is called *symmetric* if each word $\varphi(i)$ is a result of the symbol-by-symbol addition modulo $|\Sigma|$ of the word $ii \dots i$ to $\varphi(0)$.

In the rest of the paper, when speaking of a symmetric morphism, we mean an expanding symmetric morphism.

Any expanding morphism $\varphi : \Sigma^* \to \Sigma^*$ generates the mapping $\Sigma^{\mathbb{N}_0} \to \Sigma^{\mathbb{N}_0}$ that maps an infinite word $x = x_0 x_1 \dots$ to the word $\varphi(x_0)\varphi(x_1)\dots$. We denote it by the same letter φ for convenience. When $\varphi(a)$ begins with a for some $a \in \Sigma$, this mapping has fixed points, i.e. words satisfying $x = \varphi(x)$. These are exactly the words can be obtained as limits $\lim_{n\to\infty} \varphi^n(a) = \varphi^\omega(a)$ for those $a \in \Sigma$.

Example 1. The well-known Thue-Morse word $x_{TM} = 01101001100101\dots$ is a fixed point of the binary symmetric morphism φ_{TM} defined by $\varphi_{TM}(0) = 01$ and $\varphi_{TM}(1) = 10$. It can be obtained as a limit $\varphi_{TM}^\omega(0)$.

The main result of this paper is that URP contains fixed points of binary expanding symmetric morphisms.

2 Preliminaries

A *language* is a subset of Σ^*. A language is *factorial* if it contains all subwords of its words. Let us denote the language of factors of a word x by F_x and the set of symbols occurring in x by Σ_x.

Let us denote the set of infinite words with all factors in a language L by Ω_L.

In current terms the well-known Koenig lemma can be formulated as follows:

Lemma 1. *(König's lemma) If language L is infinite then Ω_L is non-empty.*

For a uniformly recurrent word x let us denote the *recurrence function* of x by R_x. It means that $R_x(n)$ is a minimal number such that all subwords of x of length n occur in each subword of x of length $R_x(n)$.

A subword u of an infinite word x is said to occur in x *non-uniformly* if x contains arbitrary long subwords where u does not occur. So, an infinite word is not uniformly recurrent if and only if it has a prefix occurring in it non-uniformly.

Let us give an example of the product of two uniformly recurrent words being not uniformly recurrent.

Example 2. Let the morphism φ be the one defined by $\varphi(0) = 011$ and $\varphi(1) = 101$. Let us consider words $x = \varphi^{\omega}(0) = 011101101101011101\ldots$ and $y = \varphi^{\omega}(1) = 101011101011101101$. They obviously are uniformly recurrent but the word $x \otimes y$ is not.

Indeed, the word $x \otimes y$ is a fixed point beginning with $\langle 0,1\rangle$ of the morphism φ' defined by

$$\begin{aligned}\varphi'(\langle 0,0\rangle) &= \langle 0,0\rangle\langle 1,1\rangle\langle 1,1\rangle\ ,\ \varphi'(\langle 1,1\rangle) = \langle 1,1\rangle\langle 0,0\rangle\langle 1,1\rangle\ ,\\ \varphi'(\langle 0,1\rangle) &= \langle 0,1\rangle\langle 1,0\rangle\langle 1,1\rangle\ ,\ \varphi'(\langle 1,0\rangle) = \langle 1,0\rangle\langle 0,1\rangle\langle 1,1\rangle\ .\end{aligned}$$

From the definition of φ' we see that there are arbitrary long subwords of $x \otimes y$ which wholly consist of symbols $\langle 0,0\rangle$ and $\langle 1,1\rangle$, i.e. letter $\langle 0,1\rangle$ occurs in $x \otimes y$ non-uniformly.

A set $I \subseteq \mathbb{N}_0$ is called *thick* if it contains arbitrary long intervals (subsets of a form $\{a, a+1, \ldots, a+n\}$). For example, if a subword occurs in an infinite word non-uniformly then the set of positions where it does not occur at is thick.

Let $x = x_0 x_1 \ldots$ be an infinite word on Σ and $I = \{i_0 < i_1 < \ldots\}$ be a subset of $\mathbb{N}_0$. The notation x_I is used for the word $x_{i_0} x_{i_1} \ldots$. The notation $x|_I$ is used for the restriction of the function $x : \mathbb{N}_0 \to \Sigma$ to I.

Infinite words x and y are *equivalent on the set* $I \subseteq \mathbb{N}_0$ if there exists a bijection $h : \Sigma_{x_I} \leftrightarrow \Sigma_{y_I}$ such that $h(x|_I) \equiv y|_I$. In other words h is a renaming of symbols such that for each $i \in I$ the equality $h(x_i) = y_i$ is true. In this case the notation $x \simeq_I y$ is used (or $x \simeq y$ if $I = \mathbb{N}_0$).

Proposition 1. *The class URP is closed under direct multiplication.*

Proof. Let infinite words x and y be in URP. The word $(x \otimes y) \otimes z \simeq x \otimes (y \otimes z)$ is uniformly recurrent for each uniformly recurrent word z. So $x \otimes y$ is in URP. ☐

Let us define a *block representation of order* m of an infinite word x on alphabet Σ as a word $x_{[m]} = x_{[m],0} x_{[m],1} \dots$ on Σ^m such that $x_{[m],i} = x_{im} x_{im+1} \dots x_{im+m-1}$.

An *arithmetical subsequence* of an infinite word x is an infinite word of the form $x_k x_{k+d} x_{k+2d} \dots$ for arbitrary initial positions $k \geqslant 0$ and differences $d \geqslant 1$.

The following lemma is folklore, the proof can be found for example in [1].

Lemma 2. *An arithmetical subsequence of a uniformly recurrent word is uniformly recurrent.*

We use its corollary.

Corollary 1. *A block representation of a word is uniformly recurrent if and only if the word is uniformly recurrent.*

Proof. "If" part. Let an infinite word $x = x_0 x_1 \dots$ on alphabet Σ be uniformly recurrent. Let us consider a word $x' = x'_0 x'_1 \dots$ on alphabet Σ^m such that $x'_i = x_i x_{i+1} \dots x_{i+m-1}$. Obviously, x' is uniformly recurrent with a recurrence function $R_{x'}(n) = R_x(n + m - 1)$. The word $x_{[m]} = x'_0 x'_m x'_{2m} \dots$ is an arithmetical subsequence of x'. Hence, by Lemma 2 it is uniformly recurrent.

"Only if" part. A word x can be obtained from the word $x_{[m]}$ by applying the evident m-uniform morphism. So x is uniformly recurrent with a recurrence function that obeys the inequality $R_x(n) \leqslant R_{x_{[m]}}(\lceil \frac{n}{m} \rceil + 1)$. ☐

An infinite word $x = x_0 x_1 \dots$ is *periodic* of a minimal period m if m is minimal such that $x_i = x_{i+m}$ for each i.

Proposition 2. *The class URP contains all periodic words.*

Proof. Let x be a periodic infinite word of a period m. Then for each uniformly recurrent word y we have $(x \otimes y)_{[m]} \simeq y_{[m]}$. Using Corollary 1 of Lemma 2 we obtain that $x \otimes y$ is uniformly recurrent. So, URP contains x. ☐

The *left shift operator* denoted by T maps an infinite word $x = x_0 x_1 \dots$ to the word $Tx = x_1 x_2 \dots$.

3 Orbits of fixed points of binary symmetric morphisms

This section is wholly devoted to proving of the next lemma:

Lemma 3. *Let x be a fixed point of an expanding binary symmetric morphism and $y \in \Omega_{F_x}$. Then for each thick set $I \subseteq \mathbb{N}_0$ the statement $x \simeq y$ follows from $x \simeq_I y$.*

Before giving the proof we should bring some more definitions.

A fixed point x of an m-uniform morphism φ with $m > 1$ is called *d-circular regarding φ* or *circular with a synchronization delay d regarding φ* (we will write just *d-circular* or *circular*) if each its subword of length at least d occurs in it at the same position modulo m. This notion was introduced in [5] and [2].

Example 3. The Thue-Morse word $x_{TM} = 01101001100101\ldots$ is d-circular with $d \leqslant R_{x_{TM}}(2)$ because each its subword of length at least $R_{x_{TM}}(2)$ contains the subword 00 occurring in x_{TM} only at odd positions.

In [3] the criterion for a fixed point of a morphism to be circular was formulated and proved. A straightforward corollary of that result is

Lemma 4. *If a fixed point of an expanding binary symmetric morphism is not circular then it is periodic.*

In the case of a periodic word Lemma 3 is obviously true. So let us speak more about circularity. Next statements follow immediately from definitions.

Proposition 3. *If subwords $u \simeq v$ of a d-circular fixed point x of a binary m-uniform symmetric morphism are of the length at least d then they occur in x at equal positions modulo m.*

Proposition 4. *If a word u is a prefix of the word $\varphi(v)$, where $v \in F_x$ for a d-circular fixed point x of a binary m-uniform symmetric morphism φ, and $|u| \geqslant d$, then u occurs in x at positions divisible by m.*

Another useful statement will be proved.

Proposition 5. *Let x be a d-circular fixed point of an m-uniform symmetric morphism φ and $y \in \Omega_{F_x}$. Then a unique word $z = z(y) \in \Omega_{F_x}$ and a unique number $t = t(y) < m$ such that $y = T^t\varphi(z)$ exist.*

Proof. The key observation is $\varphi(T^n x) = T^{mn}\varphi(x) = T^{mn}x$.

Let us consider a sequence $(n_i)_{i\in\mathbb{N}}$ of numbers such that $T^{n_i}x \to y$. Because of circularity of x, there are i_0 and $t < m$ such that for all $i \geqslant i_0$ we have $n_i = t \mod m$. So, the sequence $(T^{\frac{n_i-t}{m}}x)_{i\geqslant i_0}$ of words is well-defined.

Consider its subsequence $(T^{\frac{n_i-t}{m}}x)_{i\in I}$ which converges to some word z. Obviously, $z \in \Omega_{F_x}$ and, by observation,

$$T^t\varphi(z) = T^t\varphi(\lim_{i\in I} T^{\frac{n_i-t}{m}}x) = T^t \lim_{i\in I} \varphi(T^{\frac{n_i-t}{m}}x) = T^t \lim_{i\in I} T^{n_i-t}x = y \ .$$

Now suppose that some $z' \in \Omega_{F_x}$ and $t' < m$ satisfy $y = T^{t'}\varphi(z')$.

If we apply Proposition 4 to prefixes of $\varphi(z)$ and $\varphi(z')$ we find that $y_0y_1\dots y_{d-1}$ occurs in x at positions equal to t and t' modulo m. So, $t = t'$ by the d-circularity of x.

The morphism φ is symmetric hence $T^t\varphi(z) = T^t\varphi(z')$ implies $z = z'$. □

The main technical lemma of this part is an infinite analogue of this proposition. Let us denote for an infinite sequence $\sigma = (\sigma_i)_{i\in\mathbb{N}}$ of numbers, where $0 \leqslant \sigma_i < m$, the sum $\sum_{j=1}^{i} m^{j-1}\sigma_j$ by $S_\sigma(i)$.

Lemma 5. *Let x be a d-circular fixed point of an expanding binary m-uniform symmetric morphism φ and $y \in \Omega_{F_x}$. Then there exist a unique sequence $(\sigma_i)_{i\in\mathbb{N}}$ of numbers and a unique sequence $(y^{(i)})_{i\in\mathbb{N}}$ of words, where $0 \leqslant \sigma_i < m$ and $y^{(i)} \in \Omega_{F_x}$ for each i, such that*

$$y = T^{S_\sigma(i)}\varphi^i(y^{(i)}) \text{ for every } i \in \mathbb{N} \ . \tag{1}$$

Proof of Lemma 5. By Proposition 5, a word $y^{(1)}$ and a number σ_1 such that $y = T^{\sigma_1}\varphi(y^{(1)})$ exist and are unique. The same goes for $y^{(1)}$, there are unique $y^{(2)}$ and σ_2 such that $y^{(1)} = T^{\sigma_2}\varphi(y^{(2)})$ and so on.

We have sequences $(\sigma_i)_{i\in\mathbb{N}}$ and $(y^{(i)})_{i\in\mathbb{N}}$ such that $y = T^{\sigma_1}\varphi(y^{(1)})$ and $y^{(i)} = T^{\sigma_{i+1}}\varphi(y^{(i+1)})$ for each i, where $0 \leqslant \sigma_i < m$ and $y^{(i)} \in \Omega_{F_x}$ for each i. It is easy to see, that

$$\begin{aligned} y =& T^{\sigma_1}\varphi(y^{(1)}) = T^{\sigma_1}\varphi(T^{\sigma_2}\varphi(y^{(2)})) = T^{\sigma_1+m\sigma_2}\varphi^2(y^{(2)}) = \dots \\ &\dots = T^{S_\sigma(i)}\varphi^i(y^{(i)}) = \dots \ . \end{aligned}$$

□

We call such a sequence $(\sigma_i)_{i\in\mathbb{N}}$ a *characteristic sequence of* y and denote it by $\sigma(y)$.

Lemma 6. *Let x be a d-circular fixed point of a expanding binary m-uniform symmetric morphism φ and $y \in \Omega_{F_x}$. Then for each thick set $I \subseteq \mathbb{N}_0$ the statement $x \simeq_I y$ implies that the characteristic sequence $\sigma(y)$ consists only of* 0.

Proof. Suppose that it is not true, and let the number i be minimal such that $\sigma_i \neq 0$.

From Lemma 5 we see that a word $y' \in \Omega_{F_x}$ such that $y = T^{m^{i-1}\sigma_i}\varphi^i(y') = \varphi^{i-1}(T^{\sigma_i}\varphi(y'))$ exists and is unique. The word x is a fixed point of φ, so $x = \varphi^{i-1}(x)$.

A morphism φ is symmetric, thus $\varphi^{i-1}(T^{\sigma_i}\varphi(y')) \simeq_I \varphi^{i-1}(x)$ implies $T^{\sigma_i}\varphi(y')) \simeq_{I'} x$, where the set $I' = \{j \in \mathbb{N} \;:\; m^{i-1}j \in I\}$ is thick.

For some j the interval $B = \{jm, jm+1, \ldots, jm+d-1\}$ is a subset of I'. In this case words $u = x_B$ and $v = (T^{\sigma_i}\varphi(y'))_B$ are equivalent. So, by Proposition 3, v occurs in x at position 0 modulo m. On the other hand, Proposition 4 states that v occurs in x at position σ_i. This is a contradiction with d-circularity of x. Hence $\sigma(y)$ consists only of 0. $\square$

Let us prove the main lemma.

Proof of Lemma 3. In the case of a periodic x the lemma is obviously true so assume x to be circular. Lemma 6 states that $\sigma(y)$ consists only of 0. Lemma 5 states that for each i there is a word $y^{(i)}$ with a property $y = T^{S_\sigma(i)}\varphi^i(y^{(i)}) = \varphi^i(y^{(i)})$.

Morphism φ is symmetric and has fixed points so $y^{(i)}_0 = y_0$ for each i.

Hence, $y = \lim_{i\to\infty}\varphi^i(y_0)$ and $x \simeq y$ follows, because φ is symmetric. $\square$

We conclude this section with a remark that a little bit more general statement than Lemma 3 can be proved. Let x be a fixed point of an expanding binary symmetric morphism and $y, y' \in \Omega_{F_x}$. If $y \simeq_I y'$ for some thick set $I \subseteq \mathbb{N}_0$ then there is $t \in \mathbb{N}_0$ such that $T^t y \simeq T^t y'$.

4 URP contains fixed points of binary symmetric morphisms

First, let us prove a lemma.

Lemma 7. *If a product $x \otimes y$ of some uniformly recurrent words x and y is not uniformly recurrent and $x \simeq_I y$ for some thick set I, then there exists a word $x' \in \Omega_{F_x}$ such that $x \otimes x'$ is not uniformly recurrent and $x \simeq_I x'$.*

Proof. Let a bijection $h : \Sigma_{x_I} \leftrightarrow \Sigma_{y_I}$ be such that $h(x_i) = (y_i)$ for each $i \in I$.

The thick set I contains arbitrary large intervals of $\mathbb{N}$, in particular, larger than $R_x(1)$ and $R_y(1)$. Therefore, for each symbol a of Σ_x or Σ_y there is $i \in I$ such that $x_i = a$ or $y_i = a$. So, h is a $\Sigma_x \leftrightarrow \Sigma_y$ bijection and the uniformly recurrent word $x' = h^{-1}(y)$ is defined. By the definition of x', we have $x'|_I \equiv x|_I$.

The set I is thick thus for each n and some $i = i(n)$ it contains an interval $B = \{i, i+1, \ldots, i + R_{x'}(n) - 1\}$. Since x' is uniformly recurrent, each its subword v of length n occurs in the word x'_B. Hence, since $x'_B = x_B$, we have $v \in F_x$. It implies $x' \in \Omega_{F_x}$.

The product $x \otimes x'$ is not uniformly recurrent because $x \otimes x' \simeq x \otimes y$. $\square$

The next theorem is a main result of this paper.

Theorem 1. *A direct product $x \otimes y$ of a uniformly recurrent word y and a fixed point x of an expanding binary symmetric morphism is uniformly recurrent.*

The idea of the proof is to show that if $x \otimes y$ is not uniformly recurrent then there exists a uniformly recurrent word $x' \in \Omega_{F_x}$ such that $x \otimes x'$ is not uniformly recurrent and $x \simeq_I x'$ for some thick set I. The contradiction is that $x \otimes x'$ is uniformly recurrent in this case, since $x' \simeq x$ follows from $x' \simeq_I x$ by Lemma 3.

Proof of Theorem 1. Assume that $x = \varphi^\omega(0)$, φ is m-uniform and 1 occurs in x.

Note that $x_{[m^i]} \simeq x$ for each i because x satisfy $x = \varphi^i(x)$ as a fixed point of φ. In this case the bijection between $\Sigma_{x_{[m^i]}}$ and Σ_x that maps $\varphi^i(0)$ to 0 and $\varphi^i(1)$ to 1 is implied.

Suppose that the word $x \otimes y$ is not uniformly recurrent. Then it has a prefix p that occurs in it non-uniformly. Let k be such that $m^k \geqslant |p|$.

Let us consider the word $z = y_{[m^k]}$ which is uniformly recurrent by Corollary 1 of Lemma 2.

Suppose $(x \otimes z)_i = \langle 0, z_0 \rangle$. Since $x \otimes z \simeq x_{[m^k]} \otimes z \simeq (x \otimes y)_{[m^k]}$, it implies that p occurs in $x \otimes y$ at position im^k. Thus the symbol $\langle 0, z_0 \rangle$ occurs in $x \otimes z$ non-uniformly.

Now, let us consider the word $z' = z_{[m^l]}$, where l is such that $m^l \geqslant R_z(1)$. The word z' is uniformly recurrent by Corollary 1 of Lemma 2.

Each symbol u of $\Sigma_{z'}$ is a word $u_0 u_1 \ldots u_{m^l - 1}$ on Σ_z which contains the symbol z_0. Let us define a morphism $\psi : \Sigma_{z'}^* \to \{0,1\}^*$. Let $\psi(u) = 0$ if and only if the word $\varphi^l(0) \otimes u_0 u_1 \ldots u_{m^l-1}$ contains $\langle 0, z_0 \rangle$ and $\psi(u) = 1$ otherwise (in this case the word $\varphi^l(0) \otimes u_0 u_1 \ldots u_{m^l-1}$ contains $\langle 1, z_0 \rangle$).

Let us consider the word $y' = \psi(z')$ which obviously is uniformly recurrent. We prove that there is a thick set I such that $x \simeq_I y'$ and $x \otimes y'$ is not uniformly recurrent.

Suppose $(x \otimes y')_i = \langle 0, 0 \rangle$ for some i. In this case the word $x_{[m^l],i} \otimes z'_i = \varphi^l(0) \otimes z_{im^l} z_{im^l+1} \ldots z_{(i+1)m^l-1}$ contains the symbol $\langle 0, z_0 \rangle$ at some position $j < m^l$ by the definition of ψ. Then $(x \otimes z)_{im^l+j} = \langle 0, z_0 \rangle$.

Suppose $(x \otimes y')_i = \langle 1, 1 \rangle$ for some i. In this case we have $x_{[m^l],i} \otimes z'_i = \varphi^l(1) \otimes z_{im^l} z_{im^l+1} \dots z_{(i+1)m^l-1}$. As it was remarked, the word $z_{im^l} z_{im^l+1} \dots z_{(i+1)m^l-1}$ contains the symbol z_0 at some position $j < m^l$. By the definition of ψ, there is the symbol 1 at position j in $\varphi^l(0)$. Since the morphism φ is symmetric, $\varphi^l(1)_j$ is 0. So $(x \otimes z)_{im^l+j} = \langle 0, z_0 \rangle$.

We have proved that $\langle 0, z_0 \rangle$ occurs in $x \otimes z$ non-uniformly. By this two reasonings, it implies that the set $I \subseteq \mathbb{N}_0$ of positions i such that $(x \otimes y')_i$ is either $\langle 0, 1 \rangle$ or $\langle 1, 0 \rangle$ is a thick set.

So, $x \simeq_I y'$, where the bijection that maps 0 to 1 and 1 to 0 is implied.

Since $\langle 0, 0 \rangle = \langle x_0, y'_0 \rangle$ occurs in $x \otimes y'$ non-uniformly, $x \otimes y'$ is not uniformly recurrent.

By Lemma 7 applied to x,y' and I, there exists a word $x' \in \Omega_{F_x}$ such that $x \simeq_I x'$ and $x \otimes x'$ is not uniformly recurrent. By Lemma 3, we have $x \simeq x'$, hence $x \otimes x'$ is uniformly recurrent and this is a contradiction. So $x \otimes y$ is uniformly recurrent. □

5 Conclusion

We introduced a new class URP of words satisfying the “good” property and proved that it contains some non-trivial words. But the main question about unform recurrence of a direct product is still far from a solution.

Concerning the URP class, in subsequent papers we will prove that it contains almost all Sturmian words (and some its generalizations) and words constructed from another URP words by some “Toeplitzation”.

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